Qualifying Examination

TAH POE THEORETICAL PHYSICS
THE INSTITUTE for FUNDAMENTAL STUDY

Academic Year 2016 (2559): ROUND 1

October 2016

Name:...

Student ID:...

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 6 problems.
2. Write down detailed and clear solution to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on separate sheet.
5. Write legibly, otherwise you place yourself at a grave disadvantage .

Questions No.	Checked Questions to be marked
1	
2	
3	
4	
5	
6	

Name:...

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1) (a) Let R be be the trapezoid with vertices at $A(1,-1)$, $B(1,3)$, $C(3,2)$, $D(3,0)$, consider the following integral:

$$\int_R xy\,dA$$

Complete the following two integrals explicitly which are equal to the above double integral given by the form:

$$\int_1^3 \left(\int_?^? ?\,dy \right) dx \qquad [1]$$

$$\int_{-1}^0 \left(\int_?^? ?\,dx \right) dy + \int_0^2 \left(\int_?^? ?\,dx \right) dy + \int_2^3 \left(\int_?^? ?\,dx \right) dy \qquad [1]$$

Evaluate one of them. [3]

(b) Let R be the region in R^3 defined by $4 \le x^2 + y^2 + z^2 \le 9$ and $z \ge 0$, consider the following integral:

$$\iiint_R (x^2 + y^2)\,\mathrm{d}V$$

Write the above one using the spherical coordinate. [2]

Evaluate it. [3]

2) (a) Find solution to the following system of linear equations.

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 3, \\ -x_1 + 2x_2 + x_3 &= -1, \\ 2x_1 + 3x_2 + 8x_3 &= 4. \end{aligned}$$ [4]

(b) Determine whether each of the following matrix is invertible or not.

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 4 & 7 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$ [1]

(c) The following matrices are the above two. If these are invertible, compute the inverse.

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 4 & 7 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$ [5]

3) (a) Write the angular momentum operator in the quantum theory using $\hbar$, r and ∇ and show that its components are given by

$$l_x = -i\hbar\left(y\partial_z - z\partial_y\right), \quad l_y = -i\hbar\left(z\partial_x - x\partial_z\right), \quad l_z = -i\hbar\left(x\partial_y - y\partial_x\right). \qquad [2]$$

(b) Using the above expressions, show the following relations

$$[l^2, l_x] = 0 \quad \text{and} \quad [l_x, l_y] = i\hbar\, l_z. \qquad [4]$$

(c) Using the operator, $l_+ \equiv l_x + il_y$, show the following relations

$$[l_+, l_z] = -\hbar l_+ \quad \text{and} \quad [l^2, l_+] = 0. \qquad [4]$$

4) From the integral,

$$I = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{5 + 3\cos\theta},$$

Let $z = \exp(\mathrm{i}\theta)$.

(a) Show that the poles of the integral are at $z = -3$ and $z = -\frac{1}{3}$. [6]

(b) Find I. [4]

5) The Fourier transform of a function $f(x)$ is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\; f(x) \exp(-\mathrm{i}kx).$$

Show that the Fourier transform of a Gaussian function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\,\sigma^2}\right),$$

is

$$\hat{f}(k) = \exp\left(-\frac{1}{2}\sigma^2 k^2\right).$$ [10]

6) Find the solution of the inhomogeneous ordinary differential equation (ignoring the general solution),

$$y'' + 3y' - 4y = 2\cos x - 3\sin 2x. \qquad [10]$$

Name:..

Qualifying Examination

TAH POE THEORETICAL PHYSICS
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Academic Year 2019 (2562) : ROUND 2

Part 2: Subjects Relating to student's Research Field

Instructions and cautions

1. Choose 3 out of the following 8 problems (problem 7) to 14).
2. Write down detailed and clear solutions to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a separate sheet.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

For marker: Name.. Signature..

2.1 Relativistic Quantum Fields

1) (a) Write down the most general solution to the Klein-Gordon equation for a scalar field in 1+1 dimensional spacetime:

$$\frac{\partial^2\phi(t,x)}{\partial t^2} - \frac{\partial^2\phi(t,x)}{\partial x^2} = 0. \qquad [1]$$

(b) Rewrite the above Klein-Gordon equation by introducing speed of light c into appropriate places. [1]

(c) Suppose that

$$\phi(t,x) = e^{-i\omega t + ikx}, \qquad \omega > 0$$

is a solution to the Klein-Gordon equation given in (a). Write k as a function of ω (if you wish, you may set $c = 1$). [3]

(d) Consider the Klein-Gordon equation for a massive scalar field in 1+1 dimensional spacetime:

$$\frac{\partial^2\phi(t,x)}{\partial t^2} - \frac{\partial^2\phi(t,x)}{\partial x^2} + sm^2\phi(t,x) = 0,$$

where s is a number to be determined later, and m is the mass of the scalar field. Rewrite the equation by introducing $\hbar$ into the appropriate places (set $c = 1$). [2]

(e) Determine the value of s [3]
[Hint: $E = \hbar\omega$, $p = \hbar k$].

2) Consider a Lagrangian for a massless complex scalar field ϕ in a $d+1$-dimensional Minkowski spacetime:

$$\mathcal{L} = \partial_\mu \phi \partial_\nu \phi^* \eta^{\mu\nu},$$

where $\eta^{\mu\nu}$ is the matrix inverse of the Minkowski metric $\eta_{\mu\nu}$.

(a) Consider a transformation

$$\phi \mapsto e^{-i\alpha}\phi,$$

where α is a real constant. Write down the associated transformation for ϕ^*. [1]

(b) Show that the Lagrangian is invariant under the transformation given above [1]

(c) We want to promote the Lagrangian to have the symmetry

$$\phi \mapsto e^{-i\alpha(x)}\phi,$$

where now $\alpha(x)$ is a function of spacetime. Define $D_\mu \phi \equiv (\partial_\mu \phi + iA_\mu \phi)$, and require that

$$D_\mu \phi \mapsto e^{-i\alpha(x)} D_\mu \phi.$$

Obtain the transformation for A_μ. [3]

(d) Consider a Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Obtain the equation of motion for the field A_μ. [3]

(e) Give a brief (2-3 sentences) explanation of the significance of the equation of motion obtained in question (d)? [2]

3) Consider a system with the generic action S given by

$$S = \int_{t_0}^{t_f} L(q(t), \dot{q}(t), t)\, dt\ .$$

Consider this action invariant ($\delta S = 0$) under the symmetry transformation

$$q(t) \to q'(t) = q(t) + \delta_s q(t)\ ,$$

where

$$\delta_s q(t) = q'(t) - q(t) = \epsilon \Delta(q(t), \dot{q}(t), t)$$

is a symmetry variation with ϵ an infinitesimal parameter.
Under the symmetry assumption $\delta S = 0$, prove that there is a conserved quantity (Noether charge) given by

$$Q = \frac{\partial L}{\partial \dot{q}} \delta_s q.$$

4) Write the Yang-Mills action. From this action find the equations of motion of the theory. [10]

2.2 General Relativity & Cosmology

5) A general form of the spherically symmetric solution of Einstein field equation with prefect fluid can be written as $ds^2 = e^{2A(r)}c^2dt^2 - e^{2B(r)}dr^2 - r^2d\Omega^2$. By solving the Einstein equation, one found that functions $A(r)$ and $B(r)$ can be written as

$$e^{2B} = \left(1 - \frac{2Gm}{c^2 r}\right)^{-1},$$
$$A' = \frac{1}{r^2}\left(1 - \frac{2Gm}{c^2 r}\right)^{-1}\left(\frac{Gm}{c^2} + \frac{4\pi G p r^3}{c^4}\right),$$

where $m = m(r)$ is a gravitational mass defined as

$$m(r) \equiv \int_0^r 4\pi r^2 \rho dr.$$

(a) Explain how the gravitational mass is physically different from the bare mass. [3]

(b) Show that the conservation of the energy momentum tensor yields the equation. [3]

$$p' = A'(\rho c^2 + p).$$

(c) By substituting A' from Eq. (11) into Eq. (11), one obtains the Toman-Oppenheimer-Volkoff (TOV) equation. From this equation, find its Newtonian limit and then explain how gravitational collapse in General Relativity is qualitatively different from one in Newtonian theory. [4]

Notations/Equations you may use:

Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}.$$

The energy momentum for the perfect fluid:

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu - pg_{\mu\nu}.$$

Covariant derivatives of rank-2 tensor:

$$\nabla_\rho T_{\mu\nu} = \partial_\rho T_{\mu\nu} - \Gamma^\sigma_{\rho\mu}T_{\sigma\nu} - \Gamma^\sigma_{\rho\nu}T_{\sigma\mu}.$$

6) For linearized perturbation theory of gravity, the metric tensor can be decomposed as background and perturbation parts as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} \ll 1$. The second order action for the perturbed field $h_{\mu\nu}$ can be written as

$$S^{(2)} = \int d^4x \left(\frac{1}{4}\partial_\nu h_{\rho\beta}\partial^\nu h^{\rho\beta} - \frac{1}{2}\partial_\lambda h_{\mu\alpha}\partial^\alpha h^{\mu\lambda} - \frac{1}{4}\partial_\alpha h \partial^\alpha h + \frac{1}{2}\partial_\rho h^{\rho\nu}\partial_\nu h \right).$$

(a) Find the equation of motion for the field $h_{\mu\nu}$. [3]

(b) Show that the the equation of motion is invariant under gauge transformation

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad [3]$$

(c) Show that there are two propagating degrees of freedom for this theory. [4]

7) Let the potential of the inflaton field ϕ be

$$V(\phi) = V_0 e^{-\lambda\phi} ,$$

where V_0 and λ are the constant parameters. In this problem, we use the unit where $(8\pi G)^{-1/2} = 1$.

(a) Compute the slow-roll parameter ϵ for this inflaton. [1.5]

(b) Does the inflationary model considered in this problem has graceful exit? Why? [1.5]

(c) Write down the Friedmann equation and the evolution equation for the inflaton in the slow-roll limit.

[1.5]

(d) Solve the evolution equations in the previous item using the trial solution $\phi = C\ln(t/t_0)$, where t is the time and C as well as t_0 are the constants of the integration. Compute the expressions for C and t_0, and then compute the expression for the Huble parameter in terms of time. [2.5]

(e) Show mathematically what the flatness problem is. [3]

8) (a) Given deceleration parameter,

$$q = -\frac{\ddot{a}}{aH^2} .$$

Show that for a radiation dominated universe,

$$q_0 = \Omega_0.$$ [4]

(b) Explain in detail the Olbers' paradox and how general relativity helps solving the problem. [3]

(c) Explain in detail the problems of the Big Bang theory and how inflation help solving these problems. [3]