Qualifying Examination

TAH POE THE INSTITUTE for THEORETICAL PHYSICS FUNDAMENTAL STUDY

Academic Year 2019 (2562) : ROUND 1

August 2019

Name:..

Student ID:..

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 8 questions.
2. Write down detailed and clear solution to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
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1 Vector Calculus

Question 1

Let C be the curve $x^2 + y^2 = 1$ lying in the plane $z = 1$. Let $\vec{F} = (z - y)\hat{\imath} - y\hat{k}$

(a) Calculate $\nabla \times \vec{F}$.

(b) Calculate $\int_C \vec{F} \cdot d\vec{s}$ using a parametisation of C and a chosen orientation for C.

(c) Write $C = \partial S$ for suitable chosen surface S, applying Stoke' theorem and verify your answer in (b).

(d) Consider the sphere with radius $\sqrt{2}$ and centred at the origin. Let S' be the part of the sphere that is above the curve (i.e., lies in the region $z \leq 1$), and has C as boundary. Evaluate the surface integral of $\nabla \times \vec{F}$ over S'. Specify the orientation you are using for S'.

Question 2

Give mathematical derivations that a conservative force field is irrotational and can be expressed as a negative gradient of a scalar field U.

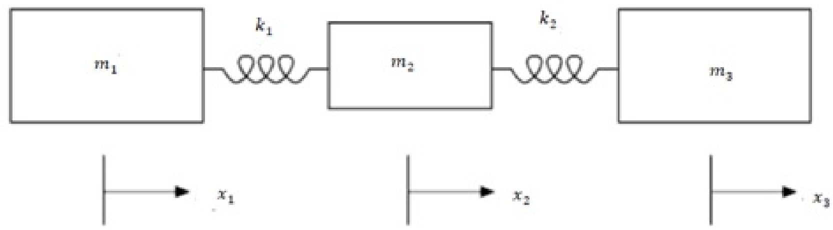

Figure 1: An oscillation system

2 Linear Algebra

Question 3 An oscillator system as shown in figure 1.

(a). Write down the Lagrange and find the equations of motion for each mass. Show these equations in the matrix form as $\ddot{x}_i = \Omega x_i$, where Ω is a 3×3 matrix and $i = 1, 2, 3$.

(b). Ω is not a Hermitian matrix in general. For given $m_1 = m$, what is the corresponding setting of m_1 and m_2 for letting Ω become a Hermitian matrix?

(c). Continue from (b) and solve the eigenvalue of Ω.

(d). Taking $k_1 = k_2 = k$, showing the simplified eigenvalues from (c). and find the corresponding normalized eigenvectors.

(e). Given a normal mode relation as $x_i = x_{i0} \times e^{i\omega t}$, explaining the motions of the normal modes with respect to the eigenvalues and eigenvectors.

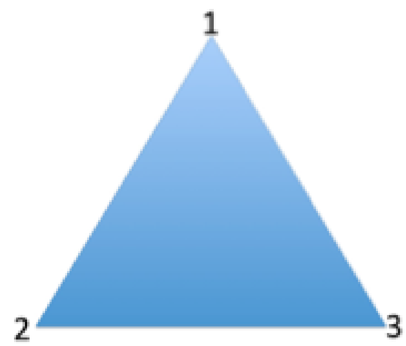

Figure 2: An equilateral triangle

3 Group Theory/Lie Algebra

Question 4

D_3 symmetry of an equilateral triangle: Given an equilateral triangle labelled at all corners as shown in figure 2. If we choose to represent above triangle with a column matrix $(1\ 2\ 3)^T$ please

(a) identify all symmetries of the equilateral triangle,

(b) find the matrix representation for each of the symmetry,

(c) construct the Cayley table for D_3 symmetry,

(d) identify all possible subgroup of the D_3.

4 Function of Complex Variables

Question 5

Show that the Function $e^x(\cos(y) + i\sin(x))$ is an analytic function. Find its derivative.

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Question 6

Evaluate the integrals

(a) $\int_c \frac{3z^2+7z+1}{z+1} dz$ where c is the circle $|z| = \frac{1}{2}$,

(b) $\int_0^{1+i} (x^2 - iy) dz$ along the path $y = x$.

5 Integral Transformations

Question 7

Find out the Fourier transform of the following function,

$$f(\boldsymbol{x}) = \left(\frac{2}{\pi a^2}\right)^{3/4} \exp\left(-\frac{r^2}{a^2}\right),$$

which is the ground state of the three-dimensional simple harmonic oscillator duly normalized. What do you infer from the result?

Hint 1: Consider the spherical polar coordinates (r, θ, φ) with z-axis taken along the momentum $\boldsymbol{k}$.

Hint 2: The Gaussian integral is:

$$\int_{-\infty}^{\infty} dx \, \exp\left(-\alpha x^2\right) = \sqrt{\frac{\pi}{\alpha}}.$$

6 Ordinary Differential Equations

Question 8

Solve the differential equation

$\frac{d^2x}{dt^2} + \frac{g}{l}x = \frac{g}{l}L$, where g, l, L are constants subjected to initial condition $x = a$, $\frac{dx}{dt} = 0$ at $t = 0$.

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Qualifying Examination

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Part 2: Subjects Relating to Student's Research Field

Instruction and cautions

1. Choose 3 out of the following 6 problem relating student's field.
2. Write down detailed and clear solution to problems of your choice.
3 Complete answer are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
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For marker: Name... Signature...

1 Research Field: General Relativity and Cosmology

1.1 Manifold Curvature Gravitation, Spherically symmetric BH

Question 1

Given the covariant derivative defined as:

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho{}_{\nu\mu} A_\rho.$$

(a). Show that $\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\rho\mu} A^\rho$.
Hint: Starting by a given scalar $\phi = A_\nu B^\nu$, and in the process you need to rename the contraction indexes.

(b). Show $\nabla_{[\mu}\nabla_{\nu]} X^\alpha = \frac{1}{2} R^\alpha{}_{\beta\mu\nu} X^\beta$ with the condition of a torsion free connection, where $R^\alpha_{\beta\mu\nu}$ is the Reimann curvature tensor, and $\nabla_{[\mu}\nabla_{\nu]} = \frac{1}{2}\left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)$.

1.2 Field Theory of Gravity

Question 2

The Einstein-Hilbert equation is given by:

$$\mathcal{S}_{EH} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}R, \quad c = 1.$$

a) Considering the Lagrangian $\mathcal{L}_{EH} = \sqrt{-g}R$ apply the least action principle to arrive at the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}_{EH}}{\partial g_{\mu\nu}} = \partial_\alpha \frac{\partial \mathcal{L}_{EH}}{\partial \partial_\alpha g_{\mu\nu}} - \partial_\alpha \partial_\beta \frac{\partial \mathcal{L}_{EH}}{\partial \partial_\alpha \partial_\beta g_{\mu\nu}}.$$

b) Derive the Einstein's equation in absence of matter from this action from the least action principle.

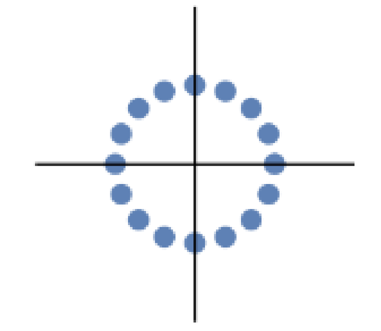
Hint: You can use the Palatini Identity:

$$\delta R_{\mu\nu} = \nabla_\alpha\left(\delta\Gamma^\alpha_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^\alpha_{\alpha\mu}\right).$$

1.3 Linearised Gravity/Gravitational Wave

Question 3

(a) How a ring distribution of test mass particles will be distorted if a plus polarized gravitational wave passes through a line which is perpendicular to the plane of the particle? (see the image)



(b) Consider a plus polarized gravitational wave of amplitude h passes through a Michelson interferometer with arm length L in a perpendicular direction to its plane. If the gravitational wave generates a change in the length of the interferometer by $\frac{\delta L}{L} \simeq h$ then to get a sensitivity $\delta L \simeq 4 \times 10^{-16}$ what should be the length of the arm of the interferometer L. The amplitude of the gravitational wave is $h \simeq 10^{-21}$ cm. Can you name any interferometer with similar arm length?

1.4 Inflation cosmology based on Mukhanov

Question 4

The energy density and pressure of the homogeneous scalar field is given by

$$\epsilon = \frac{1}{2}\dot{\phi}^2 + V(\phi) \;\; ; \;\; p = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

The Friedmann equation and the conservation law also given by

$$H^2 = \frac{8\pi}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \;\; ; \;\; \dot{\epsilon} = -3H\left(\epsilon + p\right).$$

(a) Evaluate the equations of motion of ϕ by taking $V = \frac{1}{2}m^2\phi^2$.

(b) Showing that $H^2 \approx \frac{1}{9t^2}$ in the limit of $|\dot{\phi}| \gg m\phi$.

1.5 Physics Foundations of Cosmology

Question 5

According to the CMB meaurements by the Plank satellite, the current value of matter density is $\Omega_{m,0} = 0.31$ while the dark energy density is $\Omega_{\lambda,0} = 0.69$, making the Universe exactly flat with $\Omega_0 = \Omega_{m,0} + \Omega_{\lambda,0} = 1.0$. Now show that regardless of the present-day values of $\Omega_{m,0}$, and $\Omega_{\lambda,0}$, the Universe approached an Einstein-de Sitter Universe with $\Omega_m = 1$ at high redshift.

Recall: The critical density is, $\rho_c = \frac{3H^2}{8\pi G}, \Omega_0 = 1 - \Omega_k$ and the time derivative of the scale factor is given by the Friedman equation,

$$H^2 = H_0^2(\Omega_{m,0}a^{-3} + \Omega_{\lambda,0} - (\Omega_0 - 1)a^{-2})$$

1.6 Problems of Newtonian Cosmology, Shortcomings of the Big Bang model

Question 6

(a) Why the sky is dark at night? If the universe is endless and uniformly populated with luminous stars, then every line of sight must eventually terminate at the surface of a star. Hence, contrary to observation, this argument implies that the night sky should everywhere be bright, with no dark spaces between the stars. How do you remove this paradox (Olbers' paradox)?

(b) For a spatially curved universe the total density parameter at anytime can be written as,

$$1-\Omega(t) = -\frac{H_0^2(1-\Omega_0)}{H(t)^2 a(t)^2} = -\frac{\kappa c^2}{R_0^2 H(t)^2 a(t)^2}$$

where Ω_0 is the present value of the $\Omega(t)$. Consider a Benchmark model of the universe for which the Hubble parameter can be expressed as,

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4} + \Omega_{\lambda,0} - (1-\Omega_0)a^{-2}$$

where, $\Omega_0 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\lambda,0}$. From current observations $\Omega_{r,0} = 8.4 \times 10^{-5}, \Omega_{m,0} = 0.3 \pm 0.1, \Omega_{\lambda,0} = 0.7 \pm 1$ and $1 - \Omega_0 \leq 0.2$. Find out at the Plank epoch $t_p \simeq 5 \times 10^{-44}$ and $a_p \simeq 2 \times 10^{-32}$ the deviation of $\Omega(t)$ from unity. Give your comment of the result.

2 Research Field 2: High Energy Physics

2.1 Fields Theory

Question 1: Classical Field Theory

The action is given by

$$S = \int d^4x L(\phi_r, \partial_\mu \phi_r)$$

where L is the Lagrangian density, which is a function of the fields $\phi_r(x), r = 1, ..., n$ and their first derivatives. Show that the variation of this Lagrangian density giving the Euler-Lagrange equation of motion as follows

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi_r)} \right) - \frac{\partial L}{\partial \phi_r} = 0.$$

Question 1: Quantum Field Theory

Show that the Euler-Lagrangian equations for the following Lagrangian density can be derived for the fields (Hint for the field $\bar{\psi}, \psi, \phi$)

$$L = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 - ig\bar{\psi}\gamma_5\psi\phi$$

2.2 Symmetries and Transformation

Question 3: Noether's theorem

The Lagrangian for Abelian scalar QED with a complex field is given by,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(D_\mu\varphi\right)^\dagger D^\mu\varphi - m^2\varphi^\dagger\varphi,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad A_\mu \in \mathfrak{Re}; \qquad D_\mu = \partial_\mu + ieA_\mu.$$

a) What is the Nöther current associated with the given Lagrangian subjected to the following gauge transformations:

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x), \quad \varphi(x) \to \exp\left(-i\theta(x)\right)\varphi(x), \quad \varphi^\dagger(x) \to \exp\left(i\theta(x)\right)\varphi^\dagger(x).$$

b) From the equations of motion, show that this current is conserved gauge-invariantly.

Hint 1: The Nöther current has the general expression:

$$J^\mu = \sum_k \delta\psi_k \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_k} - K^\mu,$$

where ψ_ks are different fields of the Lagrangian and K^μ is defined through the general condition that for invariance of the action, at the most, the Lagrangian changes by a total derivative $\delta\mathcal{L} = \partial_\mu K^\mu$.

Question 4: Global and Gauge Transformations

Show that the Maxwell-Chern-Simons Abelian gauge theory:

$$\mathcal{L}_{AMCS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \quad \mu,\nu,\rho = 0,1,2,$$

defined in **2+1** space-time dimensions is gauge-invariant under the $U(1)$ gauge transformation $A_\mu \to A_\mu + \partial_\mu\theta(x)$. Here, $\mu \in \mathbb{R}$. Find out the propagator for this theory and thus show that the theory is gauge-invariantly massive. Further, argue that the same gauge-invariant mass appears in the $SU(N)$ non-Abelian version:

$$\mathcal{L}_{NAMCS} = -\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \mu\epsilon^{\mu\nu\rho}\mathrm{Tr}\left[A_\mu\partial_\nu A_\rho + \frac{2}{3}\lambda A_\mu A_\nu A_\rho\right].$$

Hint 1: Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\epsilon^{\mu\nu\rho}$ is the totally anti-symmetric Levi-Civita tensor.

Hint 2: In the non-Abelian case $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right]$ and $A_\mu = A_\mu^a T^a$ with $T^a, \quad a = 1,2,\cdots(N^2-1)$ being the generator matrices of $SU(N)$. Further, adopt the convention:

$$\mathrm{Tr}\left[T^aT^b\right] = \frac{1}{2}\delta^{ab} \quad \text{and} \quad \mathrm{Tr}\left[T^aT^bT^c\right] = \frac{1}{4}\left(d^{abc} + if^{abc}\right),$$

where d^{abc} is totally symmetric whereas f^{abc} is totally anti-symmetric structure constants. Finally,

$$\left[T^a, T^b\right] = if^{abc}T^c.$$

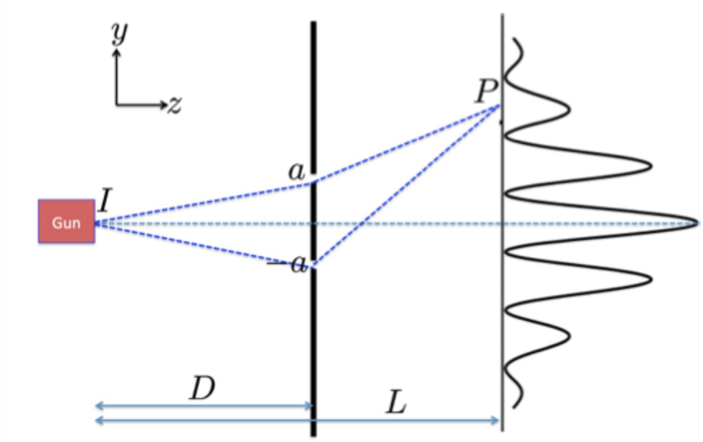


Figure 3: A set up of double-slit experiment

2.3 Quantisation

Question 5: Canonical Quantization

Double Slit Experiment: The double slit experiment of a single electron is the archetypical system used to demonstrate quantum mechanics behaviour given in figure 1. The screen is a $z = L$. The wall is located at $z = D$ and the two slits on the at $(x', y = \pm a, z = D)$. The particle propagates freely from $I = (x = 0, y = 0, z = 0, t = 0)$ to $(x', y = \pm a, z = D, t = t')$ and then to $P = (x_f, y_f, z = L, t = T)$. We know that the propagator is obtained by summing over two trajectories in which the particle goes through of the two slits. Since we don't know the limit in which the particle crossed the slit, we need to sum over all the possible intermediate times

$$\begin{aligned} K(x_f, y_f, L, T; 0,0,0,0) = \int_0^T & dt'(K(x_f, y_f, L, T; x', a, D, t')K(x', a, D, t'; 0,0,0,0) \\ & + K(x_f, y_f, L, T; x', -a, D, t')K(x', -a, D, t'; 0,0,0,0)) \end{aligned}$$

Show that the interference pattern is proportional to

$$\cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right), \quad \text{[Intensity distribution for Fraunhofer diffraction]}$$

where $2a = d$.

Hint: The free particle propagator is given by

$$K(y_f, z_f, T; y_i, z_i, 0) = \left(\frac{m}{2\pi i \hbar T}\right)^{3/2} e^{\frac{im(x_f - x_i)^2}{2\hbar T}} e^{\frac{im(y_f - y_i)^2}{2\hbar T}} e^{\frac{im(z_f - z_i)^2}{2\hbar T}}$$

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Given

$$\int_0^T \frac{1}{[t'(T-t')]^{3/2}} e^{-\frac{A}{t'}-\frac{B}{T-t'}} = \sqrt{\frac{\pi}{T^3}}\frac{\sqrt{A}+\sqrt{B}}{\sqrt{AB}} e^{-\frac{(\sqrt{A}+\sqrt{B})^2}{T}}.$$

Question 6: Path Integral Quantization

The hamiltonian for a harmonic oscillator can be written in dimensionless units ($m = \hbar = \omega = 1$) as

$$H = \hat{a}^\dagger \hat{a} + 1/2,$$

where $\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}$ and $\hat{a}^\dagger = \hat{x} - i\hat{p})/\sqrt{2}$. One unnormaised energy eigenfunction is $\psi_a = (2x^3 - 3x)e^{-x^2/2}$. Find two other (unnormalised) eigenfunctions which are closest in energy to ψ_a.