

AN UNCERTAINTY RELATION FOR QUANTUM SYSTEMS WITH AN ARBITRARILY LARGE NUMBER OF PARTICLES AND THE EXPANSION RATE OF MATTER IN THE BULK

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For any quantum system with an arbitrarily large number N of particles for which the lower end of the spectrum has a nonzero width and is bounded below by N , we rigorously derive an uncertainty relation for the product of N and the survival time T , a measure of the system resistance to change. We then use the derived inequality to investigate the highly nontrivial problem of the expansion rate of bulk matter as a function of the number N of electrons for large N . We approach the application to this problem by noting that resistance to the increase of the expansion rate can be quantum mechanically defined in terms of its survival time against such an increase. We show that a sufficient condition for matter to have a nonvanishing survival time against expansion rate increase is that the lower end of the spectrum is the lower end of an energy width for which applying the derived uncertainty relation becomes obvious. In turn, in particular, we show that if the expansion rate increases with its “size” of radius R , then the survival time decreases not faster than $1/R^3$ for large R . For completeness and consistency of the analysis, we also consider the formal zero-width limit. Because the derived uncertainty relation is general, we expect it also to have other applications.

Keywords: uncertainty relation, quantum system with arbitrarily large number of particles, expansion rate of bulk matter, survival time under expansion rate increase

For any quantum systems with an arbitrary large number N of particles for which the lower end of the spectrum is bounded below by N and is the lower end of an energy width, we rigorously derive an uncertainty relation for the product N and the survival time T of a system as a measure of resistance to any change from its initial state. We then use the corresponding inequality to investigate the critical problem of the expansion rate of matter. A fundamental property of matter is that it expands as a function of N and increases in size, and this fact can be rigorously quantified by the statement that for a nonvanishing probability of having electrons in matter within a sphere of radius R , the radius R necessarily grows not slower than $N^{1/3}$ for large N [1]. This can be easily seen from an explicit inequality given below. It is therefore unsurprising that matter occupies such a large volume. We here recall a significant historical statement by Ehrenfest [2], [3] concerning matter, when he addressed Pauli in 1931 on the occasion of awarding him the Lorentz medal for discovering that effect: “*We take a piece of metal, or a stone. When we think about it, we are astonished that this quantity of matter should occupy so large a volume.*” He then announced that the reason for that is the Pauli exclusion principle: “*Answer: only the Pauli Principle, no two electrons in the same state.*”

It is equally interesting to investigate the expansion rate of matter and infer if such an expansion slows down or speeds up as matter expands. An indirect way to investigate this is to study the resistance that

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matter may have against a change in the expansion rate and in turn to learn if this resistance decreases or increases under expansion. The resistance to change of a state can be defined in quantum physics in terms of its survival time T . With this in mind, we show that a sufficient condition for matter to have a nonvanishing survival time under an expansion rate increase is that the lower end of the spectrum of the Hamiltonian of the considered system is the lower end of a nonzero energy width, and then, in particular, we establish that if the expansion rate increases with R , then the survival time for large R necessarily decreases not faster than $1/R^3$.

We apply the derived uncertainty relation to the highly nontrivial quantum system with the Hamiltonian of bulk matter with N electrons given by

$$H = \sum_{i=1}^N \frac{\vec{p}_i^2}{2m} + \sum_{i<j}^N \frac{e^2}{|\vec{x}_i - \vec{x}_j|} - \sum_{i=1}^N \sum_{j=1}^k \frac{Z_j e^2}{|\vec{x}_i - \vec{R}_j|} + \sum_{i<j}^k \frac{Z_i Z_j e^2}{|\vec{R}_i - \vec{R}_j|}, \quad (1)$$

where m denotes the electron mass and \vec{x}_i and \vec{R}_j correspond to the respective electron and nucleus positions. We also consider neutral matter $\sum_{j=1}^k Z_j = N$. Undoubtedly, one of the most important and serious problems solved by quantum physics is the problem of the stability of matter. The Pauli exclusion principle turns out to be not only sufficient but also necessary for stability. Early investigations of the stability of matter led to the classic work of Dyson and Lenard [3], [4] and more recently Lieb and Thirring [5], [6]. The fundamental property of matter is that the lower end of the spectrum $E_N \sim N$ corresponding to the above Hamiltonian contains a single power of N in contrast to matter without the Pauli exclusion principle imposed, called “bosonic matter,” for which $E_N \sim N^\alpha$, where $\alpha > 1$ [3], [7]–[12]. Such a power-law behavior with $\alpha > 1$, unlike the behavior of ordinary matter, implies instability¹ because the formation of a single system consisting of $2N+2N$ particles is favored over two separate systems brought together, each consisting of $N+N$ particles, and the energy released when the two systems collapse into one, proportional to $(2N)^\alpha - 2N^\alpha$, is overwhelmingly large for large N , for example, for $N \sim 10^{23}$.

An elementary property, derived in detail in [1], that follows from Hamiltonian (1), which is easy to interpret physically, is that the probability $\text{Prob}[|\vec{x}_1| \leq R, \dots, |\vec{x}_N| \leq R]$ of having the N electrons within a radius R satisfies the simple inequality

$$\text{Prob}[|\vec{x}_1| \leq R, \dots, |\vec{x}_N| \leq R] \left(\frac{N}{v_R}\right)^{2/5} < \left(\frac{1}{a_0^3}\right)^{2/5} 1.846 [1 + Z^{2/3}]^{6/5}, \quad (2)$$

where $a_0 = 1/me^2$, $v_R = 4\pi R^3/3$, $Z = \max_j Z_j$, and $|e|Z_j$ denotes the charge of the j th nucleus in matter ($\hbar = 1$). We here note the inescapable fact that for a nonvanishing probability of having the electrons within a sphere of radius R , the corresponding volume v_R grows not any slower than the first power of N as $N \rightarrow \infty$ because the left-hand side of (2) would otherwise go to infinity and would contradict the finite upper bound in its right-hand side.

To derive the stated uncertainty relation under discussion, we let $-c_N N$ denote the lower end of the spectrum for all large N , where $c_N < c$ is a finite positive constant. We show that a sufficient condition for matter to have a nonvanishing survival time, i.e., a resistance to change in its expansion rate with N , is that the lower end of the spectrum is the lower end of an energy width of an interval $(-c_N, [-c_N + \Delta_N])N$, where $\Delta_N \leq \Delta$ for large N . Accordingly, we can consider the scaled Hamiltonian H/N and describe the lower end of the spectrum by a state as

$$|\psi\rangle = \int_0^\infty d\rho \chi_{\Delta_N}(\rho) \delta([H/N + c_N] - \rho) |\psi\rangle, \quad (3)$$

¹See, e.g., [12] for a pedagogical treatment of the problem of the stability of matter.

where $\chi_{\Delta_N}(\rho)$ is unity for ρ in the interval $(0, \Delta_N)$ and is zero otherwise. Multiplying this equation by $\langle\psi|$, and using the normalizability of the state $|\psi\rangle$, we obtain

$$\langle\psi|\psi\rangle = \int_0^\infty \frac{d\rho}{2\pi} \chi_{\Delta_N}(\rho) F(\rho) = 1, \quad (4)$$

$$F(\rho) = 2\pi \langle\psi|\delta([H/N + c_N] - \rho)|\psi\rangle. \quad (5)$$

The state $|\psi\rangle$ develops in time as

$$|\psi_t\rangle = e^{-iHt}|\psi\rangle = \int_0^\infty d\rho \chi_{\Delta_N}(\rho) e^{-i(-c_N+\rho)Nt} \delta([H/N + c_N] - \rho)|\psi\rangle, \quad (6)$$

$$\langle\psi|\psi_t\rangle = \int_0^\infty \frac{d\rho}{2\pi} e^{-i(-c_N+\rho)Nt} \chi_{\Delta_N}(\rho) F_N(\rho), \quad (7)$$

where $|\langle\psi|\psi_t\rangle|^2$ denotes the probability that the system at time t is found in its initial state and is called the *survival probability*. The survival time is defined by

$$T = \frac{1}{2} \int_{-\infty}^\infty dt |\langle\psi|\psi_t\rangle|^2 = \frac{1}{2N} \int_0^\infty \frac{d\rho}{2\pi} \chi_{\Delta_N}(\rho) |F_N(\rho)|^2. \quad (8)$$

On the other hand, the Schwarz inequality gives

$$1 = \left| \int_0^\infty \frac{d\rho}{2\pi} \chi_{\Delta_N}(\rho) F(\rho) \right|^2 \leq \left(\int_0^\infty \frac{d\rho}{2\pi} \chi_{\Delta_N}(\rho) \right) \left(\int_0^\infty \frac{d\rho}{2\pi} \chi_{\Delta_N}(\rho) |F_N(\rho)|^2 \right) \quad (9)$$

from (4). From (8), we then obtain the basic inequality

$$1 \leq \frac{1}{\pi} NT \Delta_N \leq \frac{1}{\pi} NT \Delta \quad (10)$$

or

$$\frac{\pi}{T \Delta} \leq N. \quad (11)$$

Expressing N in terms of T in the left-hand side of (2), from (11), we obtain

$$\text{Prob}[|\vec{x}_1| \leq R, \dots, |\vec{x}_N| \leq R] \left(\frac{1}{v_R T} \right)^{2/5} < \left(\frac{\Delta}{\pi a_0^3} \right)^{2/5} 1.846 [1 + Z^{2/3}]^{6/5}, \quad (12)$$

where the left-hand side is a lower bound of the left-hand side of (2) up to the $(\Delta/\pi)^{2/5}$ term appearing in the right-hand side of (12).

This equation is interesting in many respects. The inequality does not rule out the constancy or the increase of the survival time T with R for large R , i.e., the constancy or decrease of the expansion rate of matter. But it shows that if the expansion rate increases with R for large R , then the survival time decreases not faster than $1/R^3$ as $R \rightarrow \infty$ because the left-hand side of (12) would otherwise go to infinity, which contradicts the finite upper bound in its right-hand side for a nonzero probability. Finally, from (10)–(12), we also recover the expected quantum mechanical result in the formal limit $\Delta_N \equiv \Delta \rightarrow 0$ for a zero width because these equations say that we must simultaneously have $T \rightarrow \infty$, which corresponds to an isolated bound state, with no change in the corresponding state (up to a phase factor), and this agrees with our analysis.

Because the obtained uncertainty relation is general, we expect it to also have applications to other problems. We will consider such additional applications and also the treatment of collapsing “bosonic matter,” which requires some special tools, in a separate work.

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