Qualifying Examination

Academic Year 2015 (2558) : ROUND 3

June 2016

Name:...

Student ID:...

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Write down the detailed and clear solution to problems 1) to 5).
2. Complete answers are preferred to fragments.
3. Write on one side of the paper only and begin each answer on a separate sheet.
4. Write legibly; otherwise you place yourself at a grave disadvantage.

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1 Mathematical Methods of Physics

Question 1.

Write down the matrix representing the following transformation on $\mathbb{R}^3$

1.1) clockwise rotation of $45°$ around the x-axis. [3]

1.2) reflection in the plane $x = y$. [3]

1.3) the result of first doing 1.1) and then 1.2). [4]

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Question 2.

Consider the vector field

$$F = (-y/(x^2 + y^2), x/(x^2 + y^2), 0)$$

defined on all $\mathbb{R}^3$ except the z-axis.

2.1) Compute $\boldsymbol{\nabla} \times F$ on the region where it is defined. [5]

2.2) Let γ be the closed curve defined by the circle in xy-plane with centre $(2, 2, 0)$ and redius 1. By using Stokes' theorem, or otherwise, evaluate the line integral $\oint_\gamma F \cdot \mathrm{d}x$. [5]

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Question 3.
The Fourier transform of a function $f(x)$ is defined by

$$f(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \hat{f}(x) \exp(\mathrm{i}kx).$$

Suppose that the function $\varphi(x,t)$ satisfies the diffusion equation

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2},$$

with $|x| < \infty, t \geq 0$, and the boundary conditions: $\varphi(x,0) = h(x), |\varphi(x,t)| \to 0, |\mathrm{d}\varphi/\mathrm{d}x| \to 0$ as $x \to \pm\infty$. Consider the Fourier transform of $\varphi(x,t)$ with respect to x, which denoted $\varphi(\hat{k},t)$. Using integration by parts, or otherwise, show that function satisfies the equation

$$\frac{\partial \hat{\varphi}}{\partial t} = -k^2 \hat{\varphi}. \qquad [6]$$

Show that the solution of this equation which satisfies the boundary condition at $t = 0$ is

$$\hat{\varphi}(k,t) = \hat{h}(k) \exp\bigl(-k^2 t\bigr). \qquad [4]$$

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Question 4.

Use the Gram-Schmidt process to obtain the orthonormal basis from

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

where the inner product is defined in usual way:

$$\left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) \equiv \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 .[10]$$

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Question 5.
By using De Moivere's theorem $e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$,

5.1) Show that $\cos(\theta_1 \pm \theta_2) = \cos(\theta_1)\cos(\theta_2) \mp \sin(\theta_1)\sin(\theta_2)$. [4]

5.2) Express $\cos(5\theta)$ in terms of $\cos(\theta)$. [3]

5.3) Express $\sin(5\theta)$ in terms of $\sin(\theta)$. [3]

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Academic Year 2019 (2562) : ROUND 2

Part 2: Subjects Relating to Student's Research field

Instructions and cautions

Write down the detailed and clear solution to 5 of yours choice (choose only 5 of following 9 problems)

2 Relativistic Quantum Fields I & II

Question 1.

Write down the Dirac equation and the anti-commutation relation of the gamma matrices.

Is it possible to choose one of the gamma matrices to be simply the identity (unit) matrix? Why?

Question 2.

Which is larger (in magnitude) the bare or renormalized charge in QED? Explain why?

Question 3.

Which is larger the bare or renormalized (physical) mass of the electron in QED? Explain why?

2.1 General Relativity I & II

Before you begin read these instructions carefully.
The signature of the metric tensor is $(+,-,-,-)$. The coordinate for 4-dimensional space time is given by $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$. Our convention for the Riemann-Christoffel curvature tensor, Ricci tensor and Einstein's equation is as follows:

$$\begin{aligned} R^{\lambda}_{\mu\nu\sigma} &= \partial_\nu \Gamma^{\lambda}_{\mu\sigma} - \partial_\sigma \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\rho\nu}\Gamma^{\rho}_{\mu\sigma} - \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\rho}_{\mu\nu}, \\ R_{\mu\nu} &= R^{\lambda}_{\mu\nu\lambda}, \\ \text{and} \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= -\frac{8\pi G}{c^4} T_{\mu\nu}. \end{aligned}$$

Question 4.
Consider the metric

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2).$$

Write down the Lagrangian for the geodesic equation in terms of an affine parameter λ.

Show that the geodesic equations on this spacetime are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-2c^2\dot{t}\right) &= +2a\frac{\mathrm{d}a}{\mathrm{d}t}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2), \\ \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(2a^2\dot{r}\right) &= 2a^2 r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2), \\ \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(2a^2r^2\dot{\theta}\right) &= 2a^2r^2\sin\theta\cos\theta\dot{\phi}^2, \\ \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(2a^2r^2\sin^2\theta\dot{\phi}\right) &= 0, [6] \end{aligned}$$

where over-dots correspond to derivatives taken with respect to λ.
Now consider a geodesic along the radial direction and set $\theta = \pi/2$ and $\phi = 0$. Show that the geodesic equations can be solved to give

$$\begin{aligned} \dot{r} &= \frac{\alpha}{a^2}, \\ c^2\dot{t}^2 &= \frac{\alpha^2}{a^2} + \beta, [3] \end{aligned}$$

where α and β are integration constants. Show that for a massless particle we have $\beta = 0$.[1]

Question 5.

For the Einstein-Hilbert action,

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} + \mathcal{L}_m\right),$$

where R is the Ricci scalar, $\kappa = 8\pi G/c^4$ and $\mathcal{L}_m$ is the Lagrangian for the matter field.

5.1 Is this action invariant under the general coordinate transformations? If it is invariant, what is the associated conserved quantity? [3]

5.2 By considering the order of derivatives in this action, explain how it differs from a normal field theory and what is the consequential issue when we perform a variation of the action in order to obtain the equations of motion? [3]

5.3 By using the variational principle, show that the corresponding equations of motion are the Einstein field equations. [4]

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Question 6.

The Einstein field equations are highly nonlinear coupled differential equations. It is very difficult to solve them without imposing any assumption. The Schwarzschild solution is the first solution of the Einstein field equations in vacuum.

6.1 What are the assumptions imposing on the theory to obtain the Schwarzschild solution? [2]

6.2 The Schwarzschild solution can be expressed as [2]

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) c^2 dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\Omega^2,$$

where $\mu = GM/c^2$. What are the singularity points in this solution and what is the physical difference between them?

6.3 Explain how to find the real singularity. [3]

6.4 Pick up a set of coordinate transformations and show that one type of the singularities can be eliminated. [3]

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2.2 Cosmology I & II

Question 7.

EVOLUTION OF THE FLRW UNIVERSE

Let the line element for the spacetime of the universe be

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \tag{1}$$

where $a(t)$ is the scale factor and δ_{ij} is the Kronecker delta.

7.1 Compute the Ricci scalar and Ricci tensor from the line element in eq. (1). [3]

7.2 Use the result in the previous item with the perfect fluid model for the energy and matter to show that the component $\mu\nu = 00$of the Einstein equation yields

$$H^2 = \frac{8\pi G}{3}\rho \tag{2}$$

where $H \equiv \dot{a}/a$ a dot denotes derivative with respect to time and ρ is the energy density. [2]

7.3 Use the equation for energy conservation and eq. (2) to compute the relation between the scale factor and time during radiation, matter and A dominated epochs. [1.5]

7.4 Use the equation for energy conservation and eq. (2) to compute the relations between the scale factor and conformal time during radiation, matter and A dominated epochs. [1.5]

7.5 Write down the definition for the ΛCDM model of the universe and the contribution of each matter component to the energy density of the universe (based on observations). Then, compute the redshift at which the matter dominated and Λ dominated epochs start for this model of the universe. [2]

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Question 8.

CONSERVATION LAW IN THE FLRW UNIVERSE

8.1 Use the conservation of the energy momentum tensor $\nabla_\alpha T^\alpha_\beta = 0$ to derive

$$\dot{\rho} = -3H(\rho + p) \tag{3}$$

where $T_{\alpha\beta}$ is the energy momentum tensor for perfect fluid, a dot denotes derivative with respect to time, $H = \dot{a}/a, \rho$ is the energy density and p is pressure. [2]

8.2 Find the solution for eq.(3) in the form $\rho(a)$ using the expression $w = w_0 + w_1 a$ for the equation of state parameter. Here, w_0 and w_1 are the constant parameters. [2]

8.3 Use eq.(3) and the Friedmann equation to write $\dot{H}$ in terms of the equation of state parameter of fluid, find the conditions for cases where H is constant, decreases in time, and increase in time, and discuss how physics of each case is different. [2]

8.4 Find the expression for $H(z)$ for the case where the matter components of the universe are matter and the perfect fluid with $w = w_0 + w_1 a$. [3]

8.5 Use the result in the previous item to compute the expression for the luminosity distance. [1]

Question 9.

BASIC IDEAS OF INFLATION

9.1 Describe the problems of the Big Bang theory and how inflation can solve those problems. [6]

9.2 Describe the behavior of the Hubble horizon during the inflation period and explain how it helps to solve the problems in the Big Bang theory. [4]