ON THE CONSTRUCTION OF COHERENT STATES
OF NEWTON-EQUIVALENT HARMONIC OSCILLATOR

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Phadungkiat Kwangkaew

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Title ON THE CONSTRUCTION OF COHERENT STATES OF NEWTON-EQUIVALENT HARMONIC OSCILLATOR

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ABSTRACT

This thesis presents the study of the construction of coherent states of Newton-Equivalent Quantum Harmonic Oscillator (NEQHO). The coherent states are important in quantum mechanics. They are constructed from quantum harmonic oscillator. The uncertainty relation for the measurement of position and momentum in coherent state takes the minimal value. NEQHO stands for the model of one-parameter family of Hamiltonians alternative to standard quatum harmonic oscillator. The classical version of these Hamiltonians gives rise to the Newton's equation for a particle moving under the harmonic oscillator potential. However there is no guarantee that the quantum version of these Hamiltonians would describe the same physics. In this work, we construct the wave functions of the coherent states of NEQHO and study their uncertianty relations. We focus on the case of $|\alpha| \leq 2$, where α is a complex number characterising coherent states. Within the scope of our study, the minimal value of uncertianty relations is attained.

CHAPTER I

INTRODUCTION

Coherent states are quantum states of theoretical importance and with wide range of applications in physics. The simplest examples of coherent states are those coming from Quantum Harmonic Oscillator (QHO). A coherent state is defined as an eigenstate of the lowering operator with its corresponding complex eigenvalue. One of the interesting properties of these states is that they play a role of a ground state which is shifted in the phase space in such a way that the real part of the eigenvalue is proportional to the displacement along the x–direction, whereas the imaginary part is proportional to the displacement along the p–direction. Moreover, these states can be thought to be the most classical because the minimal value of uncertainty in measurements of position and momentum is satisfied.

As one has known that there exists the creation or annihilation operators which create or destroy one photon in a photon number state in the quantisation of the oscillation of electromagnetic wave. Hence, coherent states arise in this context as superpositions of photon number states and also being eigenstates of the annihilation operators. The probability of detecting a number of photons in a coherent state satisfies Poisson distribution. In an example of applications, a laser beam is a coherent state. Laser beams are stable because of the property of coherent states in which these state remains the same even after one photon is detected. See [1, 2, 3, 4, 5, 6, 7] for more details on theoretical importance and applications of coherent states. In this work, the issue is closely related to QHO coherent states and their uncertainty relations are focused.

Let us turn to discuss about the Hamiltonian in classical mechanics. One usually considers a Hamiltonian which is a function of generalised coordinates and conjugate momenta. The dynamics of a classical system can be described by the Hamilton's equations. It is possible to derive the Newton's equations from a given Hamiltonian for a system of a particle moving in one dimensional space. Surprisingly, the form of Hamiltonian is not unique. In other words, the same Newton's equations can be obtained from different Hamiltonians. Although there is no problem in classical physics, these Hamiltonians may lead to different physics from one another after quantisation. Hence, it is very important to be able to distinguish different physical phenomena corresponding to these Hamiltonians.

For the quantum harmonic oscillator, a one-parameter family of Hamiltonians is proposed [8]. It is found that the classical version of all the Hamiltonians in the family gives rise to the same Newton's equation for the simple harmonic oscillator. The physical quantities, e.g. the energy spectrum and the corresponding eigenstates have been investigated which are very important for studying physical phenomena relating to these Hamiltonians. Unfortunately, not much has been done along this direction. As far as we are aware, there has been several related works. For example, a mathematical extension to this work is given in [9], the explanations of related systems are given in [10, 11, 12].

To distinguish from the quantum harmonic oscillator with standard Hamiltonian and those with the one-parameter family in [8], let us keep labelling the quantum harmonic oscillator with standard Hamiltonian as QHO. But for the quantum harmonic oscillator with the one-parameter family of Hamiltonians, we will call them as Newton-equivalent quantum harmonic oscillator (NEQHO). We will give more details about the NEQHO in the main sections.

CHAPTER II

LITERATURE REVIEWS

2.1 Quantum harmonic oscillator

The harmonic oscillator model is an interesting model in physics [13, 14, 15, 16] because of a lot of systems can be reduced to or described by the simple model of simple harmonic oscillation. In the classical mechanics, we have seen many such as, spring oscillation, pendulum and so on. That model can help us to understand the simple model and can be adapted into the more complicated model. In the quantum mechanics we also start with the simple model. To study the quantum harmonic oscillator, we have two ways to analyse the system by starting from the Schrodinger's equation. The first way we study in form of hypergeometric equation and the second way we evaluate in form of eigenvalue equation by writing the position operators and momentum operators in terms of the ladder operators. Both ways give the same results. Let us proceed to show these.

The Hamiltonian of quantum harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{2.1}$$

For convenience we will write eq.(2.1) in ladder operator form. Let us start with the commutaion relation, $[\hat{x}, \hat{p}] = i\hbar$. Consider R.H.S. of eq.(2.1)

$$\begin{aligned}
\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 &= \hbar\omega\left[\frac{\hat{p}^2}{2m\hbar\omega} + \frac{m\omega}{2\hbar}\hat{x}^2 - \frac{1}{2} + \frac{1}{2}\right] \\
&= \hbar\omega\left[\frac{\hat{p}^2}{2m\hbar\omega} + \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{i}{2\hbar}i\hbar + \frac{1}{2}\right] \\
&= \hbar\omega\left[\frac{\hat{p}^2}{2m\hbar\omega} + \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{i}{2\hbar}[\hat{x}, \hat{p}] + \frac{1}{2}\right] \\
&= \hbar\omega\left[\frac{\hat{p}^2}{2m\hbar\omega} + \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{i}{2\hbar}\Big(\hat{x}\hat{p} - \hat{p}\hat{x}\Big) + \frac{1}{2}\right] \\
&= \hbar\omega\left[\frac{m\omega}{2\hbar}\hat{x}^2 + i\left(\frac{1}{4\hbar^2}\right)^{\frac{1}{2}}\hat{x}\hat{p} - i\left(\frac{1}{4\hbar^2}\right)^{\frac{1}{2}}\hat{p}\hat{x} + \frac{\hat{p}^2}{2m\hbar\omega} + \frac{1}{2}\right]
\end{aligned}$$

$$= \hbar\omega\left[\left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p}\right)\left(\sqrt{\frac{m\omega}{2\hbar}}\hat{x} + i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p}\right) + \frac{1}{2}\right]$$

$$= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{2.2}$$

where

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \tag{2.3}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right), \tag{2.4}$$

$\hat{a}$ is the lowering operator, and $\hat{a}^\dagger$ is the raising operator. When we define these operators, we use the fact that $\hat{x}$ and $\hat{p}$ are Hermitian, $\hat{x}^\dagger = \hat{x}$ and $\hat{p}^\dagger = \hat{p}$. The commutation relation of $\hat{a}$ and $\hat{a}^\dagger$ can be written as,

$$[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \tag{2.5}$$

and

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1. \tag{2.6}$$

From $\hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger - 1$, the Hamiltonian sastisfies the following commutation relations

$$[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}, \tag{2.7}$$

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger. \tag{2.8}$$

Consider the eigenvalue equation of Hamiltonian:

$$\hat{H}|n\rangle = E_n|n\rangle. \tag{2.9}$$

where E_n is the eigenvalue and $|n\rangle$ is the eigenstate.

To interpret, we rewrite an eigenvalue equation as

$$\hat{a}^\dagger\hat{a}|n\rangle = \left(\frac{E_n}{\hbar\omega} - \frac{1}{2}\right)|n\rangle. \tag{2.10}$$

Let us define $\hat{N} \equiv \hat{a}^\dagger \hat{a}$, which is the number operator. It is satisfies the commutation relations, which are

$$[\hat{N}, \hat{a}] = -\hat{a}, \tag{2.11}$$

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger. \tag{2.12}$$

Consider the eigenvalue equation of the number operator $\hat{N}$

$$\hat{N}|n\rangle = n|n\rangle. \tag{2.13}$$

Multiplying with states $\langle n|$, we obtain

$$\begin{aligned} \langle n|\hat{N}|n\rangle &= n\langle n|n\rangle \\ \left(\langle n|\hat{a}^\dagger\right)\left(\hat{a}|n\rangle\right) &= n\underbrace{\langle n|n\rangle}_{\neq 0}, \end{aligned} \tag{2.14}$$

which implies

$$n \geq 0. \tag{2.15}$$

Therefore, the lowest possible eigenstate corresponding to the eigenvalue $n = 0$ satisfies

$$\hat{a}|0\rangle = 0. \tag{2.16}$$

This definition will help us to find the ground state of wave function.

Let us show that, if the state $|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue n, then the state $a|n\rangle$ is also an eigenstate of $\hat{N}$ with eigenvalue $(n-1)$.

Proof. With the aid of eq.(2.11)

$$\begin{aligned} \hat{N}\hat{a}|n\rangle &= \left(\hat{a}\hat{N} - \hat{a}\right)|n\rangle \\ &= \hat{a}\left(\hat{N} - 1\right)|n\rangle \\ &= \left(n-1\right)\left(\hat{a}|n\rangle\right). \end{aligned} \tag{2.17}$$

Eq.(2.17) suggests that $\hat{a}|n\rangle = n_-|n-1\rangle$. To find n_-, one considers $\langle n|\hat{N}|n\rangle$,

$$\begin{aligned}\langle n|\hat{N}|n\rangle &= \langle n|n|n\rangle \\ \langle n|\hat{a}^\dagger\hat{a}|n\rangle &= n\langle n|n\rangle \\ \Big(\langle n|\hat{a}^\dagger\Big)\Big(\hat{a}|n\rangle\Big) &= n. \end{aligned} \tag{2.18}$$

One can also write

$$\begin{aligned}\Big(\langle n|\hat{a}^\dagger\Big)\Big(\hat{a}|n\rangle\Big) &= \Big(\langle n-1|n_-\Big)\Big(n_-|n-1\rangle\Big) \\ &= n_-^2. \end{aligned} \tag{2.19}$$

Comparing eq.(2.18) to (2.19), we obtained

$$n_-^2 = \sqrt{n}. \tag{2.20}$$

So we can write state of $\hat{a}|n\rangle$ as

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle. \tag{2.21}$$

Similarly, if the state $|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue n, then the state $a^\dagger|n\rangle$ is also an eigenstate of $\hat{N}$ with eigenvalue $(n+1)$.

Proof. With the aid of eq.(2.12)

$$\begin{aligned}\hat{N}\hat{a}^\dagger|n\rangle &= \Big(\hat{a}^\dagger\hat{N} + \hat{a}^\dagger\Big)|n\rangle \\ &= \hat{a}^\dagger\Big(\hat{N}+1\Big)|n\rangle \\ &= \Big(n+1\Big)\Big(\hat{a}^\dagger|n\rangle\Big). \end{aligned} \tag{2.22}$$

Eq.(2.22) suggests that $\hat{a}^\dagger|n\rangle = n_+|n+1\rangle$. To find n_+, one considers $\langle n|\hat{N}|n\rangle$,

$$\begin{aligned}\langle n|\hat{N}|n\rangle &= \langle n|n|n\rangle \\ \langle n|\hat{a}^\dagger\hat{a}|n\rangle &= n\langle n|n\rangle \\ \langle n|\Big(\hat{a}\hat{a}^\dagger - 1\Big)|n\rangle &= n\langle n|n\rangle \\ \Big(\langle n|\hat{a}\Big)\Big(\hat{a}^\dagger|n\rangle\Big) &= \Big(n+1\Big). \end{aligned} \tag{2.23}$$

One can also write

$$\Big(\langle n|\hat{a}\Big)\Big(\hat{a}^{\dagger}|n\rangle\Big) = \Big(\langle n+1|n_{+}\Big)\Big(n_{+}|n+1\rangle\Big)$$
$$= n_{+}^{2}. \tag{2.24}$$

Comparing eq.(2.23) to (2.24), we obtained

$$n_{+}^{2} = \Big(n+1\Big) \rightarrow \sqrt{n+1}. \tag{2.25}$$

So we can write $\hat{a}^{\dagger}|n\rangle$ as

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{2.26}$$

Eq.(2.16) implies that if the lowering operator acts on the ground state, one obtains zero because that the operator could not further decrease the state. On the other hand, if the raising operator acts on the ground state, it should raise up the state too. From eq.(2.26) if we operate $\hat{a}^{\dagger}$ operator on the ground state n times, the states $|n\rangle$ is given by

$$|n\rangle = \frac{(\hat{a}^{\dagger})^{n}}{\sqrt{n!}}|0\rangle. \tag{2.27}$$

By taking the inner product between eq.(2.27) and $|x\rangle$, we obtain the wave function as

$$\psi_n(x) \equiv \langle x|n\rangle$$
$$= \langle x|\frac{(a^{\dagger})^{n}}{\sqrt{n!}}|0\rangle. \tag{2.28}$$

Eq.(2.28) allows us to find the wave function of any states once we know the ground state wave function. To find the wave function of the ground state one uses eq.(2.16),

$$0 = \langle x|a|0\rangle$$
$$= \sqrt{\frac{m\omega}{2\hbar}}\left(\langle x|\hat{x}|0\rangle + \frac{i}{m\omega}\langle x|\hat{p}|0\rangle\right). \tag{2.29}$$

Then we split out $\hat{x}$ and $\hat{p}$ from the eigenvalue equation and rewrite eq.(2.29) as

$$0 = \sqrt{\frac{m\omega}{2\hbar}}\left(x\langle x|0\rangle + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\langle x|0\rangle\right)$$
$$= x\psi_0(x) + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\psi_0(x). \tag{2.30}$$

This is a first order ODE and we obtain

$$\psi_0(x) = N \exp\left(-\frac{m\omega}{2\hbar}x^2\right), \tag{2.31}$$

where N is the normalisation factor, which can be obtained by using $\langle\psi|\psi\rangle = 1$. The ground state wave function is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right). \tag{2.32}$$

We use the similar method to find the wave function of the first excited state

$$\psi_1(x) = \langle x|a^\dagger|0\rangle = \langle x|1\rangle. \tag{2.33}$$

With the aid of eq.(2.3), eq.(2.33) becomes

$$\begin{aligned}
\psi_1(x) &= \langle x|a^\dagger|0\rangle \\
&= \sqrt{\frac{m\omega}{2\hbar}}\left(\langle x|\hat{x}|0\rangle - \frac{i}{m\omega}\langle x|\hat{p}|0\rangle\right) \\
&= \sqrt{\frac{m\omega}{2\hbar}}\left(x\langle x|0\rangle - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\langle x|0\rangle\right).
\end{aligned} \tag{2.34}$$

This time we use eq. (2.31) to find $\psi_1(x)$

$$\begin{aligned}
\psi_1(x) &= \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
&= \frac{2x}{\sqrt{1}}\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right),
\end{aligned} \tag{2.35}$$

or

$$\psi_1(x) = 2x\sqrt{\frac{m\omega}{2\hbar}}\psi_0(x). \tag{2.36}$$

Similarly, we can obtain the second excited state

$$\begin{aligned}
\psi_2(x) &= \langle x|\frac{(a^\dagger)^2}{\sqrt{2!}}|0\rangle \\
&= \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{2\hbar}}\left(\langle x|\hat{x}|1\rangle - \frac{i}{m\omega}\langle x|\hat{p}|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{2\hbar}}\left(x\langle x|1\rangle - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\langle x|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)2x\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{2\hbar}}\Bigg(2x^2\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\
&-\frac{\hbar}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{\partial}{\partial x}\left(2x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right)\Bigg)\\
&= \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{2\hbar}}\Bigg(2x^2\sqrt{\frac{m\omega}{2\hbar}}\psi_0(x)\\
&-\frac{\hbar}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}\left(2x^2\Big(-\frac{m\omega}{2\hbar}2\Big)+2\right)\psi_0(x)\Bigg)\\
&= \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{2\hbar}}\left(4x^2-2\frac{\hbar}{m\omega}\right)\sqrt{\frac{m\omega}{2\hbar}}\psi_0(x)\\
&= \left(4x^2-2\frac{\hbar}{m\omega}\right)\frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{2\hbar}}\sqrt{\frac{m\omega}{2\hbar}}\psi_0(x). \qquad (2.37)
\end{aligned}
$$

For the third excited state,

$$
\begin{aligned}
\psi_3(x) &= \langle x|\frac{(a^\dagger)^3}{\sqrt{3!}}|0\rangle\\
&= \frac{1}{\sqrt{3}}\sqrt{\frac{m\omega}{2\hbar}}\Big(\langle x|\hat{x}|2\rangle-\frac{i}{m\omega}\langle x|\hat{p}|2\rangle\Big)\\
&= \frac{1}{\sqrt{3}}\sqrt{\frac{m\omega}{2\hbar}}\Big(x\langle x|2\rangle-\frac{\hbar}{m\omega}\frac{\partial}{\partial x}\langle x|2\rangle\Big)\\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\left(\frac{m\omega}{2\hbar}\right)^{\frac{3}{2}}\left(x-\frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\Big(4x^2-2\frac{\hbar}{m\omega}\Big)\psi_0(x)\\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\Big(\frac{m\omega}{2\hbar}\Big)^{\frac{3}{2}}\Big(4x^3-2x\frac{\hbar}{m\omega}-8x\frac{\hbar}{m\omega}\\
&+4x^3-2x\frac{\hbar}{m\omega}\Big)\psi_0(x)\\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\left(\frac{m\omega}{2\hbar}\right)^{\frac{3}{2}}\Big(8x^3-12x\frac{\hbar}{m\omega}\Big)\psi_0(x). \qquad (2.38)
\end{aligned}
$$

The step can be repeated upto the n^{th} order. It is also useful to change the variable x to y which is dimensionless as

$$\psi_n(y) = H_n(y)\Big(\frac{1}{2^n n!}\Big)^{\frac{1}{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\Big(-\frac{y^2}{2}\Big), \qquad (2.39)$$

where $y = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}x$ and $H_n(y)$ is Hermite polynomials

$$H_n(y) = (-1)^n e^{y^2}\frac{d^n}{dy^n}e^{-y^2}.$$

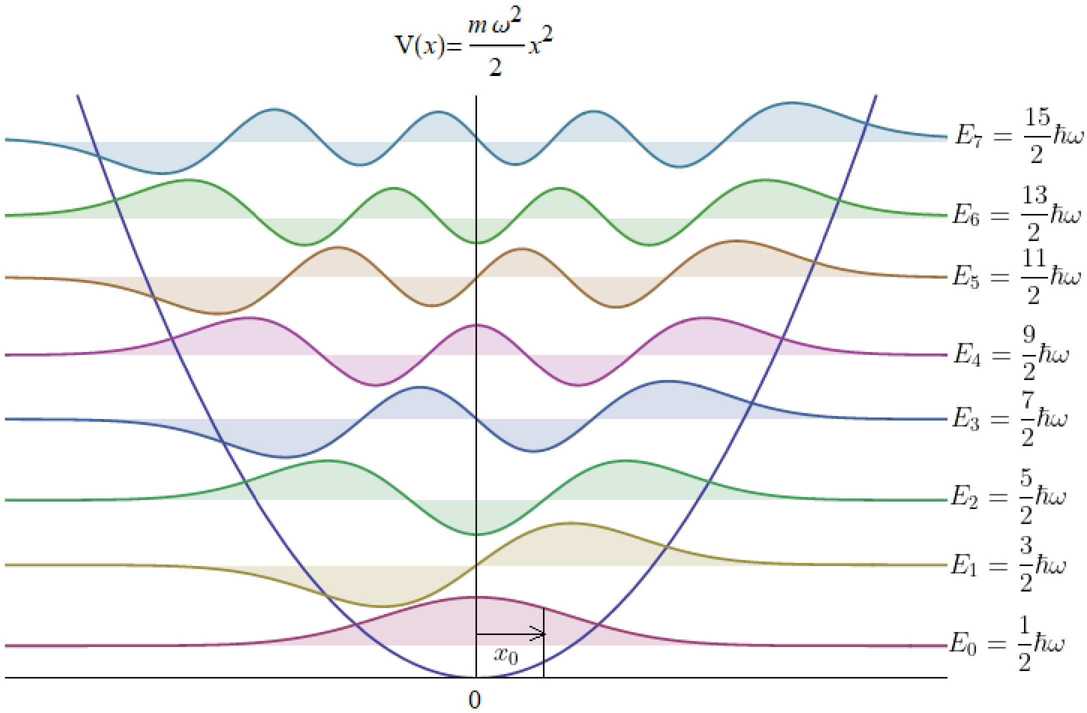

Figure 1 The plot of the wave function at each energy level with the potential V for the Quantum Harmonic oscillator, for $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ is the width of the ground state in a Gaussian distribution. This figure is adapted from [28].

Eq.(2.39) is an eigenfunction of the Hamiltonian (2.1) with the corresponding eigenvalue

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{2.40}$$

If n is zero, $E_0 = \hbar\omega/2$ which is known as the energy of the ground state. We can plot the possible energy eigenvalues (2.40) and the wave functions (2.39) of harmonic oscillator with potential $V(x) = m\omega^2 x^2/2$ in Fig1. The figure shows the discrete energy levels with the gap of $\hbar\omega$.

In this work, we are particularly intereted in the lowering operator because of its property. In particular, it has eigenstates which are interstings both in theory and in applications. These states are called the coherent states. We will discuss about these states again in the next section.

2.2 Perturbation Theory

The perturbation theory is a useful approximation to analyse the system in physics, because in practice a lot of systems are too complicated to explain without approximation [17, 18, 19, 20, 21]. Complicated systems which are slightly deviated from simple systems can be studied using the perturbation theories

To study perturbation theory, we start with the time-independent Hamiltonian

$$\hat{H} = \hat{H}^0 + \lambda \hat{H}^1, \tag{2.41}$$

where $\hat{H}^1$ is the perturbed Hamiltonian and λ is the bookkeeping parameter. The eigenvalue equation of the unperturbed system can be written as

$$\hat{H}^0 |n^0\rangle = E_n^0 |n^0\rangle. \tag{2.42}$$

The eigenvalue equation for the full Hamiltonian is

$$\hat{H}|m\rangle = E_m|m\rangle, \tag{2.43}$$

where the eigenstates can be written as a linear combination of unperturbed eigenstates as

$$|m\rangle = \sum_n C_{nm}|n^0\rangle. \tag{2.44}$$

To write the full form of perturbation for the eigenvalue equation, we use eqs.(2.41) to (2.44)

$$\begin{aligned} \left(\hat{H}^0 + \lambda \hat{H}^1\right)|m\rangle &= E_m|m\rangle \\ \sum_n C_{nm}\left(\hat{H}^0 + \lambda \hat{H}^1\right)|n^0\rangle &= \sum_n E_m C_{nm}|n^0\rangle \\ \sum_n C_{nm}\left(E_n^0 - E_m + \lambda \hat{H}^1\right)|n^0\rangle &= 0. \end{aligned} \tag{2.45}$$

We need to find the coefficient by multipling $\langle^0 j|$ with eq. (2.45),

$$\begin{aligned} \langle^0 j|\sum C_{nm}\left(E_n^0 - E_m + \lambda \hat{H}^1\right)|n^0\rangle &= 0 \\ C_{jm}\left(E_j^0 - E_m + \lambda H_{jj}^1\right) + \lambda \sum_{n\neq j} C_{nm} H_{jn}^1 &= 0, \end{aligned} \tag{2.46}$$

where $H^1_{jn} = \langle {}^0 j|\hat{H}^1|n^0\rangle$ and $\langle {}^0 j|n^0\rangle = \delta_{jn}$. Let us expand C_{jn} and E_m as

$$C_{jm} = C^0_{jm} + \lambda C^1_{jm} + \lambda^2 C^2_{jm} + \cdots\ , \tag{2.47}$$

$$E_m = E^0_m + \lambda E^1_m + \lambda^2 E^2_m + \cdots\ . \tag{2.48}$$

Subtituting eq.(2.47) and (2.48) into eq.(2.46)

$$\begin{aligned}&\Big(E^0_j - E^0_m\Big)C^0_{jm} + \lambda\Big[\Big(H^1_{jj} - E^1_m\Big)C^0_{jm} + \Big(E^0_j - E^0_m\Big)C^1_{jm} \\ &+ \sum_{n\neq j} H^1_{jn}C^0_{nm}\Big] + \lambda^2\Big[\Big(H^1_{jj} - E^1_m\Big)C^1_{jm} + \Big(E^0_j - E^0_m\Big)C^2_{jm} \\ &+ \sum_{n\neq j} C^1_{nm}H^1_{jn} - C^0_{jm}E^2_m\Big] + O(\lambda^3) = 0.\end{aligned} \tag{2.49}$$

The zeroth order of λ of eq.(2.49) is

$$\Big(E^0_j - E^0_m\Big)C^0_{jm} = 0, \tag{2.50}$$

which is trivial if $m = j$. If $m \neq j$, C^0_{jm} then

$$C^0_{jm} = \delta_{jm}. \tag{2.51}$$

Let us next consider the first order of λ of eq.(2.49). It is

$$\Big(H^1_{jj} - E^1_m\Big)C^0_{jm} + \Big(E^0_j - E^0_m\Big)C^1_{jm} + \sum_{n\neq j} H^1_{jn}C^0_{nm} = 0. \tag{2.52}$$

Considering the case $j = m$ and using $C^0_{jm} = \delta_{jm}$, it gives,

$$E^1_j = H^1_{jj}. \tag{2.53}$$

However, when $j \neq m$,

$$\begin{aligned}\Big(H^1_{jj} - E^1_m\Big)C^0_{jm} + \Big(E^0_j - E^0_m\Big)C^1_{jm} + \sum_{n\neq j} H^1_{jn}C^0_{nm} - \sum_{n=j} H^1_{jj}C^0_{jm} = 0 \\ \Big(E^0_j - E^0_m\Big)C^1_{jm} + \sum_{n\neq j} H^1_{jn}\delta^0_{nm} = 0\end{aligned}$$

then,

$$C^1_{jm} = \frac{H^1_{jm}}{E^0_m - E^0_j}. \tag{2.54}$$

Let us finally consider the second order in λ of eq.(2.49). It is

$$\Big(H^1_{jj} - E^1_m\Big)C^1_{jm} + \Big(E^0_j - E^0_m\Big)C^2_{jm} + \sum_{n\neq j} C^1_{nm}H^1_{jn} - C^0_{jm}E^2_m = 0. \tag{2.55}$$

From the result of the first order and $j = m$, we obtain

$$E^2_j = \sum_{n\neq j} \frac{H^1_{nj}H^1_{jn}}{E^0_j - E^0_n}. \tag{2.56}$$

When $j \neq m$,

$$\begin{aligned}
\Big(E^0_j - E^0_m\Big)C^2_{jm} &= -C^1_{jm}\Big(H^1_{jj} - E^1_m\Big) \\
&\quad - \sum_{n\neq j} C^1_{nm}H^1_{jn} + C^1_{jm}H^1_{jj} - C^0_{jm}E^2_m \\
\Big(E^0_j - E^0_m\Big)C^2_{jm} &= C^1_{jm}\Big(H^1_{jj} - E^1_m\Big) \\
&\quad + \sum_{n\neq j} C^1_{nm}\Big(\frac{H^1_{nj}}{E^0_j - E^0_n}\Big)H^1_{jn} + C^1_{jm}H^1_{jj} \\
\Big(E^0_j - E^0_m\Big)C^2_{jm} &= \frac{H^1_{jm}}{E^0_m - E^0_j}\Big(- E^1_m\Big) + \sum_{n\neq j} \frac{H^1_{nm}}{E^0_m - E^0_n}H^1_{jn}
\end{aligned}$$

then,

$$C^2_{jm} = \frac{H^1_{jm}H^1_{mm}}{\Big(E^0_j - E^0_m\Big)^2} + \sum_{n\neq j} \frac{H^1_{nm}H^1_{jn}}{\Big(E^0_m - E^0_n\Big)\Big(E^0_j - E^0_m\Big)}. \tag{2.57}$$

Thus,

$$E_m = E^0_m + \lambda H^1_{mm} - \lambda^2 \sum_{n\neq j} \frac{H^1_{nj}H^1_{jn}}{E^0_j - E^0_n}. \tag{2.58}$$

Actually, the pertubation can be found up to any orders but in our work we are interested only in the first order.

The pertubation theory will help us analyse the difficult system in the next chapter.

2.3 Coherent States

The coherent states are important and interested in a wide range of physics. Furthermore the properties of the coherent states are important in physics [1, 2, 3, 4, 19, 20]. This is because of the properties for those states are useful to help us evaluate a lot of things in physics. For instance, any states can be expressed as a linear combination of coherent states. The coherent states arise from the QHO. Let us consider the lowering operator with eigenvalue equation,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{2.59}$$

where $\alpha \in \mathbb{C}$ is eigenvalue of $\hat{a}$. The reason that α can be a complex number is beacuse $\hat{a}$ is not hermitian. It is in contrast to the number operator because the number operator is hermitian, yielding a real eigenvalue n . There are two ways to obtain the form of the coherent states. The first way is by expressing as a linear combination of energy eigenstates. The second way is by using the displacement operators.

So, let us discuss the first way. Consider

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle. \tag{2.60}$$

To find c_n, let us substitute eq.(2.60) into eq.(2.59). The L.H.S. is

$$\begin{aligned} \hat{a}|\alpha\rangle &= \sum_{n=0}^{\infty} c_n\hat{a}|n\rangle \\ &= \sum_{n=1}^{\infty} c_n\sqrt{n}|n-1\rangle. \end{aligned}$$

The R.H.S. of eq.(2.59)

$$\begin{aligned} \alpha|\alpha\rangle &= \alpha\sum_{n=0}^{\infty} c_n|n\rangle \\ &= \alpha\sum_{n=1}^{\infty} c_{n-1}|n-1\rangle. \end{aligned} \tag{2.61}$$

By comparing two sides, we obtain the relations of the coefficients as

$$c_n\sqrt{n} = \alpha c_{n-1}.$$

To solve for c_n, we explicitly write eq.(2.61) for first few values of n

$$\begin{aligned} n = 1, c_1 &= \frac{\alpha}{\sqrt{1}} c_0 \\ n = 2, c_2 &= \frac{\alpha}{\sqrt{2}} c_1 \\ c_2 &= \frac{\alpha}{\sqrt{2}} \frac{\alpha}{\sqrt{1}} c_0 \\ n = 3, c_3 &= \frac{\alpha}{\sqrt{3}} \frac{\alpha}{\sqrt{2}} \frac{\alpha}{\sqrt{1}} c_0 \\ c_3 &= \frac{\alpha^3}{\sqrt{3!}} c_0. \end{aligned}$$

We obtain the solution for the recurrence equation as

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0, \tag{2.62}$$

which is easy to see that this is the case. So, substituting eq.(2.62) into eq. (2.60) gives

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{2.63}$$

To find c_0, we need to normalise it, let us use normalisation condition $\langle\alpha|\alpha\rangle = 1$. From eq.(2.63), we obtain,

$$\langle\alpha| = c_0^* \sum_{n=0}^{\infty} \frac{\alpha^{*m}}{\sqrt{m!}} \langle m|. \tag{2.64}$$

Then,

$$\begin{aligned} \langle\alpha|\alpha\rangle &= |c_0|^2 \sum_{n,m=0}^{\infty} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!}\sqrt{n!}} \langle m|n\rangle \quad ; \langle m|n\rangle = \delta_{mn} \\ &= |c_0|^2 \sum_{n=0}^{\infty} \frac{\alpha^{*n}\alpha^n}{n!} \quad ; \alpha^*\alpha = |\alpha|^2 \\ &= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \quad ; \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x) \\ &= |c_0|^2 \exp\left(|\alpha|^2\right), \end{aligned} \tag{2.65}$$

By using normalisation condition, we obtain

$$|c_0|^2 = \exp\Big(-|\alpha|^2\Big)$$
$$c_0 = \exp\Big(-\frac{|\alpha|^2}{2}\Big). \tag{2.66}$$

Substituting eq.(2.66) into eq.(2.63), we obtain

$$|\alpha\rangle = \exp\Big(-\frac{|\alpha|^2}{2}\Big)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.67}$$

To check this state, we can substitue eq.(2.67) back into eq.(2.59), we obtain

$$\begin{aligned}
\hat{a}|\alpha\rangle &= \exp\Big(-\frac{|\alpha|^2}{2}\Big)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\hat{a}|n\rangle \\
&= \exp\Big(-\frac{|\alpha|^2}{2}\Big)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle \\
&= \exp\Big(-\frac{|\alpha|^2}{2}\Big)\sum_{n=1}^{\infty}\frac{\alpha^n}{\sqrt{(n-1)!}}|n-1\rangle \quad ; n-1=m \\
&= \exp\Big(-\frac{|\alpha|^2}{2}\Big)\sum_{m=0}^{\infty}\frac{\alpha^{m+1}}{\sqrt{(m)!}}|m\rangle \\
&= \alpha\exp\Big(-\frac{|\alpha|^2}{2}\Big)\sum_{m=0}^{\infty}\frac{\alpha^{m}}{\sqrt{(m)!}}|m\rangle \\
&= \alpha|\alpha\rangle.
\end{aligned} \tag{2.68}$$

If we substitute eq.(2.27) into eq.(2.67), we rewrite it as,

$$|\alpha\rangle = \exp\Big(-\frac{|\alpha|^2}{2}\Big)\sum_{n=0}^{\infty}\frac{(\alpha\hat{a}^\dagger)^n}{n!}|0\rangle. \tag{2.69}$$

Let us now discuss the second way to write the form of coherent states. Let us start from

$$|\alpha\rangle = D(\alpha)|0\rangle, \tag{2.70}$$

where $D(\alpha) = \exp\Big(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\Big)$ is "displacement operator". The coherent state written in (2.70) is also called the displacement ground state. Mathematically, from the Baker-Campbell-Hausdorff formula we can write

$$\exp(\hat{A}+\hat{B}) = \exp\Big(-\frac{1}{2}[\hat{A},\hat{B}]\Big)\exp(\hat{A})\exp(\hat{B}), \tag{2.71}$$

where $[\hat{A}, \hat{B}] \equiv c$, where c is a complex number (or $[c, \hat{A}] = [c, \hat{B}] = 0$). For the displacement operator, we set $\hat{A} = \alpha\hat{a}^\dagger$ and $\hat{B} = -\alpha^*\hat{a}$, we then have

$$\begin{aligned}
\exp\Big(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\Big) &= \exp\Big(-\frac{1}{2}[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}]\Big)\exp\Big(\alpha\hat{a}^\dagger\Big)\exp\Big(-\alpha^*\hat{a}\Big) \\
&= \exp\Big(-\frac{|\alpha|^2}{2}[\hat{a}^\dagger, -\hat{a}]\Big)\exp\Big(\alpha\hat{a}^\dagger\Big)\exp\Big(-\alpha^*\hat{a}\Big) \\
&= \exp\Big(-\frac{|\alpha|^2}{2}\Big)\exp\Big(\alpha\hat{a}^\dagger\Big)\exp\Big(-\alpha^*\hat{a}\Big),
\end{aligned} \tag{2.72}$$

where we have used the identities

$$[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}] = |\alpha|^2, \tag{2.73}$$

$$[\alpha\hat{a}^\dagger, [\alpha\hat{a}^\dagger, -\alpha^*\hat{a}]] = [-\alpha^*\hat{a}, [\alpha\hat{a}^\dagger, -\alpha^*\hat{a}] = 0. \tag{2.74}$$

Then, we rewrite the displacement operator as

$$D(\alpha) = \exp\Big(-\frac{1}{2}|\alpha|^2\Big)\exp\Big(\alpha\hat{a}^\dagger\Big)\exp\Big(-\alpha^*\hat{a}\Big), \tag{2.75}$$

The properties of the displacement operator are;

$$I)\quad D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha), \tag{2.76}$$

$$II)\ \ D^\dagger(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha, \tag{2.77}$$

$$III)\ D^\dagger(\alpha)\hat{a}^\dagger D(\alpha) = \hat{a}^\dagger + \alpha^*, \tag{2.78}$$

$$IV)\ \ D(\alpha+\beta) = D(\alpha)D(\beta)\exp\Big(-i\,\mathrm{Im}(\alpha\beta^*)\Big), \tag{2.79}$$

From eq.(2.70) and eq.(2.75), we have

$$|\alpha\rangle = \exp\Big(-\frac{|\alpha|^2}{2}\Big)\exp\Big(\alpha\hat{a}^\dagger\Big)\exp\Big(\alpha^*\hat{a}\Big)|0\rangle. \tag{2.80}$$

Using Taylor expansion in eq.(2.80) gives

$$\begin{aligned}
|\alpha\rangle &= \exp\Big(-\frac{|\alpha|^2}{2}\Big)\sum_{n=0}^{\infty}\frac{(\alpha\hat{a}^\dagger)^n}{n!}\frac{(\alpha^*\hat{a})^n}{n!}|0\rangle \\
&= \exp\Big(-\frac{|\alpha|^2}{2}\Big)\sum_{n=0}^{\infty}\frac{(\alpha\hat{a}^\dagger)^n}{n!}\Big[1 + \alpha^*\hat{a} + \frac{(\alpha^*\hat{a})^2}{2!} + \cdots\Big]|0\rangle \\
&= \exp\Big(-\frac{|\alpha|^2}{2}\Big)\Big[1 + \alpha\hat{a}^\dagger + \frac{(\alpha\hat{a}^\dagger)^2}{2!} + \cdots\Big]|0\rangle.
\end{aligned} \tag{2.81}$$

Since eq.(2.81) is exactly the same as eq.(2.69), the two ways to write coherent states are indeed coincide. Actually, describing coherent states by using displacment operators are more physically appealing than the first idea, because this way is also used to describe the coherent of photon like a laser beam. Since they are of physical importance, let us study the coherent states further by discussing their properties. The first one is the completeness. We have known that lowering operators are not hermitian so their eigenvalues are complex numbers. In fact, this also implies that the coherent states are not orthogonal. Nevertheless, it is possible to express any states as a linear combination of coherent states by using completeness relation.

$$\int \frac{d\alpha^2}{\pi}|\alpha\rangle\langle\alpha| = 1. \tag{2.82}$$

The inner product of $|\alpha\rangle$ and $|\beta\rangle$ is given by

$$\langle\beta|\alpha\rangle = \exp\Big(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2\Big)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\beta^*\alpha}{\sqrt{n!m!}}\langle m|n\rangle. \tag{2.83}$$

From this equation $\langle\beta|\alpha\rangle \neq \delta_{\beta\alpha}$. This result means that coherent states are not orthogonal. It illustrates that coherent states are overcomplete. It is possible to express a coherent state as a linear combination of other coherent states.

The wave function of a coherent state is

$$\psi_\alpha(x) = \langle x|\alpha\rangle. \tag{2.84}$$

So, if we substitute eq.(2.81) into eq.(2.84), we obtained

$$\psi_\alpha(x) = \exp\Big(-\frac{|\alpha|^2}{2}\Big)\Big[\langle x|0\rangle + \langle x|\alpha a^\dagger|0\rangle + \langle x|\frac{(\alpha a^\dagger)^2}{2!}|0\rangle + \cdots\Big]. \tag{2.85}$$

Substituting eq.(2.39) into eq.(2.85) order by order, we obtain

$$\psi_\alpha(y) = \exp\Big(-\frac{|\alpha|^2}{2}\Big)\Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}}\exp\Big(-\frac{y^2}{2}\Big)\sum_{n=0}^{\infty}\Big(\frac{\alpha^n}{n!}\Big)(n!\cdot 2)^{-\frac{n}{2}}H_n(y). \tag{2.86}$$

To simplify eq.(2.86), we use the generating functions for Hermite polynomials,

$$\exp\Big(2xt - t^2\Big) = \sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}. \tag{2.87}$$

Let

$$t^n = \Big(\frac{\alpha}{\sqrt{2}}\Big)^n, \tag{2.88}$$

then we rewrite eq.(2.86) by using generating functions as

$$\psi_\alpha(y) = \exp\Big(-\frac{|\alpha|^2}{2}\Big)\Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}}\exp\Big(-\frac{y^2}{2}\Big)\exp\Big(2y\frac{\alpha}{\sqrt{2}}-\Big(\frac{\alpha}{\sqrt{2}}\Big)^2\Big). \tag{2.89}$$

Let us now turn to the measurements of position and momentum of quantum states, especially coherent states. To do this we use the methods of classical statistics. The statistical interpretation of the wave function leads to a principal uncertainty to localise a particle. We cannot predict the definite measurement outcome for a specific particle, where it is localised at a certain time, thus we cannot assign a path to the particle. For the uncertainty of observables, we start from position operators and momentum operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\Big(\hat{a}+\hat{a}^\dagger\Big), \tag{2.90}$$

$$\hat{p} = -i\sqrt{\frac{\hbar\omega}{2}}\Big(\hat{a}-\hat{a}^\dagger\Big). \tag{2.91}$$

The mean and the mean squared of $\hat{x}$ are

$$\langle\hat{x}\rangle_\alpha \equiv \langle\alpha|\hat{x}|\alpha\rangle, \tag{2.92}$$

$$\langle\hat{x}^2\rangle_\alpha \equiv \langle\alpha|\hat{x}^2|\alpha\rangle, \tag{2.93}$$

and the mean-square deviation is

$$(\Delta x)^2_\alpha \equiv \langle\hat{x}^2\rangle_\alpha - \langle\hat{x}\rangle^2_\alpha, \tag{2.94}$$

with

$$\langle\hat{O}\rangle_\alpha \equiv \sqrt{\frac{\hbar}{m\omega}}\int_{-\infty}^{\infty}\Phi_\alpha^{(*)}(x)\hat{O}\Phi_\alpha(x)\,dx, \tag{2.95}$$

the mean-squared deviation of $\hat{x}$ is the uncertainty of an observables $\hat{x}$ in the states $|\alpha\rangle$. The mean-squared deviation of $\hat{p}$ can be defined in similar way as that of $\hat{x}$. It can be shown that the uncertainty relation for the observables $\hat{x}$ and $\hat{p}$ satisfies the inequality, which is valid for all states,

$$(\Delta\hat{x})^2_\alpha(\Delta\hat{p})^2_\alpha \geq \frac{1}{4}|\langle[\hat{x},\hat{p}]\rangle|. \tag{2.96}$$

Whenever the commutator of two observables is nonvanishing, there is an uncertainty of these observables. Since $\hat{x}$ and $\hat{p}$ do not commute, we may interpret the uncertainty relation as follows. Whenever a position measurement is accurate (i.e. precise information about the current position of a particle), the information about the momentum is inaccurate uncertain and vice versa. The coherent states attain the minimal uncertainty. So they can be interpreted as being semi-classical. To further see the semi-classical behaviour of QHO coherent state, it would be useful to investigate time evolution of these states and compare with classical harmonic oscillator. However, our work do not extend along this direction but only on the construction of coherent states and check minimal uncertainty from NEQHO, to be defined in the next section.

CHAPTER III

NEWTON-EQUIVALENT HAMILTONIANS FOR THE HARMONIC OSCILLATOR

3.1 Deriving alternative Hamiltonians from the Newton's equation

In classical mechanics, the motion of a particle under the influence of external force can be determined from the Newton's second law,

$$\vec{F} = \frac{\partial \vec{p}}{\partial t}. \tag{3.1}$$

Alternative to Newton's second law, other formulations, which are Lagrangian mechanics and Hamiltonian mechanics, can also be used to describe dynamics. These two formulations can be used to obtain Newton's second law. An important quantity in Lagrangian mechanics is a Lagrangian. Newton's equation can be obtained by substituting an appropriate Lagrangian into Euler-Lagrange equations. Similarly, an important quantity in Hamiltonian mechanics is a Hamiltonian. Newton's equation can be obtained by substituting an appropriate Hamiltonian into Hamilton's equations.

Given a Lagrangian or a Hamiltonian a unique Newton's equation is obtained. On the other hand, given a Newton's equation there can be many possible Lagrangians or Hamiltonians. The problem to determine some or all such Lagrangians or Hamiltonians is called the inverse problem, which are studied for example in [23, 24, 25, 26, 27].

The reference [8] starts by giving alternative forms of Hamiltonian for a particle of mass m moving with momentum p in one demesion under a potential $V(x)$. The one-parameter family of Hamilton functions yielding the Newton's equation of the harmonic oscillator is reviewed. For the Newton and Hamilton's equations described this system can be written respectively as

$$m\frac{d^2}{dt^2}x + \frac{d}{dx}V(x) = 0, \tag{3.2}$$

$$\dot{x} = \frac{\partial H}{\partial p}, \tag{3.3}$$

$$\dot{p} = -\frac{\partial H}{\partial \dot{x}}, \tag{3.4}$$

Here, H is the classical Hamiltonian of this system. Substituting eqs.(3.3),(3.4) to eq.(3.2), one then obtains

$$\frac{\partial^2 H}{\partial x \partial p}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial p^2}\frac{\partial H}{\partial x} + \frac{1}{m}\frac{\partial V}{\partial x} = 0. \tag{3.5}$$

To solve the Hamiltonian from this equation, it is possible to apply the method of separation of variables. In this consideration, we are interested in two ansatz which are the additive and multiplicative forms.

For the additive form, the Hamiltonian can be written as

$$H(x,p) = F(p) + G(x). \tag{3.6}$$

If we substitute eq.(3.6) into eq.(3.5), we obtain

$$-F''(p)G'(x) + \frac{1}{m}V'(x) = 0, \tag{3.7}$$

which can be separated into two equations, one involving only functions of x and another only of p. The solutions are that $F(p)$ can be written as $Ap^2 + Bp + C$ and $G(x) = V(x)/(2Am) + D$. Here A, B, C and D are arbitrary parameters, for this solution we substitute back to eq.(3.6). It can be realised as the standard classical Hamitonian form H_E.

Let us turn our attention to the multiplicative Hamiltonian,

$$H(x,p) = F(p)G(x). \tag{3.8}$$

Substituting eq.(3.8) into eq.(3.5), we obtain

$$H(x,p) = c_1 \cosh\left(c_2 p + c_3\right)\left(\frac{2V(x)}{mc_1^2 c_2^2} + c_4\right). \tag{3.9}$$

Let us choose $c_1 = 4mc^2, c_2 = \frac{1}{2mc^2}, c_3 = 0, c_4 = 1$ and redefine the Hamiltonian by subtracting $4mc^2$. As a result, the Hamiltonian in multiplicative form is expressed as

$$H_c(x,p) = 4mc^2 \cosh\left(\frac{p}{2mc}\right)\left(1 + \frac{V(x)}{2mc^2}\right)^{\frac{1}{2}} - 4mc^2. \tag{3.10}$$

In the limit of $c \to \infty$, it reduces to the standand Hamiltonian. It can be shown as follows

$$\begin{aligned} H_c(x,p) &= 4mc^2 \cosh\left(\frac{p}{2mc}\right)\left(1 + \frac{V(x)}{2mc^2}\right)^{\frac{1}{2}} - 4mc^2 \\ &= 4mc^2 \left(1 + \frac{p^2}{2^2m^2c^2 2!} + \frac{p^4}{2^4m^4c^4 4!} + \cdots\right) \\ &\left(1 + \frac{V(x)}{4mc^2} - \frac{V^2(x)}{16m^2c^4 2!} + \cdots\right) - 4mc^2 \\ &= \frac{p^2}{2m} + V(x) - \frac{V^2(x)}{8mc^2} + \frac{p^2V(x)}{8m^2c^2} - \frac{p^2V^2(x)}{32m^3c^4} + \frac{p^4}{96m^3c^2} \\ &+ \frac{p^4V(x)}{384m^4c^4} - \frac{p^4V^2(x)}{3072m^5c^6} + \cdots . \end{aligned}$$

Therefore

$$\lim_{c\to\infty}[H_c(x,p)] = \frac{p^2}{2m} + V(x) = H_E(x,p). \tag{3.11}$$

As mentioned we will focus on the harmonic oscillator system. Therefore, the potential is $V(x) = m\omega^2x^2/2$. By quantising the Hamiltonian with the chosen potential, one may obtain

$$\hat{H}_c = \frac{1}{2\beta^2 m}\cosh\left(\beta\hat{p}\right)\left(1 + \beta^2m^2\omega^2\hat{x}^2\right)^{\frac{1}{2}} + (\text{hermitian conjugate}), \tag{3.12}$$

where $\beta = (2mc)^{-1}$. In fact there are many possible ways to write the quantum Hamiltonian. This is due to the fact that $\hat{x}$ and $\hat{p}$ are not commute. Another possible expression for the Hamiltonian operator is

$$\hat{H}(\beta) = \frac{1}{2\beta^2 m}\left[(1 + i\beta m\omega\hat{x})^{\frac{1}{2}} \exp(-i\hbar\beta\partial_x)(1 - i\beta m\omega\hat{x})^{\frac{1}{2}} + (i \to -i)\right]. \tag{3.13}$$

This Hamiltonian is chosen by [8], because it is inspired by the work [22], which studies relativistic Calogero-Moser systems. For the eigenvalue equation with eq.(3.13), the ground state wave function is given by

$$\Psi_0^{(\beta)}(x) = \left[\left(\Gamma\left(\frac{1}{\hbar\beta^2 m\omega} + \frac{ix}{\hbar\beta}\right)\right)\left(\Gamma\left(\frac{1}{\hbar\beta^2 m\omega} - \frac{ix}{\hbar\beta}\right)\right)\right]^{\frac{1}{2}}, \tag{3.14}$$

where Γ is Gamma function. The wave function $\Psi_0^{(\beta)}(x)$ satisfies the eigenvalue equation,

$$\hat{H}(\beta)\Psi_0^{(\beta)}(x) = \frac{1}{\beta^2 m}\Psi_0^{(\beta)}(x), \tag{3.15}$$

which $1/\beta^2 m$ is an energy spectrum or eigenvalue. We will discuss more details about Hamiltonian operator $\hat{H}(\beta)$ later.

In classical formalism, let us note that the classical version of (2.1) can be written in form of

$$H_E = \omega a a^*, \tag{3.16}$$

which can be seen from

$$x^2 + \frac{p^2}{m^2\omega^2} = \Big(x + i\frac{p}{m\omega}\Big)\Big(x - i\frac{p}{m\omega}\Big) \tag{3.17}$$

and

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{2}{m\omega^2}\Big(x^2 + \frac{p^2}{m^2\omega^2}\Big). \tag{3.18}$$

Then, we can write a complex function from eq.(3.16) as

$$a = \Big(\frac{m\omega}{2}\Big)^{\frac{1}{2}}\Big(x + i\frac{p}{m\omega}\Big), \tag{3.19}$$

$$a^* = \Big(\frac{m\omega}{2}\Big)^{\frac{1}{2}}\Big(x - i\frac{p}{m\omega}\Big). \tag{3.20}$$

For the poisson bracket of $\{a, a^*\}$

$$\begin{aligned}\{a, a^*\} &= \Big(\frac{m\omega}{2}\Big)\Big[\frac{\partial}{\partial x}\Big(x + i\frac{p}{m\omega}\Big)\frac{\partial}{\partial p}\Big(x - i\frac{p}{m\omega}\Big) \\ &\quad - \frac{\partial}{\partial p}\Big(x + i\frac{p}{m\omega}\Big)\frac{\partial}{\partial x}\Big(x - i\frac{p}{m\omega}\Big)\Big] \\ &= \Big(\frac{m\omega}{2}\Big)\Big[(1)\Big(-\frac{i}{m\omega}\Big) - \Big(\frac{i}{m\omega}\Big)(1)\Big] \\ &= -i,\end{aligned} \tag{3.21}$$

then

$$\{a, a^*\} = -i, \tag{3.22}$$

and

$$\{a, H_E\} = -i\omega a, \tag{3.23}$$

$$\{a^*, H_E\} = i\omega a^*. \tag{3.24}$$

Let us now turn to the one-parameter family Newton-Equivalent Hamiltonians H_c. We may try to define the quatities which are similar to a and a^* defined from eq.(3.19) and eq.(3.20). The idea is that the momentum p can be expressed as $p = m\dot{x}$ and that time derivative of x can be written as $\dot{x} = \{x, H_c\}$. Therefore, the counterpart of a is

$$\begin{aligned} A &= \Big(\frac{m\omega}{2}\Big)^{\frac{1}{2}}\Big(x + i\frac{\{x, H_c\}}{\omega}\Big) \\ &= \Big(\frac{m\omega}{2}\Big)^{\frac{1}{2}}\left(x + \frac{i}{\beta m\omega}\Big(\sinh\Big(\beta p\Big)\Big(1+\beta^2 m^2\omega^2 x^2\Big)^{\frac{1}{2}}\Big)\right). \end{aligned} \tag{3.25}$$

We also write the poisson bracket relations. This can be obtained as follows

$$\begin{aligned} \{A, H_c\} &= \frac{\partial A}{\partial x}\frac{\partial H_c}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H_c}{\partial x} \\ &= \Big(\frac{m\omega}{2}\Big)^{\frac{1}{2}}\Big(\frac{1}{\beta^2 m}\Big)\Bigg[\Big(1 + \frac{i}{\beta m\omega}\Big(\sinh\Big(\beta p\Big)\frac{1}{2}2x\beta^2 m^2\omega^2 \times \\ &\Big(1+\beta^2 m^2\omega^2 x^2\Big)^{-\frac{1}{2}}\Big)\Big(\beta\sinh(\beta p)(1+\beta^2 m^2\omega^2 x^2)^{\frac{1}{2}}\Big)\Big) \\ &- \Big(0 + \frac{i}{\beta m\omega}\beta\cosh\Big(\beta p\Big)\Big(1+\beta^2 m^2\omega^2 x^2\Big)^{\frac{1}{2}}\Big)\times \\ &\Big(\cosh\Big(\beta p\Big)\frac{1}{2}2x\beta^2 m^2\omega^2\Big(1+\beta^2 m^2\omega^2 x^2\Big)^{-\frac{1}{2}}\Big)\Bigg] \\ &= \Big(\frac{m\omega}{2}\Big)^{\frac{1}{2}}\Big(\frac{1}{\beta^2 m}\Big)\Bigg[\beta\sinh\Big(\beta p\Big)\Big(1+\beta^2 m^2\omega^2 x^2\Big)^{\frac{1}{2}} \\ &+ i\beta^2 m\omega x\sinh^2\Big(\beta p\Big) - i\beta^2 m\omega x\cosh^2\Big(\beta p\Big)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{m\omega}{2}\Big)^{\frac{1}{2}}\Big(\frac{1}{\beta^2 m}\Big)\Bigg[\beta \sinh\Big(\beta p\Big)\Big(1+\beta^2m^2\omega^2x^2\Big)^{\frac{1}{2}} - i\beta^2 m\omega x\Bigg]\\
&= \Big(\frac{m\omega}{2}\Big)^{\frac{1}{2}}\Bigg[\frac{1}{\beta m}\sinh\Big(\beta p\Big)\Big(1+\beta^2m^2\omega^2x^2\Big)^{\frac{1}{2}} - i\omega x\Bigg]\\
&= -i\omega\Big(\frac{m\omega}{2}\Big)^{\frac{1}{2}}\Bigg[x + \frac{i}{\beta m\omega}\sinh\Big(\beta p\Big)\Big(1+\beta^2m^2\omega^2x^2\Big)^{\frac{1}{2}}\Bigg]\\
&= -i\omega A. \qquad (3.26)
\end{aligned}
$$

As for $\{A^*, H_c\}$, we use a similar method and we will obtain $\{A^*, H_c\} = i\omega A^*$. Moreover, the case of $\{a, a^*\}$ is generalised to $\{A, A^*\}$,

$$
\begin{aligned}
\{A, A^*\} &= \frac{\partial A}{\partial x}\frac{\partial A^*}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial A^*}{\partial x}\\
&= \Big(\frac{m\omega}{2}\Big)\Bigg[\Bigg(1 + \frac{i}{\beta m\omega}\Bigg(\sinh\Big(\beta p\Big)\frac{1}{2}2x\beta^2m^2\omega^2\times\\
&\Big(1+\beta^2m^2\omega^2x^2\Big)^{-\frac{1}{2}}\Bigg)\Bigg(0 - \frac{i}{m\omega}\cosh\Big(\beta p\Big)\Big(1+\beta^2m^2\omega^2x^2\Big)^{\frac{1}{2}}\Bigg)\\
&- \Bigg(0 + \frac{i}{m\omega}\cosh\Big(\beta p\Big)\Big(1+\beta^2m^2\omega^2x^2\Big)^{\frac{1}{2}}\Bigg)\times\\
&\Bigg(1 - \frac{i}{\beta m\omega}\Bigg(\sinh\Big(\beta p\Big)\frac{1}{2}2x\beta^2m^2\omega^2\Big(1+\beta^2m^2\omega^2x^2\Big)^{-\frac{1}{2}}\Bigg)\Bigg]\\
&= \Big(\frac{m\omega}{2}\Big)\Bigg[-\frac{i}{m\omega}\cosh\Big(\beta p\Big)\Big(1+\beta^2m^2\omega^2x^2\Big)^{\frac{1}{2}}\\
&+ x\beta\sinh\Big(\beta p\Big)\cosh\Big(\beta p\Big) - \frac{i}{m\omega}\cosh\Big(\beta p\Big)\Big(1+\beta^2m^2\omega^2x^2\Big)^{\frac{1}{2}}\\
&- x\beta\sinh\Big(\beta p\Big)\cosh\Big(\beta p\Big)\Bigg]\\
&= -i\Bigg[\cosh\Big(\beta p\Big)\Big(1+\beta^2m^2\omega^2x^2\Big)^{\frac{1}{2}}\Bigg]\\
&= -i\frac{\beta^2 m}{\beta^2 m}\Bigg[\cosh\Big(\beta p\Big)\Big(1+\beta^2m^2\omega^2x^2\Big)^{\frac{1}{2}}\Bigg]\\
&= -i\beta^2 m H_c. \qquad (3.27)
\end{aligned}
$$

Let us proceed to the quantum level, we recall the lowering operators,

$$
\hat{a} = \Big(\frac{m\omega}{2\hbar}\Big)^{\frac{1}{2}}\Big(\hat{x} + i\frac{\hat{p}}{m\omega}\Big),
$$

and all of the commutation relations for $[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{a}, \hat{H}] = \hbar\omega\hat{a}$, $[\hat{a}^\dagger, \hat{H}] = -\hbar\omega\hat{a}^\dagger$. From these commutation relations we will use the same idea, just as at the classical level. Then, we can realise that $\hat{p}$ can be written as $m\dot{\hat{x}} = -im\hbar^{-1}[\hat{x}, \hat{H}]$. So the operator form of Newton's equation is $[[\hat{x}, \hat{H}], \hat{H}] = \hbar^2\omega^2\hat{x}$. Now we have to check another one $[[\hat{x}, \hat{H}(\beta)], \hat{H}(\beta)] = \hbar^2\omega^2\hat{x}$. This is very impressive because the quantum Hamiltonians $\hat{H}$ and $\hat{H}(\beta)$ are once more Newton equivalent. Therefore,

$$\begin{aligned}\hat{A} &= \Big(\frac{m\omega}{2\hbar}\Big)^{\frac{1}{2}}\Big[\hat{x} + i\frac{1}{\hbar\omega}[\hat{x}, \hat{H}(\beta)]\Big]\\ &= \Big(\frac{m\omega}{2\hbar}\Big)^{\frac{1}{2}}\Big[\hat{x} + \frac{i}{2\beta m\omega}\Big[\Big(1 + i\beta m\omega\hat{x}\Big)^{\frac{1}{2}} \exp\Big(-i\hbar\beta\partial_x\Big)\times\\ &\Big(1 - i\beta m\omega\hat{x}\Big)^{\frac{1}{2}} - (i \to -i)\Big]\Big].\end{aligned} \tag{3.28}$$

The commutation relations of lowering operator, raising operator and hamiltonian are

$$[\hat{A}, \hat{H}(\beta)] = \hbar\omega\hat{A}, \quad [\hat{A}^\dagger, \hat{H}(\beta)] = -\hbar\omega\hat{A}^\dagger, \tag{3.29}$$

and

$$[\hat{A}, \hat{A}^\dagger] = \beta^2 m\hat{H}(\beta). \tag{3.30}$$

To evaluate these commutation relations, we need to use properties of exponential function of ∂_x to shift functions $f(x)$ to $f(x \mp a)$, so all of this is easily verified. It would be interesting to rederive the above results by using dimensionless variable

$$\lambda = \beta\Big(\hbar m\omega\Big)^{\frac{1}{2}}. \tag{3.31}$$

Then write

$$\hat{a} \to \frac{1}{\sqrt{2}}\Big(\hat{y} + i\hat{p}_y\Big), \tag{3.32}$$

$$\hat{H} \to \frac{\hbar\omega}{2}\Big(-\hat{p}_y^2 + \hat{y}^2\Big), \tag{3.33}$$

where $\hat{p}_y = \hat{p}/\sqrt{m\omega\hbar}$, $\hat{p} = -i\hbar\partial_x$.

As for the one-parameter of $\hat{H}(\beta)$, we set new symbols for convenient,

$$\hat{T}_\pm \equiv \Big(1 \pm i\lambda\hat{y}\Big)^{\frac{1}{2}} \exp\Big(\pm\lambda\hat{p}_y\Big)\Big(1 \mp i\lambda\hat{y}\Big)^{\frac{1}{2}}, \tag{3.34}$$

and the relation between $\hat{T}_{\pm}$ and $\hat{y}$ can be written in form of commutation relations as,

$$[\hat{T}_{\pm}, i\hat{y}] = \pm\lambda\hat{T}_{\pm}, \;\; [\hat{T}_{-}, \hat{T}_{+}] = 2\lambda^3 i\hat{y}. \tag{3.35}$$

To keep away from confusion, we switch one-parameter to lower case symbols as,

$$\hat{A} \to \hat{a}(\lambda), \;\; \hat{A}^{\dagger} \to \hat{a}^{\dagger}(\lambda), \;\; \frac{\hat{H}(\beta)}{\hbar\omega} \to \hat{h}(\lambda), \tag{3.36}$$

Thus, we write it as,

$$\hat{a}(\lambda) = 2^{-1/2}\Big(\hat{y} + \frac{i}{2\lambda}\Big(\hat{T}_{+} - \hat{T}_{-}\Big)\Big), \tag{3.37}$$

$$\hat{a}^{\dagger}(\lambda) = 2^{-1/2}\Big(\hat{y} - \frac{i}{2\lambda}\Big(\hat{T}_{+} - \hat{T}_{-}\Big)\Big), \tag{3.38}$$

$$\hat{h}(\lambda) = \left(2\lambda^2\right)^{-1}\left(\hat{T}_{+} - \hat{T}_{-}\right). \tag{3.39}$$

Then we rewrite eq.(3.29), (3.30) into lower case symbols

$$[\hat{a}(\lambda), \hat{h}(\lambda)] = \hat{a}(\lambda), \;\; [\hat{a}^{\dagger}(\lambda), \hat{h}(\lambda)] = -\hat{a}^{\dagger}(\lambda), \;\; [\hat{a}(\lambda), \hat{a}^{\dagger}(\lambda)] = \lambda^2\hat{h}(\lambda). \tag{3.40}$$

In order to explain the relation of $\hat{a}^{\dagger}(\lambda)$ and $\hat{h}(\lambda)$ to operators $\hat{a}^{\dagger}$ and $\hat{H}_E/\hbar\omega$, we will expand from (3.34) and limit $\lambda \to 0$, for this one we obtain

$$\lim_{\lambda\to 0}\hat{a}(\lambda) = 2^{-\frac{1}{2}}\Big(\hat{y} + i\hat{p}_y\Big) = \hat{a}, \;\; \lim_{\lambda\to 0}\hat{a}^{\dagger}(\lambda) = 2^{-\frac{1}{2}}\Big(\hat{y} - i\hat{p}_y\Big) = \hat{a}^{\dagger},$$

$$\lim_{\lambda\to 0}\hat{h}(\lambda) - \lambda^{-2} = \frac{1}{2}\Big(-\hat{p}_y^2 + \hat{y}^2 - 1\Big) = (\hbar\omega)^{-1}\hat{H}_E - \frac{1}{2}.$$

Let us now turn to the ground state wave functions of NEQHO and rewritten in form of dimensionless as

$$\psi_0^{(\lambda)}(y) = \left[\Gamma\left(\frac{1}{\lambda^2} + \frac{i\hat{y}}{\lambda}\right)\Gamma\left(\frac{1}{\lambda^2} - \frac{i\hat{y}}{\lambda}\right)\right]^{\frac{1}{2}}, \tag{3.41}$$

and for the normalisation of ground state is

$$\hat{\psi}_0^{(\lambda)}(y) = \Big(\psi_0^{(\lambda)}(y), \psi_0^{(\lambda)}(y)\Big)^{-\frac{1}{2}}\psi_0^{(\lambda)}(y), \tag{3.42}$$

where

$$\Big(\psi_0^{(\lambda)}(y), \psi_0^{(\lambda)}(y)\Big) = 2^{1-\frac{2}{\lambda}}\pi\lambda\Gamma\Big(\frac{2}{\lambda^2}\Big). \tag{3.43}$$

It converges to the normalise ground state of the standard harmonic oscillator as $\lambda \to 0$:

$$\lim_{\lambda \to 0} \hat{\psi}_0^{(\lambda)}(y) = \pi^{-\frac{1}{4}} \exp\Big(-\frac{y^2}{2}\Big). \tag{3.44}$$

In the next chaper, we will use the lowering operators from eq.(3.37) to construst the coherent states for the one-paremeter family of hamiltonians.

CHAPTER IV

ON THE CONSTRUCTION COHERENT STATE OF NEWTON-EQUIVALENT

4.1 The construction of the coherent states wave functions

Turning to our work, we will use the method and strategy from the previous chapter to construct the coherent states of the Newton-Equivalent wave functions. Actually, the closed form of excited states can be obtained after we operate raising operator on the ground state for an approprite number of times. In practice, it turns out that this kind of direct calculation for NEQHO is difficult to do. So we come up with a simpler method to be described. We will use a process to approximate the coherent state wave functions. In this chapter, we will explain and justify this process, then use the coherent state wave functions to compute uncertainty relation and then interpret.

As we mentioned, we try to find coherent states for NEQHO. Let us start with eigenvalue equations with eq.(3.37), it gives

$$2^{-1/2}\Big(\hat{y}+\frac{i}{2\lambda}\Big(\hat{T}_{+}-\hat{T}_{-}\Big)\Big)|\alpha\rangle=\alpha|\alpha\rangle. \tag{4.1}$$

From this equation the state $|\alpha\rangle$ should be the functions of λ and we will also find coefficient as the functions of λ. We are going to restrict to small values of λ. To find coefficient, we need to expand $T_{\pm}$ in λ as

$$2^{-1/2}\Bigg[\hat{y}+\frac{i}{2\lambda}\Bigg[\Big(1+i\lambda\hat{y}\Big)^{\frac{1}{2}}\exp\Big(+\lambda\hat{p}_y\Big)\Big(1-i\lambda\hat{y}\Big)^{\frac{1}{2}} \\ -\Big(1-i\lambda\hat{y}\Big)^{\frac{1}{2}}\exp\Big(-\lambda\hat{p}_y\Big)\Big(1+i\lambda\hat{y}\Big)^{\frac{1}{2}}\Bigg]\Bigg]=\alpha|\alpha\rangle. \tag{4.2}$$

Expanding in λ gives

$$2^{-1/2}\left(\hat{y}+i\hat{p}_y+i\frac{\lambda^2}{6}\Big(\hat{p}_y^3-3\hat{p}_y-3i\hat{y}+3\hat{y}^2\hat{p}_y\Big)+O(\lambda^3)\right)|\alpha\rangle=\alpha|\alpha\rangle \quad (4.3)$$

$$2^{-1/2}\left(\hat{y}+i\hat{p}_y+i\frac{\lambda^2}{6}\Big(\hat{p}_y^3-3\hat{p}_y-3i\hat{y}+3\hat{y}^2\hat{p}_y\Big)\right)\Big(c_0|0\rangle+c_1|1\rangle+c_2|2+...\Big)$$
$$=\alpha\Big(c_0|0\rangle+c_1|1\rangle+c_2|2\rangle+...\Big), \quad (4.4)$$

for convenient, we multiply with $\langle y|$ and written as,

$$2^{-1/2}\left(y+\partial_y+\frac{\lambda^2}{6}\Big(-\partial_{yyy}-3\partial_y+3y+y^2\partial_y\Big)\right)\Big(c_0\psi_0(y)+c_1\psi_1(y)$$
$$+c_2\psi_2(y)+...\Big)=\alpha\Big(c_0\psi_0(y)+c_1\psi_1(y)+c_2\psi_2(y)+...\Big). \quad (4.5)$$

In this equation, we have made a perturbative expansion of the operator and keep up to order λ^2. Let us make a further approximation of the coherent state by expressing it as a linear combination from the ground state $\psi_0(y)$ up to state $\psi_N(y)$. For instance, if we are interested in the $N=2$ truncation, we can see that R.H.S. of eq.(4.5) is a linear combination of $\psi_0(y)$, $\psi_1(y)$, $\psi_2(y)$. However, the L.H.S. is a linear combination of $\psi_0(y)$, $\psi_1(y)$, $\psi_2(y)$, $\psi_3(y)$, $\psi_4(y)$, $\psi_5(y)$. So in the case, eq.(4.5) gives 6 conditions for 3 unknowns c_1, c_2, c_3. So to proceed, let us some extra conditions by ignoring the conditions corresponding to the coeficients of $\psi_3(y)$, $\psi_4(y)$, $\psi_5(y)$. The removal of these conditions as explained might seem arbitrary. But we expect that as N increases, our method would not lead to any problem. By following the process, we obtain

$$c_1\to\frac{\alpha(\alpha^2(-\lambda^2)+2\alpha^2+3\lambda^2+4)}{(\alpha^2+2)(\alpha^2+2)}c_0,\ c_2\to\frac{(\sqrt{2}\alpha^4+2\sqrt{2}\alpha^2\lambda^2+2\sqrt{2}\alpha^2-\sqrt{2}\lambda^2)}{(\alpha^2+2)(\alpha^2+2)}c_0.$$

Note that this relation works only for the $N=2$ truncation. If we consider larger N, we should find the similar relations again.

As a result of this, we substitute into the definition of the coherent state wave functions as,

$$\psi_\alpha^2(y)\equiv\langle y|\alpha\rangle$$

$$
\begin{aligned}
&= \frac{e^{-\frac{y^2}{2}}}{\pi^{\frac{1}{4}}} c_0 \Bigg[\frac{2\left(\alpha^2 y^2 + \sqrt{2}\alpha y + 1\right)}{\left(\alpha^2+2\right)} \\
&- \lambda^2 \frac{\left(2\alpha^2 - 4\alpha^2 y^2 + 2y^2 + \sqrt{2}\alpha^3 y - 3\sqrt{2}\alpha y - 1\right)}{\left(\alpha^2+2\right)^2} \Bigg],
\end{aligned} \tag{4.6}
$$

then we can compute the variance of position and momentum for the state described by this wave function by using

$$
(\Delta y)_\alpha^{(\lambda,2)} = \sqrt{\langle \hat{y}^2 \rangle_\alpha^{(\lambda,2)} - \left(\langle \hat{y} \rangle_\alpha^{(\lambda,2)}\right)^2}, \tag{4.7}
$$

$$
(\Delta p_y)_\alpha^{(\lambda,2)} = \sqrt{\langle \hat{p}_y^2 \rangle_\alpha^{(\lambda,2)} - \left(\langle \hat{p}_y \rangle_\alpha^{(\lambda,2)}\right)^2}. \tag{4.8}
$$

As an example result, for consider the case $N = 2$, $\alpha = 0.5$. We obtain, presenting up to 3 significant figures,

$$
(\Delta y)_{0.5}^{(\lambda,2)} = 0.681 - 0.0234\lambda^2, \tag{4.9}
$$

$$
(\Delta p)_{0.5}^{(\lambda,2)} = 0.741 - 0.0169\lambda^2, \tag{4.10}
$$

then if we write the uncertainty relation from two variance, we obtain

$$
(\Delta y)_{0.5}^{(\lambda,2)} (\Delta p)_{0.5}^{(\lambda,2)} = 0.504 - 0.0059\lambda^2. \tag{4.11}
$$

It can be expected that the result of eq.(4.11) does not accurately respresent the uncertainty relation of the NEQHO coherent states because $\psi_\alpha^2(y)$ is expressed as a linear combination of only the states $|0\rangle, |1\rangle, |2\rangle$. If we need more accuracy, we should add more state into the linear combination. If we include states up to $|N\rangle$, we may follow above steps which eventually give

$$
(\Delta y)_\alpha^{(\lambda,N)} = \sqrt{\langle \hat{y}^2 \rangle_\alpha^{(\lambda,N)} - \left(\langle \hat{y} \rangle_\alpha^{(\lambda,N)}\right)^2}, \tag{4.12}
$$

$$
(\Delta p)_\alpha^{(\lambda,N)} = \sqrt{\langle \hat{p}^2 \rangle_\alpha^{(\lambda,N)} - \left(\langle \hat{p} \rangle_\alpha^{(\lambda,N)}\right)^2}, \tag{4.13}
$$

where the expectation value of the operator is

$$
\langle \hat{O} \rangle_\alpha = \left(\sqrt{\frac{\hbar}{m\omega}}\right) \int_{-\infty}^{\infty} dy \psi_\alpha^{(\lambda,N)*}(y) \hat{O} \psi_\alpha^{(\lambda,N)}(y). \tag{4.14}
$$

By following the algorithm outlined, we have computed $(\Delta y)_\alpha^{(\lambda,N)}$, $(\Delta p_y)_\alpha^{(\lambda,N)}$, and $(\Delta y)_\alpha^{(\lambda,N)}(\Delta p_y)_\alpha^{(\lambda,N)}$ for various α and N. For any fixed N, these quantities are expressible as Taylor series expansion in λ up to the order of λ^2 such that each coefficient is a function of α. After substituting in a numerical value for α, these coefficients are given by some numerical values. Since these coefficients are mathematical functions, their values can be given precisely (i.e. up to any significant figures), as long as we give a precise value of α. Correspondingly, the programming language that we have used in the calculation allows arbitrary precision, and we have made the full use of this capability.

As an example result, let us consider the cases with fixed $\alpha = 0.5 + 0.7i$ and vary order of truncation $N = 2, 4, 6, ..., 26$. These results can be read from Table 1,

Table 1 The values of $(\Delta y)_\alpha^{(\lambda,N)} - (\Delta y)_\alpha^{(\lambda,N-2)}$ and $(\Delta p_y)_\alpha^{(\lambda,N)} - (\Delta p_y)_\alpha^{(\lambda,N-2)}$ for $\alpha = 0.5 + 0.7i$ with various values of N

N	$(\Delta y)_\alpha^{(\lambda,N)} - (\Delta y)_\alpha^{(\lambda,N-2)}$	$(\Delta p_y)_\alpha^{(\lambda,N)} - (\Delta p_y)_\alpha^{(\lambda,N-2)}$
4	$-8.97 \times 10^{-2} + 1.65 \times 10^{-1}\lambda^2$	$-2.02 \times 10^{-2} - 1.41 \times 10^{-1}\lambda^2$
6	$-7.34 \times 10^{-3} + 1.14 \times 10^{-1}\lambda^2$	$1.29 \times 10^{-3} - 7.66 \times 10^{-2}\lambda^2$
8	$-1.86 \times 10^{-4} + 8.58 \times 10^{-3}\lambda^2$	$7.31 \times 10^{-5} - 4.30 \times 10^{-3}\lambda^2$
10	$-2.31 \times 10^{-6} + 2.62 \times 10^{-4}\lambda^2$	$1.20 \times 10^-6 - 1.08 \times 10^{-4}\lambda^2$
12	$-1.71 \times 10^{-8} + 4.41 \times 10^{-6}\lambda^2$	$1.03 \times 10^{-8} - 1.86 \times 10^{-6}\lambda^2$
14	$-8.31 \times 10^{-11} + 4.54 \times 10^{-8}\lambda^2$	$5.51 \times 10^{-11} - 2.18 \times 10^{-8}\lambda^2$
16	$-2.87 \times 10^{-13} + 3.05 \times 10^{-10}\lambda^2$	$2.03 \times 10^{-13} - 1.66 \times 10^{-10}\lambda^2$
18	$-7.40 \times 10^{-16} + 1.42 \times 10^{-12}\lambda^2$	$5.48 \times 10^{-16} - 8.56 \times 10^{-13}\lambda^2$
20	$-1.48 \times 10^{-18} + 4.79 \times 10^{-15}\lambda^2$	$1.13 \times 10^{-18} - 3.12 \times 10^{-15}\lambda^2$
22	$-2.35 \times 10^{-21} + 1.22 \times 10^{-17}\lambda^2$	$1.85 \times 10^{-21} - 8.44 \times 10^{-18}\lambda^2$
24	$-3.04 \times 10^{-24} + 2.42 \times 10^{-20}\lambda^2$	$2.45 \times 10^{-24} - 1.75 \times 10^{-20}\lambda^2$
26	$-3.27 \times 10^{-27} + 3.84 \times 10^{-23}\lambda^2$	$2.69 \times 10^{-27} - 2.89 \times 10^{-23}\lambda^2$

From the results, it can also be seen that the values of $(\Delta y)^{(\lambda,N)}_{0.5+0.7i} - (\Delta y)^{(\lambda,N-2)}_{0.5+0.7i}$, $(\Delta p_y)^{(\lambda,N)}_{0.5+0.7i} - (\Delta p_y)^{(\lambda,N-2)}_{0.5+0.7i}$ converge as N is increased.

Table 2 The values of $(\Delta y)^{(\lambda,N)}_{\alpha}$ and $(\Delta p_y)^{(\lambda,N)}_{\alpha}$ for $\alpha = 0.5+0.7i$ with various values of N

N	$(\Delta y)^{(\lambda,N)}_{\alpha}$	$(\Delta p_y)^{(\lambda,N)}_{\alpha}$
2	$0.804 - 0.026\lambda^2$	$0.726 - 0.040\lambda^2$
4	$0.715 + 0.139\lambda^2$	$0.706 - 0.181\lambda^2$
6	$0.707 + 0.253\lambda^2$	$0.707 - 0.257\lambda^2$
8	$0.707 + 0.261\lambda^2$	$0.707 - 0.262\lambda^2$
10	$0.707 + 0.262\lambda^2$	$0.707 - 0.262\lambda^2$
12	$0.707 + 0.262\lambda^2$	$0.707 - 0.262\lambda^2$
14	$0.707 + 0.262\lambda^2$	$0.707 - 0.262\lambda^2$
16	$0.707 + 0.262\lambda^2$	$0.707 - 0.262\lambda^2$
18	$0.707 + 0.262\lambda^2$	$0.707 - 0.262\lambda^2$
20	$0.707 + 0.262\lambda^2$	$0.707 - 0.262\lambda^2$
22	$0.707 + 0.262\lambda^2$	$0.707 - 0.262\lambda^2$
24	$0.707 + 0.262\lambda^2$	$0.707 - 0.262\lambda^2$
26	$0.707 + 0.262\lambda^2$	$0.707 - 0.262\lambda^2$

We have checked that for other values of α with $|\alpha| \leq 1$, the behaviours are also qualitatively the same. That is, when N is large enough the values of $(\Delta y)^{(\lambda,N)}_{\alpha}$, $(\Delta p_y)^{(\lambda,N)}_{\alpha}$ and $(\Delta y)^{(\lambda,N)}_{\alpha}(\Delta p_y)^{(\lambda,N)}_{\alpha}$ converge as Table 2 and Table 3. We suppose that other values of α also share this behaviour.

We therefore choose the value of N sufficiently large enough. In particular, we choose $N = 25$ which is useful to study coherent state wave functions with $|\alpha| \leq 2$. Larger values of α can also be studied accurately, provided that we increase N to an

Table 3 The values of and $(\Delta y)_\alpha^{(\lambda,N)}(\Delta p_y)_\alpha^{(\lambda,N)}$ for $\alpha = 0.5 + 0.7i$ with various values of N

N	$(\Delta y)_\alpha^{(\lambda,N)}(\Delta p_y)_\alpha^{(\lambda,N)}$
2	$0.584 - 5.12 \times 10^{-2}\lambda^2 + O(\lambda^3)$
4	$0.504 - 3.10 \times 10^{-2}\lambda^2 + O(\lambda^3)$
6	$0.500 - 3.20 \times 10^{-3}\lambda^2 + O(\lambda^3)$
8	$0.500 - 1.12 \times 10^{-4}\lambda^2 + O(\lambda^3)$
10	$0.500 - 1.83 \times 10^{-6}\lambda^2 + O(\lambda^3)$
12	$0.500 - 1.68 \times 10^{-8}\lambda^2 + O(\lambda^3)$
14	$0.500 - 9.88 \times 10^{-11}\lambda^2 + O(\lambda^3)$
16	$0.500 - 4.00 \times 10^{-13}\lambda^2 + O(\lambda^3)$
18	$0.500 - 1.18 \times 10^{-15}\lambda^2 + O(\lambda^3)$
20	$0.500 - 2.67 \times 10^{-18}\lambda^2 + O(\lambda^3)$
22	$0.500 - 4.73 \times 10^{-21}\lambda^2 + O(\lambda^3)$
24	$0.500 - 6.77 \times 10^{-24}\lambda^2 + O(\lambda^3)$
26	$0.500 - 7.97 \times 10^{-27}\lambda^2 + O(\lambda^3)$

appropriate value. It is a well-known result that coherent states for standard QHO has the minimal value of uncertainty. See Equation (2.96). As for the coherent states of NEQHO, however, it is non-trivial whether these states give the minimal value of uncertainty. So we study the values $(\Delta y)_\alpha^{(\lambda,N)}(\Delta p_y)_\alpha^{(\lambda,N)}$ for $N = 25$, $|\alpha| \leq 2$, and compare with the minimal value of uncertainty, which is 0.5. In particular, Table 4 demonstrates the difference in each of the cases where $\alpha = 0, 0.2, 0.4, \ldots, 2$. The differences for other cases with $|\alpha| \leq 2$ (recall that α is a complex number) also share the same feature as those presented in Table 4. That is, the coefficients of the 0^{th} and 2^{nd} order of λ are very small. As a result of Table 4, we see that when α are increased, the difference between

Table 4 The difference between the uncertainty $(\Delta y)_\alpha^{(\lambda,N)}(\Delta p_y)_\alpha^{(\lambda,N)}$ with $N = 25$, $\alpha = 0, 0.2, 0.4, ..., 2$, and the minimal value 0.5 of uncertainty

α	$(\Delta y)_\alpha^{(\lambda,25)}(\Delta p_y)_\alpha^{(\lambda,25)} - 0.5$
0.0	0
0.2	$2.79 \times 10^{-62} - 2.22 \times 10^{-55}\lambda^2 + O(\lambda^3)$
0.4	$1.11 \times 10^{-46} - 5.66 \times 10^{-41}\lambda^2 + O(\lambda^3)$
0.6	$1.31 \times 10^{-37} - 1.36 \times 10^{-32}\lambda^2 + O(\lambda^3)$
0.8	$3.11 \times 10^{-26} - 1.07 \times 10^{-26}\lambda^2 + O(\lambda^3)$
1.0	$2.37 \times 10^{-26} - 3.51 \times 10^{-22}\lambda^2 + O(\lambda^3)$
1.2	$2.00 \times 10^{-22} - 1.51 \times 10^{-18}\lambda^2 + O(\lambda^3)$
1.4	$3.60 \times 10^{-19} - 1.56 \times 10^{-15}\lambda^2 + O(\lambda^3)$
1.6	$2.05 \times 10^{-16} - 5.52 \times 10^{-13}\lambda^2 + O(\lambda^3)$
1.8	$4.75 \times 10^{-14} - 8.43 \times 10^{-11}\lambda^2 + O(\lambda^3)$
2.0	$5.32 \times 10^{-12} - 6.51 \times 10^{-9}\lambda^2 + O(\lambda^3)$

the uncertainties are further away from the minimal value 0.5 of uncertainty. We expect these differences are just errors which result from using truncation $N = 25$. If N is increased, we expect that these differences would be decreased and eventually disappears as $N \to \infty$. We propose that the greater error for greater value of α can be interpreted as follows. For the standard coherent states, the real part of α is the expectation value in position of the coherent state, whereas the imaginary part is the expectation value in momentum of the coherent state. We expect that this would be similar for our case. So this means that the errors would be related to the distance of the expectation values from the origin of phase space.

CHAPTER V

CONCLUSIONS

We may conclude that in order to obtain coherent state wave function, we write lowering operator up to the second order in λ. NEQHO wave function coherent states are obtained as eigenstate of lowering operator. According to QHO, the uncertainty between position and momentum of coherent state satisfies minimum value.

According to NEQHO coherent states with $|\alpha| \leq 2$, the uncertainties are approximately equal to the minimal value, see table 3. So by using this consideration, it is not a simple matter to distinguish them from the coherent states for QHO.

We expect that even for $|\alpha| \geq 2$, the uncertainties of NEQHO coherent states are still approximately equal to the minimal value. Nevertheless, this should still be investigated. Because, error between $(\Delta y)_\alpha^{(\lambda,25)}(\Delta p_y)_\alpha^{(\lambda,25)} - 0.5$ tends to be larger for $|\alpha| \geq 2$, see table 4. We leave this verification to future works.

It is also interesting to go beyond the second order in λ to see whether the uncertainties of NEQHO coherent states still have the minimal value. We also leave this varification to future work. Furthermore, other physical phenomena relating to coherent states should also be investigated to see whether it is possible to distinguish NEQHO coherent states from their QHO counterparts. One of the phenomena to be checked is the noise in the time evolution of coherent states.

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