Vainshtein mechanism in modified gravity theories

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+ $\alpha$
Introduction

Accelerated expansion of the universe

- Implication of cosmological constant?
  \[ G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \]

- Observationally, \( \Lambda \) is fine!

- Cosmological constant problem
  \(~120 \text{ orders} of magnitude differences\)

Can *modification of gravity* solve this puzzle???
Modification of gravity

\[ G_{\mu\nu} \approx 8\pi G T_{\mu\nu} \]

In order to explain an accelerated expansion of the universe, **ONLY** long range modification is needed.
Modification of gravity

Example: Brans-Dicke theory  
Brans & Dicke (1961)

\[ S = \int d^4 x \sqrt{-g} \left[ \varphi R - \frac{\omega_{BD}}{\varphi} (\nabla \varphi)^2 \right] \]

- Self-accelerating solution
  \[ \omega_{BD} < -\frac{3}{2} \]

- Solar system constraint
  \[ \omega_{BD} > 4 \times 10^4 \]

Inconsistent with \textit{long} range and \textit{short} range
How can modification of gravity reduce General relativity at short distances??
Galileon theory

✓ Galileon (Nicolis et al. ’09)

\[ \mathcal{L} \supset (\partial \varphi)^2 \Box \varphi \]

second derivative with respect to space-time

✓ Galileon term contains the second derivative term, but ...

coupling between scalar and curvature

\[ \text{EOM} \supset (\Box \varphi)^2 - (\nabla_\mu \nabla_\nu \varphi)^2 - \Box_{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi \]

No higher-order derivative terms in EOM !!
Vainshtein mechanism

✓ Galileon example
(Deffayet et al. ’10)

\[ \mathcal{L}_\phi = -\frac{1}{2} (\partial \phi)^2 - \frac{r_c^2}{2M_{Pl}} (\partial \phi)^2 \Box \phi \]

Vainshtein radius
\[ r_V \sim (r_s r_c^2)^{1/3} \]

Solar system scale
Horizon scale

“Nonlinear”

\[ F_\phi \sim \frac{GM}{r^2} \left( \frac{r}{r_V} \right)^{3/2} \ll \frac{GM}{r^2} \]

φ is effectively weakly coupled to matter

“Linear”

\[ F_\phi \sim \frac{GM}{r^2} \]

φ is strongly coupled to matter

Inside the Vainshtein radius, general relativity can be recovered

self-accelerating solution
\[ r_c \sim \mathcal{O}(H_0^{-1}) \]
Horndeski theory

Horndeski found the most general Lagrangian whose EOM is second-order differential equation for $\phi$ and $g_{\mu\nu}$ (also known as **Generalized galileon**)


\[ \mathcal{L}_2 = K(\phi, X) \]  
\[ \mathcal{L}_3 = - G_3(\phi, X) \Box \phi \]  
\[ \mathcal{L}_4 = G_4(\phi, X) R + G_{4,X} \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) \right] \]  
\[ \mathcal{L}_5 = G_5(\phi, X) G_{\mu\nu}(\nabla^\mu \nabla^\nu \phi) \]  
\[ - \frac{1}{6} G_{5,X} \left[ (\Box \phi)^3 - 3(\Box \phi)(\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) + 2(\nabla^\mu \nabla_\alpha \phi)(\nabla^\alpha \nabla_\beta \phi)(\nabla^\beta \nabla_\mu \phi) \right] \]

\[ \mathcal{L}_2 \supset (\partial \phi)^2, \quad V(\phi) \]

\[ \mathcal{L}_3 \supset (\partial \phi)^2 \Box \phi \]

\[ \mathcal{L}_4 \supset (M_{P1}^2/2) R \]

\[ \mathcal{L}_5 \supset G^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \]

(Germani et al. 2011; Gubitosi, Linder 2011)

\[ X = -(\partial \phi)^2/2, \quad G_{iX} = \partial G_i / \partial X \]
Question:

Does Vainshtein mechanism work in Horndeski theory?
Formulation

✓ Newtonian gauge and perturbations on *cosmological background*

\[ ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Psi)d\mathbf{x}^2 \]

\[ \phi \to \phi(t) + \delta \phi(t, \mathbf{x}), \quad Q \equiv (H/\dot{\phi})\delta \phi \]

\[ \rho_m \to \rho_m(t)[1 + \delta(t, \mathbf{x})] \]

\[ \epsilon = \Psi, \Phi, \text{ and } Q \ll 1 \]

✓ In field equations

\[ \text{EOM} \supset \left\{ \begin{array}{l}
\text{"mass terms"}, \quad \text{"time derivative terms"}, \\
(L(t)^2 \partial^2 \epsilon)^n, \quad (L(t) \partial \epsilon)^m \\
\end{array} \right\} \]

Neglect

Quasi-static approximation

\[ \partial_t \ll \partial_x \]

\[ L(t) \sim \mathcal{O}(H^{-1}) \]

higher-order terms

Picking up the terms like

\[ \partial^2 \epsilon, \ (\partial^2 \epsilon)^2, \ (\partial^2 \epsilon)^3, \ (\partial^2 \epsilon)^4, \ \delta \]
Field equations

✓ Traceless part of the Einstein equations

\[
\nabla^2 \left( \mathcal{F}_T \Psi - G_T \Phi - A_1 Q \right) = \frac{B_1}{2a^2H^2} Q^{(2)} + \frac{B_3}{a^2H^2} \left( \nabla^2 \Phi \nabla^2 Q - \partial_i \partial_j \Phi \partial^i \partial^j Q \right)
\]

✓ 00 component of the Einstein equation

\[G_T \nabla^2 \Psi = \frac{a^2}{2} \rho_m \delta - A_2 \nabla^2 Q\]

\[= - \frac{B_2}{2a^2H^2} Q^{(2)} - \frac{B_3}{a^2H^2} \left( \nabla^2 \Phi \nabla^2 Q - \partial_i \partial_j \Psi \partial^i \partial^j Q \right) - \frac{C_1}{3a^4H^4} Q^{(3)}\]

✓ Scalar field equation

\[A_0 \nabla^2 Q - A_1 \nabla^2 \Psi - A_2 \nabla^2 \Phi + \frac{B_0}{a^2H^2} Q^{(2)} - \frac{B_1}{a^2H^2} \left( \nabla^2 \Psi \nabla^2 Q - \partial_i \partial_j \Psi \partial^i \partial^j Q \right)\]

\[- \frac{B_2}{a^2H^2} \left( \nabla^2 \Phi \nabla^2 Q - \partial_i \partial_j \Phi \partial^i \partial^j Q \right) - \frac{B_3}{a^2H^2} \left( \nabla^2 \Phi \nabla^2 \Psi - \partial_i \partial_j \Phi \partial^i \partial^j \Psi \right)\]

\[- \frac{C_0}{a^4H^4} Q^{(3)} - \frac{C_1}{a^4H^4} \mathcal{U}^{(3)} = 0\]

\[Q^{(2)} \equiv (\nabla^2 Q)^2 - (\partial_i \partial_j Q)^2\]

\[Q^{(3)} \equiv (\nabla^2 Q)^3 - 3 \nabla^2 Q (\partial_i \partial_j Q)^2 + 2 (\partial_i \partial_j Q)^3\]

\[\mathcal{U}^{(3)} \equiv Q^{(2)} \nabla^2 \Phi - 2 \nabla^2 Q \partial_i \partial_j Q \partial^i \partial^j \Phi + 2 \partial_i \partial_j Q \partial^i \partial^j \partial^k Q \partial_k \partial^i \Phi\]

\[A_i, B_i, \text{ and } C_i \text{ are functions of } K, G_3, G_4, G_5\]
\[ F_T := 2 \left[ G_4 - X \left( \dot{\phi}G_{5X} + G_{5\phi} \right) \right], \]
\[ G_T := 2 \left[ G_4 - 2XG_{4X} - X \left( H\dot{\phi}G_{5X} - G_{5\phi} \right) \right], \]
\[ \Theta := -\dot{\phi}XG_{3X} + 2HG_4 - 8HXG_{4X} - 8HX^2G_{4XX} + \dot{\phi}G_{4\phi} + 2X\dot{\phi}G_{4\phi}X - H^2\dot{\phi} \left( 5XG_{5X} + 2X^2G_{5XX} \right) + 2HX \left( 3G_{5\phi} + 2XG_{5\phi}X \right), \]
\[ A_0 := \frac{\dot{\Theta}}{H^2} + \frac{\Theta}{H} + F_T - 2G_T - 2\frac{\dot{G}_T}{H} - \frac{\mathcal{E} + \mathcal{P}}{2H^2}, \]
\[ A_1 := \frac{1}{H} \frac{dG_T}{dt} + G_T - F_T, \]
\[ A_2 := G_T - \frac{\Theta}{H}, \]
\[ B_0 := \frac{X}{H} \left\{ \dot{\phi}G_{3X} + 3 \left( \dot{X} + 2HX \right) G_{4XX} + 2X\dot{X}G_{4XXX} - 3\dot{\phi}G_{4\phi}X + 2\dot{\phi}XG_{4\phi}XX \right. \]
\[ + \left( \dot{H} + H^2 \right) \dot{\phi}G_{5X} + \dot{\phi} \left[ 2H\dot{X} + \left( \dot{H} + H^2 \right) X \right] G_{5XX} + H\dot{\phi}X\dot{X}G_{5XXX} - 2 \left( \dot{X} + 2HX \right) G_{5\phi}X \]
\[ - \dot{\phi}XG_{5\phi}X - X \left( \dot{X} - 2HX \right) G_{5\phiXX} \}, \]
\[ B_1 := 2X \left[ G_{4X} + \dot{\phi} \left( XG_{5X} + XG_{5XX} \right) \right] - G_{5\phi} + XG_{5\phi}X, \]
\[ B_2 := -2X \left( G_{4X} + 2XG_{4XX} + H\dot{\phi}G_{5X} + H\dot{\phi}XG_{5XX} - G_{5\phi} - XG_{5\phi}X \right), \]
\[ B_3 := H\dot{\phi}XG_{5X}, \]
\[ C_0 := 2X^2G_{4XX} + \frac{2X^2}{3} \left( 2\dot{\phi}G_{5XX} + \dot{\phi}XG_{5XXX} - 2G_{5\phi}X + XG_{5\phiXX} \right), \]
\[ C_1 := H\dot{\phi}X \left( G_{5X} + XG_{5XX} \right). \]

*All coefficients are determined by the background*
Spherical symmetric case

✓ EOM for gravity and scalar field can be integrated once,

\[
c_h^2 \frac{\Psi'}{r} - \frac{\Phi'}{r} - \alpha_1 \frac{Q'}{r} = \frac{\beta_1}{H^2} \left( \frac{Q'}{r} \right)^2 + 2 \frac{\beta_3}{H^2} \frac{\Phi' Q'}{r} \\
\frac{\Psi'}{r} + \alpha_2 \frac{Q'}{r} = \frac{1}{8\pi G_T} \frac{\delta M(t,r)}{r^3} - \frac{\beta_2}{H^2} \left( \frac{Q'}{r} \right)^2 - 2 \frac{\beta_3}{H^2} \frac{\Psi' Q'}{r} - 2 \frac{\gamma_1}{3 H^4} \left( \frac{Q'}{r} \right)^3 \\
\alpha_0 \frac{Q'}{r} - \alpha_1 \frac{\Psi'}{r} - \alpha_2 \frac{\Phi'}{r} = 2 \left[ - \frac{\beta_0}{H^2} \left( \frac{Q'}{r} \right)^2 + \frac{\beta_1}{H^2} \frac{\Psi' Q'}{r} + \frac{\beta_2}{H^2} \frac{\Phi' Q'}{r} + \frac{\beta_3}{H^2} \frac{\Phi' \Psi'}{r} \\
+ \frac{\gamma_0}{H^4} \left( \frac{Q'}{r} \right)^3 + \frac{\gamma_1}{H^4} \frac{\Phi' \left( \frac{Q'}{r} \right)^2} \right]
\]

where

\[
c_h^2 \equiv \mathcal{F}_T / G_T
\]

(Propagation speed of the gravitational waves)

Case 1: \[ G_{4X} = 0, \ G_5 = 0 \]

\[ a \ Q'^2 + b \ Q' + c = 0 \]

Case 2: \[ G_{5X} = 0 \]

\[ a \ Q'^3 + b \ Q'^2 + c \ Q' + d = 0 \]

Case 3: \[ G_{5X} \neq 0 \]

\[ a \ Q'^6 + b \ Q'^5 + c \ Q'^4 + d \ Q'^3 + e \ Q'^2 + f \ Q' + g = 0 \]
For sufficiently large $r$, (linearize $\psi$, $\Phi$, and $\varphi$)

\begin{align*}
\Phi' &= \frac{1}{8\pi G_T} \frac{c_h^2 \alpha_0 - \alpha_1^2}{\alpha_0 + (2\alpha_1 + c_h^2 \alpha_2) \alpha_2} \frac{\delta M}{r^2} \\
\Psi' &= \frac{1}{8\pi G_T} \frac{\alpha_0 + \alpha_1 \alpha_2}{\alpha_0 + (2\alpha_1 + c_h^2 \alpha_2) \alpha_2} \frac{\delta M}{r^2} \\
Q' &= \frac{1}{8\pi G_T} \frac{\alpha_1 + c_h^2 \alpha_2}{\alpha_0 + (2\alpha_1 + c_h^2 \alpha_2) \alpha_2} \frac{\delta M}{r^2}
\end{align*}

In general, the gravitational coupling $G_{\text{eff}}$ in the Poisson equation is different from Newton’s constant on cosmological scales.
Case 1: \( G_{4X} = 0, \ G_5 = 0 \)

- Kinetic gravity braiding with non-minimal coupling (Deffayet et al. 2010)

\[
\mathcal{L} = G_4(\phi)R + K(\phi, X) - G_3(\phi, X)\Box \phi
\]

- Scalar field solution

\[
\frac{Q'}{r} = \frac{H^2}{B} \left( \sqrt{1 + \left(\frac{r_V}{r}\right)^3} - 1 \right)
\]

- Inside the Vainshtein radius \( r_V \)

\[
Q' \simeq \frac{H}{B} \sqrt{\frac{2BC\mu}{r}} \ll \frac{GM}{r^2}
\]

Gravitational potentials

\[
\Psi' = \Phi' \simeq \frac{1}{16\pi G_4(t)} \frac{\delta M}{r^2}
\]

Newton’s constant \( G_N(t) \)

PPN parameter \( \gamma \equiv \Psi' / \Phi' = 1 \)

Propagation speed of the gravitational waves \( c_h^2 = 1 \)

Vainshtein radius

\[
r_V = \left(\frac{2BC\mu}{H^2}\right)^{1/3}
\]

\( r_V \sim 100 \text{ pc for sun} \)

Friedmann equation

\[
3H^2 = 8\pi G_{\cos} (\rho_m + \rho_\phi)
\]

\( G_{\cos} = G_N \)

Time-dependence of \( G_N \) can be tested by BBN

\[
\left| 1 - \frac{G_N|_{\text{BBN}}}{G_N|_{\text{now}}} \right| \lesssim 0.1
\]

(Uzan 2011)
Case 2: \( G_5 X = 0 \)

✓ Lagrangian

\[
\mathcal{L} = K(\phi, X) - G_3(\phi, X) \Box \phi + G_4(\phi, X) R + G_{4X} \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right] + G_5(\phi) G_{\mu \nu} \nabla^\mu \nabla^\nu \phi
\]

does not depend on the kinetic term \( X \)

✓ Scalar field equations

\[
(Q')^3 + C_2 H^2 r (Q')^2 + \left( \frac{C_1}{2} H^4 r^2 - H^2 C_\beta \frac{\mu}{r} \right) Q' - \frac{H^4 C_\alpha \mu}{2} = 0
\]

\( C_1, C_2, C_\alpha, \text{ and } C_\beta \) are functions of \( K, G_3, G_4, G_5 \)

✓ 3 possible solutions at short distance can be matched to the linear solution

\[
Q' \simeq \pm H \sqrt{C_\beta \frac{\mu}{r}}, \quad -\frac{C_\alpha}{C_\beta} \frac{H^2 r}{2}
\]
**Case 2:** \( G_{5X} = 0 \)

- **Solution I**
  \[ Q' \simeq \pm H \sqrt{C_{\beta} \frac{\mu}{r}} \]
  \( \Psi' = \Phi' \simeq \frac{C_{\Psi}(t)}{8\pi G_{T}(t)} \frac{\delta M}{r^2} \)
  \( \gamma \equiv \Psi'/\Phi' = 1 \)

- **Solution II**
  \[ Q' \simeq -\frac{C_{\alpha}}{C_{\beta}} \frac{H^2 r}{2} \]
  \( \Phi' \simeq c_h^2 \Psi' \simeq \frac{c_h^2}{8\pi G_{T}} \frac{\delta M}{r^2} \)
  \( \gamma \equiv \frac{\Psi'}{\Phi'} = \frac{1}{c_h^2} (\neq 1) \)

**Friedmann equation**
\[ 3H^2 = 8\pi G_{cos} (\rho_m + \rho_\phi) \]
\[ G_{cos} = G_N \]
**Time-dependence of** \( G_N \)
**can be tested by BBN**

**Solar-system tests** (Will 2005)
\[ |1 - \gamma| < 2.3 \times 10^{-5} \]

**propagation speed of gravitational waves**
At sufficiently small scales, the inverse square law for the gravitational force is not a solution anymore!

\( G_{5X} \neq 0 \)

\[ \Psi'(r), \Phi'(r) \propto \frac{1}{r^2} \]

is not a solution anymore!

The coupling between the Einstein tensor and the kinetic term \( XG_{\mu\nu} \) leads to a strong modification of gravity even at short distances.

The Vainshtein mechanism \textit{no longer works} in the presence of \( G_{5X} \)!!
Current Problems

How can we determine arbitrary functions in Horndeski theory?

- There is no principle to determine arbitrary functions
- Probably, they are determined from the effective theory of a fundamental theory
- Need to construct such a theory
Massive gravity

de Rham, Gabadadze, Tolley (2011)

✓ The action

\[
S = M^2_{\text{Pl}} \int d^4 x \sqrt{-g} \left[ \frac{R}{2} + m^2 \left( \mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4 \right) + \mathcal{L}_m [g_{\mu\nu}, \psi] \right]
\]

where

\[
\mathcal{U}_2(\mathcal{K}) = \frac{1}{2!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\rho\sigma} \mathcal{K}_{\alpha}^{\mu} \mathcal{K}_{\beta}^{\nu} \quad \mathcal{K}_{\mu}^{\mu} = \delta_{\mu}^{\nu} - (\sqrt{g^{-1} f})_{\mu}^{\nu}
\]

\[
\mathcal{U}_3(\mathcal{K}) = \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\sigma} \mathcal{K}_{\alpha}^{\mu} \mathcal{K}_{\beta}^{\nu} \mathcal{K}_{\gamma}^{\rho}
\]

\[
\mathcal{U}_4(\mathcal{K}) = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} \mathcal{K}_{\alpha}^{\mu} \mathcal{K}_{\beta}^{\nu} \mathcal{K}_{\gamma}^{\rho} \mathcal{K}_{\delta}^{\sigma}
\]

(BD) **Ghost free** massive gravity
The action in the decoupling limit (extracting the scalar mode of graviton)

\[
S = \int d^4 x \left[ -\frac{1}{2} \tilde{h}_{\mu\nu} \mathcal{E}^{\mu\nu,\alpha\beta} \tilde{h}_{\alpha\beta} + \sum_{n=2}^{5} \frac{\beta_n}{(\Lambda_3^3)^{n-2}} \mathcal{L}^{(n)}_{\text{galileon}} + \frac{\beta_X}{\Lambda_3^6} \tilde{h}^{\mu\nu} \hat{X}^{(3)}_{\mu\nu} + \frac{1}{M_P} \hat{h}_{\mu\nu} T^{\mu\nu} + \frac{1}{M_P} \hat{\phi} T + \frac{\beta_T}{\Lambda_3^3 M_P} \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} T^{\mu\nu} \right]
\]

\[
\mathcal{L}^{(2)}_{\text{galileon}} = -\frac{1}{2} (\partial \phi)^2
\]

\[
\mathcal{L}^{(3)}_{\text{galileon}} = -\frac{1}{2} (\partial \phi)^2 [\Pi]
\]

\[
\mathcal{L}^{(4)}_{\text{galileon}} = -\frac{1}{2} (\partial \phi)^2 ([\Pi]^2 - [\Pi]^{2})
\]

\[
\mathcal{L}^{(5)}_{\text{galileon}} = -\frac{1}{2} (\partial \phi)^2 ([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3])
\]

\[\beta_i\] are functions of model parameters \(\alpha_i\)

Horndeski theory includes the decoupling limit of dRGT massive gravity
Massive gravity is a candidate of the galileon effective theory, but

- homogeneous and isotropic cosmological solution suffers from

  ghost instabilities

Gumrukcuoglu, Lin, Mukohyama (2011)
Gumrukcuoglu, Lin, Mukohyama (2012)
Bi-gravity theory

The action

\[ S = M_f^2 \int d^4 x \sqrt{-f} \frac{R[f]}{2} \]

\[ + M_g^2 \int d^4 x \sqrt{-g} \left[ \frac{R[g]}{2} + m^2 (U_2 + \alpha_3 U_3 + \alpha_4 U_4) + L_m [g_{\mu \nu}, \psi] \right] \]

where

\[ K^\mu_\nu = \delta^\mu_\nu - (\sqrt{g^{-1} f})^\mu_\nu \]

Bigravity = \textit{massive} graviton + \textit{massless} graviton

There is a viable cosmological (self-accelerating) solution without ghost-instabilities

Hassan and Rosen (2011)

Strauss et al. (2011)
Decoupling limit of massive gravity

- Scalar mode of graviton can be extracted from Stuckelberg field

\[ f_{\mu\nu} = \eta_{ab} \partial_\mu Y^a \partial_\nu Y^b \]

- Explicit relation with galileon theory

Decoupling limit of bi-gravity theory

- UNKNOWN...
- Does really Vainshtein mechanism operate ??
- How can we extract the scalar mode of massive graviton ??
Vainshtein screening successfully operates in Horndeski theory, but

- Newton’s constant $G=G(t)$
- constrained from PPN and BBN
- inverse-square law *can not be reproduced* at small scales if $G_{5X} \neq 0$

Bigravity theory

- Can we extract the *scalar* mode of massive graviton?
- What about *Vainshtein mechanism*?

I’m still working ...