

# Massive Gravity: Nonlinear Instability of a Homogeneous and Isotropic Universe

Antonio De Felice

*ThEP's CRL, NEP, The Institute for Fundamental Study, Naresuan University, Phitsanulok 65000, Thailand and Thailand Center of Excellence in Physics, Ministry of Education, Bangkok 10400, Thailand*

A. Emir Gümrükçüoğlu and Shinji Mukohyama

*Kavli Institute for the Physics and Mathematics of the Universe, Todai Institutes for Advanced Study, University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8583, Japan*

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We argue that all homogeneous and isotropic solutions in nonlinear massive gravity are unstable. For this purpose, we study the propagating modes on a Bianchi type-I manifold. We analyze their kinetic terms and dispersion relations as the background manifold approaches the homogeneous and isotropic limit. We show that in this limit, at least one ghost always exists and that its frequency tends to vanish for large scales, meaning that it cannot be integrated out from the low energy effective theory. This ghost mode is interpreted as a leading nonlinear perturbation around a homogeneous and isotropic background.

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*Introduction.*—The concept of the mass has been central in many areas of physics. Gravitation is not an exception, and it is one of the simplest but yet unanswered questions whether the graviton, a spin-2 particle that mediates gravity, can have a nonvanishing mass or not. This question is relevant not only from a theoretical viewpoint, but also from a phenomenological viewpoint, since a nonzero graviton mass may lead to late-time acceleration of the Universe and thus may be considered as an alternative to dark energy.

Recently Refs. [1,2] proposed the first example of a fully nonlinear massive gravity theory, where the so-called Boulware-Deser (BD) ghost [3], which had been one of the major obstacles against a stable nonlinear gravity theory with a nonvanishing graviton mass, is removed by construction. Because of the theoretical and phenomenological motivations mentioned above, this theory has been attracting significant interest.

The first homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) solution for this theory was presented in Ref. [4] for a Minkowski fiducial metric and then extended to more general fiducial metrics in Ref. [5]. The analysis of linear perturbations in this general setup was also carried out in Ref. [5]. Although a massive spin-2 particle generically has 5 propagating degrees of freedom, it was found that the number of degrees of freedom in the gravity sector was 2, the same as in general relativity (GR). This is due to the vanishing of the kinetic terms for the expected additional degrees. (Backgrounds with additional symmetries which remove the extra degrees were introduced in Ref. [6].) This feature may extend to other setups: Reference [7] obtained a vanishing kinetic term on spherically symmetric inhomogeneous backgrounds.

The goal of this Letter is to determine the fate of the extra degrees of freedom. We find that in the nonlinear

massive gravity, all cosmological solutions that respect homogeneity and isotropy have a ghost, i.e., an excitation with a wrong sign kinetic term. Therefore, the universe in this theory should have either inhomogeneities [8] or anisotropy [9]. We note that the ghost mode found in this Letter is among 5 degrees of freedom of the massive spin-2 field and, thus, is not the BD ghost.

*The model and the background.*—Imposing the absence of the BD ghost, the massive gravity action, in vacuum, can be constructed as [2]

$$S = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} [R - 2\Lambda + 2m_g^2 \mathcal{L}_{\text{MG}}], \quad (1)$$

with  $\mathcal{L}_{\text{MG}} = \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4$ , where

$$\mathcal{L}_2 = \frac{1}{2} ([\mathcal{K}]^2 - [\mathcal{K}^2]),$$

$$\mathcal{L}_3 = \frac{1}{6} ([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]),$$

$$\mathcal{L}_4 = \frac{1}{24} ([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]),$$

the square brackets denote the trace operation, and

$$\mathcal{K}_\nu^\mu = \delta_\nu^\mu - (\sqrt{g^{-1}f})_\nu^\mu. \quad (2)$$

Here,  $g_{\mu\nu}$  and  $f_{\mu\nu}$  are physical and fiducial metrics, respectively. Since we are interested in the stability of the gravity sector only, it is sufficient to consider a vacuum configuration, with a cosmological constant  $\Lambda$ .

The physical metric is chosen to be the simplest anisotropic extension of FLRW, namely, the axisymmetric Bianchi type-I metric

$$ds^2 = -N^2 dt^2 + a^2 (e^{4\sigma} dx^2 + e^{-2\sigma} \delta_{ij} dy^i dy^j), \quad (3)$$

where  $N$ ,  $a$ , and  $\sigma$  are functions of the time variable  $t$ . In the rest of the Letter, greek indices span the space-time coordinates, while the indices  $i, j = 2, 3$  correspond to the coordinates on the  $y$ - $z$  plane, with  $y^2 = y$ ,  $y^3 = z$ . Since our goal is to obtain the stability conditions of this metric in the isotropic limit, the whole system in this limit needs to reduce to the general cosmological solutions given in Refs. [4,5]. For this reason, we consider a fiducial metric to be in the flat FLRW form,

$$f_{\mu\nu} = -n^2 \partial_\mu \phi^0 \partial_\nu \phi^0 + \alpha^2 (\partial_\mu \phi^1 \partial_\nu \phi^1 + \delta_{ij} \partial_\mu \phi^i \partial_\nu \phi^j), \quad (4)$$

where both  $n$  and  $\alpha$  are functions of the time-Stückelberg field  $\phi^0$ .

The equations of motion for the background can be calculated by varying the action with respect to the Stückelberg fields and the metric. As a result, we obtain three independent equations as

$$\begin{aligned} 3(H^2 - \Sigma^2) - \Lambda &= m_g^2 [-(3\gamma_1 - 3\gamma_2 + \gamma_3) + \gamma_1(2e^\sigma + e^{-2\sigma})X - \gamma_2(e^{2\sigma} + 2e^{-\sigma})X^2 + \gamma_3 X^3], \\ \frac{3\dot{\Sigma}}{N} + 9H\Sigma &= m_g^2 (e^{-2\sigma} - e^\sigma)X[\gamma_1 - \gamma_2(e^\sigma + r)X + \gamma_3 r e^\sigma X^2], \\ J_\phi^{(x)}(H + 2\Sigma - H_f e^{-2\sigma} X) + 2J_\phi^{(y)}(H - \Sigma - H_f e^\sigma X) &= 0, \end{aligned} \quad (5)$$

where

$$\begin{aligned} J_\phi^{(x)} &\equiv \gamma_1 - 2\gamma_2 e^\sigma X + \gamma_3 e^{2\sigma} X^2, \\ J_\phi^{(y)} &\equiv \gamma_1 - \gamma_2(e^{-2\sigma} + e^\sigma)X + \gamma_3 e^{-\sigma} X^2, \end{aligned} \quad (6)$$

and

$$\begin{aligned} \gamma_1 &\equiv 3 + 3\alpha_3 + \alpha_4, & \gamma_2 &\equiv 1 + 2\alpha_3 + \alpha_4, & \gamma_3 &\equiv \alpha_3 + \alpha_4 \\ H &\equiv \dot{a}/(aN), & H_f &\equiv \dot{\alpha}/(\alpha n), & \Sigma &\equiv \dot{\sigma}/N, \\ X &\equiv \alpha/a, & r &\equiv an/(\alpha N). \end{aligned} \quad (7)$$

We note that, in the isotropic limit ( $\sigma, \Sigma \rightarrow 0$ ), we have  $J_\phi^{(x)} = J_\phi^{(y)}$ , so that the Stückelberg equation of motion, Eq. (5), at leading order, gives

$$\gamma_1 - 2\gamma_2 X + \gamma_3 X^2 \simeq 0, \quad (8)$$

that is  $X \rightarrow \text{const}$ , which corresponds to the FLRW result found in Ref. [5]. In the same limit, we can also see that  $H \rightarrow \text{const}$ , as expected.

*Even modes.*—Let us now consider the perturbations which transform as 2D scalars under a spatial rotation in the  $y$ - $z$  plane (also referred to as even modes). Then, the perturbed metric for the even sector can be written as [10]

$$\begin{aligned} ds^2 &= -N^2(1 + 2\Phi)dt^2 + 2aNdt[e^{2\sigma}\partial_x \chi dx + e^{-\sigma}\partial_i B dy^i] \\ &+ a^2 e^{4\sigma}(1 + \psi)dx^2 + 2a^2 e^\sigma \partial_x \partial_i \beta dx dy^i \\ &+ a^2 e^{-2\sigma}[\delta_{ij}(1 + \tau) + \partial_i \partial_j E]dy^i dy^j, \end{aligned} \quad (9)$$

while the even-type perturbations of Stückelberg fields read

$$\phi^0 = t + \pi^0, \quad \phi^1 = x + \partial_x \pi^1, \quad \phi^i = y^i + \partial^i \pi. \quad (10)$$

We can then define gauge invariant combinations as follows

$$\begin{aligned} \hat{\Phi} &= \Phi - \frac{1}{2N} \partial_t \left( \frac{\tau}{H - \Sigma} \right), \\ \hat{\chi} &= \chi + \frac{\tau e^{-2\sigma}}{2a(H - \Sigma)} - \frac{a e^{2\sigma}}{N} \partial_t \left[ e^{-3\sigma} \left( \beta - \frac{e^{-3\sigma}}{2} E \right) \right], \\ \hat{B} &= B + \frac{e^\sigma}{2a(H - \Sigma)} \tau - \frac{a e^{-\sigma}}{2N} \dot{E}, \\ \hat{\psi} &= \psi - \frac{H + 2\Sigma}{H - \Sigma} \tau - e^{-3\sigma} \partial_x^2 (2\beta - e^{-3\sigma} E), \\ \hat{\tau}_\pi &= \pi^0 - \frac{\tau}{2N(H - \Sigma)}, \\ \hat{\beta}_\pi &= \pi^1 - e^{-3\sigma} \left( \beta - \frac{e^{-3\sigma}}{2} E \right), \\ \hat{E}_\pi &= \pi - \frac{1}{2} E. \end{aligned} \quad (11)$$

The first four definitions do not refer to the Stückelberg perturbations and are thus already present in GR [11]. However, the additional three degrees arise from the breaking of general coordinate invariance by the nonzero expectation value of the Stückelberg fields.

In order to find the behavior of the perturbations, we proceed as usual by expanding the action at second order in the perturbation fields, then by employing the Fourier plane-wave decompositions, as in  $\exp[i(k_L x + k_i y^i)]$ . The degrees of freedom arising from the  $g_{0\mu}$  perturbations, namely  $\hat{\Phi}$ ,  $\hat{B}$ , and  $\hat{\chi}$ , are nondynamical, and thus can be integrated out. Furthermore, the kinetic term for the  $\hat{\tau}_\pi$  is proportional to the background equations of motion, so that this degree of freedom is also nondynamical. We interpret this field as the would-be BD ghost, which is eliminated in this theory by construction.

In the massless theory (i.e., GR), using the constraint equations also removes the degrees  $\hat{\beta}_\pi$ ,  $\hat{E}_\pi$ , leaving only  $\hat{\psi}$  in the action, which becomes one of the two gravity wave (GW) polarizations in the isotropic limit. However, in our

case, due to the nonzero mass of the graviton, these two degrees of freedom are dynamical, in general.

Thus, the Lagrangian for even-type perturbations in vacuum has three physical propagating modes  $\mathcal{V}_a$  ( $a = 1, 2, 3$ ). Assuming small deviation from FLRW, with  $|\sigma| \ll 1$  and  $|\Sigma/H| \ll 1$ , we study the kinetic matrix  $\mathcal{K}_{ab}$

$$S_{\text{even}}^{(2)} \ni \frac{M_p^2}{2} \int N dt dk_L d^2 k_T a^3 \left( \frac{\dot{\mathcal{V}}_a^*}{N} \mathcal{K}_{ab} \frac{\dot{\mathcal{V}}_b}{N} \right). \quad (12)$$

Thanks to the 2D rotational symmetry on the  $y$ - $z$  plane, the action depends on  $k_T \equiv \sqrt{k_2^2 + k_3^2}$  instead of the individual components. The eigenvalues of  $\mathcal{K}_{ab}$ , at leading order in small anisotropy expansion, are

$$\kappa_1 \simeq \frac{p_T^4}{8p^4}, \quad \kappa_2 \simeq -\frac{2a^4 M_{\text{GW}}^2 p_L^2}{1-r^2} \sigma, \quad \kappa_3 \simeq -\frac{p_T^2}{2p_L^2} \kappa_2, \quad (13)$$

where we defined  $M_{\text{GW}}^2 \equiv m_g^2(1-r)X^2(\gamma_2 - \gamma_3 X)$  and introduced the physical momenta

$$p_L \equiv \frac{k_L}{ae^{2\sigma}} \simeq \frac{k_L}{a}, \quad p_T \equiv \frac{k_T}{ae^{-\sigma}} \simeq \frac{k_T}{a}, \quad p^2 \equiv p_L^2 + p_T^2. \quad (14)$$

The kinetic term  $\kappa_1$ , which is the only eigenvalue that does not vanish in the isotropic limit, corresponds to one of the gravity wave polarizations in FLRW. Once small but non-vanishing anisotropy is introduced, two additional even modes acquire nonzero kinetic terms at quadratic order. More importantly, from (13), we see that  $\kappa_2$  and  $\kappa_3$  have opposite signs, regardless of the parameters of the theory. Thus, we conclude that in the isotropic limit one of the new degrees is always a ghost. Assuming that  $\sigma(1-r) > 0$  (which turns out to be the condition for stability in the odd sector, as we show later), the ghost mode is associated with the eigenvalue  $\kappa_2 < 0$ .

We conclude the discussion of the even modes by presenting their dispersion relations. We first make a field redefinition into new field basis fields  $\mathcal{W}_a$  defined such that the kinetic action can be written as

$$S_{\text{even}}^{(2)} \ni \frac{1}{2} \int N dt dk_L d^2 k_T a^3 \left( \frac{\dot{\mathcal{W}}_a^*}{N} \eta_{ab} \frac{\dot{\mathcal{W}}_b}{N} \right), \quad (15)$$

where  $\eta_{ab} = \text{diag}(1, -1, 1)$ . The mass spectrum can be determined either by studying the equation for the frequency discriminant, or, equivalently, by performing a Lorentz transformation to diagonalize the frequency matrix. Eventually, we find

$$\begin{aligned} \omega_1^2 &\simeq p^2 + M_{\text{GW}}^2, \\ \omega_2^2 &\simeq -\frac{1-r^2}{24\sigma} \left[ \sqrt{(10p^2 + p_T^2)^2 - 8p_L^2 p_T^2} - (2p^2 + 3p_T^2) \right], \\ \omega_3^2 &\simeq -\omega_2^2 + \frac{1-r^2}{12\sigma} (2p^2 + 3p_T^2), \end{aligned} \quad (16)$$

with  $\omega_2^2 \omega_3^2 < 0$  in general, and  $\omega_2^2 < 0$  by assuming  $\sigma(1-r) > 0$ . We note that the dispersion relation corresponding to the ghost  $\omega_2^2$  becomes smaller at larger scales. Therefore, at sufficiently large scales, this mode cannot be integrated out from the low energy effective theory. This feature makes the FLRW background unstable for massive gravity. As a consequence, the homogeneous and isotropic cosmology cannot be accommodated in the nonlinear massive gravity theory.

*Odd modes.*—Let us now discuss the odd sector (i.e., the divergenceless part of the modes which transform as 2D vectors under a rotation in the  $y$ - $z$  plane). The perturbed metric we consider is

$$ds^2 = -N^2 dt^2 + 2ae^{-\sigma} N v_i dt dy^i + 2a^2 e^\sigma \partial_x \lambda_i dx dy^i + a^2 e^{4\sigma} dx^2 + a^2 e^{-2\sigma} (\delta_{ij} + \partial_{(i} h_{j)}) dy^i dy^j, \quad (17)$$

where  $\partial_{(i} h_{j)} \equiv (\partial_i h_j + \partial_j h_i)/2$  and  $\partial^i v_i = \partial^i \lambda_i = \partial^i h_i = 0$ . For the Stückelberg fields, we consider instead

$$\phi^0 = t, \quad \phi^1 = x, \quad \phi^i = y^i + \pi^i, \quad (18)$$

where  $\partial_i \pi^i = 0$ . Since the vectors are defined on the 2D  $y$ - $z$  plane, the transverse condition can be used to reduce each of these vectors to a single degree of freedom

$$v_i = \epsilon_i^j \partial_j v, \quad \lambda_i = \epsilon_i^j \partial_j \lambda, \quad h_i = \epsilon_i^j \partial_j h, \quad \pi_i = \epsilon_i^j \partial_j \pi_{\text{odd}},$$

where  $\epsilon_i^j$  is a unit antisymmetric tensor with  $\epsilon_2^3 = -\epsilon_3^2 = 1$ . Additionally, for the odd modes we can introduce gauge invariant combinations as follows:

$$\hat{v} = v - \frac{ae^{-2\sigma}}{2N} \dot{h}, \quad \hat{\lambda} = \lambda - \frac{e^{-3\sigma}}{2} h, \quad \hat{h}_\pi = \pi_{\text{odd}} - \frac{1}{2} \dot{h}. \quad (19)$$

Using these fields, the second-order resulting action depends on the three perturbations  $(\hat{v}, \hat{\lambda}, \hat{h}_\pi)$ . Among these,  $\hat{v}$  does not have any time derivatives and can be removed by solving its own constraint equation. In GR, this operation also removes  $\hat{h}_\pi$  and the final action can be written in terms of  $\hat{\lambda}$  only. However, in this nonlinear theory of massive gravity, we expect the field  $\hat{h}_\pi$  to remain in the action as an extra degree of freedom coming from the Stückelberg sector.

After a further field redefinition,

$$\mathcal{Q}_1 \equiv -e^{3\sigma} \hat{\lambda}, \quad \mathcal{Q}_2 \equiv \frac{2e^{3\sigma} p_L^2}{p^2} \hat{\lambda} - 2\hat{h}_\pi, \quad (20)$$

the quadratic action, for small anisotropy, takes the following form:

$$S_{\text{odd}}^{(2)} \simeq \frac{M_{\text{Pl}}^2}{2} \int N dt dk_L d^2 k_T a^3 \left[ K_{11} \frac{|\dot{\mathcal{Q}}_1|^2}{N^2} - \Omega_{11}^2 |\mathcal{Q}_1|^2 + K_{22} \frac{|\dot{\mathcal{Q}}_2|^2}{N^2} - \Omega_{22}^2 |\mathcal{Q}_2|^2 \right], \quad (21)$$

where the two modes decouple at leading order in the small anisotropy expansion, with coefficients

$$K_{11} = \frac{a^4 p_L^2 p_T^4}{2p^2}, \quad K_{22} = \frac{a^4 p_T^2 M_{GW}^2}{4(1-r^2)} \sigma, \quad (22)$$

$$\frac{\Omega_{11}^2}{K_{11}} = p^2 + M_{GW}^2, \quad \frac{\Omega_{22}^2}{K_{22}} = c_{\text{odd}}^2 p^2,$$

and  $c_{\text{odd}}^2 = (1-r^2)/(2\sigma)$ . Thus, at leading order, we identify the mode  $\mathcal{Q}_1$  with one of the gravity wave polarizations in the FLRW background [5]. The extra degree of freedom  $\mathcal{Q}_2$  is massless and has sound speed  $c_{\text{odd}}$ . In order for this mode to be stable, we require the kinetic term for  $\mathcal{Q}_2$  to be positive, that is,

$$(1-r)\sigma > 0. \quad (23)$$

In this case,  $c_{\text{odd}}^2$  also becomes positive, and the odd mode  $\mathcal{Q}_2$  is, in general, free from ghost instabilities.

Conclusions.—In the search for a theory which could explain the dark energy enigma, the nonlinear massive gravity, recently introduced in Ref. [1], has raised a lot of interest among both theoretical and experimental physicists, thanks to its implications in our understanding of fundamental forces, if it is theoretically consistent and observationally viable.

This theory admits homogeneous and isotropic solutions, and it has been shown in Ref. [5] that, out of the five modes which would be typically expected in this theory, only two actually propagate at the linearized level. Therefore, it is of interest to investigate the reason for this unexpected feature.

We propose here that this phenomenon is due to the high symmetry structure of the FLRW background. Accordingly, we have studied the small anisotropy limit of the Bianchi-I manifold and found that there is always a ghost mode in the even sector, with a propagation speed that diverges in the isotropic limit. Furthermore, this mode does not have a mass gap; its frequency tends to zero for small values of the momentum. Therefore, at sufficiently large scales, the frequency cannot be considered as large compared to the ultraviolet cutoff of the theory. As a consequence, the ghost mode cannot be integrated out in general, and the almost-isotropic background becomes unstable under production of negative energy quanta.

Although our analysis is linear, the terms in the quadratic action with coefficients proportional to the small anisotropy can be interpreted as the leading order nonlinear perturbations in a pure FLRW universe. The presence of a mode with negative kinetic term indicates that a homogeneous and isotropic universe in nonlinear massive gravity is unstable.

Our conclusion about ghost instability is far more general than it appears, despite the simplicity of the analysis presented above. (Partially massless gravity [12] may evade our conclusion, but it is a different theory. Also, nonlinear completion is not known.) This is because, whenever a quadratic kinetic term vanishes, the leading kinetic term is generically cubic and thus can easily become negative, signaling the existence of ghost at the nonlinear level. Moreover, the other type of homogeneous and isotropic solutions (in the nonself-accelerating branch) suffer from ghost instability already at the linearized level [13], as expected from the classical work of Higuchi [14]. Therefore, all homogeneous and isotropic backgrounds, as well as most (if not all) of the known spherically symmetric inhomogeneous solutions, are unstable.

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