Qualifying Examination

TAH POE THEORETICAL PHYSICS
THE INSTITUTE for FUNDAMENTAL STUDY

Round 1/2564 (1/2021)

27 August 2021

Name:..

Student ID:...

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 8 questions. (Each questions is for 10 points/marks)
2. Write down detailed and clear solution to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	
7	
8	

Name:..

1 Vector Calculus

Question 1. The fluid has its density function in the form of $\rho(x, y, z, t)$ moving with velocity $\vec{v}$ and it doesn't increase or decrease its mass. Show that

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{1}$$

where $\vec{J} = \rho\vec{v}$ (Hint: Find the fluid continuity equation). [10 marks]

2 Linear Algebra

Question 2. Solve the system of equations

$$x - z = 5 \tag{1}$$

$$-2x + 3y = 1 \tag{2}$$

$$x - 3y + 2z = -10 \tag{3}$$

Hint: Use matrix multiplication $Mr = k,$ so $r = M^{-1}k$ and, $M^{-1} = \frac{1}{\det M}C^T$ where $C_{ij} =$ cofactor of m_{ij}. [10 marks]

3 Group Theory/Lie Algebra

Question 3. Given $U \in SU(N)$, where $SU(N)$ is a group of the unitary $N \times N$ matrices, i.e., $U^\dagger U = I$ and $\det U = 1$.

3.1) Consider a transformation of N-vector valued field Ψ under $SU(N)$ such that $\Psi' = U\Psi$. Show that $\Psi^\dagger \Psi$ is invariant under an $SU(N)$ transformation. [2 marks]

3.2) The element U can be expressed as an exponential

$$U(\lambda_1, \lambda_2, ..., \lambda_{N^2-1}) = e^{i\sum_{j=1}^{N^2-1} \lambda_j G_j} \ ,$$

where λ_j are real-valued and continuous, and G_j are the generators of the group. Show that, in order to make U unitary, all generators must be Hermitian [4 marks].

3.3) Show that, in order to make U possessing unit determinant, all generators must be traceless. [2 marks]

3.4) Given the relation $U = U_2^{-1} U_1^{-1} U_2 U_1 \in SU(N)$. Show that all generators form the Lie-algebra $[T_a, T_b] = i f_{abc} T_c$, where $f_{abc} \in \mathrm{R}$ are anti-sysmmetric structure constant. [2 marks]

4 Group Theory/Lie Algebra

Question 4. Give $u \in SU(2)$, where

$$\begin{pmatrix} a & b \\ -b & a^* \end{pmatrix} \tag{1}$$

with $|a|^2 + |b|^2 = 1$.

4.1) Show that the manifold associated with $SU(2)$ is 3-sphere S^3. [2 marks]

4.2) Show that $SU(2)$ possesses a set of generators is $\{\sigma_1, \sigma_2, \sigma_3\}$, where σ_j are the Pauli matrices given by [2 marks]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{2}$$

4.3) Show that, with a given set of generators in 4.2), one can construct any element $u \in SU(2)$. [6 marks]

5 Function of Complex Variables

Question 5. We consider the Feymman propagator Δ

$$\Delta(x-y) = \langle 0|\mathbf{T}\hat{\phi}(x)\hat{\phi}(y)|0\rangle \ , \tag{1}$$

where $\mathbf{T}$ is time-ordering operator such that

$$\mathbf{T}\hat{\phi}(x)\hat{\phi}(y) = \theta(x^0-y^0)\hat{\phi}(x)\hat{\phi}(y) + \theta(y^0-x^0)\hat{\phi}(y)\hat{\phi}(x) \ , \tag{2}$$

and $\theta(t)$ is the Heaviside function such that

$$\theta(t) = \begin{cases} 1 & \text{if } t0 \\ 0 & \text{if } t < 0 \end{cases} \ . \tag{3}$$

Here $\hat{\phi}(x)$ is the field operator given by

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(\hat{a}_{\mathbf{p}}e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p}}e^{ip\cdot x}) \ , \ \text{ where } \ \omega_p = |\sqrt{\mathbf{p}^2+m^2}| \ , \tag{4}$$

and $\hat{a}^\dagger_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}$ are creation and annihilation operators, respectively. Using the Wightman function

$$D(x-y) = \langle 0|\hat{\phi}(x)\hat{\phi}(y)\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip(x-y)} \ , \tag{5}$$

the Feynman propagator (1) can be expressed as

$$\Delta(x-y) = \theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x) \ . \tag{6}$$

Show that in the momentum-energy space, the Feynman propagator $\Delta(p)$ given by

$$\Delta(p) = \frac{1}{p^2-m^2+i\epsilon} \ , \tag{7}$$

where ϵ is an infinitesimal positive real number, is completely consistent with the space-time coordinate form (6) [10marks].

Hint: The relation between $\Delta(x-y)$ and $\Delta(p)$ is given by

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)\Delta(p)} \ .$$

6 Integral Transforms

Question 6. (6.1). Find the Fourier transform, $F(t)$, of the function $f(x) = e^{-K|x|}$, where $K > 0$ is a constant. [2 marks]

(6.2). Find the Fourier transform, $G(t)$, of the delta function $\delta(x)$. [2 marks]

(6.3). A function $f(x)$ and $f(\pm\infty) = 0$, the corresponding Fourier Transform is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx.$$

Suppose the higher order derivatives of the function and the corresponding Fourier transforms are $\frac{d^n f(x)}{dx^n}$ and $g_n(\omega)$, respectively, where n is positive integer.

Prove that

$$g_n(\omega) = (-i\omega)^n g(\omega).$$

Note that you need to use integration by part and present the first, second, third and n-th order derivatives as well as the corresponding conditions in the process. [6 marks]

7 Integral Transforms

Question 7. Using the derivative form of Fourier transform to solve the following questions. (7.1). A heat flow partial differential equation is given by

$$\frac{\partial \psi(x,t)}{\partial t} = a^2 \frac{\partial^2 \psi(x,t)}{\partial x^2},$$

where $\psi(x,t)$ is the temperature at position x and time t, solve $\psi(x,t)$. [5 marks]

(7.2). A 1-dimensional neutron diffusion equation with a source is

$$-D\frac{d^2\psi(x)}{dx^2} + K^2 D\psi(x) = Q\delta(x),$$

where $\psi(x)$ is the neutron flux, $\delta(x)$ is the delta function, and D and K^2 are constants, solve $\psi(x)$. [5 marks]

hint (7.1) and (7.2): One may use the results in the Question 6. to solve the equation in the transform space, and transform your solution back to find the solutions.

hint (7.1): One need to use the Gaussian integral $\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2} dx$.

8 Ordinary Differential Equations

Question 8. A linear, second-order, homogeneous ODE of the general form

$$y'' + P(x)y' + Q(x)y = 0,$$

where $y = y(x)$ and the prime means the derivative to x. Let y_1 and y_2 be two independent solutions, then the Wronskian is given by definition as

$$W = y_1 y_2' - y_1' y_2.$$

(8.1). Show that $W' = -P(x)W$. [3 marks]

(8.2). Show that the relation between y_1 and y_2 shall be

$$y_2(x) = y_1(x)\frac{y_2(b)}{y_1(b)} + y_1(x)W(a)\int_b^x \frac{e^{-\int_a^{x_2} P(x_1)dx_1}}{[y_1(x_2)]^2} dx_2,$$

where a and b are arbitrary constants. [7 marks]

Qualifying Examination

TAH POE THE INSTITUTE IF THEORETICAL PHYSICS *for* FUNDAMENTAL STUDY

Round 1/2564 (1/2021)

30 August 2021

Name:..

Student ID:...

Part 2: Subjects Relating to student's Research Field : General Relativity and Cosmology

Instructions and cautions

1. Choose 3 out of the following 6 problem relating student's field.
2. Write down detailed and clear solution to problems of your choice.
3. Complete answer are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	

Name:...

1 Manifold, Curvature, Gravitation, Spherically symmetric solution, Black holes

Question 1. One of the most popular solutions of Einstein field equaition is a static and spherically-symmetric solution.

1.1) For 4-dimensional spacetime, wrrite down a general form of the metric solution which is respect to the static condition and spherical symmetry as well as explain how to obtain it.

[3 points]

1.2) The vacuum solution can be written as

$$ds^2 = -\left(1 - \frac{2\mu}{r}\right) c^2 dt^2 + \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where μ is a constant. Indentify the singularity points and then explain how they are different.

[3 points]

1.3) In case of spherically symmetric solution (does not need to be static), explain how the solution should be by comparing to the static one. [4 points]

2 Field theory of gravity

Question 2. The Einstein field equation can be obtained in terms of classical field theory associated with the Einstein-Hilbert action as follows

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} + L_m\right),$$

2.1) Find the form of the energy momentum tensor to satisfy the Einstein field equation according to the variational principle. [4 points]

2.2) By requiring that the action must be invariant under general coordinate transformation, show that the energy momentum tennsor will be covariantly conserved. [6 points]

3 Linearized gravity, Gravitational wave

Question 3. It is well-known that the solution for the linearized gravitation is a wave solution called gravitational wave.

3.1) Identify the number of propagating degrees of freedom of the gravitational wave and then explain how to obtain such a number according to the symmetry of the theory. [5 points]

3.2) One of the crucial differences between the electromagnetic wave and gravitational wave is that it is not possible to generate gravitational wave by using a dipole source while it is poosible for electromagnetic wave. Explain how/why they are different. [5 points]

4 Inflationary cosmology: Based on the book of V. Mukhanov

Question 4. One of the important keys of the inflationary model in cosmology which make it differs from other accelerated expansion of the universe is that the ”graceful exit”.

4.1) Explain what is the ”graceful exit” in this context then give the simple potential form for a scalar field to characterize the model which is graceful exit and is not graceful exit. [4 points]

4.2) Inflationary model can be classified into three classes; Old inflation, New inflation and Chaotic inflation. Explain how these three classes are different. [6 points]

5 Physical Foundations of Cosmology, chapter 5, and lecture on inflation universe for Cosmology I course, Acceleration equation

Question 5. One of generalized versions of scalar field inflation is the scalar field with non-canonical kinetic terms called $k-$inflation. The action of such a model can be written as

$$S = \int d^4x\sqrt{-g}\left(P(X,\phi)\right), \quad X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi.$$

5.1) Find the energy momentum tensor of the scalar field. [4 points]

5.2) Indentify the energy density and pressure of the scalar field by comparing to the perfect fluid one and then find the condition on the form of P to obtain the acceleration phase of the universe. [3 points]

5.3) Find the conditions on the form of P to obtain the graceful exit. [3 points]

6 Problems of Newtonian cosmology, shortcomings of the Big Bang model

Question 6. In 1927, Lemaitre proposed a model of the evolution of the universe. This model provides the evolution of the universe classified into 3 stages; Cosmic expansion, very slow expansion and fast expansion. Later on, this model is known as "hot Big Bang" model.

6.1) Explain in details how the universe evolves in each stage and what is the dominant content in the universe at each stage. [6 points]

6.2) Explain how this model cannot be used to explain the universe nowadays and then find the proper solutions to this shortcomings. [4 points]

Qualifying Examination

TAH POE THE INSTITUTE IF THEORETICAL PHYSICS for FUNDAMENTAL STUDY

Round 1/2564 (1/2021)

30 August 2021

Name:...

Student ID:..

Part 2: Subjects Relating to student's Research Field : Complex system and Network Theory

Instructions and cautions

1. Choose 3 out of the following 6 problem relating student's field.
2. Write down detailed and clear solution to problems of your choice.
3. Complete answer are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	

1 Network basics

Question 1. For a undirected network with $N = \{1, 2, 3, 4, 5, 6, 7\}$ as set of nodes and edges set E= $\{$(1,2), (1,3), (1,4), (2,3), (2,5), (3,5) , (3,1), (3,2), (4,5), (5,6), (2,7), (4,7) $\}$. Find the

1.1) Adjacency matrix [2 marks]

1.2) Network density [2 marks]

1.3) Degree of each node, degree distribution and average degree of the network. [4 marks]

1.4) Distance matrix. [2 marks]

2 Network connectivity and walks

Question 2 Consider an undirected network $G(N, E)$ with 5 nodes ($N = \{1, 2, 3, 4, 5\}$) and set of edges as
$E = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 5), (3, 5), (3, 1), (3, 2), (4, 5)\}$.

2.1) Draw the above network. Also give the adjacency matrix. [2 marks]

2.2) How do you estimate the number of total possible walks of length K between any two given nodes in a network from the adjacency matrix ? Find number of all possible walks of length 4 between the following pairs of nodes $(1, 2)$ and $(1, 5)$. [3 marks]

2.3) Find all the closed paths of length 3 from each node. [2 marks]

2.4) What is a complete network ? Calculate the number of edges in a complete network (directed and undirected) having N vertices. How many more edges are needed to make it G(N,E) as a complete network ? [1 marks]

2.5) Find the number of edges in network H which is the Complement of G. Find adjacency matrix of network H. [2 marks]

3 Random networks

Question 3.1) Derieve the expression for the degree distribution of Erdos Renyii graph $ER(N, p)$, where N is the number of nodes in the network and p is the probability with which any two nodes in the network are connected. [5 marks]

3.2) Find the average degree of a random graph (Erdos Renyii) $ER(N, p)$, where N is the number of nodes in the network and p is the probability with which any two nodes in the network are connected. [5 marks]

4 Spectral properties of Network

Question 4.1) Define the energy of a network. [2 marks]

4.2) How to evaluate the robustness and the flow of information in a network ? [3 marks]

4.3) What is the Fiedler Eigenvalue and Eigenvector?
Give their physical significance. [3 marks]

4.4) Define spectral gap of a network. [2 marks]

5 Network Laplacian and centrality

Question 5.1) Define the Laplacian matrix of a network. What are the bounds for the eigen values of the Laplacian matrix? [2 marks]

5.2) Define the Subgraph centrality of a node in a network. How can we model the effect of external parameter say β on the sub graph centrality. Using subgraph centrality, discuss the Estrada index of a network. [5 marks]

5.3) How can we represent a network in the form of Density matrix? Define the Von-Neumann Entropy for a network? [3 marks]

6 Minimum spanning Tree

Question 6.1) What is the difference between a spanning tree and minimum spanning tree?

[3 marks]

6.2) Given the weighted network in Figure 1, find the minimum spanning tree (MST) of the network. [4 marks]

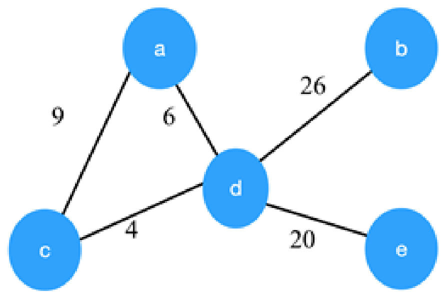

Figure 1: Weighted network

6.3) Consider a network of 3 vertices with adjacency matrix as

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1}$$

How many different minimum spanning trees are possible? What is the weight cost of each MST ? [3 marks]