Qualifying Examination

Academic Year 2016 (2559) : ROUND 2

February 2017

Name:..

Student ID:..

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 6 problems.
2. Write down detailed and clear solution to problem of your choice.
3. Complete answers or solution are preferred to fragments.
4. Write on one side of the paper only and begin the rest of answer or solution on a separate sheet.
5. Write legibly, otherwise you place yourself at a grave disadvantage

1 Mathematical Methods of Physics

1.1 Vector Calculus

Question 1.

By using index notation or other methods, prove the following identities for vectors and operators in three dimensional flat space. [**Hint 1:** It might be helpful to use the component from of a cross product (and similar identity for curl), i.e. $(\vec{a}\times\vec{b})_k = \epsilon_{kij}a_j b_j$, where ϵ_{kij} is a Lavi-Civita tensor] [**Hint 2 :** You may wish to use the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.]

1.1) $\nabla^2(f(\vec{x})g(\vec{x})) = \nabla^2(f(\vec{x}))g(\vec{x}) + 2\vec{\boldsymbol{\nabla}}f(\vec{x})\cdot\vec{\boldsymbol{\nabla}}g(\vec{x}) + f(\vec{x})\nabla^2 g(\vec{x})$. [4]

1.2) $\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{\nabla}}\times\vec{A}(\vec{x})) = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}\cdot\vec{A}(\vec{x})) - \nabla^2\vec{A}(\vec{x})$. [6]

1.2 Linear Algebra

Question 2.

We wish to diagonalise a matrix

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

by using eigenvalue method. Let us follow the following steps. [10]

2.1) Obtain the eigenvalues and eigenvectors of A. Make sure that the eigenvectors are normalised. [6]

2.2) Suppose the eigenvectors obtained from previous subquestion are

$$\vec{v_1} = \begin{pmatrix} a \\ b \end{pmatrix}, \vec{v_2} = \begin{pmatrix} c \\ d \end{pmatrix}.$$

Define a matrix

$$P = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Obtain P and make sure that $\det(P) = 1$. [**Hint :**If you obtain $\det(P) \neq 1$, you may wish to check the normalisation of the eigenvectors and/or rename the eigenvectors] [2]

2.3) Compute $P^T AP$. Your answer should be a diagonal matrix. [2]

1.3 Group Theory / Lie Algebra

Question 3.

In this question, we investigate some relationships between finite and infinitesimal transformation. [10 points in total]

3.1) Write down the Taylor series for $\sin x, \cos x$, and e^x around $x = 0$. Each of the series expansion should be up to include the term of order x^6. [3]

3.2) Show that

$$e^{i\theta\sigma} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where

$$\sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Note that $e^{i\theta\sigma}$ is defined using the Taylor series expansion. [**Hint:** as a way to save your time, compute $(i\sigma)^2$, realise some pattern, and then quote a formula] [5]

3.3) Consider a group of $N \times N$ orthogonal matrices (under matrix multiplication). A matrix M is said to be orthogonal if

$$MM^T = \mathbb{I},$$

where $\mathbb{I}$ is $N \times N$ identity matrix. Let us expand the matrix M around $\epsilon = 0$ and keep the expansion up to and including the order ϵ. That is

$$M = \mathbb{I} + i\epsilon R + \mathcal{O}(\epsilon^2),$$

where R is an $N \times N$ matrix. By substituting the above expression of M into the definition of orthogonality and expand the equation up to and including the order ϵ, show that R has to be antisymmetric. [2]

1.4 Function of Complex Variables

Question 4.

A complex function $f(z)$ is analytic at a point z_0 if and only if the Cauchy-Riemann equation

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \quad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}$$

are satisfied at z_0 and the partial derivative are continuous at z_0 ,where $f(z) = u(x,y) + iv(x,y)$ and $z = x + iy$. Determine if the following complex function is analytic everywhere

$$f = (2x + x^2 - y^2) + i2(1 + x)y,$$

if you answer is no, explain why? [10]

1.5 Integral Transforms

Question 5.

The Fourier transform of a function $f(x)$ is defined as

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx) \mathrm{d}x \tag{1}$$

while an inverse Fourier transform also exists

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) \exp(ikx) \mathrm{d}k \tag{2}$$

5.1) The wave equation propagating in one spatial direction is given by

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{3}$$

Show that Fourier transform reduces the above partial differential equation to the following ordinary differential equation

$$\frac{\partial^2 \tilde{u}(k,t)}{\partial t^2} = -c^2 k^2 \tilde{u}(k,t) \tag{4}$$

for a fixed value of f, where $\tilde{u}(k,t)$ is the Fourier transform of $u(x,t)$ with respect to spatial coordinate x. [4]

5.2) Solve the ordinary differential equation of (4) and get a general solution for $\tilde{u}(k,t)$. [**Hint :** Notice that k is considered as a fixed constant.] [6]

1.6 Ordinary Differential Equations

Question 6.

Determine the following problems

6.1) Solve the following second-order ordinary differential equation and obtain the general solution,

$$y''(x) - 4y'(x) + 3y(x) = 0$$

where $y'(x) = \mathrm{d}y/\mathrm{d}x$. [4]

6.2) Solve the following second-order ordinary differential equation and obtain the general solution,

$$y''(x) - 4y'(x) + 4y(x) = 0$$

where $y'(x) = \mathrm{d}y/\mathrm{d}x$. [6]

Name:...

Qualifying Examination

TAH POE THEORETICAL PHYSICS
THE INSTITUTE for FUNDAMENTAL STUDY

Academic Year 2016 (2559) : ROUND 2

Part 2: Subjects Relating to student's Research Field

Instructions and cautions

1. Choose 3 out of the following 6 problems.
2. Write down detailed and clear solutions to problems of your choice.
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For marker: Name... Signature...

2 General Relativity & Cosmology

2.1 Manifold, Curvature, Gravitation, Spherically symmetry solution, Black holes

Question 1.

Rotating solution or Kerr solution of the Einstein equation can be written as

$$\begin{aligned} ds^2 &= c^2\left(1-\frac{2\mu r}{\rho^2}\right)dt^2+\frac{4\mu acr\sin^2\theta}{\rho^2}dt\,d\phi-\frac{\rho^2}{\Delta}dr^2-\rho^2d\theta^2-\left(r^2+a^2+\frac{2\mu ra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2, \\ \rho^2 &= r^2+a^2\cos^2\theta, \quad \Delta=r^2-2\mu r+a^2. \end{aligned}$$

1.1) Find the radius of the Stationary Limit Surface (SLS) and the Event Horizon (EH).
. [3]

1.2) Give the physical meaning for both SLS and EH and explain how they are physically different. [3]

1.3) Write down the condition to have the naked singularity and then explain significant consequence for the existence of this singularity. [4]

2.2 Field theory of gravity

Question 2.

For the Einstein-Hilbert action,

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}R,$$

where R is the Ricci scalar.

2.1) Find the equations of motion by varying the action with respect to the metric tensor.
. [4]

2.2) How many the dynamical fields in this action for Palatini formalism and what is the the dynamical fields. [2]

2.3)What is the assumption (postulate) in metric formalism while it is not assumed in Palatini formalism. [2]

2.4) Explain how to obtain the assumption in the previous item in Palatini formalism. [2]

2.3 Linearized gravity, Gravitational wave

Question 3.

For linearized perturbation theory of gravity, the metric tensor can be decomposed as background and perturbation parts as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} \ll 1$. The first order perturbation of the Einstein tensor can be written as

$$\begin{aligned} G^{(1)}_{\mu\nu} &= \frac{1}{2}\left[\Box\bar{h}_{\mu\nu} - 2\partial_\rho\partial_{(\nu}\bar{h}^{\rho}_{\mu)} + \eta_{\mu\nu}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma}\right], \\ \bar{h}_{\mu\nu} &= h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}. \end{aligned}$$

3.1) By choosing the suitable gauge condition, show that the first order perturbation of Einstein equation in vacuum is a wave equation of $\bar{h}_{\mu\nu}$. [3]

3.2) Show that this equation corresponds to the transverse wave in which propagation speed is equal to the speed of light. [3]

3.3) Explain the concept how to observe the gravitational wave in both direct and indirect methods. [4]

2.4 Inflationary cosmology: Based on the book of V. Mukhanov

Question 4.

Explain in detail the problems of the Big Bang theory and how inflation help solving these problems. [10]

Question 5.

Why do we need a dark energy in our present model of the universe? [10]

Question 6.

Describe distinction of dark energy models and modified gravities. [10]