Qualifying Examination

TAH POE THE INSTITUTE IF THEORETICAL PHYSICS for FUNDAMENTAL STUDY

Academic Year 2019 (2562) : ROUND 2

January 2020

Name:..

Student ID:...

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 8 questions.
2. Write down detailed and clear solution to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	
7	
8	

1 Vector Calculus

Question 1.

Evalaute the $\iint_S \mathbf{F}.\mathbf{dS}$, where $\mathbf{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and $\mathbf{S}$ is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

2 Linear Algebra

Question 2.

2.1 Let

$$A = \begin{bmatrix} -4 & -6 & -12 \\ -2 & -1 & -4 \\ 2 & 3 & 6 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} 6 \\ 5 \\ -3 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

1. Express the vector **u** as a linear combination of **v** and **w**.
2. Compute $A^3\mathbf{v}$.
3. Compute $A^3\mathbf{w}$.
4. Compute $A^3\mathbf{u}$.

2.2 If A and B are two unitary matrices, show that AB is also unitary.

3 Group Theory

Question 3.

Consider the following multiplication table for a group G and answer the following questions.

×	**a**	**b**	**c**	**d**	**f**
a	**b**	**f**	**d**	**a**	**c**
b	**f**	**c**	**a**	**b**	**d**
c	**d**	**a**	**f**	**c**	**b**
d	**a**	**b**	**c**	**d**	**f**
f	**c**	**d**	**b**	**f**	**a**

(a) Which element is the identity element?

(b) Is the group commutative?

(c) Is there some x G such that $x^3 = d$?

Question 4.

Let G be a group with elements a and b in G. Show the equivalence

$$a^k = e \Longleftrightarrow \left(bab^{-1}\right)^k = e$$

Can you draw any conclusion regarding order (a) and order (bab^{-1})?

4 Function of Complex Variables

Question 5.

Evaluate the contour integral

$$\int_C f(z)dz$$

using the parametric representations for C, where

$$f(z) = \frac{z^2 - 1}{z}$$

and the curve C is

(a) the semicircle $z = 2e^{i\theta}(0 \leq \theta \leq \pi)$

(b) the semicircle $z = 2e^{i\theta}(\pi \leq \theta \leq 2\pi)$

(c) the circle $z = 2e^{i\theta}(0 \leq \theta \leq 2\pi)$

5 Integral Transform

Question 6.

The Hamiltonian of a harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2.$$

The time independent Schrödinger equation for the lowest energy state is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi(x) + \frac{1}{2}k^2x^2\phi(x) = \sqrt{\frac{k}{m}}\frac{\hbar}{2}\phi(x).$$

Hint: The Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

(a). The approximate solution of the wave function is

$$\phi(x) \sim e^{-\frac{\sqrt{mk}}{2\hbar}x^2}.$$

Find the normalized $\phi(x)$.

(b). Derive $\phi(p)$ by using Fourier Transform.

(c). Is the wave function $\phi(p)$ normalized?

(d). Show that $\phi(p)$ satisfy the Schrödinger equation in momentum basis.

Question 7.

The Parseval's relation is given by

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega.$$

Where $F(\omega)$ and $G(\omega)$ were the Fourier Transform of $f(t)$ and $g(t)$ respectively.
(a) Proof the Parseval's relation
(b) Obtain the integral $\int_{-\infty}^{\infty} \frac{1}{\omega^2+\alpha^2} d\omega$ by using the Parseval's relation with the function

$$u(t) = \begin{cases} e^{-\alpha t} & , \quad t \geq 0 \\ 0 & , \quad t \leq 0 \end{cases} \quad ; \quad \alpha > 0.$$

6 Ordinary Differential Equations

Question 8.

Solve the following ODEs,

$$\text{a)} \qquad x\frac{dy}{dx} - y = \frac{x^3}{y}e^{\frac{y}{x}}$$

$$\text{b)} \qquad y'' - 3y' + 2y = \sin x$$

Name:..

Qualifying Examination

TAH POE THEORETICAL PHYSICS
THE INSTITUTE for FUNDAMENTAL STUDY

Academic Year 2019 (2562) : ROUND 2

Part 2: Subjects Relating to student's Research Field

Instruction and cautions

1. Choose 3 out of the following 6 problem relating student's field.
2. Write down detailed and clear solution to problems of your choice.
3 Complete answer are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	

For marker: Name... Signature...

1 Manifold, Curvature, Gravitation, Spherically symmetric solution, Black holes

Question 1.

A space of constant curvature is given by

$$R_{\alpha\beta\gamma\delta} = K\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right).$$

a). A 3-dimensional spherically symmetric space is given by

$$d\sigma^2 = e^{\lambda(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

the non-vanishing Ricci curvatures are given by

$$R_{rr} = \frac{\partial_r\lambda(r)}{r} \quad ; \quad R_{\theta\theta} = \csc^2\theta R_{\phi\phi} = 1 + \frac{1}{2}re^{-\lambda(r)}\partial_r\lambda(r) - e^{-\lambda(r)},$$

show that the metric for a three dimensional spherically symmetric space with constant curvature is

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2\left(dr^2 + \sin^2\theta d\phi^2\right).$$

b). Show this metric on the conformally flat form by taking this transformation.

$$r = \frac{\bar{r}}{1 + \frac{1}{4}K\bar{r}^2}.$$

2 Inflationary cosmology: Base on the book of V. Makhanov

Question 2.

The Friedmann equation and the conservation law are given by

$$H^2 = \frac{8\pi}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$

and

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0,$$

where ϕ is a scalar field and "dot" denotes the derivative to time.

a). The slow-roll condition is give by

$$|\dot{\phi}^2| \ll |V| \;\; ; \;\; |\ddot{\phi}| \ll 3H\dot{\phi} \sim |V,_\phi|.$$

The Hubble parameter defined as $H \equiv \frac{d\ln a}{dt}$, where a is the time dependent scale factor, show that

$$a(t) = a_i \exp\left(8\pi \int_{\phi(t)}^{\phi_i} \frac{V}{V,_\phi} d\phi\right),$$

where "i" represents the initial value.

b). For a power-low potential $V = \frac{1}{n}\phi^n$, how does the scalar field change with time if we need a e-fold expansion?

3 Field Theory of Gravity

Question 3.

The Einstein equation is given as,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu},$$

where $\kappa = 8\pi G/c^4$ and $T_{\mu\nu}$ is the energy-momentum tensor [EMT] of the matter field.

Derive the Einstein equation for the action,

$$\mathcal{L}_S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right],$$

and show that the scalar equation of motion (EOM) is Harmonic:

$$\partial^2 \phi = 0, \quad \text{where} \quad \partial^2 = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \right).$$

Hint: Consider the formulae:

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}, \qquad \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu} \sqrt{-g}.$$

4 Linearized gravity, Gravitational wave

Question 4.

In the linear approximation the covariant metric tensor is expanded about the Minkowski background as,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| << |\eta_{\mu\nu}|.$$

a) Show that the contravariant counterpart must take the form,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}.$$

b) Show that in this linear theory the scalar curvature is give by,

$$R = \partial_\mu \partial_\nu h^{\mu\nu} - \partial^2 h, \quad h = h^\mu_\mu.$$

Hint: Useful formulae:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu} \right),$$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \quad R_{\sigma\nu} = R^\rho{}_{\sigma\mu\nu}\delta^\mu_\rho, \quad R = R_{\mu\nu} g^{\mu\nu}.$$

5 Physical Foundation of Cosmology

Question 5.

According to the Hubble's Law, the velocity of any two galaxies in the universe moving apart from each other is

$$\text{Velocity} = H_0 \times \text{Distance}.$$

a) We can encounter a situation in which the very faraway galaxies are receding from each other with a velocity more than the velocity of light very faraway. Give your comment on this. Do you think special theory of relativity is violated in this case?

Explain the difference between co-moving distance and physical distance, and their connection to cosmological redshift. You only need a couple sentences here, to show a qualitative understanding.

6 Questions of Newtonian cosmology

Question 6.

The mass within the sun's 8 kpc orbital radius about the Milky Way is $\sim 10^{11} M \odot$ ($M_{\odot} =$ mass of the sun $\approx 2 \times 10^{33}$g). By what factor does the average density in this sphere exceed the mean density of matter in the Universe? (Assume the values preferred at the moment: $\Omega_m = \rho_{\text{matter}} / \rho_{\text{crit}} = 0.25, H_0 = 70 \text{kms}^{-1} \text{Mpc}^{-1}$).