COSMOLOGICAL SCALING SOLUTIONS IN DEGENERATE HIGHER-ORDER SCALAR-TENSOR THEORY AND COUPLED DARK ENERGY MODEL

WITTAYA THIPAKSORN

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by Wittaya Thipaksorn
has been approved by the Graduate School as partial fulfillment of the requirements for the Doctor of Philosophy in Theoretical Physics of Naresuan University.

Oral Defense Committee

. . . . Advisor
(Associate Professor Khamphee Karwan, Ph.D.)

. . . . Committee
(Assistant Professor Pitayuth Wongjun, Ph.D.)

. . . . Committee
(Associate Professor Pichet Vanichchapongjaroen, Ph.D.)

. . . . Committee
(Assistant Professor Teeraparb Chantavat, Ph.D.)

. . . . External Examiner
(Associate Professor Phongpichit Channuie, Ph.D.)

Approved

. . . .
(Associate Professor Krongkarn Chootip, Ph.D.)
Dean of the Graduate School

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Wittaya Thipaksorn

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Author	Wittaya Thipaksorn
Advisor	Associate Professor Khamphee Karwan, Ph.D.
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ABSTRACT

In this thesis, we study cosmological scaling solutions in two cosmological models. We first consider the scaling solutions in the modified theory of gravity, and then study this type of solution in the coupled dark energy model. The modified theory of gravity used in our study is a subclass of Degenerate Higher-Order Scalar-Tensor (DHOST) theory which satisfies gravitational wave constraints. The coupled dark energy model used in our work is constructed from the general conformal transformation in which the coefficient of the conformal transformation depends on both the scalar field and its kinetic term. Under this transformation, the action for the subclass of the DHOST theory mentioned above is related to the Einstein-Hilbert action.

We use autonomous system to analyze the cosmic evolution. We study the DHOST theory having the scaling solutions in which the Lagrangians have been derived in literature. To obtain the scaling solutions satisfying the cosmic acceleration at late time, we assume coupling between a scalar degree of freedom in the DHOST theory and matter. In this model, the coupling term is inspired from simple conformal transformation. We find that for some ranges of the parameters, both scaling and field dominated points can be attractors. The deviation from the Einstein theory of gravity needs to be small to prevent the density parameter of dark matter larger than unity. Similar to coupled dark energy model in Einstein gravity, the background universe cannot evolve from radiation

domination through ϕ-matter-dominated epoch (ϕMDE) towards acceleration epoch, the coincidence problem in this DHOST model cannot be alleviated.

In the coupled dark energy model with general conformal coupling, we find that the late-time scaling point can be an attractor, while ϕMDE can be a saddle point for some choices of parameters. In this model, the cosmic evolution from radiation domination through ϕMDE towards acceleration epoch can be realized. Therefore coincidence problem can be alleviated. Based on our results, the coincidence problem cannot be alleviated in both DHOST theory and coupled dark energy model if coupling term is inspired from conformal transformation. For the general conformal transformation, it is possible to alleviate the coincidence problem.

CHAPTER I

INTRODUCTION

1.1 Background and Motivation

The General Relativity (GR) was proposed by Albert Einstein in 1915 [1]. This theory is a geometrical theory of gravity in four-dimensional spacetime that has changed our viewpoint about nature of gravity. This theory successfully describes phenomena in laboratory and Solar system, but cannot describe the acceleration of the present universe without introducing mysterious form of energy that has a negative pressure called dark energy. The simplest model of dark energy is cosmological constant. By putting a positive cosmological constant into the Einstein-Hilbert action, the acceleration of the present universe can be realized. However, the value of cosmological constant is extremely fine-tuned which is known as fine-tuning problem or cosmological constant problem. To avoid the mentioned problem, dark energy has to evolve in time. However for evolving dark energy models, there is the coincidence problem that questions the moment in the cosmic history at which the accelerated expansion occurs [2]. The coincidence problem can be alleviated if the evolution of dark energy has suitable fixed points in phase space [2, 3]. For alternative way, the accelerating universe can be achieved by assuming that physics of gravity on large scale deviates from Einstein theory. These theories are called the modified theory of gravity. The simplest modified theory of gravity can be constructed by adding the scalar field into the action for gravity. These theories are called scalar-tensor theories of gravity. The additional scalar field can be field of gravity, if scalar field, ϕ, couples non-minimally to Ricci scalar, R. This is a fundamental construction of Brans Dickes theory. Besides non-minimally coupling between ϕ and R, there is non-minimal derivative coupling between ϕ and curvature tensors. The most general actions in four-dimensional spacetime for scalar-tensor theories containing both forms of non-minimally coupling and having second order equations of motion (EOMs) is Horndeski theory [4]. Because this theory contains up to second order derivative in EOMs, it is free from Ostrogradsky instability. The simple example for such Lagrangians are $\mathcal{L} \propto f(\phi)R$ and $\mathcal{L} \propto G^{\mu\nu}\phi_{\mu\nu}$. The full Lagrangians of Horndeski theory will present in section 3.2.

In the modern view points, the Horndeski theory can be also reconstructed from Galileon theory in four dimensions [5, 6]. However, in more general case, there also exist viable theories which do not suffer from the Ostrogradsky instability even though the corresponding field equations are higher order. It will be presented in section 3.3 that this theory can be constructed by suitable combination of Galileon actions. This theory is beyond Horndeski or GLPV theory [7]. Such theories have interesting consequences for cosmology and astrophysics. In particular, it leads to a breaking of the Vainshtein mechanism inside matter, which can modify the structure of nonrelativistic stars as well as that of relativistic ones. Theories which the Lagrangians are degenerate, have been studied. Although the equations of motion are higher order derivative, these theories propagate at most three degrees of freedom without Ostrogradsky instability, because extra degrees of freedom can be eliminated by constraints arising from degeneracy conditions. These theories is Degenerate Higher-Order Scalar-Tensor (DHOST) theories in which Horndeski and GLPV theories are included [8].

The detection of the gravitational wave (GW) emitted from neutron binary stars, shows that the speed of gravitational waves, c_{GW}, is the same as the speed of light ($c_{\text{light}} = 1$) [9, 10]. This implies very restrictions on form of scalar-tensor theories in particular DHOST theories. Hence, DHOST theories satisfying the constraint $c_{GW} = 1$, have the corresponding Lagrangians as [11]

$$\begin{aligned} L^{\text{DHOST}}_{c_{GW}=1} &= P + Q\Box\phi + f^{(4)}R + \alpha_3\phi^\mu\phi^\nu\phi_{\mu\nu}\Box\phi \\ &+ \frac{1}{8f}\left[48f_{,X}^2 - 8(f - Xf_{,X})\alpha_3 - X^2\alpha_3^2\right]\phi^\mu\phi_{\mu\nu}\phi_\lambda\phi^{\lambda\nu}\big) \\ &+ \frac{1}{2f}\left(4f_{,X} + X\alpha_3\right)\alpha_3(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2\,, \end{aligned} \tag{1.1}$$

where P, Q, f and α_3 are arbitrary functions depending on scalar field, ϕ and $X \equiv$

$-\nabla_\mu\phi\nabla^\mu\phi$. The terms P, Q, $f^{(4)}R$ and α_3 represent the Lagrangian of scalar field, the coefficient of cubic Galileon Lagrangian, non-minimally coupling term and the coefficient of L_3 in higher-order scalar-tensor Lagrangians respectively. ${}^{(4)}R$ is four-dimensional Ricci scalar. A subscript $_{,X}$ denotes derivative with respect to X. Moreover, we have used $\phi_\mu \equiv \nabla_\mu\phi$, $\phi_{\mu\nu} \equiv \nabla_\nu\nabla_\mu\phi$ and $\Box\phi \equiv \nabla_\mu\nabla^\mu\phi$.

There are many attempts to study whether these theories are suitable as dark energy candidates [11]. The scaling solutions during the matter dominated epoch and de Sitter solutions at late time have been found without the cosmological constant, realizing self-acceleration. The quasi-static perturbations around the self-accelerating solutions are evaluated. It is shown that, for this theory the stricted constraints coming from astrophysical objects and gravitational waves can be satisfied.

Besides the constraint on propagation speed, there is another constraint which come from the requirement that GW in DHOST theories does not decay to dark energy perturbations, i.e., graviton is stable [12]. This constraint together with the constraint on propagation speed of GW tightly constrain form of the Lagrangian for the DHOST theories.

Scaling behaviour for the cosmic evolution is the interesting feature arising in some models of dark energy and modified theories of gravity, because it is possible scenarios, among many others, that could lead to fixed points which become attractors for some ranges of model parameters [13, 14, 15, 16, 17, 18, 19, 20]. The scaling behaviour is the constancy of the ratio between energy density of dark energy and dark matter during some period of time. Since scaling behaviour could lead to fixed point corresponding to matter domination and attractor corresponding to late-time acceleration that satisfies the observations, the coincidence problem could be alleviated. The scaling point that can represent the matter dominated epoch is the ϕMDE point in which there is a small fraction of dark energy during matter domination. The coincidence problem could be alleviated if the universe can evolve from radiation domination through ϕMDE toward acceleration epoch at late time.

Possible dark energy models having scaling behaviour can be constructed by assuming interaction between dark energy and dark matter [21, 22, 23, 24, 25]. Due to such interaction, the ratio of the energy density of dark energy to that of dark matter can be constant with time during the scaling regime. Possible models of coupled dark energy are inspired from the frames transformation in theories of gravity. The interaction between dark energy and cold dark matter (CDM) can be inspired from the conformal transformation [26, 27, 28, 29, 30] and disformal transformation [31, 32, 33, 34, 35, 36, 37].

Scaling solutions in the DHOST theories which satisfy the above two constraints on GW have been discussed. Demanding the existence of the scaling solution, the suitable form of the Lagrangians has been derived [38]. The scaling solutions can satisfy the cosmic acceleration at late time if the coupling between a scalar degree of freedom in the DHOST theory and dark matter is assumed. In this model, the coupling term between the scalar degree of freedom and dark matter is inspired from conformal transformation in which the coefficient of the conformal transformation depends only on scalar field. The evolution of background universe for this class of DHOST theories is studied in this thesis [39]. Our results show that, for this DHOST model, cosmic evolution cannot evolve from radiation domination through ϕMDE towards acceleration epoch. This behaviour is similar to that of coupled dark energy model in Einstein gravity [23] with the same coupling between dark components as in the DHOST model considered here. This implies that the coincidence problem cannot be alleviated in this form of coupling. The sequence of cosmic evolution through suitable fixed points can be realized in the coupled dark energy model in which the coupling term consists of $\mathcal{Z} \equiv u^\mu \partial_\mu \phi$, where such term can lead to pure momentum transfer between the dark components [25]. Here, u^μ is a four velocity of CDM and $\partial_\mu \phi$ is a derivative of scalar field. We will show in this thesis that the proper sequence of cosmic evolution can be achieved in the model of coupled dark energy inspired from the general conformal transformation in which the coefficient of the transformation depends on both the scalar field and its kinetic term [40].

1.2 Objectives

1.2.1 To study background evolution of the universe in DHOST theories with scaling solutions.

1.2.2 To study cosmic evolution in coupled dark energy model with general conformal coupling.

1.3 Frameworks

The scope of this work is to perform dynamical analysis for DHOST theories with scaling solutions and coupled dark energy model with general conformal coupling. The cosmic evolutions based on the fixed points found in the analysis are discussed.

CHAPTER II

FUNDAMENTAL COSMOLOGY

2.1 The Einstein Theory of Gravity

Let us start from the Einstein-Hilbert action in the form

$$S_{EH} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R\,, \tag{2.1}$$

where $\kappa \equiv \sqrt{8\pi G} \equiv 1/M_p$, G is Newton's gravitational constant, M_p is reduced Planck mass, g is a determinant of the metric tensor $g_{\alpha\beta}$, R is the Ricci scalar defined by $R \equiv g^{\alpha\beta}R_{\alpha\beta}$, and $R_{\alpha\beta}$ is the Ricci tensor defined by

$$R_{\alpha\beta} = \partial_\alpha\Gamma^\lambda_{\ \beta\lambda} - \partial_\lambda\Gamma^\lambda_{\ \alpha\beta} + \Gamma^\sigma_{\ \alpha\lambda}\Gamma^\lambda_{\ \beta\sigma} - \Gamma^\lambda_{\ \alpha\beta}\Gamma^\sigma_{\ \lambda\sigma}\,. \tag{2.2}$$

The Christoffel symbol, $\Gamma^\lambda_{\ \alpha\beta}$ can be computed by the relation

$$\Gamma^\lambda_{\ \alpha\beta} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\beta g_{\alpha\rho} + \partial_\alpha g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right)\,, \tag{2.3}$$

where $g^{\lambda\rho}$ is the inverse of metric tensor. In order to describe matter in gravity, we add a matter action to the above action. Now we have

$$S = S_{EH} + S_m = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R + \mathcal{L}_m\left[g_{\alpha\beta},\psi\right]\right\}\,, \tag{2.4}$$

where ψ denotes the matter field. Using the variational method with respect to $g^{\alpha\beta}$, it follows that

$$\delta S = \delta S_{EH} + \delta S_m = \int d^4x\left\{\frac{1}{2\kappa^2}\delta\left(\sqrt{-g}R\right) + \frac{\delta}{\delta g^{\alpha\beta}}\left(\sqrt{-g}\mathcal{L}_m\right)\delta g^{\alpha\beta}\right\}\,. \tag{2.5}$$

The energy-momentum tensor, $T_{\alpha\beta}$ is defined by

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\alpha\beta}} \quad \text{or} \quad T^{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\alpha\beta}}\,. \tag{2.6}$$

Considering the variation of the Einstein-Hilbert action in Eq. (2.5), we obtain

$$\delta S_{EH} = \frac{1}{2\kappa^2}\int d^4x\left\{\underbrace{\delta\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta}}_{\delta S_1} + \underbrace{\sqrt{-g}\delta g^{\alpha\beta}R_{\alpha\beta}}_{\delta S_2} + \underbrace{\sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta}}_{\delta S_3}\right\}\,. \tag{2.7}$$

Using the relation

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}\,, \tag{2.8}$$

the first and second terms (δS_1 and δS_2) in Eq. (2.7) can be written as

$$\delta S_1 + \delta S_2 = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right\}\delta g^{\alpha\beta}\,. \tag{2.9}$$

In order to calculate the expression of the last term (δS_3), we need to derive the variation of the Ricci tensor $\delta R_{\alpha\beta}$ as follows

$$\begin{aligned}
\delta R_{\alpha\beta} &= \delta\left(\partial_\alpha \Gamma^\lambda{}_{\beta\lambda} - \partial_\lambda \Gamma^\lambda{}_{\alpha\beta} + \Gamma^\sigma{}_{\alpha\lambda}\Gamma^\lambda{}_{\beta\sigma} - \Gamma^\lambda{}_{\alpha\beta}\Gamma^\sigma{}_{\lambda\sigma}\right) \\
&= \partial_\alpha \delta\Gamma^\lambda{}_{\beta\lambda} - \partial_\lambda \delta\Gamma^\lambda{}_{\alpha\beta} + \delta\Gamma^\sigma{}_{\alpha\lambda}\Gamma^\lambda{}_{\beta\sigma} + \Gamma^\sigma{}_{\alpha\lambda}\delta\Gamma^\lambda{}_{\beta\sigma} - \delta\Gamma^\lambda{}_{\alpha\beta}\Gamma^\sigma{}_{\lambda\sigma} - \Gamma^\lambda{}_{\alpha\beta}\delta\Gamma^\sigma{}_{\lambda\sigma} \\
&= \partial_\alpha \delta\Gamma^\lambda{}_{\beta\lambda} + \Gamma^\sigma{}_{\alpha\lambda}\delta\Gamma^\lambda{}_{\beta\sigma} - \Gamma^\lambda{}_{\alpha\beta}\delta\Gamma^\sigma{}_{\lambda\sigma} - \left(\partial_\lambda \delta\Gamma^\lambda{}_{\alpha\beta} - \delta\Gamma^\sigma{}_{\alpha\lambda}\Gamma^\lambda{}_{\beta\sigma} + \delta\Gamma^\lambda{}_{\alpha\beta}\Gamma^\sigma{}_{\lambda\sigma}\right) \\
&= \partial_\alpha \delta\Gamma^\lambda{}_{\beta\lambda} + \Gamma^\sigma{}_{\alpha\lambda}\delta\Gamma^\lambda{}_{\beta\sigma} - \Gamma^\lambda{}_{\alpha\beta}\delta\Gamma^\sigma{}_{\lambda\sigma} - \Gamma^\lambda{}_{\alpha\sigma}\delta\Gamma^\sigma{}_{\lambda\beta} \\
&\quad - \left(\partial_\lambda \delta\Gamma^\lambda{}_{\alpha\beta} - \delta\Gamma^\sigma{}_{\alpha\lambda}\Gamma^\lambda{}_{\beta\sigma} + \delta\Gamma^\lambda{}_{\alpha\beta}\Gamma^\sigma{}_{\lambda\sigma} - \Gamma^\lambda{}_{\alpha\sigma}\delta\Gamma^\sigma{}_{\lambda\beta}\right) \\
&= \nabla_\beta \delta\Gamma^\lambda{}_{\alpha\lambda} - \nabla_\lambda \delta\Gamma^\lambda{}_{\alpha\beta}\,,
\end{aligned} \tag{2.10}$$

where $\delta\Gamma^\lambda{}_{\alpha\beta}$ can be written as

$$\delta\Gamma^\lambda{}_{\alpha\beta} = \frac{1}{2}g^{\lambda\rho}\left(\nabla_\beta \delta g_{\alpha\rho} + \nabla_\alpha \delta g_{\beta\rho} - \nabla_\rho \delta g_{\alpha\beta}\right)\,. \tag{2.11}$$

Now we rewrite the δS_3 as

$$\begin{aligned}
\delta S_3 &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}g^{\alpha\beta}\left[\nabla_\beta \delta\Gamma^\lambda{}_{\alpha\lambda} - \nabla_\lambda \delta\Gamma^\lambda{}_{\alpha\beta}\right], \\
&= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\nabla_\sigma\left[g^{\alpha\sigma}(\delta\Gamma^\lambda{}_{\alpha\lambda}) - g^{\alpha\beta}\delta(\Gamma^\sigma{}_{\alpha\beta})\right]\,, \\
&= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\nabla_\rho\left[\delta^\rho_\alpha \nabla_\beta \delta g^{\alpha\beta} - g_{\alpha\beta}g^{\mu\rho}\nabla_\mu \delta g^{\alpha\beta}\right]\,, \\
&= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\nabla_\rho \xi^\rho\,, \\
&= \frac{1}{2\kappa^2}\oint dS_3\sqrt{-g}\xi^\rho n_\rho\,,
\end{aligned} \tag{2.12}$$

where $\xi^\rho = \delta^\rho_\alpha \nabla_\beta \delta g^{\alpha\beta} - g_{\alpha\beta}g^{\mu\rho}\nabla_\mu \delta g^{\alpha\beta}$ is a unit vector normal to hypersurface enclosing entire spacetime. The quantity $\xi^\sigma n_\sigma$ in the last line is evaluated on the hypersurface.

Hence, the term in the above equation corresponds to a boundary contribution at infinity which can be set to zero by demanding $\nabla_\beta \delta g^{\alpha\beta} = 0$ in stead of $\delta g^{\alpha\beta} = 0$. Finally, we obtain the Einstein field equation as

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = G_{\alpha\beta} = \kappa^2 T_{\alpha\beta}\,, \tag{2.13}$$

where $G_{\alpha\beta}$ is Einstein tensor which represents the gravitational interaction in terms of the spacetime curvature. This equation describes the dynamics of the gravity due to the matter field. We take the covariant derivative of the above equation,

$$\nabla_\alpha G^{\alpha\beta} = \kappa^2 \nabla_\alpha T^{\alpha\beta}\,. \tag{2.14}$$

Since the left-hand side of this equation is zero because of the twice contracted Bianchi identity, the right-hand side satisfies

$$\nabla_\alpha T^{\alpha\beta} = 0\,. \tag{2.15}$$

This corresponds to the energy and momentum conservation of the total matter field. In general, the energy-momentum tensors conserve if system is invariant under coordinate transformation.

2.2 The Friedmann-Lemaître-Robertson-Walker Metric

In cosmology, cosmological principle provides the symmetries of spacetime. Cosmological principle is the notion that the universe is homogeneous and isotropic in three-dimensional space. Isotropy means that the universe we look at does not have special direction. Homogeneity means the average density on large scales is about the same everywhere in the universe. The metric that satisfies homogeneity and isotropy of the three-dimensional space is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. To ensure that the homogeneity and isotropy of the three-dimensional space are time invariance, we write the time-dependent part of the spatial metric in the form of time-dependent factor as

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij} dx^i dy^j\,, \tag{2.16}$$

where $a(t)$ is scale factor and γ_{ij} is metric tensor in spatial components. We start with a three-dimensional space in Cartesian coordinate embedded in four-dimensional space

$$x^2 + y^2 + z^2 \pm \alpha^2 = \pm\mathcal{R}^2 \,, \tag{2.17}$$

where $\mathcal{R}$ is a constant radius of a three-dimensional space in Cartesian coordinate and α is the fourth component of Cartesian coordinate. The above equation corresponds to flat space if $\mathcal{R} \to \infty$, corresponds to 3-sphere if the sign in front of $\mathcal{R}^2$ is positive and becomes hyperboloid if the sign in front of $\mathcal{R}^2$ is negative. From the above expression, we can rewrite as

$$r^2 \pm \alpha^2 = \pm\mathcal{R}^2 \,, \tag{2.18}$$

or

$$\alpha^2 = \pm\left(-r^2 \pm \mathcal{R}^2\right) \,, \tag{2.19}$$

where $r^2 \equiv x^2 + y^2 + z^2$. The differentiation of Eq. (2.18) gives

$$2rdr = \mp 2\alpha d\alpha \quad \to \quad \frac{r}{\alpha}dr = \mp d\alpha \,. \tag{2.20}$$

Squaring both sides of the above equation, then we get

$$\frac{r^2}{\alpha^2}dr^2 = d\alpha^2 \,. \tag{2.21}$$

Inserting Eq. (2.19) into the above equation, it yields

$$\pm\frac{r^2}{-r^2 \pm \mathcal{R}^2}dr^2 = d\alpha^2 \,. \tag{2.22}$$

From Eq. (2.16), we can write

$$ds^2 = -dt^2 + a^2 dl^2 \,, \tag{2.23}$$

where for Cartesian coordinate, we have

$$dl^2 \equiv dx^2 + dy^2 + dz^2 \pm d\alpha^2 \,, \tag{2.24}$$

and for spherical coordinate, we have

$$dl^2 \equiv dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \pm d\alpha^2 \,. \tag{2.25}$$

Inserting Eq. (2.22) into the above equation, we then obtain

$$\begin{aligned} dl^2 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + \frac{r^2}{-r^2 \pm \mathcal{R}^2} dr^2 \,. \\ &= \left(1 + \frac{r^2}{-r^2 \pm \mathcal{R}^2}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,, \\ &= \frac{\pm\mathcal{R}^2}{-r^2 \pm \mathcal{R}^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,, \\ &= \frac{dr^2}{\left(1 - \frac{r^2}{\mp\mathcal{R}^2}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,. \end{aligned} \tag{2.26}$$

Considering the case of flat space $\mathcal{R} \to \infty$, the above equation reduces to

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,. \tag{2.27}$$

Inserting the above expression into Eq. (2.23), we obtain the FLRW metric for spatially flat space in spherical coordinate as

$$ds^2 = -dt^2 + a^2 \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] \,, \tag{2.28}$$

where δ_{ij} is the Kronecker delta. From the above equation, the dimension of radial coordinate r is length while the scale factor $a(t)$ is dimensionless. we also write the FLRW metric for spatially flat space in Cartesian coordinate as

$$ds^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j \,. \tag{2.29}$$

We now consider the case of $\pm\mathcal{R}^2$ corresponding to 3-sphere and hyperboloid. For convenience we can rescale

$$\tilde{r} \to \frac{r}{\pm\mathcal{R}} \,. \tag{2.30}$$

Eq. (2.26) becomes

$$d\tilde{l}^2 = \frac{d\tilde{r}^2}{(1 - k\tilde{r}^2)} + \tilde{r}^2 d\theta^2 + \tilde{r}^2 \sin^2\theta d\phi^2 \,, \tag{2.31}$$

where $\tilde{l} = l/\mathcal{R}$ and $k = 1, -1$ represent closed (spherical) space and open (hyperbolic) space respectively. From Eqs. (2.23) and (2.31), the FLRW metric for curved space in spherical coordinate yields

$$ds^2 = -dt^2 + \tilde{a}(t)^2 \left[\frac{d\tilde{r}^2}{(1 - k\tilde{r}^2)} + \tilde{r}^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \,, \tag{2.32}$$

where $\tilde{a}(t) = a(t)\mathcal{R}$. From the above equation, the radial coordinate $\tilde{r}$ is dimensionless while the scale factor $\tilde{a}(t)$ is length. In general, we can write Eqs. (2.28) and (2.32) in unified form by omitting tilde from a and r as

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{(1-kr^2)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] , \tag{2.33}$$

where dimensionless $k = 1, 0, -1$ represent closed space, flat space and open space respectively. If $k = 0$, Eq. (2.33) satisfies Eq. (2.28) which the scale factor $a(t)$ is dimensionless while dimension of radial coordinate r is length. If $k = 1$ and -1, Eq. (2.33) satisfies Eq. (2.32) which dimension of the scale factor $a(t)$ is length while the radial coordinate r is dimensionless. The metric tensor $g_{\alpha\beta}$ in Eq. (2.33) can be read as

$$g_{\alpha\beta} = diag\left(-1, \frac{a^2}{1-kr^2}, a^2r^2, a^2r^2\sin^2\theta \right) . \tag{2.34}$$

From the FLRW metric that we have mentioned, we can compute the Christoffel symbols using Eq. (2.3). From the relation

$$g_{\alpha\beta}g^{\beta\rho} = \delta^{\rho}_{\alpha} , \tag{2.35}$$

where δ^{ρ}_{α} is kronecker delta, the inverse of metric tensor for the FLRW metric can be read as

$$g^{\alpha\beta} = diag\left(-1, (\frac{a^2}{1-kr^2})^{-1}, (a^2r^2)^{-1}, (a^2r^2\sin^2\theta)^{-1} \right) . \tag{2.36}$$

From Eqs. (2.3), (2.34) and (2.36), we obtain the only non-zero Christoffel symbols as

$$\begin{aligned}
\Gamma^{0}{}_{ij} &= \frac{\dot{a}}{a}g_{ij} = Hg_{ij}, \quad \Gamma^{i}{}_{0j} = \frac{\dot{a}}{a}\delta^{i}_{j} = H\delta^{i}_{j} , \\
\Gamma^{1}{}_{11} &= \frac{kr}{1-kr^2}, \quad \Gamma^{1}{}_{22} = -r(1-kr^2), \quad \Gamma^{1}{}_{33} = -r(1-kr^2)\sin^2\theta , \\
\Gamma^{2}{}_{12} &= \Gamma^{2}{}_{21} = \Gamma^{3}{}_{13} = \Gamma^{3}{}_{31} = \frac{1}{r} , \\
\Gamma^{2}{}_{33} &= -\sin\theta\cos\theta, \quad \Gamma^{3}{}_{23} = \Gamma^{3}{}_{32} = \cot\theta ,
\end{aligned} \tag{2.37}$$

where a dot denotes derivative with respect to time t, $H \equiv \dot{a}/a$ is Hubble parameter and index $i = 1, 2, 3$ represented coordinate r, θ, ϕ respectively.

From Eq. (2.2), we can compute the non-zero components of the Ricci tensor for the FLRW metric as

$$
\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}\,, && (2.38)\\
R_{11} &= \frac{2a\ddot{a}+2\dot{a}^2+2k}{1-2kr^2}\,, && (2.39)\\
R_{22} &= r^2(2a\ddot{a}+2\dot{a}^2+2k)\,, && (2.40)\\
R_{33} &= r^2(2a\ddot{a}+2\dot{a}^2+2k)\sin^2\theta\,, && (2.41)
\end{aligned}
$$

and we can obtain the Ricci scalar for the FLRW metric as

$$R = 6\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right) = \left(6\dot{H}+12H^2+\frac{6k}{a^2}\right)\,. \tag{2.42}$$

From the observational data, the universe is quite spatially flat ($k \approx 0$) [41].

2.3 Perfect Fluid

The energy-momentum tensor $T_{\alpha\beta}$ that satisfies the requirement of the isotropy and homogeneity is in the form of a perfect fluid. It can be written as

$$T_{\alpha\beta} = (\rho+p)u_\alpha u_\beta + p g_{\alpha\beta}\,, \tag{2.43}$$

or

$$T^\alpha_\beta = (\rho+p)u^\alpha u_\beta + p\delta^\alpha_\beta\,, \tag{2.44}$$

where ρ and p are energy density and pressure of the fluid respectively and u_μ is four-velocity of the fluid. To satisfy the homogeneity of space, both ρ and p are only functions of time t. Moreover, the spatial components of the four-velocity have to vanish. In the comoving flame, $u^\mu = (1,0,0,0)$. A perfect fluid is idealized fluid in which shear stress, viscosity or heat transfer are neglected such that $T_{0i} = T_{i0} = T_{ij} = 0$. These requirements satisfy the isotropy of space. Therefore, we can write Eqs. (2.43) and (2.44) as

$$T_{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & pg_{11} & 0 & 0 \\ 0 & 0 & pg_{22} & 0 \\ 0 & 0 & 0 & pg_{33} \end{pmatrix} \quad \text{and} \quad T^{\alpha}_{\beta} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} . \tag{2.45}$$

Additionally, we can calculate the trace of $T_{\alpha\beta}$ as

$$T = T^{\alpha}_{\alpha} = -\rho + 3p \, . \tag{2.46}$$

The evolution of energy density can be calculated by the conservation equation for energy-momentum tensor as follows

$$\nabla_{\alpha} T^{\alpha}_{\beta} = 0 \, . \tag{2.47}$$

The component $\beta = 0$ of the above equation corresponds to the energy conservation while $\beta = i$ corresponds to the momentum conservation. Since $T^{i}_{0} = 0$ and $\partial_i p = 0$ in the background universe, the evolution of the energy density is only considered by the component $\beta = 0$

$$\nabla_{\alpha} T^{\alpha}_{0} = 0 \, . \tag{2.48}$$

From the expression of the covariant derivative, the above equation can be written as

$$\partial_{\alpha} T^{\alpha}_{0} + \Gamma^{\alpha}{}_{\alpha\lambda} T^{\lambda}_{0} - \Gamma^{\lambda}{}_{\alpha 0} T^{\alpha}_{\lambda} = 0 \, . \tag{2.49}$$

Because of the isotropy of the space, $T^{i}_{0} = 0$. Eq. (2.49) reduces to

$$\partial_0 T^{0}_{0} + \Gamma^{i}{}_{i0} T^{0}_{0} - \Gamma^{i}{}_{j0} T^{j}_{i} = 0 \, . \tag{2.50}$$

Using Eqs. (2.37) and (2.45), the above equation becomes

$$\begin{aligned} -\dot{\rho} - H\delta^{i}_{i}\,\rho - H\delta^{i}_{j}\,T^{j}_{i} &= 0 \, , \\ \dot{\rho} + H\delta^{i}_{i}\,\rho + HT^{i}_{i} &= 0 \, , \\ \dot{\rho} + 3H(\rho + p) &= 0 \, , \end{aligned} \tag{2.51}$$

where δ_i^i is trace of kronecker delta in three-dimensional space. This is the conservation equation of the energy density for the perfect fluid in the spatially flat Friedmann universe. If we know the relation between ρ and p, we can solve Eq. (2.51) in terms of $\rho(a)$. For the simple perfect fluid, the relation between ρ and p is characterized by the equation of state

$$p = w\rho \quad , \tag{2.52}$$

where w is the equation of state parameter. Substituting Eq. (2.52) into Eq. (2.51), we obtain

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \,. \tag{2.53}$$

For simplicity, we set w to be a constant value. Then Eq. (2.53) yields

$$\rho \propto a^{-3(1+w)} \,. \tag{2.54}$$

We know how the energy density changes when the universe expands or shrinks through determining the equation of state parameter w. The universe is filled by a mixture of different matters. The classification of such matters depends on w as follows.

- Matter, $w = w_m = 0$

  For all forms of the matter, the pressure is much less than the energy density $|p| \ll \rho$. The main matter components in the universe are dark matter and baryons which are non-relativistic particles. The energy density of the matter obeys

  $$\rho_m \propto a^{-3} \,, \tag{2.55}$$

  where the subscript $_m$ denotes the matter.

- Radiation, $w = w_\gamma = 1/3$

  The radiation is relativistic particles such as photons and neutrinos. For the radiation, the energy density obeys

  $$\rho_\gamma \propto a^{-4} \,, \tag{2.56}$$

  where the subscript $_\gamma$ denotes the radiation.

- Dark energy, $w = w_d \approx -1$

  The dark energy is unknown form of energy with negative pressure. The dark energy is introduced for describing the accelerated expansion of the universe at late time. In general, w_d is not necessarily constant. For simple cases, the dark energy can be in the form of the cosmological constant Λ ($w_\Lambda = -1$) or the scalar field ϕ. In the following notations, we denote the subscript $_d$ as the dark energy and the subscript $_\Lambda$ as the cosmological constant.

Since scalar field is one of candidates for the dark energy, we will study the energy-momentum tensor of the scalar field. For the scalar field in the gravity, its dynamics is described by the action

$$S_\phi = \int d^4x\sqrt{-g}\left(-\frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi - V(\phi)\right) , \tag{2.57}$$

where $V(\phi)$ is the potential of the scalar field. We note that the metric signature is $(-, +, +, +)$. The energy-momentum tensor for the scalar field can be derived from Eq. (2.6) as

$$T^\phi_{\alpha\beta} = \nabla_\alpha\phi\nabla_\beta\phi + \left(-\frac{1}{2}\nabla_\rho\phi\nabla^\rho\phi - V(\phi)\right) g_{\alpha\beta} \,. \tag{2.58}$$

The EOM of the scalar field can be calculated by varying Eq. (2.57) with respect to ϕ. Hence, we obtain the EOM of the scalar field as

$$\nabla_\alpha\nabla^\alpha\phi - V_{,\phi} = 0 \,, \tag{2.59}$$

where subscript $_{,\phi}$ denotes derivative with respect to ϕ. This equation can be also derived from the conservation of the energy-momentum tensor for the scalar field. If the scalar field is static and space independent, the energy-momentum tensor can mimic the energy-momentum tensor for the cosmological constant,

$$T^\phi_{\alpha\beta} = -g_{\alpha\beta}V(\phi) \,, \tag{2.60}$$

where in this case $V(\phi)$ is constant both space and time. In the Friedmann universe, the scalar field satisfies the properties of perfect fluid. Hence, Eq. (2.44) and Eq. (2.58) give

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad \text{and} \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \,, \tag{2.61}$$

where ρ_ϕ and p_ϕ are the energy density and pressure of the scalar field respectively.

2.4 The Coupled Fluid

In general, the total energy-momentum tensor conserves according to diffeomorphism invariance. If all matters interact only through the gravity, the energy-momentum tensor of all matters conserves separately. If matter interacts with other matter additional to the gravity, the energy-momentum tensor of each coupled matter does not conserve,

$$\nabla_\alpha T^{\alpha\beta}_{(I)} = Q^\beta_{(I)} \,, \tag{2.62}$$

where subscript $_{(I)}$ denotes matter involving the additional interaction. $Q^\beta_{(I)}$ describes the energy and momentum transfer between matter. In the Friedmann universe, $Q^j_{(I)}$ corresponding to the momentum transfer has to vanish because of the isotropy of the space. For component $\beta = 0$, Eq. (2.62) gives

$$\nabla_\alpha T^{\alpha 0}_{(I)} = Q^0_{(I)} \,, \tag{2.63}$$

where $Q^0_{(I)}$ describes the energy transfer between matter. According to the conservation of total energy-momentum tensor, we have

$$\nabla_\alpha T^{\alpha 0} = \sum_I \nabla_\alpha T^{\alpha 0}_{(I)} + \sum_U \nabla_\alpha T^{\alpha 0}_{(U)} = 0 \,, \tag{2.64}$$

where index U runs over uncoupled matter. Since the energy-momentum tensors of uncoupled matter separately conserved, e.g., $\nabla_\alpha T^{\alpha 0}_{(U)} = 0$, Eqs. (2.63) and (2.64) give

$$\sum_I \nabla_\alpha T^{\alpha 0}_{(I)} = \sum_I Q^0_{(I)} = 0 \,. \tag{2.65}$$

Since the properties of the dark energy and the dark matter have not been clearly known, one could assume direct interaction between them. From the above equation, we obtain

$$Q \equiv Q^0_m = -Q^0_d \,. \tag{2.66}$$

For this reason, Eq. (2.64) gives

$$\nabla_\alpha T^{\alpha 0}_m = Q \quad \text{and} \quad \nabla_\alpha T^{\alpha 0}_d = -Q \,. \tag{2.67}$$

Hence, in the Friedmann universe, the above equations become

$$\dot{\rho}_m + 3H(\rho_m + p_m) = Q \quad \text{and} \quad \dot{\rho}_d + 3H(\rho_d + p_d) = -Q\,. \tag{2.68}$$

2.5 The Friedmann Equation

From the Einstein equation Eq. (2.13), the Ricci tensor, and the Ricci scalar for the FLRW metric in Eqs. (2.38) - (2.42), we obtain for the component $\alpha\beta = 00$ as

$$G_{00} = \kappa^2 T_{00}\,. \tag{2.69}$$

By using Eqs. 2.13) and ((2.45), the above equation can be written as

$$\begin{aligned}
R_{00} - \frac{1}{2}g_{00}R &= \kappa^2 T_{00}\,,\\
-3\frac{\ddot{a}}{a} - \frac{1}{2}(-1)\left(6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + \frac{6k}{a^2}\right) &= \kappa^2\rho\,,\\
3\frac{\dot{a}^2}{a^2} + \frac{3k}{a^2} &= \kappa^2\rho\,,\\
3H^2 + \frac{3k}{a^2} &= \kappa^2\rho\,,\\
H^2 &= \frac{\kappa^2}{3}\rho - \frac{k}{a^2}\,,
\end{aligned} \tag{2.70}$$

where ρ in this equation is the total energy density of all matters in the universe. For the component $\alpha\beta = ii$, we obtain

$$\begin{aligned}
G_{ii} &= \kappa^2 T_{ii}\,,\\
R_{ii} - \frac{1}{2}g_{ii}R &= \kappa^2 T_{ii}\,,\\
\left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\frac{g_{ii}}{a^2} - \frac{1}{2}g_{ii}\left(6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + \frac{6k}{a^2}\right) &= \kappa^2 g_{ii}p\,,\\
\left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\frac{1}{a^2} - \frac{1}{2}\left(6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + \frac{6k}{a^2}\right) &= \kappa^2 p\,,\\
-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} &= \kappa^2 p\,,\\
-\frac{2\ddot{a}}{a} - H^2 - \frac{k}{a^2} &= \kappa^2 p\,,\\
\frac{2\ddot{a}}{a} + H^2 &= -\kappa^2 p - \frac{k}{a^2}\,,
\end{aligned} \tag{2.71}$$

where p is the total pressure of all matters in the universe. Inserting Eq. (2.70) into Eq. (2.71), we obtain the acceleration equation written as

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\left(\rho + 3p\right) . \tag{2.72}$$

Usually, we call Eq. (2.70) as the Friedmann equation and call Eq. (2.72) as the acceleration equation. We use these equations to describe the evolution of the universe in which the spacetime is FLRW metric and the matters are modeled by the perfect fluid. This is the Friedmann universe. Using Eq. (2.52), the above equation becomes

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\rho\left(1 + 3w_\tau\right) , \tag{2.73}$$

where w_τ is the total equation of state parameter defined by

$$w_\tau = \frac{p_\tau}{\rho_\tau} = \frac{p_\gamma + p_m + p_d}{\rho_\gamma + \rho_m + p_d} . \tag{2.74}$$

From Eq. (2.73), we can see that the total equation of state parameter has to be less than $-1/3$ to obtain an accelerated expansion of the universe. Differentiating Eq. (2.70) with respect to time and using Eq. (2.53), we obtain

$$\dot{H} = -\frac{\kappa^2}{2}\rho\left(1 + w_\tau\right) + \frac{k}{a^2} . \tag{2.75}$$

Ignoring the last term, this equation tells us that if $\dot{H} < 0$, $\dot{H} = 0$ and $\dot{H} > 0$, the universe is dominated by the ordinary matter ($w_\tau > -1$), contains only the cosmological constant ($w_\tau = -1$) and is dominated by the phantom field ($w_\tau < -1$) respectively. The Friedmann equation Eq. (2.70) can be written as

$$\Omega_\tau + \Omega_k = 1 , \tag{2.76}$$

where

$$\Omega_\tau \equiv \frac{\kappa^2}{3H^2}\rho \quad \text{and} \quad \Omega_k \equiv \frac{k}{(aH)^2} . \tag{2.77}$$

Here Ω_k is the density parameter of the curvature, while Ω_τ is the total density parameter of all matters which can be written as

$$\Omega_\tau = \Omega_\gamma + \Omega_m + \Omega_d , \tag{2.78}$$

where

$$\Omega_\gamma \equiv \frac{\kappa^2}{3H^2}\rho_\gamma \quad , \quad \Omega_m \equiv \frac{\kappa^2}{3H^2}\rho_m \quad \text{and} \quad \Omega_d \equiv \frac{\kappa^2}{3H^2}\rho_d \,. \tag{2.79}$$

From observations, the universe is filled with the radiation, matter and dark energy which

$$|\Omega_{k\,0}| \sim 0.01 \quad , \quad \Omega_{\gamma\,0} \sim 10^{-4} \quad , \quad \Omega_{m\,0} \sim 0.31 \quad \text{and} \quad \Omega_{d\,0} \sim 0.69 \,. \tag{2.80}$$

The subscript $_0$ denotes the values at the present time. From observations, the present universe is in the dark energy dominated epoch. However, the dark energy slowest decays and the radiation fastest decays. Hence, when we look back in the past, we first see the matter dominated epoch in which the energy density $\rho_m > \rho_d > \rho_\gamma$. If we further look back in the past, we will see the radiation dominated epoch in which $\rho_\gamma > \rho_m > \rho_d$.

CHAPTER III

REVIEWS OF THE LITERATURE

3.1 Galileon Theories

Mostly, the fundamental theories in physics provide equations of motion up to second-order time derivative of dynamical variables such as Newton's law because their Lagrangians depend on first-order time derivative of dynamical variables. In general, if Lagrangian includes second-order time derivative of dynamical variables, the equations of motion become fourth-order time derivative of dynamical variables. This leads to the Ostrogradsky instability because the corresponding Hamiltonian contains linear conjugate momentum. However for some special forms of Lagrangians, the equations of motion are still second order differential equation even though second order derivatives appear in the Lagrangians. The theories which have such properties are for example Galileon theory and some of its extension.

Galileon theories are the most general scalar field theories in flat Minkowski spacetime which action contains second order time derivative of scalar field but can provide the equations of motion up to second order derivative of scalar field. The theories have under the following transformation [42]

$$\phi(x) \ \rightarrow \ \phi(x) + b_\mu x^\mu + c \,, \tag{3.1}$$

where b_μ and c are arbitrary constants. It follows from the above transformation that

$$\phi_\mu \ \rightarrow \ \phi_\mu + b_\mu \,, \tag{3.2}$$

where $\phi_\mu \equiv \partial_\mu \phi$ in flat Minkowski spacetime.

This symmetry suggests that the the Lagrangians of Galileon theories are invariant under the above transformation, i.e.,

$$\phi_{\mu\nu} \ \rightarrow \ \partial_\mu \left(\phi_\nu + b_\nu\right) = \phi_{\mu\nu} \,, \tag{3.3}$$

where $\phi_{\mu\nu} \equiv \partial_\nu \partial_\mu \phi$. Let us first start our consideration from the case where the Lagrangian depends on the generic function of $\phi_{\mu\nu}$ as

$$\mathcal{L} = \mathcal{L}\left(\phi, \phi_\mu, \phi_{\mu\nu}\right) \,, \tag{3.4}$$

so that the Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_\mu} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} = 0 \,. \tag{3.5}$$

Generally if Lagrangians depend on second order derivative of dynamical variables, the EOMs become fourth order derivative of dynamical variables. This leads to the Ostrogradsky instability. However for some special forms of Lagrangian, the EOMs are still second order differential equation even though second order derivatives appear in the Lagrangians. To prove the above viewpoint, we suppose $\mathcal{L}$ is quadratic in $\phi_{\mu\nu}$ and takes the form

$$\mathcal{L} = \mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2} \phi_{\alpha_1\beta_1} \phi_{\alpha_2\beta_2} \,. \tag{3.6}$$

For simplicity, we suppose tensor $\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}$ depends on only ϕ_μ. To guarantee that the Euler-Lagrange equation is second order differential equation, we require

$$\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2} \equiv \begin{cases} \text{totally antisymmetric under } \alpha_1 \leftrightarrow \alpha_2 \text{ and } \beta_1 \leftrightarrow \beta_2 \,, \\ \text{symmetric under } \alpha_1 \leftrightarrow \beta_1 \text{ and } \alpha_2 \leftrightarrow \beta_2 \,. \end{cases}$$

Inserting the Eq. (3.6) into Eq. (3.5), we obtain

$$\begin{aligned} 0 &= \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_\mu} \,, \\ &= \partial_\mu \partial_\nu \frac{\partial \left(\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2} \phi_{\alpha_1\beta_1} \phi_{\alpha_2\beta_2}\right)}{\partial \phi_{\mu\nu}} - \partial_\mu \frac{\partial \left(\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2} \phi_{\alpha_1\beta_1} \phi_{\alpha_2\beta_2}\right)}{\partial \phi_\mu} \,. \end{aligned} \tag{3.7}$$

From Eqs. (A.1) and (A.2) in appendix, Eq. (3.7) becomes

$$0 = 2\phi_{\mu\beta_2} \phi_{\nu\alpha_2} \phi_{\alpha_1\beta_1} \left(\frac{\partial^2 \mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial \phi_\mu \partial \phi_\nu}\right) - \phi_{\alpha_1\beta_1} \phi_{\alpha_2\beta_2} \phi_{\mu\nu} \left(\frac{\partial^2 \mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial \phi_\mu \partial \phi_\nu}\right) \,. \tag{3.8}$$

Eq. (3.8) can be up to second order derivative if tensor $\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}$ takes the form

$$\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}\left(\phi_\mu\right) \equiv \mathcal{A}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3} \phi_{\alpha_3} \phi_{\beta_3} \,, \tag{3.9}$$

where $\mathcal{A}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}$ is tensor that is defined as general form in Eq. (3.12). Inserting the above expression into Eq. (3.8) and following calculation in Eq. (A.3), we obtain

$$0 = -4\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\mathcal{A}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3} \,. \tag{3.10}$$

From Eq. (A.3), one can extend to D-dimensional flat spacetime in which Galileon theories can be defined in several ways. We start from the Galileon Lagrangian given by

$$\mathcal{L}_N^{\text{Gal},1} = \left(\mathcal{A}_{(2n+2)}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}\phi_{\alpha_{n+1}}\phi_{\beta_{n+1}}\right)\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n} \,. \tag{3.11}$$

This is called the type-1 Galileon Lagrangian. The $2m$-contravariant tensor $\mathcal{A}_{(2m)}$ is defined by

$$\mathcal{A}_{(2m)}^{\alpha_1\alpha_2\ldots\alpha_m\beta_1\beta_2\ldots\beta_m} \equiv \frac{1}{(D-m)!}\,\varepsilon^{\alpha_1\alpha_2\ldots\alpha_m\sigma_1\sigma_2\ldots\sigma_{D-m}}\,\varepsilon^{\beta_1\beta_2\ldots\beta_m}{}_{\sigma_1\sigma_2\ldots\sigma_{D-m}} \,, \tag{3.12}$$

and the totally antisymmetric Levi-Civita tensor is given by

$$\varepsilon^{\alpha_0\alpha_1\ldots\alpha_{D-1}} = -\frac{1}{\sqrt{-g}}\delta_0^{[\alpha_0}\delta_1^{\alpha_1}\ldots\delta_{D-1}^{\alpha_{D-1}]} \,, \tag{3.13}$$

where N indicates N-times multiplication of ϕ, while n indicates the number of $\phi_{\mu\nu}$. Then we have

$$N = n + 2 \,. \tag{3.14}$$

Since the maximum number of the indices of Levi-Civita tensor are restricted by D indices, we obtain

$$n + 1 \leq D \qquad \rightarrow \qquad N \leq D + 1 \,. \tag{3.15}$$

When $D = 4$, N can take four possible values, e.g., $N = 2, 3, 4, 5$. Thus there are only four possible non-trivial Galileon Lagrangians of the above form in four dimensions, and these were shown respectively in [43]. To obtain the EOM of the type-1 Galileon Lagrangian, we replace Lagrangian in Eq. (3.11) into Eq. (3.5) and follow Eq. (A.4). The result is

$$0 = -\left(2 + n\right)\mathcal{A}_{(2n+2)}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}\beta_{n+1}} \,. \tag{3.16}$$

Other forms of Galileon Lagrangians can be written as

$$\mathcal{L}_N^{\text{Gal},2} = \left(\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_1}\phi_\lambda\phi^\lambda_{\beta_1}\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,, \tag{3.17}$$

$$\mathcal{L}_N^{\text{Gal},3} = \left(\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_\lambda\phi^\lambda\right)\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\,. \tag{3.18}$$

We call the above equations as the type-2 and type-3 Galileon Lagrangian respectively. We will also perform Euler-Lagrange equation to make sure that their EOMs are still up to second order derivative. Let us start from the type-2 Galileon Lagrangian. Substituting Eq. (3.17) into Eq. (3.5), we obtain

$$\begin{aligned}
0 &= \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial\phi_{\mu\nu}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\phi_\mu}\,,\\
&= \partial_\mu\partial_\nu\frac{\partial\left[\left(\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_1}\phi_\lambda\phi^\lambda_{\beta_1}\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right]}{\partial\phi_{\mu\nu}}\\
&\quad -\partial_\mu\frac{\partial\left[\left(\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_1}\phi_\lambda\phi^\lambda_{\beta_1}\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right]}{\partial\phi_\mu}\,.
\end{aligned} \tag{3.19}$$

Substituting Eqs. (A.6) and (A.7) in appendix into Eq. (3.19), there are several terms which cancel each other. Finally, we obtain

$$\begin{aligned}
0 &= -n\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi^\lambda_{\beta_1}\phi_{\lambda\alpha_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\\
&\quad +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi^\lambda_\lambda\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,.
\end{aligned} \tag{3.20}$$

In order to obtain the EOM of the type-3 Galileon Lagrangian, we replace Eq. (3.18) into Eq. (3.5). We obtain

$$\begin{aligned}
0 &= n\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi^\lambda_{\alpha_1}\phi_{\lambda\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\\
&\quad -\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi^\lambda_\lambda\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,.
\end{aligned} \tag{3.21}$$

We can see that three types of Galileon have purely second order derivative equations of motion on flat spacetime that have already shown in Eq. (3.16), Eq. (3.20) and Eq. (3.21). They definitely have invariance under the Galileon symmetry in Eq. (3.2). In fact, there are relations among three types of Galileon Lagrangian that were shown in

[44]. To prove these relations, let us define

$$J^\alpha_N = X\mathcal{A}_{(2n)}^{\alpha\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,. \tag{3.22}$$

Here in this chapter we define $X \equiv \phi_\mu\phi^\mu$. Following in Eq. (A.10) in the appendix, we get

$$\partial_\alpha J^\alpha_N \;=\; 2\mathcal{L}_N^{\text{Gal,2}} + \mathcal{L}_N^{\text{Gal,3}}\,. \tag{3.23}$$

From the above equation, we can see that

$$\mathcal{L}_N^{\text{Gal,2}} = -\frac{1}{2}\mathcal{L}_N^{\text{Gal,3}} + \frac{1}{2}\partial_\alpha J^\alpha_N\,. \tag{3.24}$$

This shows that $\mathcal{L}_N^{\text{Gal,2}}$ is equal to $-\frac{1}{2}\mathcal{L}_N^{\text{Gal,3}}$ up to a total derivative. Moreover, By using Eq. (3.12) together with Eq. (3.13), we can rewrite

$$\begin{aligned}
\mathcal{A}_{(2n)}{}^{\alpha_1\alpha_2\ldots\alpha_n}{}_{\beta_1\beta_2\ldots\beta_n} &= -\frac{1}{(D-n)!}\,\delta^{\alpha_1\alpha_2\ldots\alpha_n\sigma_1\sigma_2\ldots\sigma_{D-n}}_{\beta_1\beta_2\ldots\beta_n\sigma_1\sigma_2\ldots\sigma_{D-n}}\,,\\
&= -\delta^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_n}\,.
\end{aligned} \tag{3.25}$$

Likewise, we can rewrite

$$\begin{aligned}
\mathcal{A}_{(2n+2)}{}^{\alpha_1\alpha_2\ldots\alpha_{n+1}}{}_{\beta_1\beta_2\ldots\beta_{n+1}} &= -\delta^{\alpha_1\alpha_2\ldots\alpha_{n+1}}_{\beta_1\beta_2\ldots\beta_{n+1}}\,,\\
&= -\sum_{i=1}^{n+1}(-1)^{i-1}\,\delta^{\alpha_1}_{\beta_i}\delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_1\beta_2\ldots\beta_{i-1}\beta_{i+1}\ldots\beta_{n+1}}\,,\\
&= -\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_2\beta_3\ldots\beta_{n+1}}\\
&\quad -\sum_{i=2}^{n+1}(-1)^{i-1}\,\delta^{\alpha_1}_{\beta_i}\delta^{\alpha_2\alpha_3\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_{n+1}}_{\beta_1\beta_2\ldots\beta_{i-1}\beta_{i+1}\ldots\beta_{n+1}}\,.
\end{aligned} \tag{3.26}$$

On using the above expression, $\mathcal{L}_N^{\text{Gal,1}}$ in Eq. (3.11) can be rewritten as

$$\begin{aligned}
\mathcal{L}_N^{\text{Gal,1}} &= -\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_2\beta_3\ldots\beta_{n+1}}\phi_{\alpha_1}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}\\
&\quad -\sum_{i=2}^{n+1}(-1)^{i-1}\,\delta^{\alpha_1}_{\beta_i}\delta^{\alpha_2\alpha_3\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_{n+1}}_{\beta_1\beta_2\ldots\beta_{i-1}\beta_{i+1}\ldots\beta_{n+1}}\phi_{\alpha_1}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}\,.
\end{aligned} \tag{3.27}$$

Considering the first term on the right-hand side of the above equation which is shown in Eq. (A.11) in the appendix, we can obtain

$$-\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_2\beta_3\ldots\beta_{n+1}}\phi_{\alpha_1}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}} \;=\; \mathcal{L}_N^{\text{Gal,3}}\,. \tag{3.28}$$

For the second term of Eq. (3.27), starting at $i=2$ we now obtain

$$\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_1\beta_3\ldots\beta_{n+1}}\phi_{\alpha_1}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}} = -\mathcal{L}^{\text{Gal},2}_N . \tag{3.29}$$

At $i=3$, it yields

$$-\delta^{\alpha_1}_{\beta_3}\delta^{\alpha_2\alpha_3\alpha_4\ldots\alpha_{n+1}}_{\beta_1\beta_2\beta_4\ldots\beta_{n+1}}\phi_{\alpha_1}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\phi^{\beta_3}_{\alpha_3}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}} = -\mathcal{L}^{\text{Gal},2}_N . \tag{3.30}$$

Eq. (3.29) and Eq. (3.30) are shown the calculation in Eq. (A.12) and Eq. (A.13). While $i=4,5,6,\ldots,n+1$, we still obtain the same results. Since the second term in Eq. (3.27) is sum from $i=2$ to $i=n+1$ that are n times, it can be rewritten as

$$-\sum_{i=2}^{n+1}(-1)^{i-1}\,\delta^{\alpha_1}_{\beta_i}\delta^{\alpha_2\alpha_3\ldots\alpha_{i-1}\alpha_{i+1}\ldots\alpha_{n+1}}_{\beta_1\beta_2\ldots\beta_{i-1}\beta_{i+1}\ldots\beta_{n+1}}\phi_{\alpha_1}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}} = -n\mathcal{L}^{\text{Gal},2}_N . \tag{3.31}$$

Substituting Eq. (3.28) and Eq. (3.31) into Eq. (3.27), we obtain the following relation

$$\mathcal{L}^{\text{Gal},1}_N = \mathcal{L}^{\text{Gal},3}_N - n\mathcal{L}^{\text{Gal},2}_N . \tag{3.32}$$

Substituting Eq. (3.14) into the above relation, it yields

$$\mathcal{L}^{\text{Gal},1}_N = \mathcal{L}^{\text{Gal},3}_N - (N-2)\,\mathcal{L}^{\text{Gal},2}_N . \tag{3.33}$$

We can write $\mathcal{L}^{\text{Gal},3}_N$ in terms of $\mathcal{L}^{\text{Gal},1}_N$ and $\mathcal{L}^{\text{Gal},2}_N$ as

$$\mathcal{L}^{\text{Gal},3}_N = \mathcal{L}^{\text{Gal},1}_N + (N-2)\,\mathcal{L}^{\text{Gal},2}_N , \tag{3.34}$$

or $\mathcal{L}^{\text{Gal},2}_N$ in terms of $\mathcal{L}^{\text{Gal},1}_N$ and $\mathcal{L}^{\text{Gal},3}_N$ as

$$\mathcal{L}^{\text{Gal},2}_N = \frac{1}{N-2}\left(\mathcal{L}^{\text{Gal},3}_N - \mathcal{L}^{\text{Gal},1}_N\right) . \tag{3.35}$$

Substituting the last two equations into Eq. (3.24), Thus we respectively have

$$\mathcal{L}^{\text{Gal},1}_N = -N\mathcal{L}^{\text{Gal},2}_N + \partial_\alpha J^\alpha_N , \tag{3.36}$$

$$\mathcal{L}^{\text{Gal},1}_N = -\frac{N}{2}\mathcal{L}^{\text{Gal},3}_N + \frac{N-2}{2}\partial_\alpha J^\alpha_N . \tag{3.37}$$

From Eq. (3.24), Eq. (3.36) and Eq. (3.37), we can see that three types of Galileon Lagrangian are related up to total derivative. Since the type-3 Galileon Lagrangian is more compact, we are going to use it via this thesis.

As we have already mentioned at the beginning of this section, there are 4 possible Lagrangians in case $D = 4$. Therefore, we are going to write down for 4 possible ones from the simplest Galileon Lagrangian in Eq. (3.18) together with Eq. (3.25).

For $N = 2$, it yields

$$\mathcal{L}^{\text{Gal,3}}_{N=2} = X \,. \tag{3.38}$$

For $N = 3$, its expression is

$$\begin{aligned}\mathcal{L}^{\text{Gal,3}}_{N=3} &= \mathcal{A}_{(2n=2)}{}^{\alpha_1}{}_{\beta_1}\phi_\lambda\phi^\lambda\phi^{\beta_1}_{\alpha_1}\,,\\ &= -X\delta^{\alpha_1}_{\beta_1}\phi^{\beta_1}_{\alpha_1}\,,\\ &= -X\phi^{\alpha_1}_{\alpha_1}\,,\\ &= -X\Box\phi\,.\end{aligned} \tag{3.39}$$

This is called the cubic Galileon Lagrangian.

For $N = 4$, its expression is

$$\begin{aligned}\mathcal{L}^{\text{Gal,3}}_{N=4} &= \mathcal{A}_{(2n=4)}{}^{\alpha_1\alpha_2}{}_{\beta_1\beta_2}\phi_\lambda\phi^\lambda\phi^{\beta_1}_{\alpha_1}\phi^{\beta_2}_{\alpha_2}\,,\\ &= -X\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2}\phi^{\beta_1}_{\alpha_1}\phi^{\beta_2}_{\alpha_2}\,,\\ &= -X\left(\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}-\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\right)\phi^{\beta_1}_{\alpha_1}\phi^{\beta_2}_{\alpha_2}\,,\\ &= -X\underbrace{\left(\phi^{\alpha_1}_{\alpha_1}\phi^{\alpha_2}_{\alpha_2}-\phi^{\alpha_2}_{\alpha_1}\phi^{\alpha_1}_{\alpha_2}\right)}_{\alpha_1,\alpha_2\to\alpha,\beta}\,,\\ &= X\left(\phi_{\alpha\beta}\phi^{\alpha\beta}-\Box\phi^2\right)\,.\end{aligned} \tag{3.40}$$

This is called the quartic Galileon Lagrangian.

For $N = 5$, its expression is

$$\begin{aligned}\mathcal{L}^{\text{Gal,3}}_{N=5} &= \mathcal{A}_{(2n=6)}{}^{\alpha_1\alpha_2\alpha_3}{}_{\beta_1\beta_2\beta_3}\phi_\lambda\phi^\lambda\phi^{\beta_1}_{\alpha_1}\phi^{\beta_2}_{\alpha_2}\phi^{\beta_3}_{\alpha_3}\,,\\ &= -X\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}\phi^{\beta_1}_{\alpha_1}\phi^{\beta_2}_{\alpha_2}\phi^{\beta_2}_{\alpha_2}\,,\\ &= -X\left(\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\delta^{\alpha_3}_{\beta_3}-\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_3}\delta^{\alpha_3}_{\beta_2}+\delta^{\alpha_1}_{\beta_3}\delta^{\alpha_2}_{\beta_1}\delta^{\alpha_3}_{\beta_2}\right.\end{aligned}$$

$$
\begin{aligned}
& -\delta^{\alpha_1}_{\beta_3}\delta^{\alpha_2}_{\beta_2}\delta^{\alpha_3}_{\beta_1} + \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_3}\delta^{\alpha_3}_{\beta_1} - \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\delta^{\alpha_3}_{\beta_3}\big)\,\phi^{\beta_1}_{\alpha_1}\phi^{\beta_2}_{\alpha_2}\phi^{\beta_3}_{\alpha_3}\,, \\
= \;& -X\,\underbrace{\left(\phi^{\alpha_1}_{\alpha_1}\phi^{\alpha_2}_{\alpha_2}\phi^{\alpha_3}_{\alpha_3} - 3\phi^{\alpha_1}_{\alpha_1}\phi^{\alpha_3}_{\alpha_2}\phi^{\alpha_2}_{\alpha_3} + 2\phi^{\alpha_2}_{\alpha_1}\phi^{\alpha_3}_{\alpha_2}\phi^{\alpha_1}_{\alpha_3}\right)}_{\alpha_1,\alpha_2,\alpha_3\to\lambda,\alpha,\beta}\,, \\
= \;& -X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right)\,. \qquad (3.41)
\end{aligned}
$$

This is called the quintic Galileon Lagrangian. From the above resulting Lagrangians, the general form of Galileon Lagrangian for four dimensions on flat spacetime, can now written as

$$
\mathcal{L} = \sum_{N=2}^{5} c_N \mathcal{L}^{\rm G}_N\,, \qquad (3.42)
$$

where c_N's are constant and

$$
\begin{aligned}
\mathcal{L}^{\rm G}_2 &= X\,, \\
\mathcal{L}^{\rm G}_3 &= X\Box\phi\,, \\
\mathcal{L}^{\rm G}_4 &= X\left(\Box\phi^2 - \phi_{\alpha\beta}\phi^{\alpha\beta}\right)\,, \\
\mathcal{L}^{\rm G}_5 &= X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right)\,. \qquad (3.43)
\end{aligned}
$$

As we have described, the Galileon theories have equations of motion containing only second order derivative. Generalizing the Galileon theories to the most general scalar theory with equations of motion order 2 or lower in derivatives, one can just add function f depending on ϕ and X or let constant c_N's depend on ϕ and X. Then the generalized Galileon Lagrangian is given by

$$
\begin{aligned}
\mathcal{L}_n\{f\} &= f(\phi, X) \times \mathcal{L}^{\rm Gal,3}_{N=n+2}\,, \\
&= f(\phi, X)\left(X\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\right)\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\,. \qquad (3.44)
\end{aligned}
$$

Clearly, the Galileon symmetry given in Eq. (3.1) is broken for the generalized Galileons because $f(\phi, X)$ is not invariant under the Galileon transformation.

3.2 Covariant Galileon and Horndeski Theories

To generalize Galileon theories on flat spacetime to the theories in curved spacetime, we replace all partial derivatives appearing in the Lagrangians by covariant derivatives. As a result, we obtain minimally covariantized theories. However, variations of $\mathcal{L}_4$ and $\mathcal{L}_5$ with respect to ϕ give the EOMs which contain higher order derivatives. These terms can lead to the Ostrogradsky instability. Let us consider variation with respect to ϕ for $\mathcal{L}_{N=4}^{\text{Gal},3}$ in the version of minimally covariantization. It yields

$$\begin{aligned} \delta\mathcal{L}_{N=4}^{\text{Gal},3} &= \delta\left[X\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right)\right] \,, \\ &= \delta\left[\phi_\lambda\phi^\lambda\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right)\right] \,, \\ &= \delta\left(\phi_\lambda\phi^\lambda\right)\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right) \\ &\quad +\phi_\lambda\phi^\lambda\delta\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right) \,. \end{aligned} \tag{3.45}$$

Considering the first term in the above expression, we obtain

$$\begin{aligned} \delta\left(\phi_\lambda\phi^\lambda\right)\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right) \,, &= 2\left(\phi^\lambda\nabla_\lambda\delta\phi\right)\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right) \,, \\ &= 2\nabla_\lambda\left[\phi^\lambda\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right)\delta\phi\right] \\ &\quad -2\nabla_\lambda\left[\phi^\lambda\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right)\right]\delta\phi \,, \qquad (3.46) \\ &= -2\nabla_\lambda\left[\phi^\lambda\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right)\right]\delta\phi \,. \qquad (3.47) \end{aligned}$$

Note that, the first term in Eq. (3.46) is zero because it is the surface term when we write it in the action form. For the second term in Eq. (3.45), we obtain

$$\phi_\lambda\phi^\lambda\delta\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right) = X\left(2\phi^{\alpha\beta}\nabla_\beta\nabla_\alpha\delta\phi - 2\Box\phi\nabla^\lambda\nabla_\lambda\delta\phi\right) \,. \tag{3.48}$$

After ignoring the surface terms, the above equation becomes

$$\phi_\lambda\phi^\lambda\delta\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right) = 2\left[\nabla_\alpha\nabla_\beta\left(X\phi^{\alpha\beta}\right) - \nabla^\lambda\nabla_\lambda\left(X\Box\phi\right)\right]\delta\phi \,. \tag{3.49}$$

Substituting Eq. (3.47) and Eq. (3.49) into Eq. (3.45), we now obtain

$$\begin{aligned} \delta\mathcal{L}_{N=4}^{\text{Gal},3} &= -2\nabla_\lambda\left[\phi^\lambda\left(\phi_{\alpha\beta}\phi^{\alpha\beta} - \Box\phi^2\right)\right]\delta\phi \\ &\quad +2\left[\nabla_\alpha\nabla_\beta\left(X\phi^{\alpha\beta}\right) - \nabla^\lambda\nabla_\lambda\left(X\Box\phi\right)\right]\delta\phi \,. \end{aligned} \tag{3.50}$$

We can see that the third and fourth order derivatives of ϕ appear in the above result. Moreover, the third order derivatives of metric tensor $g_{\mu\nu}$ also appear in Eq. (3.50). We call those higher order derivative terms as dangerous terms. As we have showed above, the minimally covariantized theories from Galileon Lagrangians are not sufficient to construct the healthy covariant theories that are free from the Ostrogradsky instability. In order to construct the healthy covariant ones, we have to add the correction term into the actions to exactly cancel all such higher derivatives. The calculation for the correction terms is shown in the appendix (B). The Lagrangians for the healthy theory are

$$\mathcal{L}_4^H = G_4(\phi, X)R - 2G_{4,X}\left(\Box\phi^2 - \phi_{\alpha\beta}\phi^{\alpha\beta}\right) , \tag{3.51}$$

which is the quartic Horndeski Lagrangian, and

$$\mathcal{L}_5^H = G_5(\phi, X)G_{\alpha\beta}\phi^{\alpha\beta} + \frac{1}{3}G_{5,X}\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi_{\beta\lambda}\right) , \tag{3.52}$$

which is the quintic Hondeski Lagrangian. Also, the case of the cubic Horndeski Lagrangian is the Lagrangian taking the form

$$\mathcal{L}_3^H = G_3(\phi, X)\Box\phi\,. \tag{3.53}$$

Additionally, the variation with respect to the metric $g_{\mu\nu}$ for the covariant generalized Galileons, had been derived in [45]. The resulting equations of motion after adding the correction terms, are also second order derivatives of the metric. Obviously, covariant generalized Galileons are no longer invariant under Eq. (3.1) but they can be generalized to Eq. (3.2) and

$$\phi \to \phi + c\,, \tag{3.54}$$

where c is constant. According to Eqs. (3.51)-(3.53), we can get re-discovered Horndeski theories which can be expressed in terms of an arbitrary linear combination of the Lagrangians as

$$\mathcal{L} = \sum_{N=2}^{5}\mathcal{L}_N^{\mathrm{H}}\,, \tag{3.55}$$

where

$$
\begin{aligned}
\mathcal{L}_2^{\mathrm{H}} &= G_2(\phi, X)\,, \\
\mathcal{L}_3^{\mathrm{H}} &= G_3(\phi, X)\Box\phi\,, \\
\mathcal{L}_4^{\mathrm{H}} &= G_4(\phi, X)R - 2G_{4,X}\left(\Box\phi^2 - \phi_{\alpha\beta}\phi^{\alpha\beta}\right)\,, \\
\mathcal{L}_5^{\mathrm{H}} &= G_5(\phi, X)G_{\alpha\beta}\phi^{\alpha\beta} + \frac{1}{3}G_{5,X}\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi_{\beta\lambda}\right)\,. \quad (3.56)
\end{aligned}
$$

where G_2, G_3, G_4 and G_5 are arbitrary functions of ϕ and X and $\phi_{\mu\nu}$ in curved spacetime is $\nabla_\nu\nabla_\mu\phi$. We have set the reduced Planck mass $M_p \equiv 1/\sqrt{8\pi G} = 1$. From the above calculation, the covariant generalized Galileons are the Horndeski theory. If the coefficients G_2, G_3, G_4 and G_5 depend on only X, the Horndeski theory becomes the covariant Galileons or the extended Galileons.

3.3 Beyond Horndeski and GLPV Theories

As presented above, the Horndeski theory is the most general scalar-tensor theory in four dimensions of spacetime leading to covariant second order equations of motion for both scalar field and metric tensor. Hence, it definitely propagates three degrees of freedom, e.g., two-tensor and one scalar degrees of freedom. It returns to the question whether it is possible to extend the Horndeski theory to be more general, whose equations of motion could contain higher order derivative of dynamical fields without Ostrogradsky ghost. Currently, such theories have been found. One of possible theories is the beyond Horndeski or GLPV theory. Adding the minimally covariantized Galileon of type-1 term into the L_4 and L_5 of the Horndeski theory, we obtain the GLPV theory which still has the same degrees of freedom as the Horndeski theory. The Lagrangians of this theory can be described by

$$L_2^\phi \equiv G_2(\phi, X) \,, \tag{3.57}$$

$$L_3^\phi \equiv G_3(\phi, X)\, \Box\phi \,, \tag{3.58}$$

$$\begin{aligned} L_4^\phi &\equiv G_4(\phi, X)\, {}^{(4)}\!R - 2G_{4,X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}) \\ &\quad + F_4(\phi, X)\epsilon^{\mu\nu\rho}{}_\sigma\, \epsilon^{\mu'\nu'\rho'\sigma}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'} \,, \end{aligned} \tag{3.59}$$

$$\begin{aligned} L_5^\phi &\equiv G_5(\phi, X)\, {}^{(4)}\!G_{\mu\nu}\phi^{\mu\nu} \\ &\quad + \frac{1}{3}G_{5,X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^\nu{}_\sigma) \\ &\quad + F_5(\phi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\phi_{\sigma\sigma'} \,, \end{aligned} \tag{3.60}$$

where F_4 and F_5 are coefficients of the minimally covariantized type-1 Galileon Lagrangian [46]. Hence, the Horndeski theory is a subset of the above theory by restricting the conditions

$$F_4(\phi, X) = 0, \qquad F_5(\phi, X) = 0, \tag{3.61}$$

which guarantee that the equation of motion is only second order derivatives in dynamical fields. It has been shown that GLPV theory propagates the same degrees of freedom as the Horndeski theory using Hamiltonian analysis [47].

3.4 Degenerate Higher-Order Scalar-Tensor (DHOST) Theories

General action of scalar-tensor theories which contains the second order derivative of scalar field can lead to higher order of time derivative in the EOM. However, if there is the suitable symmetry of the second order derivative of scalar field in the action for example that appear in the Horndeski and GLPV theories, the EOMs are still second order of time derivative, i.e., the propagating degrees of freedom in the theories are single scalar and two-tensor. In addition to such symmetry, the general action can also lead to three propagating degrees of freedom if Lagrangians are degenerate. Base on this idea, the generalization of the Horndeski and GLPV theory have been constructed. These theories are Degenerate Higher-Order Scalar-Tensor (DHOST) theories.

3.4.1 Higher-Order Scalar-Tensor Theories

In this section, we consider scalar-tensor theories whose action is given by the general form

$$S = S_g + S_\phi \,, \tag{3.62}$$

where S_g depends on the Ricci scalar R of the metric $g_{\mu\nu}$,

$$S_g \equiv \int d^4x \sqrt{-g}\, f(\phi, X) R \,, \tag{3.63}$$

and S_ϕ depends on the quadratic term of second derivatives of the scalar field ϕ

$$S_\phi \equiv \int d^4x \sqrt{-g}\, C^{\mu\nu,\rho\sigma}\, \nabla_\mu \nabla_\nu \phi\, \nabla_\rho \nabla_\sigma \phi \,, \tag{3.64}$$

where $C^{\mu\nu,\rho\sigma}$ is an arbitrary tensor depending only on ϕ and $\nabla_\mu\phi$. In principle, S_ϕ can depend on the cubic term of second derivatives of the scalar field which is the generalization of L_5 in the Horndeski and GLPV theory. The construction of theories that includes this term is presented in [48]. We will see in the following section that this term in the theories is ruled out because of the constraint on the propagation speed of GW. Then we concentrate on the quadratic term of second derivatives of the scalar field. Note that when the function f is $1/2$, S_g reduces into the familiar Einstein-Hilbert action. We now consider $C^{\mu\nu,\rho\sigma}$. In general, we can write $C^{\mu\nu,\rho\sigma}$ in terms of symmetric and antisymmetric parts. For simplicity, we start by considering the first pair of indices,

$$C^{\mu\nu,\rho\sigma} = \frac{1}{2}\left(C^{\mu\nu,\rho\sigma} + C^{\nu\mu,\rho\sigma}\right) + \frac{1}{2}\left(C^{\mu\nu,\rho\sigma} - C^{\nu\mu,\rho\sigma}\right) . \tag{3.65}$$

Since the antisymmetric part contracting with $\nabla_\mu\, \nabla_\nu \phi$ vanishes, we can ignore the antisymmetric part. Applying the same consideration to the second pair of indices $\rho\sigma$, one can check that under swapping indices $\rho\sigma$ the antisymmetric part of $C^{\mu\nu,\rho\sigma}$ can also be ignore. Hence, $C^{\mu\nu,\rho\sigma}$ satisfies the relations, [11]

$$C^{\mu\nu,\rho\sigma} = C^{\nu\mu,\rho\sigma} = C^{\mu\nu,\sigma\rho} = C^{\rho\sigma,\mu\nu} \,. \tag{3.66}$$

According to these properties, $C^{\mu\nu,\rho\sigma}$ can be expressed in the most general form of metric tensor and first order derivative of scalar field as

$$
\begin{aligned}
C^{\mu\nu,\rho\sigma} &= \frac{1}{2}\alpha_1(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) + \alpha_2 g^{\mu\nu}g^{\rho\sigma} + \frac{1}{2}\alpha_3(\phi^{\mu}\phi^{\nu}g^{\rho\sigma} + \phi^{\rho}\phi^{\sigma}g^{\mu\nu}) \\
&+ \frac{1}{4}\alpha_4(\phi^{\mu}\phi^{\rho}g^{\nu\sigma} + \phi^{\nu}\phi^{\rho}g^{\mu\sigma} + \phi^{\mu}\phi^{\sigma}g^{\nu\rho} + \phi^{\nu}\phi^{\sigma}g^{\mu\rho}) + \alpha_5\phi^{\mu}\phi^{\nu}\phi^{\rho}\phi^{\sigma}\,, \quad (3.67)
\end{aligned}
$$

where the α_I are five arbitrary functions of ϕ and X. Since $C^{\mu\nu,\rho\sigma}$ is contracted with $\nabla_\mu\nabla_\nu\phi\nabla_\rho\nabla_\sigma\phi$, we can obtain the five possible Lagrangians quadratic in second derivatives,

$$
\begin{aligned}
L_1^{\phi} &\equiv \phi^{\mu\nu}\phi_{\mu\nu}\,, \quad L_2^{\phi} \equiv (\phi_\mu{}^{\mu})^2\,, \quad L_3^{\phi} \equiv \phi_\mu{}^{\mu}\,\phi^{\rho}\phi_{\rho\sigma}\phi^{\sigma}\,, \\
L_4^{\phi} &\equiv \phi^{\mu}\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho}\,, \quad L_5^{\phi} \equiv (\phi^{\rho}\phi_{\rho\sigma}\phi^{\sigma})^2\,. \qquad (3.68)
\end{aligned}
$$

Now the action S_ϕ from Eq. (3.64) can be written as

$$
S_\phi = \int d^4x\sqrt{-g}\,\left(\alpha_1 L_1^{\phi} + \alpha_2 L_2^{\phi} + \alpha_3 L_3^{\phi} + \alpha_4 L_4^{\phi} + \alpha_5 L_5^{\phi}\right) \equiv \int d^4x\sqrt{-g}\,\alpha_I L_I^{\phi}\,, \qquad (3.69)
$$

where index I runs over $I = 1, 2, \ldots, 5$. The action in Eq. (3.62) includes a particular case of the quartic Horndeski term

$$
L_4^{\mathrm{H}} = G_4(\phi, X)^{(4)}R - 2G_{4,X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu})\,. \qquad (3.70)
$$

Indeed, the above Lagrangian is of the form of the Lagrangians in Eqs. (3.62)-(3.69) by setting

$$
f = G_4, \quad \alpha_1 = -\alpha_2 = G_{4,X}, \quad \alpha_3 = \alpha_4 = \alpha_5 = 0. \qquad (3.71)
$$

The action (3.62) also includes the GLPV theory which can be written as

$$
L_4^{bH} = F_4(\phi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\,\epsilon^{\mu'\nu'\rho'\sigma}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\,. \qquad (3.72)
$$

This corresponds to the action (3.62) with

$$
\alpha_1 = -\alpha_2 = XF_4, \quad \alpha_3 = -\alpha_4 = 2F_4, \quad \alpha_5 = 0. \qquad (3.73)
$$

3.4.2 Degeneracy of Lagrangian

In general, the action (3.62) contains one scalar mode, two tensor modes and also extra scalar mode called Ostrogradsky ghost leading to Ostrogradsky instability. When the systems have Ostrogradsky degrees of freedom, the dynamical variables associated with these degrees of freedom can be infinite for the finite value of Hamiltonian. We can avoid this instability by imposing the constraints on function f and α_I from the requirement that the Lagrangians have to be degenerate. To understand this degeneracy idea more, we use very simple toy model based on classical point of views. Let us consider the Lagrangian of the form

$$L = \frac{1}{2}a\,\ddot{\phi}^2 + b\,\ddot{\phi}\,\dot{q} + \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}c\,\dot{q}^2 - V(\phi, q)\,, \tag{3.74}$$

where a, b and c are constant coefficients and $V(\phi, q)$ is some potential. This Lagrangian involves the acceleration of ϕ but not velocity of q. If $a \neq 0$, the term that is proportional to a generates fourth-order equations of motion for ϕ, whereas, if $a = 0$ but $b \neq 0$, one obtains third-order equations of motion for ϕ and q respectively.

To compute the degree of freedom, it is convenient to work with a more familiar Lagrangian containing only velocities, let us introduce new auxiliary variable

$$Q \equiv \dot{\phi}\,, \tag{3.75}$$

leading to the new Lagrangian

$$L = \frac{1}{2}a\,\dot{Q}^2 + b\,\dot{Q}\,\dot{q} + \frac{1}{2}c\,\dot{q}^2 + \frac{1}{2}Q^2 - V(\phi, q) - \lambda(Q - \dot{\phi})\,, \tag{3.76}$$

which does not include any acceleration. We now introduce the kinetic matrix called the Hessian matrix defined by [8]

$$M \equiv \left(\frac{\partial^2 L}{\partial v^a \partial v^b}\right) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\,, \tag{3.77}$$

where the symbol v^a denotes the velocities, i.e., $v^a \equiv \{\dot{Q}, \dot{q}\}$. In the generic case, if M is invertible, $\ddot{Q}$ and $\ddot{q}$ can be separated independently. Then the differential system requires six initial conditions which are $Q, \dot{Q}, q, \dot{q}, \lambda$ and ϕ. The six initial conditions correspond to the existence of three degrees of freedom including the extra degree of freedom associated with Ostrogradsky degree of freedom. To avoid the presence of extra degree of freedom, we have to impose the Hessian matrix, M, to be degenerate, i.e.,

$$\det M = ac - b^2 = 0\,. \tag{3.78}$$

This implies that $\ddot{Q}$ and $\ddot{q}$ cannot be separated independently. Since we can write $\ddot{Q}$ in terms of $\ddot{q}$, two initial conditions decrease. Then there are only four initial conditions. This means that this system consists of two degrees of freedom. The extra mode Q associated with $\dot{\phi}$ is eliminated when M is degenerate. Since the EOM of Q contains third order derivative of ϕ, Ostrogradsky degree of freedom is killed when Ostrogradsky degree of freedom associated with Q is eliminated. In this situation, the initial Lagrangian (3.74) is degenerate. In general, the number of degrees of freedom can also be determined by using a Hamiltonian analysis. When the Lagrangian is degenerate, the conjugate momenta implies the existence of primary constraint. When we perform time evolution of this constraint, one finds that it leads to a secondary constraint in phase space. These two constraints which are second class constraint, kill one degree of freedom in agreement with the analysis based on the equations of motion. In principle, If some dynamical variables are constrained, their EOMs become constraint equations. This means that some degrees of freedom are eliminated. If two constrained variables form a pair of canonical variables, These two constraints are second class constraints which can eliminate one degree of freedom of the system.

3.4.3 Kinetic Matrix

In order to study Hamiltonian analysis or analysis of constraints we have to write H in terms of p and $\dot{q}$. We then need to separate space and time by performing 3+1 decomposition[7]. We assume the existence of a slicing of spacetime with three-dimensional spacelike hypersurfaces. We introduce their normal unit vector n^μ, which is time-like, and satisfies the normalization condition $n_\mu n^\mu = -1$. Using this normal vector, we can define the projection tensor as

$$\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu \,. \tag{3.79}$$

This tensor projects any tensor into spacelike hypersurfaces. The spatial components of this projection tensor are the metric tensor on spacelike hypersurfaces denoted by h_{ij}. It is convenient to define the spatial projection of $A_\mu \equiv \nabla_\mu \phi$,

$$\hat{A}_\mu \equiv \gamma_\mu^\nu A_\nu \,, \tag{3.80}$$

and its normal projection

$$A_* \equiv A_\mu n^\mu \,. \tag{3.81}$$

The extrinsic curvature K_{ij} defined by

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - D_i N_j - D_j N_i\right), \tag{3.82}$$

where N is the lapse function, N^i the shift vector and D_i denotes the three-dimensional covariant derivative compatible with h_{ij}. Using the above definitions and the property $\nabla_\mu A_\nu = \nabla_\nu A_\mu$, where ∇_μ is the four-dimensional covariant derivative compatible with $g_{\mu\nu}$, we can find that the 3+1 covariant decomposition of $\nabla_\mu A_\nu$ is given by [8, 48]

$$\nabla_\mu A_\nu \to D_i \hat{A}_j - A_* \, K_{ij} + n_{(i}\left(K_{j)k}\hat{A}^k - D_{j)} A_*\right) + \, n_i n_j \left(V_* - \hat{A}_k \, a^k\right) \,, \tag{3.83}$$

where the Latin indices run over $1, 2, 3$, a^k is spatial component of the acceleration vector defined by $a^\nu = n^\mu \, \nabla_\mu \, n^\nu$ and

$$V_* \equiv n^\mu \nabla_\mu A_* = \frac{1}{N}\left(\dot{A}_* - N^i D_i A_*\right) \,. \tag{3.84}$$

We note that the term on the left-hand side of the arrow in Eq. (3.83) is evaluated in four-dimensional spacetime, while the terms on the right-hand side of the arrow are evaluated on spacelike hypersurfaces. In Eq. (3.83), there is only time derivatives appear for the three-dimensional metric h_{ij} (inside the extrinsic curvature) and for the component A_*

(inside V_*). V_* plays for A_* the same role that K_{ij} plays for h_{ij}. From Eq. (3.83), we can write the relevant kinetic part of the Lagrangian on spacelike hypersurfaces as

$$(\nabla_\mu A_\nu)_{\text{kin}} \to \lambda_{ij}\,\dot{A}_* + \Lambda_{ij}{}^{kl}\,K_{kl}\,, \tag{3.85}$$

with

$$\lambda_{ij} \equiv \frac{1}{N} n_i n_j\,, \qquad \Lambda_{ij}{}^{kl} = -A_*\,h^k_{(i}h^l_{j)} + 2\,n_{(i}h^{(k}_{j)}\hat{A}^{l)}\,. \tag{3.86}$$

Strictly speaking, only the $\dot{h}_{ij}$ term is relevant but we will keep K_{ij} for convenience. We thus find that the kinetic part of the quadratic Lagrangian in $\nabla_\mu A_\nu$ can be written on spacelike hypersurface as

$$L^{(\phi)}_{\text{kin}} = C^{ij,kl}\lambda_{ij}\,\lambda_{kl}\,\dot{A}_*^2 + 2C^{ij,kl}\Lambda_{ij}{}^{mn}\lambda_{kl}\,\dot{A}_*K_{mn} + C^{ij,kl}\Lambda_{ij}{}^{mn}\Lambda_{kl}{}^{pq}K_{mn}K_{pq}\,, \tag{3.87}$$

which is similar to the Lagrangian (3.74), with A_* and K_{ij} (or $\dot{h}_{ij}$) playing the role of Q and $\dot{q}$, respectively. Then we can compute the analogs of the coefficients a, b and c in the Lagrangian (3.74) directly by substituting the explicit expressions for $C^{ij,kl}$, λ_{ij} and $\Lambda_{ij}{}^{kl}$. Hence, the first kinetic coefficient is given by

$$\mathcal{A} \equiv C^{ij,kl}\lambda_{ij}\lambda_{kl} = \frac{1}{N^2}[\alpha_1 + \alpha_2 - (\alpha_3 + \alpha_4)A_*^2 + \alpha_5 A_*^2]\,, \tag{3.88}$$

while the coefficients of the mixed terms can be written as

$$\mathcal{B}^{mn} \equiv C^{ij,kl}\Lambda_{ij}{}^{mn}\lambda_{kl} = \beta_1 h^{mn} + \beta_2\,\hat{A}^m\hat{A}^n\,, \tag{3.89}$$

with

$$\beta_1 = \frac{A_*}{2N}\left(2\alpha_2 - \alpha_3 A_*^2\right)\,, \qquad \beta_2 = -\frac{A_*}{2N}\left(\alpha_3 + 2\alpha_4 - 2\alpha_5 A_*^2\right)\,. \tag{3.90}$$

Finally, the kinetic coefficient for the purely metric part is given by

$$\mathcal{K}^{mn,pq} \equiv C^{ij,kl}\Lambda_{ij}{}^{mn}\Lambda_{kl}{}^{pq}\,. \tag{3.91}$$

Substituting the explicit expressions for Eq. (3.67) and Eq. (3.86) into above equation, we obtain

$$\mathcal{K}^{ij,kl} \;=\; \kappa_1 h^{i(k}h^{l)j} + \kappa_2\,h^{ij}h^{kl} + \frac{1}{2}\kappa_3\left(\hat{A}^i\hat{A}^j h^{kl} + \hat{A}^k\hat{A}^l h^{ij}\right)$$

$$+\frac{1}{2}\kappa_4\left(\hat{A}^i\hat{A}^{(k}h^{l)j}+\hat{A}^j\hat{A}^{(k}h^{l)i}\right)+\kappa_5\hat{A}^i\hat{A}^j\hat{A}^k\hat{A}^l\,, \tag{3.92}$$

with

$$\kappa_1=\alpha_1A_*^2\,,\quad \kappa_2=\alpha_2A_*^2\,,\quad \kappa_3=-\alpha_3A_*^2\,,\quad \kappa_4=-2\alpha_4\,,\quad \kappa_5=\alpha_5A_*^2-\alpha_4\,. \tag{3.93}$$

To obtain the full kinetic part of the action, we also have to consider the gravitational term $f^{(4)}R$. We start with

$$\begin{aligned}\int d^4x\sqrt{-g}\,f^{(4)}R &= \int d^4xN\sqrt{h}\left\{f\left[K_{ij}K^{ij}-K^2+{}^{(3)}R\right]+2D_if\left(a^i-Kn^i\right)\right.\\ &\left.\quad -2K\frac{1}{N}\left(\dot{f}-N^iD_if\right)\right\}\,,\end{aligned} \tag{3.94}$$

where ${}^{(4)}R$ is four-dimensional Ricci scalar and ${}^{(3)}R$ is three-dimensional Ricci scalar. Here, we write

$$\dot{f}=2f_{,X}(\hat{A}_i\dot{\hat{A}}^i-A_*\dot{A}_*)+f_{,\phi}\dot{\phi}\,. \tag{3.95}$$

From Eqs. ((3.94)) and (3.95), we can see that the second term on the right-hand side gives the mixed kinetic terms. We can write the coefficient $\mathcal{B}^{ij}_{\rm grav}$ for the mixed kinetic terms $\mathcal{B}^{j}_{\rm grav}K_{ij}\dot{A}_*$ as

$$\mathcal{B}^{ij}_{\rm grav}=2f_{,X}\frac{A_*}{N}\,h^{ij}\,. \tag{3.96}$$

For the terms that are second-order in K_{ij}, i.e., $\mathcal{K}^{ij,kl}_{\rm grav}K_{ij}K_{kl}$, we obtain the corresponding coefficient as

$$\mathcal{K}^{ij,kl}_{\rm grav}=\gamma_1h^{i(k}h^{l)j}+\gamma_2\,h^{ij}h^{kl}+\frac{1}{2}\gamma_3\left(\hat{A}^i\hat{A}^jh^{kl}+\hat{A}^k\hat{A}^lh^{ij}\right)\,, \tag{3.97}$$

with

$$\gamma_1=-\gamma_2=f,\qquad \gamma_3=4f_{,X}\,. \tag{3.98}$$

In summary, the coefficients that we obtained from the total action are

$$\tilde{\mathcal{B}}^{ij}=\mathcal{B}^{ij}+\mathcal{B}^{ij}_{\rm grav},\qquad \tilde{\mathcal{K}}^{ij,kl}=\mathcal{K}^{ij,kl}+\mathcal{K}^{ij,kl}_{\rm grav}\,, \tag{3.99}$$

The coefficients $\mathcal{A}$, $\tilde{\mathcal{B}}^{ij}$ and $\tilde{\mathcal{K}}^{ij,kl}$ play the same role as a, b and c in the toy model respectively.

3.4.4 Degeneracy Conditions

The full kinetic matrix associated with Eq. (3.87) can be written as [8, 48]

$$\begin{pmatrix} \mathcal{A} & \tilde{B}^{kl} \\ \tilde{B}^{ij} & \tilde{\mathcal{K}}^{ij,kl} \end{pmatrix} . \tag{3.100}$$

Hence, the theory is degenerate if above matrix is not invertible, i.e., its determinant vanish, which can occur when at least one of eigenvalue is zero. Requiring the determinant of this matrix to vanish, we obtain the condition which can be written in the form

$$D_0(X) + D_1(X)A_*^2 + D_2(X)A_*^4 = 0\,, \tag{3.101}$$

with

$$D_0(X) \equiv -4(\alpha_2+\alpha_1)\left[Xf(2\alpha_1 + X\alpha_4 + 4f_{,X}) - 2f^2 - 8X^2f_{,X}^2\right]\,, \tag{3.102}$$

$$\begin{aligned} D_1(X) \;\equiv\;& 4\left[X^2\alpha_1(\alpha_1+3\alpha_2) - 2f^2 - 4Xf\alpha_2\right]\alpha_4 + 4X^2f(\alpha_1+\alpha_2)\alpha_5 \\ & +8X\alpha_1^3 - 4(f+4Xf_{,X} - 6X\alpha_2)\alpha_1^2 - 16(f+5Xf_{,X})\alpha_1\alpha_2 \\ & +4X(3f-4Xf_{,X})\alpha_1\alpha_3 - X^2f\alpha_3^2 + 32f_{,X}(f+2Xf_{,X})\alpha_2 \\ & -16ff_{,X}\alpha_1 - 8f(f-Xf_{,X})\alpha_3 + 48ff_{,X}^2\,, \end{aligned} \tag{3.103}$$

$$\begin{aligned} D_2(X) \;\equiv\;& 4\left[2f^2 + 4Xf\alpha_2 - X^2\alpha_1(\alpha_1+3\alpha_2)\right]\alpha_5 + 4\alpha_1^3 + 4(2\alpha_2 - X\alpha_3 - 4f_{,X})\alpha_1^2 \\ & +3X^2\alpha_1\alpha_3^2 - 4Xf\alpha_3^2 + 8(f+Xf_{,X})\alpha_1\alpha_3 - 32f_{,X}\alpha_1\alpha_2 + 16f_{,X}^2\alpha_1 \\ & +32f_{,X}^2\alpha_2 - 16ff_{,X}\alpha_3\,. \end{aligned} \tag{3.104}$$

Note that the terms $\hat{A}^2$ in these expressions have already replaced by $X + A_*^2$. Since the determinant have to vanish for any value of A_*, we can obtain the degenerate theories that are characterized by the three degeneracy conditions

$$D_0(X) = 0, \qquad D_1(X) = 0, \qquad D_2(X) = 0\,. \tag{3.105}$$

3.4.5 Classification of Degenerate Theories

The solutions of the above degeneracy conditions can be classified by considering possible solution of $D_0(X) = 0$. The possible solutions of $D_0(X) = 0$ are $\alpha_1 + \alpha_2 = 0$, $Xf(2\alpha_1 + X\alpha_4 + 4f_{,X}) - 2f^2 - 8X^2 f_{,X}^2 = 0$, and $f = 0$ which corresponding to class I, class II and class III of DHOST theories respectively. For classes II and III, the square of the propagation speed of tensor modes and that of the scalar mode have opposite sign, which implies that a gradient instability develops in either the scalar or tensor sector [49]. We only focus on class I, which includes Horndeski and beyond Horndeski and does not suffer from this instability. This class contains four independent functions of ϕ and X. Considering class $\alpha_1 + \alpha_2 = 0$ or $\alpha_1 = -\alpha_2$, we can then use the condition $D_1(X) = 0$ to write α_4 in terms of α_2 and α_3:

$$\begin{aligned}\alpha_4 \;=\; & \frac{1}{8(f + X\alpha_2)^2}\left[16X\alpha_2^3 + 4(3f + 16Xf_{,X})\alpha_2^2 + (16X^2 f_{,X} - 12Xf)\alpha_3\alpha_2\right. \\ & \left. -X^2 f\alpha_3^2 + 16f_{,X}(3f + 4Xf_{,X})\alpha_2 + 8f(Xf_{,X} - f)\alpha_3 + 48ff_{,X}^2\right] . \end{aligned} \tag{3.106}$$

Similarly, the condition $D_2(X) = 0$ yields

$$\alpha_5 = \frac{(4f_{,X} + 2\alpha_2 + X\alpha_3)\left(-2\alpha_2^2 + 3X\alpha_2\alpha_3 - 4f_{,X}\alpha_2 + 4f\alpha_3\right)}{8(f + X\alpha_2)^2} . \tag{3.107}$$

In summary, degenerate theories in this class depend on three arbitrary functions α_2, α_3 and f. In the special case, the theories without dynamics of metric satisfy $\mathcal{A} = 0$. It is required the additional conditions as $\alpha_3 + \alpha_4 = 0$ and $\alpha_5 = 0$. We find the relation

$$4f_{,X} + 2\alpha_2 + X\alpha_3 = 0 . \tag{3.108}$$

This means that α_2 and α_3 are not independent. Theory satisfying these conditions is the GLPV theory by setting

$$f = G_4 , \qquad \alpha_1 = -\alpha_2 = 2G_{4,X} + XF_4 , \qquad \alpha_3 = -\alpha_4 = 2F_4 . \tag{3.109}$$

3.5 Constraints from Gravitational Wave

According to the previous section, the DHOST theories are the most general scalar-tensor theories which are free from Ostrogradsky instability. To avoid the Laplacian instability, theories satisfying class II and III are ruled out. In this section, we will study other constraints on the DHOST theories. Recently, the detection of the gravitational wave emitted from neutron binary stars, shows that the speed of gravitational waves, c_g, is the same as the speed of light, within deviations of order 10^{-15}. This result puts a tight constraint to the DHOST theories. For the quadratic DHOST Lagrangian, the speed of gravitational waves can be computed in the units where $c_{\text{light}} = 1$ from

$$c_{GW}^2 = \frac{G_4}{G_4 - X\alpha_1} \,. \tag{3.110}$$

We denote that f is replaced by G_4 in this section onwards. However, for the cubic DHOST theories, the propagation speed of gravitational wave is background-dependent so that this form of DHOST is ruled out by the result from gravitational wave detection. Hence, we do not consider the cubic DHOST theories in this thesis. To satisfy $c_{GW} = 1$, α_1 has to vanish. From the class I of the DHOST theories which have already discussed, the condition $\alpha_1 = 0$ gives

$$\alpha_1 = \alpha_2 = 0 \,. \tag{3.111}$$

Replacing the above relation into Eqs. (3.103) and (3.104), we respectively obtain

$$\begin{aligned} \alpha_4 &= \frac{1}{8G_4}\left[48G_{4,X}^2 - 8(G_4 - XG_{4,X})\alpha_3 - X^2\alpha_3^2\right] \,, \\ \alpha_5 &= \frac{1}{2G_4}\left(4G_{4,X} + X\alpha_3\right)\alpha_3 \,. \end{aligned} \tag{3.112}$$

We now consider another constraint on the the DHOST theories. It has been shown that the GW in DHOST theories can decay to scalar perturbations which implies that the GW is unstable. To avoid such decay, we require [12]

$$\alpha_3 = 0 \,. \tag{3.113}$$

Inserting the conditions from Eq. (3.113) into Eq. (3.112), we get

$$\alpha_5 = 0\,, \quad \text{and} \quad \alpha_4 = \frac{6G_{4,X}^2}{G_4}\,. \tag{3.114}$$

Hence, the action for quadratic DHOST theories in which the propagation speed of GW is equal to speed of light and the GW do not decay to dark energy perturbations can be written in the form

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\mathcal{L} + S_{\mathcal{M}}\ , \\ &= \int d^4x\sqrt{-g}\left\{G_2 + G_3\Box\phi + G_4 R + \frac{6G_{4,X}^2}{G_4}\phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu\right\} + S_{\mathcal{M}}\ , \end{aligned} \tag{3.115}$$

where $S_{\mathcal{M}} = S_\gamma + S_m$ is the action for the total matter and S_γ is the action for the radiation and S_m is the action for the matter in the universe.

3.6 Scaling Solutions

Since the evolution of dark matter and dark energy are completely unrelated and are at different time scales, it is a puzzle why the density parameter of the dark matter and the dark energy are the same order of magnitude at late time. This is the coincidence problem. To solve this problem, the dark energy and matter should follow the same evolution, at least for some period of time. This demands that the energy density of dark energy is proportional to that of matter such that the ratio ρ_d/ρ_m is constant with time. A solution that leads to the constant ratio of the matter and dark energy densities is a scaling solution. To drive the accelerated expansion of the present universe, the equation of state parameter of dark energy has to be less than $-1/3$. During the scaling regime, ρ_m is no longer scale as a^{-3} but the effective equation of state parameter of matter is negative as the dark energy. To realize such property of ρ_m, one assumes that there is an interaction between the matter and the dark energy.

In the models of the dark energy from classes of the Horndeski theories, there are self-accelerating solutions in which w_d undergoes a tracking solution with constant value. This tracking solution corresponds to $\dot{\phi} \propto H^p$ with p is a constant [38]. For example, the covariant Galileon [42, 45] gives $w_d = -2$ with $p = -1$ during the matter dominated [50, 51], but it does not satisfy the observational data [52]. The extended Galileon proposed in Ref. [53] give the trackig solution which w_d is nearly -1. This model satisfies the observational data [19]. In the beyond Horndeski theories such as the DHOST theories, the tracking solutions can exist for particular models [11, 54], but the general conditions for its existence have been unknown. The existence of the tracking solution could alleviate the coincidence problem because the tracking solution is attractor [3].

In Horndeski theories, there is the scaling solution which is a special kind of tracking solution [55, 13, 56, 57, 58, 59, 60, 21, 61, 23, 62, 63, 64, 65]. In addition to the constant ratio of ρ_d to ρ_m, the scaling solution satisfies $\dot{\phi} \propto H$. If the scalar field has a constant coupling to matter, the scaling solution exists for the cubic-order Horndeski Lagrangian $L = Xg_2(Y) - g_3(Y)\Box\phi$, where g_2, g_3 are arbitrary functions of $Y = Xe^{\lambda\phi}$ (λ is a constant) [66]. In this model, there is ϕMDE which is the scaling solution that can be used to describe the matter dominated epoch. The coincidence problem could be alleviated if the universe can evolve from the radiation dominated epoch through the ϕMDE which should be a saddle point and then reach the attractor corresponding to cosmic acceleration at late time.

The existence of the ϕMDE potentially resolves the H_0 tension as follows. The H_0 tension is the discrepancy of the estimated H_0 from CMB [67] and that from the local measurements of the expansion rate of the universe. The H_0 from CMB data analysis which is based on ΛCDM is lower than that from local measurements by more than 3σ [68]. Hence, to solve the H_0 tension, the dynamics of the universe should be different from that for ΛCDM. The resolutions from modification of the late-time expansion of the universe [69, 70, 71] are tightly constrained by baryon acoustic oscillations (BAO) [72, 73, 74]. Potential resolution of the H_0 tension is based on the modification of the dynamics of the universe during the last scattering epoch and matter domination by early dark energy [75, 76, 77]. In these models, the sound horizon at the last scattering is reduced and therefore the CMB acoustic peaks shift to smaller angular scales. Then the location of the acoustic peaks can shift to the larger angular scales and match with the data when H_0 increases [77].

For coupled dark energy models with ϕMDE, a small fraction of energy density for dark energy during the ϕMDE rises the effective equation of state parameter $w_{\rm eff} = \Omega_\phi w_\phi = \Omega_\phi$ to slightly positive. Here, Ω_ϕ and w_ϕ are the density parameter and equation of state parameter of scalar-field dark energy. The positive effective equation of state parameter during matter domination can also shift the CMB acoustic peaks to smaller angular scales leading to a higher H_0 [25].

Scaling and tracking behaviours for the cosmic evolution are the interesting features arising in some models of dark energy and modified theories of gravity, because they could lead to attractors (stable fixed points) in the phase space of the cosmic evolution which could satisfy the observational constraints [13, 14, 15, 16, 17, 18, 19, 20]. Scaling and tracking solutions in the DHOST theories those satisfy the above two constraint on GW have been discussed. In Chapter IV, we will consider the DHOST theories with scaling solution and we will consider coupled dark energy model from general conformal transformation in Chapter V.

CHAPTER IV

THE DHOST THEORIES WITH SCALING SOLUTION

4.1 The Lagrangians Having Scaling Solution

In this section, we will present the DHOST theories which have scaling solution. Since the scaling solution is behaviour of the background evolution, we concentrate on the Friedmann universe. Starting from the condition for the scaling solution,

$$\frac{\rho_\phi}{\rho_m} = \text{constant}\,, \tag{4.1}$$

where ρ_m is energy density of matter which equation of state parameter, w_m is not necessarily zero. We differentiate the above equation with respect to time t yielding

$$\begin{aligned}\frac{\dot{\rho}_\phi}{\rho_m} - \frac{\rho_\phi \dot{\rho}_m}{\rho_m^2} &= 0\,,\\ \rho_m \dot{\rho}_\phi - \rho_\phi \dot{\rho}_m &= 0\,,\\ \frac{\dot{\rho}_\phi}{\rho_\phi} - \frac{\dot{\rho}_m}{\rho_m} &= 0\,.\end{aligned} \tag{4.2}$$

The conservation equation adding the phenomenological interaction term on the right-hand side can be written as

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -Q\rho_m\dot{\phi}\,, \tag{4.3}$$

where in general Q can be function of ϕ, ρ_ϕ and p_ϕ. Supposing that the scalar field has a direct coupling to matter, and the total energy density of the scalar field and matter is conserved, we have

$$\dot{\rho}_m + 3H(\rho_m + p_m) = Q\rho_m\dot{\phi}\,. \tag{4.4}$$

Dividing Eqs. (4.3) and (4.4) by ρ_ϕ and ρ_m respectively, we obtain

$$\frac{\dot{\rho}_m}{\rho_m} + 3H(1 + w_m) = Q\dot{\phi}\,, \tag{4.5}$$

$$\frac{\dot{\rho}_\phi}{\rho_\phi} + 3H(1 + w_\phi) = -Q\dot{\phi}\,\frac{\Omega_m}{\Omega_\phi}\,. \tag{4.6}$$

Using Eq. (4.2), the above two equations become

$$\begin{aligned} 3H(w_\phi - w_m) &= Q\dot{\phi}\left(\frac{\Omega_m + \Omega_\phi}{\Omega_\phi}\right), \\ \frac{\dot{\phi}}{H} &= \frac{3\Omega_\phi(w_m - w_\phi)}{Q}. \end{aligned} \tag{4.7}$$

If we impose additional condition that w_ϕ is constant during the scaling regime, $\rho_\phi \propto \dot{\phi}^2$. This means that $\rho_\phi/\rho_m \propto \dot{\phi}^2/\rho_m$ is constant. Then from the Friedmann equation we obtain

$$H^2 = \frac{\rho_m}{3}\left(1 + \frac{\rho_\phi}{\rho_m}\right). \tag{4.8}$$

This gives $H^2 \propto \rho_m$ during the scaling regime so that $\rho_\phi/\rho_m \propto \dot{\phi}^2/H^2$. This means that at the scaling regime $\dot{\phi}/H$ is constant. At late time, we ignore the contribution from the radiation. From Eq. (2.75), we obtain

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}(1 + w_\tau), \tag{4.9}$$

where in this case

$$w_\tau = w_\phi \Omega_\phi. \tag{4.10}$$

Since ρ_ϕ/ρ_m and w_ϕ are constant, $w_\tau = w_{eff}$ is also constant during the scaling regime. For convenience, we define

$$\frac{\dot{\phi}}{H} = \frac{2h}{\lambda}, \tag{4.11}$$

where

$$h = -\frac{\dot{H}}{H^2} \quad \text{and} \quad \lambda = -\frac{2hQ}{3\Omega_\phi w_\phi}. \tag{4.12}$$

Hence, at the scaling regime, $\lambda \equiv$ constant. In the DHOST theories Eq. (3.115), the energy-momentum tensor of the scalar field in Eq. (2.6) depend on L_ϕ such that $T_{\alpha\beta} = -g_{\alpha\beta}L_\phi + \cdots$. Hence, ρ_ϕ and p_ϕ depend on $L_\phi + \cdots$. Here, we can write $\mathcal{L}$ in Eq. (3.115) as

$$\mathcal{L} = \mathcal{L}_\phi, \tag{4.13}$$

where $\mathcal{L}_\phi$ is the effective Lagrangian of the scalar field. Following from Eq. (3.115), we can write

$$\mathcal{L}_\phi = G_2 + G_3\Box\phi + G_4 R + A_4 Z, \tag{4.14}$$

where the quantity $Z \equiv \phi^\mu \phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu$ and the function $A_4 \equiv \frac{6G_{4,X}^2}{G_4}$. Since $\rho_\phi \propto p_\phi \propto H^2$ during the scaling regime and $R \propto H^2$ in the Friedmann universe, the existence of the scaling solution can be ensured if we demand that

$$\mathcal{L}_\phi \propto H^2 \,. \tag{4.15}$$

We next study whether some additional conditions are required for the existence of the scaling solution. Since $\mathcal{L}_\phi$ depends on $\phi, X, \Box\phi, R$ and Z, we can write

$$\frac{\dot{\mathcal{L}}_\phi}{H} = -2h\mathcal{L}_\phi \,, \tag{4.16}$$

$$\frac{\partial \mathcal{L}_\phi}{\partial \phi}\frac{\dot{\phi}}{H} + \frac{\partial \mathcal{L}_\phi}{\partial X}\frac{\dot{X}}{H} + \frac{\partial \mathcal{L}_\phi}{\partial \Box\phi}\frac{\dot{\Box\phi}}{H} + \frac{\partial \mathcal{L}_\phi}{\partial R}\frac{\dot{R}}{H} + \frac{\partial \mathcal{L}_\phi}{\partial Z}\frac{\dot{Z}}{H} = -2h\mathcal{L}_\phi \,. \tag{4.17}$$

The terms associated with the time derivatives of ϕ such that $X = \dot{\phi}^2, \Box\phi = -\ddot{\phi} - 3H\dot{\phi}, R = 6(2H^2 + \dot{H})$ and $Z = -\dot{\phi}^2\ddot{\phi}^2$ can be written as

$$\frac{\dot{\phi}}{H} = \frac{2h}{\lambda} \,, \tag{4.18}$$

$$\frac{\dot{X}}{H} = -2hX \,, \tag{4.19}$$

$$\frac{\dot{\Box\phi}}{H} = -2h\Box\phi \,, \tag{4.20}$$

$$\frac{\dot{R}}{H} = -2hR \,, \tag{4.21}$$

$$\frac{\dot{Z}}{H} = -2h(3Z) \,. \tag{4.22}$$

The partial derivative of $\mathcal{L}_\phi$ with respect to $\phi, X, \Box\phi, R$ and Z can be respectively expressed as

$$\frac{\partial \mathcal{L}_\phi}{\partial \phi} = G_{2,\phi} + G_{3,\phi}\Box\phi + G_{4,\phi}R + A_{4,\phi}Z \,, \tag{4.23}$$

$$\frac{\partial \mathcal{L}_\phi}{\partial X} = G_{2,X} + G_{3,X}\Box\phi + G_{4,X}R + A_{4,X}Z \,, \tag{4.24}$$

$$\frac{\partial \mathcal{L}_\phi}{\partial \Box\phi} = G_3 \,, \tag{4.25}$$

$$\frac{\partial \mathcal{L}_\phi}{\partial R} = G_4 \,, \tag{4.26}$$

$$\frac{\partial \mathcal{L}_\phi}{\partial Z} = A_4 \,. \tag{4.27}$$

Substituting Eqs. (4.18)-(4.22) and Eqs. (4.23)-(4.27) into Eq. (4.17), we can obtain the general expression as

$$XG_{,X} - \frac{1}{\lambda}G_{,\phi} + sG = 0\,, \tag{4.28}$$

where s is constant given by

$$s \equiv \begin{cases} 1 & \text{for} \quad G = G_2, \\ 1 & \text{for} \quad G = G_3, G_4, \\ -2 & \text{for} \quad G = A_4\,. \end{cases}$$

To solve Eq. (4.28), we set

$$G(\phi, X) = X^s g(\phi, X)\,. \tag{4.29}$$

Then we get

$$G_{,X} = sX^{s-1}g + X^s g_{,X}\,, \tag{4.30}$$

$$G_{\phi} = X^s g_{,\phi}\,. \tag{4.31}$$

Replacing Eqs. (4.29)-(4.31) into Eq. (4.28), when $x^s \neq 0$ we now obtain

$$Xg_{,X} - \frac{1}{\lambda}g_{,\phi} = 0\,. \tag{4.32}$$

The simplest way to solve the above equation is setting

$$g(\phi, X) = x(X)f(\phi)\,. \tag{4.33}$$

We now obtain

$$g_{,X} = x_{,X}f \quad \text{and} \quad g_{,\phi} = xf_{,\phi}\,. \tag{4.34}$$

Replacing the above partial derivative of g into Eq. (4.32), it reads

$$X\frac{x_{,X}}{x} = \frac{1}{\lambda}\frac{f_{,\phi}}{f}\,, \tag{4.35}$$

$$\frac{d\ln x}{d\ln X} = \frac{1}{\lambda}\frac{f_{,\phi}}{f}\,. \tag{4.36}$$

Easily, Eq. (4.36) can be integrated by setting

$$\frac{d\ln x}{d\ln X} = \frac{1}{\lambda}\frac{f_{,\phi}}{f} = c\,, \tag{4.37}$$

where c is constant. Finally, we obtain the non-trivial solutions of x and f given as

$$x = c_1 X^c \quad \text{and} \quad f = c_2 e^{c\lambda\phi}\,, \tag{4.38}$$

where c_1 and c_2 are constant. One see that

$$g = c_1 c_2 \left(X e^{\lambda\phi}\right)^c\,, \tag{4.39}$$

for arbitrary c. This result comes from a simple calculation. However one can show that solution of Eq. (4.32) can be written in the form

$$g = g(Y)\,, \tag{4.40}$$

where g is an arbitrary function of

$$Y = X e^{\lambda\phi}\,. \tag{4.41}$$

Each coefficients in Eq. (3.115) can be respectively written as

$$G_2(\phi, X) = X g_2(Y)\,, \tag{4.42}$$

$$G_3(\phi, X) = g_3(Y)\,, \tag{4.43}$$

$$G_4(\phi, X) = g_4(Y)\,, \tag{4.44}$$

$$A_4(\phi, X) = X^{-2} a_4(Y)\,. \tag{4.45}$$

Here, the function $a_4(Y)$ is determined from $g_4(Y)$.

4.2 Cosmology in the DHOST Theories with Scaling Solution

4.2.1 Evolution Equations for the Background Universe

To study the evolution of the background universe in the DHOST theories described by the action (3.115), we use FLRW metric for the spatially flat universe in the form

$$ds^2 = -n^2(t)dt^2 + a^2(t)\gamma_{ij}dx^i dx^j\,, \tag{4.46}$$

where $n(t)$ is an auxiliary function which will be set to unity after the evolution equations are obtained. Using the above line element and homogeneity of the scalar field in the background universe, the action (3.115) becomes

$$S = \int dt a^3 n \left\{ G_2 - 6G_{4,\phi} H \frac{\dot{\phi}}{n^2} - 6G_4 \left[\frac{H}{n} + \frac{G_{4,X}}{G_4} \frac{\dot{\phi}}{n^2} \frac{d}{dt} \left(\frac{\dot{\phi}}{n} \right) \right]^2 \right\} + S_{\mathcal{M}} \,, \quad (4.47)$$

where we have set $G_3 = 0$ for simplicity.

Variation of the action (4.47) with respect to n and a yield

$$\begin{aligned} \rho_{\mathcal{M}} \;=\; & E_{00} \equiv \frac{1}{G_4^2} \Big[-G_4 X \left(-6\dot{\phi} \left(-2G_{4,X}^2 \dddot{\phi} - 6HG_{4,X}^2 \ddot{\phi} \right) \right. \\ & +G_4 \left(12 \left(2H^2 + \dot{H} \right) G_{4,X} + 2G_{2,X} \right) + 6G_{4,X}^2 \ddot{\phi}^2 \Big) \\ & +G_4^2 \left(6G_4 H^2 + 6H\dot{\phi} \left(2G_{4,X} \ddot{\phi} + G_{4,\phi} \right) + G_2 \right) \\ & +12X^2 G_{4,X} \ddot{\phi} \left(\left(G_{4,X}^2 - 2G_4 G_{4,XX} \right) \ddot{\phi} - 2G_4 G_{4,\phi X} \right. \\ & \left. +G_{4,X} G_{4,\phi} \right) \Big] \,, \end{aligned} \quad (4.48)$$

and

$$\begin{aligned} -p_{\mathcal{M}} \;=\; & E_{ii} \equiv \frac{1}{G_4} \Big[G_4 \left(4\dot{\phi} \left(G_{4,X} \dddot{\phi} + 2HG_{4,X} \ddot{\phi} + HG_{4,\phi} \right) + 6G_4 H^2 \right. \\ & \left. +4G_4 \dot{H} + 4G_{4,X} \ddot{\phi}^2 + 2G_{4,\phi} \ddot{\phi} + G_2 \right) \\ & +X \left(\left(8G_4 G_{4,XX} - 6G_{4,X}^2 \right) \ddot{\phi}^2 + 8G_4 \ddot{\phi} G_{4,\phi X} + 2G_4 G_{4,\phi\phi} \right) \Big] \,, \end{aligned} \quad (4.49)$$

where $\rho_{\mathcal{M}}$ and $p_{\mathcal{M}}$ are the energy density and pressure of the total matter fluid which is supposed to be perfect fluid. Subscript $_{,XX}$ and $_{,\phi\phi}$ denote the secnd derivatives with respect to X and ϕ while subscript $_{,X\phi}$ denotes the derivatives with respect to both X and ϕ. The quantities, $\rho_{\mathcal{M}}$ and $p_{\mathcal{M}}$, are obtained from variation of the action for the matter with respect to metric. Then Eqs. (4.48) and (4.49) can be combined to eliminate $\dot{H}$ as

$$\begin{aligned} 0 \;=\; & \frac{1}{G_4^2} \Big[G_4 X \left(-6G_4 H^2 G_{4,X} + 6H\dot{\phi} \left(2G_{4,X} G_{4,\phi} - 2G_{4,X}^2 \ddot{\phi} \right) + 6G_{4,X}^2 \ddot{\phi}^2 \right. \\ & \left. +6G_{4,X} G_{4,\phi} \ddot{\phi} - 2G_4 G_{2,X} + 3G_2 G_{4,X} \right) \\ & +G_4^2 \left(6G_4 H^2 + 6H\dot{\phi} \left(2G_{4,X} \ddot{\phi} + G_{4,\phi} \right) + G_2 \right) - G_4 \rho_{\mathcal{M}} \left(G_4 - 3XG_{4,X} w_{\mathcal{M}} \right) \\ & +3X^2 G_{4,X} \left(-2G_{4,X}^2 \ddot{\phi}^2 + 4G_{4,X} G_{4,\phi} \ddot{\phi} + 2G_4 G_{4,\phi\phi} \right) \Big] \,. \end{aligned} \quad (4.50)$$

In the above equation, $w_{\mathcal{M}} \equiv p_{\mathcal{M}}/\rho_{\mathcal{M}}$ is the equation of state parameter of the total matter. Varying the action (4.47) with respect to scalar field ϕ, we get the evolution equation for scalar field which can be written in the form

$$F(\ddddot{\phi}, \dddot{\phi}, \ddot{\phi}, \dot{\phi}, \phi, \ddot{H}, \dot{H}, H) = Q\,, \tag{4.51}$$

where Q is the interaction term arisen from the variation of the matter action $S_{\mathcal{M}}$ with respect to the scalar field ϕ. In principle, the form of the interaction term Q depends on the form of S_m. If S_m does not depend on scalar field ϕ, Q vanishes. We can see later that if Q does not vanish, the universe is accelerated in the scaling regime. For simplicity, we use here the phenomenological form of the interaction term studied in the literature. Hence, we write the function F in the above equation in the form of the conservation equation for the effective energy density of the scalar field as $F \to \dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0$. Then we add the phenomenological interaction term on the right-hand side of the conservation equation as

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -Q\rho_m\dot{\phi}\,, \tag{4.52}$$

where Q is constant, ρ_ϕ and p_ϕ are the effective energy density and the effective pressure of the scalar field ϕ. Supposing that the scalar field has a direct coupling to matter, and the total energy density of the scalar field and matter is conserved, we have

$$\dot{\rho}_m + 3H\rho_m = Q\rho_m\dot{\phi}\,. \tag{4.53}$$

The effective energy density and pressure of the scalar field are defined such that Eqs. (4.48) and (4.49) take the form of the usual Friedmann and acceleration equations when they are written in terms of these effective quantities as $3H^2 = \rho_{\mathcal{M}} + \rho_\phi$ and $2\dot{H} + 3H^2 = -p_{\mathcal{M}} - p_\phi$. The expressions for ρ_ϕ and p_ϕ can be read from Eqs. (4.48) and (4.49) as

$$\rho_\phi \equiv 3H^2 - E_{00}\,, \quad p_\phi \equiv E_{ii} - 2\dot{H} - 3H^2\,. \tag{4.54}$$

From the above expressions, the effective equation of state parameter of the scalar field can be defined as $w_\phi \equiv p_\phi/\rho_\phi$.

Evolution of the background universe can be studied using the dynamical analysis. To compute the autonomous equation describing the evolution of the background universe, we compute the expression for $\dot{H}/H^2$ as follows: differentiating Eq. (4.49) with respect to time, eliminating $\ddddot{\phi}$ from the resulting equation using Eq. (4.51), and then eliminating the remaining $\dddot{\phi}$ terms using Eq. (4.49). Finally, we obtain

$$0 = \tilde{E}_i(\ddot{\phi}, \dot{\phi}, \phi, H, \rho_{\mathcal{M}}, w_{\mathcal{M}})\,. \tag{4.55}$$

Differentiating the above equation with respect to time and eliminating $\dddot{\phi}$ terms using Eq. (4.49), we get

$$\frac{\dot{H}}{H^2} = -h(\ddot{\phi}, \dot{\phi}, \phi, H, \rho_{\mathcal{M}}, w_{\mathcal{M}})\,. \tag{4.56}$$

In this work, we explore features of the scaling solutions in the model described by Eqs. (4.42) and (4.44) by setting

$$G_2 = X\left(\tilde{c}_2 Y^{n_2} - \tilde{c}_6 Y^{n_6}\right)\,, \tag{4.57}$$

$$G_4 = \frac{1}{2} + \tilde{c}_4 Y^{n_4}\,, \tag{4.58}$$

where $\tilde{c}_2$, $\tilde{c}_4$ and $\tilde{c}_6$ are constant and n_2, n_4 and n_6 are constant integer. When the coupling between scalar field and matter is constant, the scaling solutions can give

$$\lambda = -\frac{2hQ}{3\Omega_\phi w_\phi}\,. \tag{4.59}$$

4.2.2 The Autonomous Equations

To compute the autonomous equations from the evolution equations presented in the previous subsection, we define the dimensionless variables as

$$x \equiv \frac{\dot{\phi}}{M_p H}\,, \quad y \equiv \frac{M_p^2 \mathrm{e}^{\frac{-\lambda\phi}{M_p}}}{H^2}\,, \quad z \equiv \frac{\ddot{\phi}}{\dot{\phi} H}\,, \quad \Omega_m \equiv \frac{\rho_m}{3M_p^2 H^2}\,, \quad \Omega_\gamma \equiv \frac{\rho_\gamma}{3M_p^2 H^2}\,, \tag{4.60}$$

where Ω_m and Ω_γ are the density parameter of matter and radiation, respectively. We note that $\Omega_{\mathcal{M}} = \Omega_m + \Omega_\gamma$. For convenience, we normalize the variables x, y and z by their values at scaling fixed point, such that

$$x_r \equiv \frac{x}{x_s}\,, \quad y_r \equiv \frac{y}{y_s}\,, \quad \text{and} \quad z_r \equiv \frac{z}{z_s}\,, \tag{4.61}$$

where subscript $_s$ denotes the quantities at the scaling fixed point, The scaling fixed point in this case is the fixed point that x satisfies the condition in Eq. (4.11) and Q satisfies Eq. (4.59). To compute x_s and z_s, we compute derivative of x with respect to $N \equiv \ln a$ as

$$x' = zx - x\frac{\dot{H}}{H^2}\,, \tag{4.62}$$

which is a possible form of the autonomous equation. Here, a prime denotes derivative with respect to $N = \ln a$. From the condition in Eq. (4.11), we have

$$h_s = \left.\frac{\dot{\phi}\lambda}{2H}\right|_s \equiv \frac{x_\lambda}{2}\,, \tag{4.63}$$

where $x_\lambda \equiv x_s\lambda$. Inserting this solution into Eq. (5.28), we get $z_s = -h_s = -x_\lambda/2$. In terms of dimensionless variables, the constraint equations (4.50) and (4.55) are given by Eqs. (C.2) and (C.5) in the appendix. We see that these constraint equations can be solved for z and Ω_m in terms of x and y. Here we are interested in the evolution of the late-time universe so that we set $\Omega_\gamma = 0$. Hence, the late-time dynamics of the background universe can be described by two dynamical variables $x$and y.

Using definitions of x_r and y_r, we can write the autonomous equations as

$$x'_r = -\frac{x_\lambda z_r x_r}{2} - x_r\frac{\dot{H}}{H^2}\,, \tag{4.64}$$

$$y'_r = -x_\lambda x_r y_r - 2y_r\frac{\dot{H}}{H^2}\,, \tag{4.65}$$

where z_r is computed from the constraint equations which the solutions are shown in Eqs. (C.7)-(C.9). When the autonomous equations are written in these forms, the coupling constant Q in the autonomous equations is always divided by λ so that dynamics of the background universe depend on Q/λ rather than Q. In the numerical integration for the evolution of the universe discussed below, we concentrate on the cases where z_r is the first solution given in Eq. (C.7) to avoid the contributions from the imaginary parts of the solution. We note that the solution that gives $z_r = x_r = y_r = 1$ is not necessarily be the solution in Eq. (C.7) unless $n_4 = \pm 1$. Hence, in our numerical integration for the cosmic evolution, we choose the models where $n_4 = \pm 1$. According to Eq. (4.56),

$\dot{H}/H^2$ also depends on Ω_m. However Ω_m in this expression can be eliminated using the constraint equations Eq. (C.2).

To compute the fixed points of this system, we set x_r, y_r and z_r in the constraint equations Eqs. (C.1) and (C.2) to be unity and then we solve for the parameters as

$$\begin{aligned} c_2 &= -\frac{1}{2\left(2c_4+1\right)^2\left(n_2-n_6\right)}\Big[-6c_4^2\big(-2\left(\Omega_{m\,s}+2n_6\left(x_\lambda-3\right)+x_\lambda-6\right)+2n_4^3x_\lambda^2 \\ &\quad -n_4^2x_\lambda\left(n_6x_\lambda+x_\lambda-6\right)+4n_4\left(x_\lambda-4\right)\big)+6c_4\left(2\Omega_{m\,s}-n_4\left(x_\lambda-4\right)\right. \\ &\quad \left.+2n_6\left(x_\lambda-3\right)+x_\lambda-6\right)-4c_4^3\big(3n_4^3x_\lambda^2-3n_4^2x_\lambda\left(n_6x_\lambda+x_\lambda-6\right) \\ &\quad +6n_4\left(x_\lambda-4\right)-2\left(2n_6\left(x_\lambda-3\right)+x_\lambda-6\right)\big)+3\Omega_{m\,s}+2n_6\left(x_\lambda-3\right) \\ &\quad +x_\lambda-6\Big]\,, \end{aligned} \tag{4.66}$$

$$\begin{aligned} c_6 &= -\frac{1}{2\left(2c_4+1\right)^2\left(n_2-n_6\right)}\Big[6c_4^2\big(2\left(\Omega_{m\,s}+x_\lambda-6\right)-2n_4^3x_\lambda^2+n_4^2\left(x_\lambda-6\right)x_\lambda \\ &\quad -4n_4\left(x_\lambda-4\right)+n_2\left(n_4^2x_\lambda^2+4x_\lambda-12\right)\big)+6c_4\left(2\Omega_{m\,s}-n_4\left(x_\lambda\right.\right. \\ &\quad \left.\left.-4\right)+2n_2\left(x_\lambda-3\right)+x_\lambda-6\right)-4c_4^3\big(3n_4^3x_\lambda^2-3n_4^2\left(x_\lambda-6\right)x_\lambda \\ &\quad +6n_4\left(x_\lambda-4\right)+n_2\left(-3n_4^2x_\lambda^2-4x_\lambda+12\right)-2\left(x_\lambda-6\right)\big)+3\Omega_{m\,s} \\ &\quad +2n_2\left(x_\lambda-3\right)+x_\lambda-6\Big]\,, \end{aligned} \tag{4.67}$$

where $\Omega_{m\,s}$ is Ω_m at the scaling fixed point, and we redefine the coefficients as

$$c_2 \equiv \tilde{c}_2 x_s^2 Y_s^{n_2}\,, \quad c_4 \equiv \tilde{c}_4 Y_s^{n_4}\,, \quad \text{and} \quad c_6 \equiv \tilde{c}_6 x_s^2 Y_s^{n_6}\,. \tag{4.68}$$

We set $h_s = x_\lambda/2$ and $x_r = y_r = 1$ and substitute c_2 and c_6 from Eq. (4.66) and (4.67) into Eq. (4.56) as

$$\frac{x_\lambda}{2} = h(\ddot{\phi},\dot{\phi},\phi,H,\rho_m)|_s = h(x_r,y_r,z_r,\Omega_m)|_s = h(1,1,1,\Omega_{m\,s})\,. \tag{4.69}$$

This relation yields

$$0 = \frac{18c_4\left(2c_4+1\right)^4 n_4\Omega_{m\,s}(Q_\lambda-2)x_\lambda^{13}\left(Q_\lambda x_\lambda+x_\lambda-3\right)}{\lambda^{12}}\,, \tag{4.70}$$

where $Q_\lambda = Q/\lambda$. The interesting conditions required by the above equation are

$$\Omega_{m\,s} = 0\,, \quad Q_\lambda x_\lambda + x_\lambda - 3 = 0\,, \quad \text{or} \quad c_4 = 0\,. \tag{4.71}$$

We can see that $Q_\lambda - 2 = 0$ is the special case of the condition $Q_\lambda x_\lambda + x_\lambda - 3 = 0$. These conditions lead to three classes of fixed point as follows : (1) $Q_\lambda x_\lambda + x_\lambda - 3 = 0$ corresponding to scaling fixed point where Q satisfies Eq. (4.59), (2) $\Omega_{m\,s} = 0$ corresponding to the field dominated point where Q does not necessarily satisfy Eq. (4.59), and (3) $c_4 = 0$ yielding $y_r = 0$ for negative n_4. These fixed points have been found in [38]. The stabilities of these fixed points will be discussed in the next section.

4.2.3 Fixed Points and Stabilities

To investigate stabilities of the fixed points, we linearize the autonomous equations around the fixed point and check the sign of the eigenvalues of the Jacobian matrix defined by

$$J_{ij} = \left.\frac{\partial x_i'}{\partial x_j}\right|_{\text{fixed point}} , \tag{4.72}$$

where $x_i = (x_r, y_r)$.

(a) Scaling Fixed Point

The scaling fixed point corresponds to the condition

$$x_\lambda = \frac{3}{Q_\lambda + 1} . \tag{4.73}$$

From $h_s = x_\lambda/2$, we have

$$w_{\text{eff}} = -\frac{Q_\lambda}{Q_\lambda + 1} . \tag{4.74}$$

We see that if the coupling term disappears, $w_{\text{eff}} = 0$ because for the scaling solution ρ_ϕ/ρ_m is constant. Using the relation $w_{\text{eff}} = \Omega_\phi w_\phi$ and Eq. (4.74), we can compute Ω_ϕ as well as Ω_m at the fixed point if w_ϕ at the fixed point is specified. Inserting the relations for the scaling fixed point into the Jacobian matrix, we obtain the polynomial equation for the eigenvalues of the fixed points. For the sufficiently large c_4, the eigenvalues of the Jacobian matrix depend only on x_λ and given by

$$E_{a\,l} = \left\{\frac{x_\lambda - 6}{2}, 0\right\} . \tag{4.75}$$

Since one of the eigenvalues is zero, the stabilities of this fixed point cannot be determined using the linear stability analysis. Non-linear stability analysis can be performed using the center manifold method, but we will not consider the non-linear analysis in this work. If c_4 is not too large, the eigenvalues of the Jacobian matrix can be written as

$$E_a = \{\mu_1, \mu_2\} \ . \tag{4.76}$$

To describe the accelerated expansion of the late-time universe required by observations, we demand $x_\lambda < 1$. The eigenvalues μ_1 and μ_2 can be computed from the equation

$$a_2\mu^2 + a_1\mu + a_0 = 0 \,, \tag{4.77}$$

where a_2, a_1 and a_0 are complicated functions of $x_\lambda, \Omega_{m\,s}, c_2, c_4, c_6, n_2, n_4$ and n_6. Here,

$$\mu_1 = \frac{x_\lambda - 6}{4}\left(1 - \sqrt{1 + \frac{8a_0}{a_1(x_\lambda - 6)}}\right) \,, \quad \mu_2 = \frac{x_\lambda - 6}{4}\left(1 + \sqrt{1 + \frac{8a_0}{a_1(x_\lambda - 6)}}\right) \,. \tag{4.78}$$

In the above expressions, the relation $a_1/(2a_2) = (6 - x_\lambda)/4$ is used. It follows from the relations for μ_1 and μ_2 that the real part of μ_2 is always negative for $x_\lambda < 6$, while real part of μ_1 can be either negative or positive. Hence, the fixed point is stable when the real part of μ_1 is negative and becomes saddle when the real part of μ_1 is positive. Due to the lengthy expressions of a_0, a_1 and a_2, we compute μ_1 numerically and plot the results as a function of c_4.

The real part of μ_1 for some choices of the parameters is plotted in Fig. (1). In all plots, x_λ and $\Omega_{m\,s}$ are chosen such that w_{eff} satisfies observational constraints. For $\Omega_{m\,s} = 0.3$, we set $x_\lambda = 0.92$ and $x_\lambda = 0.69$ which correspond to $w_\phi = -0.99$ and $w_\phi = -1.10$, respectively. From Fig. (1) and Eq. (4.78), we see that the stabilities of the fixed point depend on x_λ which controls the value of w_{eff} through the relation $x_\lambda = -3(1 + w_{\text{eff}})$ at the fixed points. In the plot, when x_λ decreases, the fixed point of some models, e.g., the models with $n_6 = -1$, can become saddle points. According to Fig. (1), the fixed point is stable for the wide range of c_4 if n_6 is positive. For $n_6 = -3$, the fixed point can be either saddle or stable depending on the value of c_4. From the plot, we see that the real part of μ_1 reaches zero when c_4 is sufficiently large independent of n_2, n_4, n_6 and x_λ.

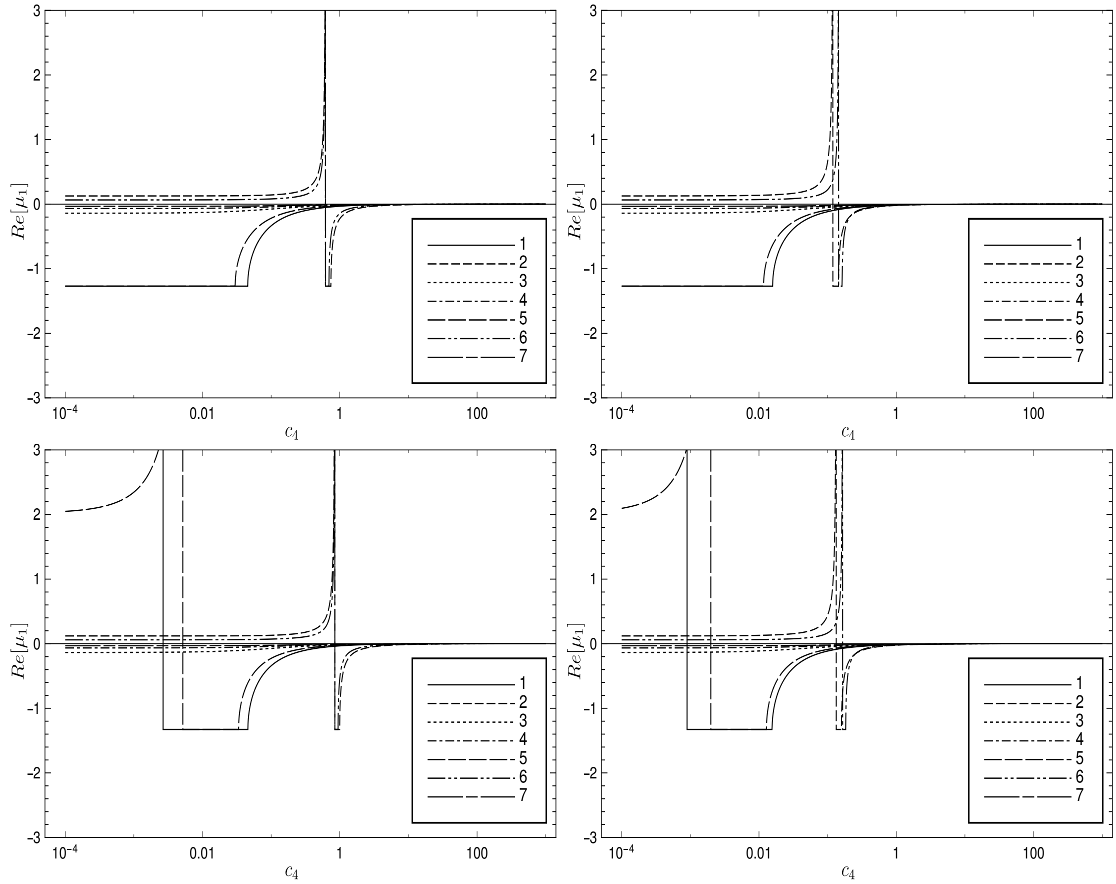

Figure 1 Plots of the real part of μ_1 as a function of c_4. The upper left, upper right, lower left and lower right panels correspond to $(x_\lambda, n_4) = (0.92, -1), (0.92, -2), (0.69, -1)$ and $(0.69, -2)$, respectively. In the plots, lines 1, 2 ,3 , 4, 5, 6 and 7 represent the cases of $(n2, n6)$ = (0,-1), (0,-3), (0,1), (0,3), (1,-1), (1,-3) and (1,3).

(b) Field Dominated Point

In Eq. (4.70), we have shown that $\Omega_m = 0$ is a possible fixed point of the system. To obtain this equation, we set $h = x_\lambda/2$ at the fixed point according to Eq. (4.11). Nevertheless, the condition $h = x_\lambda/2$ can be relaxed if x_r, y_r and z_r are not equal to unity at the fixed point, where the condition $x_r = y_r = z_r = 1$ defines the scaling fixed point. From Eqs. (4.64) and (4.65), we see that the fixed points exist when

$$h = \frac{x_\lambda}{2} z_r = \frac{x_\lambda}{2} x_r \,, \tag{4.79}$$

where the expressions for x_r and z_r at the fixed point can be solved from Eqs. (C.1), (C.2) and (C.5). For the fixed point $\Omega_m = 0$, the expressions for x_r and z_r are complicated and strongly depend on n_2, n_4 and n_6 because Eqs. (C.1), (C.2) and (C.5) contain $x_r^{n_2}$, $x_r^{n_4}$ and $x_r^{n_6}$. However, we can substitute Eq. (4.79) into Eq. (4.9) to obtain

$$w_\phi = w_{\text{eff}} = -1 + \frac{x_\lambda x_{r\,b}}{3} \,, \tag{4.80}$$

where subscript $_b$ denotes evaluation at the field dominated point. We note that for this fixed point there is no any requirement on Q_λ. This follows from Eqs. (4.52) and (4.53) that the effect of the coupling Q disappears when $\Omega_m = 0$. According to this fixed point, the eigenvalues computed from the Jacobian matrix are given by

$$E_b = \{\frac{x_\lambda x_{r\,b} - 6}{2}, x_\lambda x_{r\,b}(Q_\lambda + 1) - 3\} \,. \tag{4.81}$$

It follows from Eq. (4.80) that observational data require $x_\lambda x_{r\,b} < 1$ so that the first eigenvalue in Eq. (4.81) is always negative. We see that if Q_λ does not satisfy Eq. (4.73), the second eigenvalue in Eq. (4.81) is negative when $Q_\lambda < 3/(x_\lambda x_{r\,b}) - 1$ for positive $x_\lambda x_{r\,b}$ and $Q_\lambda > -3/|x_\lambda x_{r\,b}| - 1$ for negative $x_\lambda x_{r\,b}$. These results are the same as in [23]. In the case where Q_λ satisfies Eq. (4.73), one of the eigenvalues vanishes. In this case, the eigenvalues for the field dominated point are similar to those for the scaling fixed point which $c_4 \to \infty$. Since one of the eigenvalues vanishes, we cannot use the linear dynamical analysis to estimate the stabilities of the fixed point. However we will not go beyond

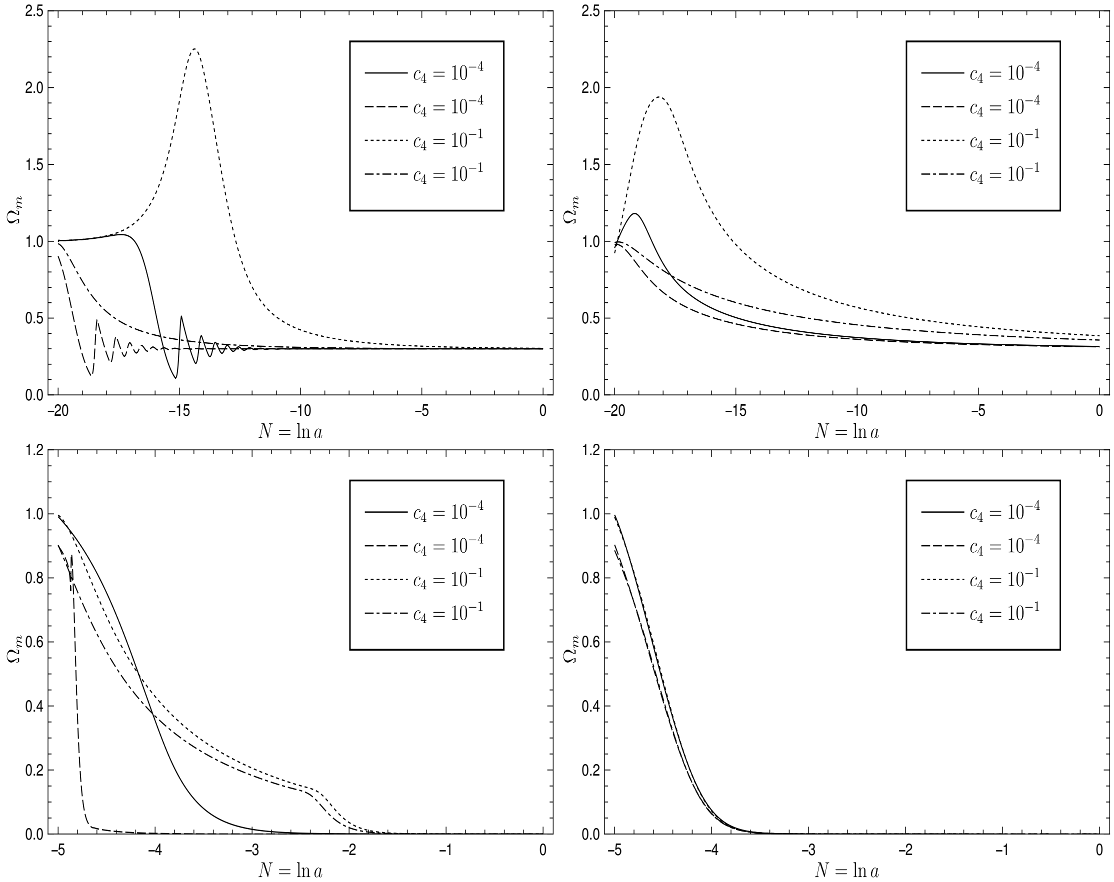

Figure 2 Plots of Ω_m as a function of N. The upper two panels represent the cases $x_r > 0$ during the matter domination, while the lower two panels represent the cases $x_r < 0$ during the matter domination. The left two panels and the right two panels correspond to the model of $(n_2, n_4, n_6) = (0, -1, -1)$ and $(0, -1, 1)$, respectively.

the linear analysis in this work. We check the stabilities of this fixed point by integrating numerically the cosmic evolution as shown in Fig. (2). For a given value of x_λ which could make the field dominated point stable, we can choose n_2, n_4, n_6 and c_4 such that the scaling fixed point is also stable. The question is that the cosmic evolution will reach the scaling fixed point at late time in what situation. Since it is difficult to make the analytical analysis for answering this question, we solve the autonomous equations numerically and plot the evolution of Ω_m in Fig. (2) for some values of the model parameters. According to Fig. (2), the cosmic evolution will reach the scaling fixed point at late time if $x_r > 0$ during the matter domination. For $x_r < 0$ during the matter domination, the cosmic evolution will evolve towards the field dominated point. This result is consequences of a positive x_λ of the scaling points given by Eq. (4.63), and the fact that the evolution of x cannot cross $x = 0$. This implies that although one of the eigenvalues vanishes, the field dominated point can be stable. Since the scaling fixed points we consider in the plots are stable points, these points should be reached for wide ranges of initial conditions. However, if c_4 is large enough and the initial condition for y_r significantly differs from its value at the fixed point, the value of Ω_m can be larger than unity before reaching the fixed point. This implies that Ω_ϕ can be negative, so that the definitions in Eq. (4.54) may cannot be interpreted as the energy density and pressure of dark component. These cases are shown in Fig. (2). We note that in Fig. (2) the numerical integration cannot be started from radiation dominated epoch due to numerical instability. In the top left panel of Fig. (2), the initial values for x_r and y_r during the matter domination for the solid, long-dash, dash, and dash-long-dash lines are $(x_r, y_r) = (0.55, 10^{-5}), (0.05, 0.24), (0.1, 10^{-8})$, and $(0.79, 0.7)$ respectively. In the top right panel of Fig. (2), the initial values for x_r and y_r during the matter domination for the solid, long-dash, dash, and dash-long-dash lines are $(x_r, y_r) = (0.4, 0.2), (0.74, 0.8), (0.18, 0.01)$, and $(0.85, 0.8)$ respectively. For the cases where y_r significantly differs from their values at the fixed point, the maximum value of Ω_m during the cosmic evolution increases when c_4 increases. Since c_4 quantifies the deviation from the Einstein gravity, this suggests that the deviation from the Einstein gravity should not be large to avoid unphysical value of Ω_m during the cosmic evolution. Moreover, even though the initial values of x_r and y_r during the matter domination are in the same order of magnitude of the value at fixed point, the cosmic evolution reaches the fixed point slowly for positive initial x_r compared with the negative initial value of x_r.

(c) $y_r = 0$: ϕMDE Point

According to Eq. (4.70), the other fixed point corresponds to $y_r = 0$. It follows from Eq. (4.65) that $y_r' = 0$ when $y_r = 0$. If we consider Eq. (4.64) in addition, we see that $x_r' = 0$ when $z_r = 2h/x_\lambda$. Here, h for this fixed point is not necessarily equal to $x_\lambda/2$ because x_λ is evaluated at the scaling fixed point (fixed point a). From the definitions of G_2 and G_4 in Eqs. (4.57) and (4.58) as well as the definition of y in Eq. (4.60), we see that the existence of the fixed point $y_r = 0$ requires $n_2 \leq 0, n_6 < 0$ and $n_4 < 0$. Here, we demand that $n_2 \neq n_6$ and $n_4 \neq 0$. Inserting $z_r = 2h/x_\lambda$ and $\Omega_\gamma = 0$ into Eqs. (C.1), (C.2) and (C.5) and then taking the limit $y_r \to 0$, we respectively obtain

$$h|_c = \frac{3 + c_2 x_{r\,c}^2}{2}\,, \quad \Omega_{m\,c} = 1 - \frac{c_2 x_{r\,c}^2}{3} \quad \text{and} \quad x_{r\,c} = -\frac{Q_\lambda x_\lambda}{c_2}\,, \tag{4.82}$$

where the subscript $_c$ denotes evaluation at ϕMDE point. Substituting the above $x_{r\,c}$ into the expression for $\Omega_{m\,c}$, we get

$$\Omega_{m\,c} = 1 - \frac{Q_\lambda^2 x_\lambda^2}{3c_2}\,. \tag{4.83}$$

This equation shows that c_2 has to be positive otherwise $\Omega_{m\,c}$ is larger than unity. The eigenvalues for this fixed point are

$$E_c = \{-\frac{3}{2} + \frac{Q_\lambda^2 x_\lambda^2}{2c_2}, 3 + \frac{Q_\lambda(1+Q_\lambda)x_\lambda^2}{c_2}\}\,. \tag{4.84}$$

These eigenvalues coincide with those in [23]. The first eigenvalue can be written as $-3\Omega_{m\,c}/2$, so that it is always negative. The second eigenvalue becomes positive when $Q_\lambda > 0$ or $Q_\lambda < -1$ for positive c_2. Since x_λ is evaluated at the scaling fixed point, it follows from Eq. (4.73) that $Q_\lambda < 1$ yields $x_\lambda < 0$ corresponding to phantom expansion. We now check how the evolution of the universe can move from this fixed point during matter domination to the scaling fixed point at late time. Let us first consider $x_{r\,c}$ in Eq. (4.82). We can use Eq. (4.73) to write $x_{r\,c} = (x_\lambda - 3)/c_2$. The scaling fixed point can lead to the acceleration of the universe if $x_\lambda < 2$. Hence, $x_{r\,c}$ is negative. Since $x_{r\,c}$ is the value of x_r during matter domination in our consideration, the universe will evolve towards the field dominated fixed point rather than the scaling fixed point as presented in the previous section. For illustration, we plot evolution of Ω_m in Fig. (2). For given values of x_λ, Q_λ, and $\Omega_{m\,c}$, the value of c_2 can be computed from Eq. (4.83). From

Table 1 The models used in the plots. We set $\Omega_{m\,c} = 0.95$ for Model I-IV and $\Omega_{m\,c} = 0.93$ for Model V. The column w_{eff} shows the value of w_{eff} at the field dominated point.

Model	(n_2, n_4, n_6)	Q_λ	$x_{r\,c}$	c_4	w_{eff}
I	(0,-1,-1)	-10	-0.045	7.7	-0.88
II	(0,-1,-1)	2	-0.075	1.7	-1.28
III	(0,-1,-1)	2/3	-0.125	0.67	-1.44
IV	(0,-1,-1)	1/6	-0.49	5.6×10^{-3}	-1.47
V	(0,-1,-2)	2	-0.075	4.0	-1.17

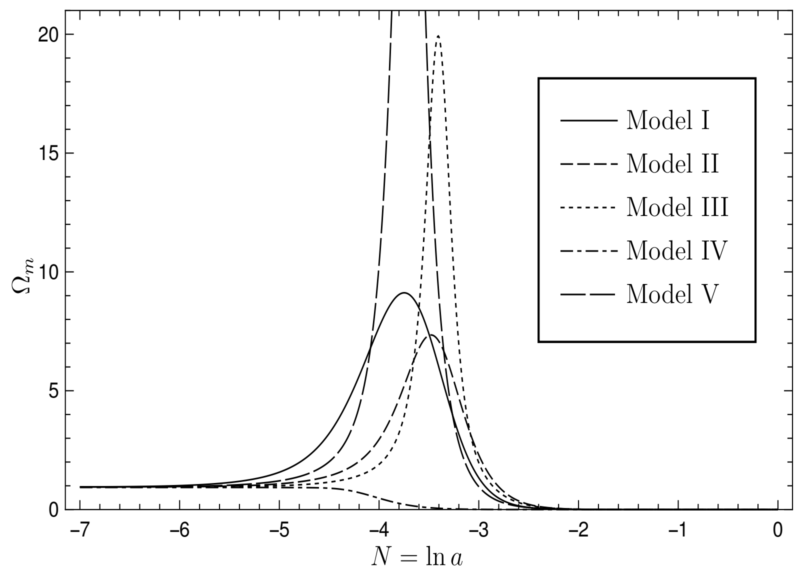

Figure 3 Plots of Ω_m as a function of N for models I-V given in Tab. 1.

the values of x_λ, Q_λ and c_2, we can compute $x_{r\,c}$ from Eq. (4.82) and compute c_4 from Eq. (4.66) by setting $\Omega_{m\,s} = 0.3$. Finally, c_6 can be computed from Eq. (4.67). The models used in the plots are shown in Tab. 1.

From Fig. (3), we see that Ω_m evolves towards the field dominated point for various values of Q_λ which correspond to various w_{eff} at late time. In the plots, we initially set $y_r = 10^{-11}$ according to the ϕMDE point, so that the value of Ω_m can be larger than unity before reaching the field dominated point. However, if c_4 is sufficiently small, e.g., $c_4 = 5.6 \times 10^{-3}$ for model IV, Ω_m can be less than unity through out the evolution of the universe. By definition, c_4 quantifies how large of the deviation from the Einstein gravity. The above results suggest that the deviation from the Einstein gravity should not be large to avoid the case $\Omega_m > 1$ during the cosmic evolution. From the analysis of the Vainshtein mechanism, the bound on the difference between the gravitational constant of the gravitational source and the gravitational coupling for GW gives [78]

$$\left|\frac{XG_{4,X}}{G_4}\right| < \mathcal{O}(10^{-3})\,. \tag{4.85}$$

In terms of c_4, $|XG_{4,X}| = |n_4 c_4|$ at the scaling fixed point. Hence, the small c_4 seems to agree with the above bound.

CHAPTER V

COUPLED DARK ENERGY MODEL FROM GENERAL CONFORMAL TRANSFORMATION

5.1 The Model

In this section, we consider the general conformal transformation which is defined by

$$\bar{g}_{\mu\nu} = C(X,\phi) g_{\mu\nu} \,, \quad \text{and} \quad \bar{g}^{\mu\nu} = \frac{1}{C(X,\phi)} g^{\mu\nu} \,, \tag{5.1}$$

where $C(X,\phi)$ is the coefficient of the conformal transformation depending on the scalar field ϕ and its kinetic term $X \equiv -\nabla_\mu\phi\nabla^\mu\phi/2$. Using this form of the conformal transformation, the Einstein-Hilbert action is transformed as the action of DHOST theories in Eq. (3.115) [79, 12]. In order to construct the coupled dark energy model inspired from the conformal transformation, we suppose that the dark energy is in the form of a scalar field ϕ involving the conformal transformation. Therefore the interaction between the dark energy and the dark matter arises when the Lagrangian of the dark matter depends on the metric $\bar{g}_{\mu\nu}$ defined in Eq. (5.1). Hence, the model of coupled dark energy can be described by the action in which the gravitational part of the action is written in terms of the metric $g_{\mu\nu}$ while the part of the coupled matter is written in terms of $\bar{g}_{\mu\nu}$ as

$$S = \int d^4x \Big[\sqrt{-g}\Big(\frac{1}{2}R + P(X,\phi) + \mathcal{L}_{\mathcal{M}}(g_{\mu\nu})\Big) + \sqrt{-\bar{g}}\mathcal{L}_m(\bar{g}_{\mu\nu},\psi,\psi_{,\mu})\Big] \,, \tag{5.2}$$

where $P(X,\phi) \equiv X - V(\phi)$ and $\mathcal{L}_{\mathcal{M}}$ is the ordinary matter Lagrangian including baryon and radiation, $\mathcal{L}_m$ is the dark matter Lagrangian, ψ is the matter field and $\psi_{,\mu}$ is the partial derivative of the field. Using the variational method with respect to $g_{\alpha\beta}$, we obtain the Einstein equation in the form

$$G^{\alpha\beta} = T^{\alpha\beta}_\phi + T^{\alpha\beta}_m + T^{\alpha\beta}_{\mathcal{M}} \,, \tag{5.3}$$

where $G^{\alpha\beta}$ is the Einstein tensor computed from $g_{\mu\nu}$, and the energy-momentum tensors for scalar field and matter are defined in unbarred frame as

$$T_{\phi}^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}P(\phi,X))}{\delta g_{\mu\nu}}\,, \quad T_{\mathcal{M}}^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathcal{M}})}{\delta g_{\mu\nu}}\,, \tag{5.4}$$

$$T_{m}^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-\bar{g}}\mathcal{L}_{m}\right)}{\delta g_{\mu\nu}}\,. \tag{5.5}$$

From these definitions of the energy-momentum tensor and $\nabla_\alpha G^{\alpha\beta} = 0$ as well as the conservation of the energy-momentum tensor for the ordinary matter, we have $\nabla_\alpha(T_\phi^{\alpha\beta} + T_m^{\alpha\beta}) = 0$. Here, ∇_α is the covariant derivative compatible with the metric $g_{\alpha\beta}$. Since the dark matter Lagrangian depends on field ϕ, the energy-momentum tensors of dark energy and dark matter do not separately conserve. From the action (5.2), we see that the metric tensor does not depend on ψ. Hence, variation of the action (5.2) with respect to ψ yields

$$\bar{\nabla}_\alpha \bar{T}_m^{\alpha\beta} = 0\,, \tag{5.6}$$

where $\bar{\nabla}_\alpha$ is defined from barred metric. This implies that $\bar{T}_m^{\alpha\beta}$ conserves in the barred frame. The relation of energy-momentum tensor between the barred frame and the unbarred frame defined in Eq. (5.5) can be written as

$$T_m^{\alpha\beta} = \frac{\sqrt{-\bar{g}}}{\sqrt{-g}}\frac{\partial \bar{g}_{\rho\sigma}}{\partial g_{\alpha\beta}}\frac{2}{\sqrt{-\bar{g}}}\frac{\delta\left(\sqrt{-\bar{g}}\mathcal{L}_m\right)}{\delta \bar{g}_{\rho\sigma}} = \frac{\sqrt{-\bar{g}}}{\sqrt{-g}}\frac{\partial \bar{g}_{\rho\sigma}}{\partial g_{\alpha\beta}}\bar{T}_m^{\rho\sigma}\,. \tag{5.7}$$

Varying the action (5.2) with respect to the field ϕ, we obtain the evolution equation for scalar field as

$$\nabla_\alpha\nabla^\alpha\phi - V_{,\phi} + Q = 0\,, \tag{5.8}$$

where Q is the coupling term coming from variation of the dark matter action $\int d^4x\sqrt{-\bar{g}}\mathcal{L}_m$ in Eq. (5.2) with respect to ϕ. The variation of this part of the action can be computed as

$$\delta\int d^4x\sqrt{-\bar{g}}\mathcal{L}_m = \int d^4x\,\delta\phi\left\{\frac{\sqrt{-\bar{g}}}{2}\bar{T}_m^{\alpha\beta}C_{,\phi}g_{\alpha\beta} + \frac{1}{2}\nabla_\sigma\left(\sqrt{-\bar{g}}\bar{T}_m^{\alpha\beta}g_{\alpha\beta}C_{,X}\phi^\sigma\right)\right\}\,. \tag{5.9}$$

Using Eq. (5.7), we have

$$\begin{aligned}\sqrt{-g}T_m^{\alpha\beta} &= \left(C\delta_\rho^\alpha\delta_\sigma^\beta - \frac{1}{2}C_{,X}\phi^\alpha\phi^\beta g_{\rho\sigma}\right)\sqrt{-\bar{g}}\bar{T}_m^{\rho\sigma}\,,\\ &= C\sqrt{-\bar{g}}\bar{T}_m^{\alpha\beta} - \frac{1}{2}C_{,X}\phi^\alpha\phi^\beta\sqrt{-\bar{g}}g_{\rho\sigma}\bar{T}_m^{\rho\sigma}\,.\end{aligned} \tag{5.10}$$

Contracting $g_{\alpha\beta}$ to the both sides of the above equation, and setting $T_m \equiv g_{\alpha\beta}T_m^{\alpha\beta}$, The above equation can be written as

$$\sqrt{-g}T_m = (C + C_{,X}X)\sqrt{-\bar{g}}g_{\rho\sigma}\bar{T}_m^{\rho\sigma}\,, \tag{5.11}$$

which yields

$$\sqrt{-\bar{g}}g_{\alpha\beta}\bar{T}_m^{\alpha\beta} = \frac{\sqrt{-g}T_m}{C + C_{,X}X}\,. \tag{5.12}$$

Substituting the above relation into Eq. (5.9), we obtain

$$\begin{aligned} &\delta\int d^4x\sqrt{-\bar{g}}\mathcal{L}_m \\ &= \int d^4x\sqrt{-g}\,\delta\phi\left\{\frac{C_{,\phi}}{2\,(C + C_{,X}X)}T_m + \frac{1}{2}\nabla_\alpha\left(\frac{C_{,X}}{C + C_{,X}X}\phi^\alpha T_m\right)\right\}. \end{aligned} \tag{5.13}$$

Combining the above equation with Eq. (5.8), we obtain

$$\nabla_\alpha\nabla^\alpha\phi - V_{,\phi} = -\Gamma T_m - \nabla_\alpha\left(\Xi\phi^\alpha T_m\right) \equiv -Q\,, \tag{5.14}$$

where $\Gamma \equiv C_{,\phi}/[2(C + C_{,X}X)]$and $\Xi \equiv C_{,X}/[2(C + C_{,X}X)]$. Multiplying ϕ_β to both sides of th above equation, we can obtain the equation in the form as

$$\nabla_\alpha T^\alpha_{\beta\,\phi} = -\Gamma\phi_\beta T_m - \nabla_\alpha\left(\Xi\phi^\alpha T_m\right)\phi_\beta \equiv -Q\phi_\beta\,, \tag{5.15}$$

where $T^\alpha_{\beta\,\phi}$ is the energy-momentum tensor of the scalar field. Hence, the general conformal transformation induces the coupling term between dark energy and dark matter in the form

$$Q = \Gamma T_m + \nabla_\alpha\left(\Xi\phi^\alpha T_m\right)\,. \tag{5.16}$$

According to the conservation of the total energy-momentum tensor, Eq. (5.15) gives

$$\nabla_\alpha T^\alpha_{\beta\,m} = Q\phi_\beta\,. \tag{5.17}$$

In the case that we consider, if the conformal coefficient C depends only on the field ϕ, then $C_{,X}$ vanishes. Therefore Eq. (5.14) reduces to the equation for the case of usual conformal transformation. When $C_{,X}$ does not vanish, the coupling term Q contains coupling between the field derivative and the energy density as well as between the field derivative and the derivative of energy density of CDM. The latter form of the coupling can lead to different effects on cosmic evolution compared with the usual conformal coupling case.

5.2 Evolution of the Background Universe

In this section, The effects of the interaction between dark energy and dark matter due to the general conformal transformation are studied on the evolution of the background universe. Using the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, we have

$$ds^2 = -dt^2 + a^2\delta_{ij}dx^i dx^j \,. \tag{5.18}$$

Supposing that the scalar field is homogeneous and other matter components in the universe are described by perfect fluid, Eqs. (5.14) and (5.17) become

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = \bar{Q} \,, \tag{5.19}$$

and

$$\dot{\rho}_m + 3H\rho_m = -\bar{Q}\dot{\phi} \,, \tag{5.20}$$

where $\bar{Q}$ is the coupling term evaluated at the the background universe. In the following calculation, we consider the case where matter include co-dark matter and baryon. The energy density of dark matter is denoted by ρ_m and the coupling term in Eq. (5.16) becomes

$$\bar{Q} = -\Gamma\rho_m + \left(\ddot{\phi} + 3H\dot{\phi}\right)\Xi\rho_m + 2\Xi_{,\phi}X\rho_m + 2\Xi_{,X}\ddot{\phi}X\rho_m + \Xi\dot{\phi}\dot{\rho}_m \,. \tag{5.21}$$

We see that the interaction term $\bar{Q}$ in the above equation depends on $\ddot{\phi}$ and $\dot{\rho}_m$. Hence, we combine Eqs. (5.19) and (5.20) to write the evolution equations for ϕ and ρ_m in the forms

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = Q_0 \,, \quad \text{and} \quad \dot{\rho}_m + 3H\rho_m = -\dot{\phi}Q_0 \,, \tag{5.22}$$

where Q_0 is the interaction term which has already eliminated $\ddot{\phi}$ and $\dot{\rho}_m$. The interaction term Q_0 can be written as

$$Q_0 = \frac{(\Theta V_{,\phi} + 3H\Theta\dot{\phi} - 2X\Xi_{,\phi} + \Gamma)\rho_m}{\Theta\rho_m - 2X\Xi - 1}\,, \tag{5.23}$$

where $\Theta \equiv \Xi + 2X\Xi_{,X}$. Since the energy-momentum tensors of baryon and radiation separately conserve, the conservation of these energy-momentum tensors in the background universe yields

$$\dot{\rho}_b = -3H\rho_b\,, \quad \text{and} \quad \dot{\rho}_r = -4H\rho_r\,, \tag{5.24}$$

where ρ_b and ρ_r are the energy density of baryon and radiation.

5.2.1 Autonomous Equations

To compute the autonomous equations, we define the dimensionless dynamical variables as

$$\begin{aligned} x &= \frac{\dot{\phi}}{\sqrt{6}H}, \quad y = \frac{V}{3H^2}, \quad \Omega_m = \frac{\rho_m}{3H^2}\,, \\ \Omega_b &= \frac{\rho_b}{3H^2}\,, \quad \Omega_r = \frac{\rho_r}{3H^2}\,, \end{aligned} \tag{5.25}$$

and the dimensionless functions as

$$\begin{aligned} &z = \frac{C_{,X}}{C}H^2\,, \quad \lambda = \frac{V_{,\phi}}{V}\,, \\ &\gamma = \Gamma\,, \quad \chi = \Xi H^2\,. \end{aligned} \tag{5.26}$$

In terms of the above dimensionless variables, the Friedmann equation gives

$$1 = x^2 + y + \Omega_m + \Omega_b + \Omega_r\,. \tag{5.27}$$

Using the definitions of x, y, z and Ω_m from Eq. (5.22), we obtain autonomous equations as

$$
\begin{aligned}
x' =& - x\frac{\dot{H}}{H^2} \\
&+ \frac{\begin{bmatrix} \sqrt{6}\left(\gamma\Omega_m + \lambda y\right) + 6x + 3\sqrt{6}\left(\gamma z\Omega_m - 2\Omega_m\chi_{,\phi} + 2\lambda yz\right)x^2 \\ +36zx^3 - 18\sqrt{6}z\Omega_m\chi_{,\phi}x^4 \end{bmatrix}}{36x^2\Omega_m\left(3x^2z+1\right)\chi_{,X} + 3z\Omega_m - 12x^2z - 2}\,, \qquad (5.28)\\
y' =& \sqrt{6}\lambda xy - 2y\frac{\dot{H}}{H^2}\,, \qquad (5.29)\\
z' =& 6x\left(\frac{C_{,X}}{C}\right)_{,X}\left(x\frac{\dot{H}}{H^2} + x'\right) + \sqrt{6}x\left(\frac{C_{,X}}{C}\right)_{,\phi} + 2z\frac{\dot{H}}{H^2}\,, \qquad (5.30)\\
\Omega_m' =& - 2\Omega_m\frac{\dot{H}}{H^2} \\
&- \frac{36x^3(3x^2z+1)(6x+\sqrt{6}\lambda y)\chi_{,X} + 3\sqrt{6}\lambda xyz - 2\sqrt{6}\gamma x(3x^2z+1)}{36x^2\Omega_m\left(3x^2z+1\right)\chi_{,X} + 3z\Omega_m - 12x^2z - 2}\Omega_m \\
&+ \frac{12\sqrt{6}x^3\left(3x^2z+1\right)\chi_{,\phi} + 18x^2z + 6}{36x^2\Omega_m\left(3x^2z+1\right)\chi_{,X} + 3z\Omega_m - 12x^2z - 2}\Omega_m \\
&- \frac{9\left[12\left(3x^4z + x^2\right)\chi_{,X} + z\right]}{36x^2\Omega_m\left(3x^2z+1\right)\chi_{,X} + 3z\Omega_m - 12x^2z - 2}\Omega_m^2\,, \qquad (5.31)
\end{aligned}
$$

where a prime denotes a derivative with respect to $N \equiv \ln a$ and

$$\frac{\dot{H}}{H^2} = \frac{1}{2}\left(\Omega_m - 2x^2 + 4y - 4\right)\,. \qquad (5.32)$$

From Eq. (5.24), we get

$$
\begin{aligned}
\Omega_b' &= -3\Omega_b - 2\frac{\dot{H}}{H^2}\Omega_b\,, \qquad (5.33)\\
\Omega_r' &= -4\Omega_r - 2\frac{\dot{H}}{H^2}\Omega_r\,. \qquad (5.34)
\end{aligned}
$$

Considering the denominator of all terms except the first term which is proportional to $\dot{H}/H^2$ in Eqs. (5.28) and (5.31). The denominators are the same and can vanish when

$$\chi_{,X} = -\frac{3z\Omega_m - 12x^2z - 2}{36x^2\Omega_m\left(3x^2z+1\right)}\,. \qquad (5.35)$$

This suggests that x' and Ω_m' can be infinite when the above equation is satisfied. To ensure that the background universe properly evolves, we have to avoid the situations in which the above equation is satisfied.

To perform further analysis, we choose the potential of the scalar field and coefficient C in the forms

$$V(\phi) = V_0 \mathrm{e}^{\lambda\phi} , \quad C(\phi, X) = C_0 \mathrm{e}^{\lambda_1 \phi} \left[1 + \mathrm{e}^{\lambda_2 \phi} \left(\frac{X}{\Lambda_0} \right)^{\lambda_3} \right] , \tag{5.36}$$

where $C_0, \lambda_1, \lambda_2$ and λ_3 are dimensionless constants, while V_0 and Λ_0 are constants with the same dimension as X. According to Eq. (5.26), inserting this form of C into Eq. (5.35), we obtain

$$\lambda_3 = \frac{27x^2z^2\Omega_m + 3z\Omega_m + 36x^4z^2 + 18x^2z + 2}{6z\Omega_m} . \tag{5.37}$$

For the case of positive λ_3, we get $z > 0$ according to the definition in Eq. (5.26). This suggests that the above equation can be satisfied if $\lambda_3 > 0$. This implies that x' and Ω'_m can be infinite at some time during the evolution of the universe if λ_3 is positive. Based on the numerical investigation, the divergence of x' and Ω'_m can be avoided if $\lambda_3 < 1$.

5.2.2 Fixed Points

In the dynamical analysis, the contribution from the radiation energy density is ignored because we focus on the fixed points corresponding to the matter-dominated epoch and the late-time accelerating universe. Since Eq. (5.33) has fixed points at $\Omega_b = 0$ and at $\dot{H}/H^2 = -3/2$, we also drop the contribution from baryon. The first fixed point can be reached in the future while the second fix point involves the matter dominated epoch. Since the ϕMDE requires $\dot{H}/H^2 = -3(1 + w_{\text{eff}})/2 \lesssim -3/2$ during matter domination, the second fixed point is not exactly compatible with ϕMD. Hence, in order to study the ϕMDE point in the dynamical analysis, the contribution from the baryon energy density is dropped . However, we will show the numerical integration that the inclusion of baryon energy density does not obstruct the existence of ϕMDE, because we still get $\Omega'_b \sim 0$ when $\dot{H}/H^2 \lesssim -3/2$.

Since we ignore contributions from radiation and baryon energy density, Eq. (5.27) becomes

$$\Omega_m = 1 - x^2 - y . \tag{5.38}$$

Substituting this equation into Eq. (5.32), we obtain

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}\left(x^2 - y + 1\right) . \tag{5.39}$$

At fixed point we set $y' = 0$, Eq. (5.29) gives two solutions corresponding to the fixed points $y_c = 0$ and

$$\frac{\dot{H}}{H^2} = \sqrt{\frac{3}{2}}\lambda x_c\,, \tag{5.40}$$

where the subscript $_c$ denotes the evolution at the fixed point.

(a) Field Dominated Point and Scaling Point

Let us first consider the fixed point $y_c \neq 0$. We match Eq. (5.39) with Eq. (5.40). Therefore we obtain

$$y_c = \sqrt{\frac{2}{3}}\lambda x_c + x_c^2 + 1\,. \tag{5.41}$$

Inserting C from Eq. (5.36) together with Eq. (5.38), (5.40) and (5.41) into Eqs. (5.28) and (5.30), we obtain the following equations for the fixed points after setting $x' = z' = 0$,

$$\begin{aligned} 0 = &-\sqrt{6}\lambda\lambda_3 + \left(\lambda\lambda_1 - 2\left(\lambda^2+3\right)\right)\lambda_3 x_c + \sqrt{6}\lambda_3\left(\lambda_1 - \lambda(9z_c+2)\right)x_c^2 \\ &+ 3\left(-2\lambda^2\lambda_3^2 + \left(-5\lambda^2 + \lambda_1\lambda - 2\lambda_2\lambda - 18\right)\lambda_3 + \lambda\lambda_2\right)z_c x_c^3 \\ &+ 3\sqrt{6}z_c\left(-2\lambda_3\lambda_2 + \lambda_2 + \lambda_3\left(\lambda_1 - \lambda\left(2\lambda_3 + 6z_c + 5\right)\right)\right)x_c^4 \\ &- 9\left(\left(\lambda^2+12\right)\lambda_3 - 3\lambda\lambda_2\right)z_c^2 x_c^5 + 9\sqrt{6}\left(3\lambda_2 - \lambda\lambda_3\right)z_c^2 x_c^6\,, \end{aligned} \tag{5.42}$$

$$0 = \frac{\sqrt{6}}{\lambda_3}\left(\lambda_2 + \lambda\lambda_3\right)x_c z_c\left(\lambda_3 - 3x_c^2 z_c\right) . \tag{5.43}$$

Solving Eq. (5.43), we obtain the solutions for z_c as

$$z_c = 0 \quad \text{and} \quad z_c = \frac{\lambda_3}{3x_c^2}\,. \tag{5.44}$$

Since the $z_c = 0$ solution corresponds to the case where the kinetic dependence of C is negligible, i.e., $z = C_{,X}/C = 0$, we focus only on the second solution. Besides, the condition $\lambda_2 + \lambda\lambda_3 = 0$ is also the solution of Eq. (5.43). However, we will not discuss this case in detail.

Substituting the second fixed point of z of Eq. (5.44) into Eq. (5.42), we get two fixed points of variable x as

$$x_c = \left\{ -\frac{\lambda}{\sqrt{6}}, \frac{\sqrt{6}\,(2\lambda_3+1)}{\lambda_1+\lambda_2-\lambda\,(3\lambda_3+2)} \right\} . \tag{5.45}$$

Substituting two fixed points of x_c from above equation into Eq. (5.41), we obtain two fixed points of variable y as

$$y_c = \left\{ 1-\frac{\lambda^2}{6}, 1+\frac{6\,(2\lambda_3+1)^2}{(\lambda_1+\lambda_2-\lambda\,(3\lambda_3+2))^2}+\frac{2\lambda\,(2\lambda_3+1)}{\lambda_1+\lambda_2-\lambda\,(3\lambda_3+2)} \right\} . \tag{5.46}$$

Substituting x_c and y_c into definitions $\Omega_\phi \equiv x^2+y$ and $w_\phi \equiv (x^2-y)/\Omega_\phi$, we obtain the density parameter and equation of state of scalar field at the fixed as

$$\Omega_{\phi c} = \left\{ 1, \frac{12\,(2\lambda_3+1)^2}{(\lambda_1+\lambda_2-\lambda\,(3\lambda_3+2))^2}+\frac{2\lambda\,(2\lambda_3+1)}{\lambda_1+\lambda_2-\lambda\,(3\lambda_3+2)}+1 \right\} , \tag{5.47}$$

$$w_{\phi c} = \left\{ \frac{1}{3}\left(\lambda^2-3\right), -\frac{\lambda_1+\lambda_2+\lambda\lambda_3}{(\lambda_1+\lambda_2-\lambda\,(3\lambda_3+2))\,\sigma} \right\} , \tag{5.48}$$

where

$$\sigma = \left(\frac{12\,(2\lambda_3+1)^2}{(\lambda_1+\lambda_2-\lambda\,(3\lambda_3+2))^2}+\frac{2\lambda\,(2\lambda_3+1)}{\lambda_1+\lambda_2-\lambda\,(3\lambda_3+2)}+1 \right) . \tag{5.49}$$

We see that the first pair of (x_c, y_c) corresponds to the field dominated point, while the second pair corresponds to the scaling point. For the case of field dominated point, λ can be expressed in terms of $w_{\phi c}$ as

$$\lambda = \sqrt{3(w_{\phi c}+1)} , \tag{5.50}$$

which is the same as that for the field-dominated solution for uncoupled quintessence with exponential potential. For the case of the scaling point, From Eq (5.47) and (5.48) we can solve for λ and λ_1 and write them in terms of $\Omega_{\phi c}$ and $w_{\phi c}$ as

$$\lambda = \mp\frac{\sqrt{3}(w_{\phi c}\Omega_{\phi c}+1)}{\sqrt{(w_{\phi c}+1)\,\Omega_{\phi c}}} , \tag{5.51}$$

$$\lambda_1 = -\lambda_2 \pm \frac{\sqrt{3}\,(-3\lambda_3 w_{\phi c}\Omega_{\phi c}-2w_{\phi c}\Omega_{\phi c}+\lambda_3)}{\sqrt{(w_{\phi c}+1)\,\Omega_{\phi c}}} . \tag{5.52}$$

From the above equations, we see that the values of λ and λ_1 can be computed from the values of λ_2, λ_3, $w_{\phi c}$ and $\Omega_{\phi c}$. Based on observational constraints, we can be specify the values of $w_{\phi c}$ and $\Omega_{\phi c}$, i.e., if we suppose that the scaling point corresponds to the late-time universe, we can set $w_{\phi c} = -0.99$ and $\Omega_{\phi c} = 0.7$. This suggests that to perform further analysis, we need to specify only the parameters λ_2 and λ_3 instead of all parameters of the model λ, λ_1, λ_2 and λ_3. Therefore, we can exclude the cases where the fixed points do not correspond to the observational constraints in our analysis. Substituting λ and λ_1 from the above equations into Eqs. (5.45) and (5.46), we obtain x_c and y_c in terms of $w_{\phi c}$ and $\Omega_{\phi c}$ as

$$x_c = \pm\sqrt{\frac{1}{2}\Omega_{\phi c}\left(1+w_{\phi c}\right)} \quad \text{and} \quad y_c = \frac{1}{2}\Omega_{\phi c}\left(1-w_{\phi c}\right) . \tag{5.53}$$

(b) Kinetic Dominated Point and ϕMDE Point

We now consider the fixed point $y_c = 0$. Then Eqs. (5.38) and (5.39) respectively give

$$\Omega_m = 1 - x^2 , \tag{5.54}$$

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}\left(x_c^2 + 1\right) . \tag{5.55}$$

Substituting $y_c = 0$ and the above two equations into Eqs. (5.28) and (5.30) and performing the same steps as those for Eqs. (5.42) and (5.43), we obtain

$$\begin{aligned} 0 &= \left(1-x_c^2\right)\Big[\sqrt{6}\lambda_1\lambda_3 + 3\lambda_3\left(\left(6\lambda_3-3\right)z_c+2\right)x_c \\ &\quad +3\sqrt{6}\left(-2\lambda_3\lambda_2+\lambda_2+\lambda_1\lambda_3\right)z_c x_c^2 + 9\lambda_3 z_c\left(2\lambda_3-9z_c+5\right)x_c^3 \\ &\quad +27\sqrt{6}\lambda_2 z_c^2 x_c^4 + 27\lambda_3 z_c^2 x_c^5\Big] , \qquad (5.56) \\ 0 &= -\frac{1}{\lambda_3} z_c\left(3\lambda_3\left(x_c^2+1\right) - \sqrt{6}\lambda_2 x_c\right)\left(\lambda_3 - 3x_c^2 z_c\right) . \qquad (5.57) \end{aligned}$$

In the following consideration, we denote that a superscript $^{(\phi)}$ is used for the quantities corresponding to the fixed point $y_c = 0$. This fixed point will play a role of ϕMDE in the subsequent consideration. From Eq. (5.57), we can solve for z at the fixed point as

$$z_c^{(\phi)} = 0 \quad \text{and} \quad z_c^{(\phi)} = \frac{\lambda_3}{3(x_c^{(\phi)})^2} . \tag{5.58}$$

These solutions of $z_c^{(\phi)}$ are similar to the case of scaling point. Inserting the second solution for $z_c^{(\phi)}$ into Eq. (5.56), we can solve for $x_c^{(\phi)}$ as

$$x_c^{\text{kinetic}} = \pm 1\,, \quad \text{and} \quad x_c^{(\phi)} = -\frac{\lambda_1+\lambda_2}{\sqrt{6}\,(3\lambda_3+2)} \mp \frac{\sqrt{\lambda_1^2+2\lambda_2\lambda_1+\lambda_2^2+6\lambda_3\,(3\lambda_3+2)}}{\sqrt{6}\,(3\lambda_3+2)}\,. \tag{5.59}$$

The first two solutions correspond to kinetic-dominated points, while the other solutions correspond to ϕMDE points. Inserting $x_c^{(\phi)}$ into the definition of Ω_ϕ in Eq. (5.54), we obtain expression for Ω_ϕ at $y_c = 0$ in the form

$$\Omega_{\phi c}^{(\phi)} = \Bigg[1, 1, \frac{\left(\lambda_1+\lambda_2+\sqrt{\lambda_1^2+2\lambda_2\lambda_1+\lambda_2^2+6\lambda_3\,(3\lambda_3+2)}\right)^2}{6(3\lambda_3+2)^2}, \frac{\left(\lambda_1+\lambda_2-\sqrt{\lambda_1^2+2\lambda_2\lambda_1+\lambda_2^2+6\lambda_3\,(3\lambda_3+2)}\right)^2}{6(3\lambda_3+2)^2}\Bigg]\,. \tag{5.60}$$

From the definitions $w_\phi \equiv (x^2-y)/\Omega_\phi$ and $y = 0$ at these fixed points, we obtain $w_{\phi c}^{(\phi)} = 1$. Hence, the effective equation of state parameter $w_{\text{eff}} = \Omega_\phi w_\phi = \Omega_{\phi c}^{(\phi)}$ is slightly positive during the ϕMDE. We can write λ_1 in terms of $\Omega_{\phi c}^{(\phi)}$, λ_2 and λ_3 using Eq. (5.60) as

$$\lambda_1^{(\phi)} = -\lambda_2 \mp \sqrt{\frac{3}{2}}\frac{\left|3\lambda_3\Omega_{\phi c}^{(\phi)}+2\Omega_{\phi c}^{(\phi)}-\lambda_3\right|}{\sqrt{\Omega_{\phi c}^{(\phi)}}}\,, \tag{5.61}$$

which are similar to scaling fixed point. In the following consideration, we use the subscripts $_-$ and $_+$ to indicate the selected sign in the expressions which contain $\pm$ or $\mp$. As an example, if we apply this notation to Eq. (5.59), we get

$$x_{c+}^{(\phi)} = -\frac{\lambda_1+\lambda_2}{\sqrt{6}\,(3\lambda_3+2)} + \frac{\sqrt{\lambda_1^2+2\lambda_2\lambda_1+\lambda_2^2+6\lambda_3\,(3\lambda_3+2)}}{\sqrt{6}\,(3\lambda_3+2)}\,. \tag{5.62}$$

Using such notation, the possible expressions of λ and λ_1 for the scaling points can be expressed as follows: according to Eqs. (5.51) and (5.52), there are two possible forms of λ and λ_1 such that $(\lambda, \lambda_1) = (\lambda_-, \lambda_{1+})$ and $(\lambda_+, \lambda_{1-})$. For ϕMDE point, Eq. (5.59) shows that there are two possible forms of $x_c^{(\phi)}$, i.e., $x_{c-}^{(\phi)}$ and $x_{c+}^{(\phi)}$. Each of them leads to two possible choices of λ_1 given in Eq. (5.61).

5.2.3 Stability

To investigate the stabilities of the fixed points considered in the previous section, we linearize the autonomous equations (5.28) - (5.30) around the fixed points. Before performing the linearization, we insert Ω_m from Eq. (5.38) and C from Eq. (5.36) into the autonomous equations. We estimate the stability of these fixed points by checking the signs of the eigenvalues of the Jacobian matrix defined by

$$J_{ij} = \left.\frac{\partial x_i'}{\partial x_j}\right|_{\text{fixed point}} , \tag{5.63}$$

where $x_i = (x, y, z)$.

(a) Field dominated Point

Let us first consider the field dominated point in which x and y at fixed point given by the first solution in Eqs. (5.45) and (5.46), while z at the fixed point is the second solution in Eq. (5.44). The eigenvalues of the Jacobian matrix for this case can be written as

$$\begin{aligned} \mu_1 &= 3\lambda_3(1+w_{\phi c}) + \lambda_2\sqrt{3(1+w_{\phi c})}\,, \\ \mu_2 &= -\frac{3}{2}\left(1-w_{\phi c}\right)\,, \\ \mu_3 &= \frac{\lambda_3\left(9w_{\phi c}-3\right) + 6w_{\phi c} - \sqrt{3}\left(\lambda_1+\lambda_2\right)\sqrt{1+w_{\phi c}}}{4\lambda_3+2}\,, \end{aligned} \tag{5.64}$$

where we have expressed λ in terms of w_ϕ for the case of this fixed point using Eq. (5.50). One can check that the field dominated point is stable when both of the following conditions are satisfied

$$\lambda_3 < -\frac{\lambda_2}{\sqrt{3(1+w_{\phi c})}}\,, \tag{5.65}$$

$$\lambda_1 \quad \begin{cases} < -\frac{2w_{\phi c}\left(2\sqrt{3}\lambda_2-3\sqrt{w_{\phi c}+1}\right)}{\sqrt{3}\left(w_{\phi c}+1\right)} & \text{for } \lambda_3 < -1/2 \\ > -\frac{2w_{\phi c}\left(2\sqrt{3}\lambda_2-3\sqrt{w_{\phi c}+1}\right)}{\sqrt{3}\left(w_{\phi c}+1\right)} & \text{for } \lambda_3 > -1/2 \end{cases} . \tag{5.66}$$

Since μ_2 is always negative when $w_{\phi c} < 1$ which is the case for scalar field with standard kinetic term, the field dominated points cannot be unstable.

(b) Scaling Fixed Point

For the expressions of x_c and y_c at the scaling point given by Eq. (5.53), the eigenvalues can be written as

$$\begin{aligned}
\mu_1 &= 3\lambda_3\left(1+w_{\phi c}\Omega_{\phc}\right) \mp \lambda_2\sqrt{3\Omega_{\phi c}(1+w_{\phi c})}\,, \\
\mu_2 &= -\frac{3}{4}\left(1-w_{\phi c}\Omega_{\phi c}\right)+3\sqrt{\frac{r_a}{r_b}}\,, \\
\mu_3 &= -\frac{3}{4}\left(1-w_{\phi c}\Omega_{\phi c}\right)-3\sqrt{\frac{r_a}{r_b}}\,, \qquad (5.67)
\end{aligned}$$

where

$$\begin{aligned}
r_a &= \lambda_3\left(w_{\phi c}^2\left(2w_{\phi c}+1\right)\Omega_{\phi c}^3+\left(-3w_{\phi c}^2-18w_{\phi c}+16\right)\Omega_{\phi c}^2+\left(16w_{\phi c}-15\right)\Omega_{\phi c}+1\right) \\
&\quad +\Omega_{\phi c}\left(w_{\phi c}^2\left(w_{\phi c}+1\right)\Omega_{\phi c}^2-2\left(w_{\phi c}^2+5w_{\phi c}-4\right)\Omega_{\phi c}+9w_{\phi c}-7\right)\,, \qquad (5.68) \\
r_b &= 16\left(\lambda_3\Omega_{\phi c}+2\lambda_3 w_{\phi c}\Omega_{\phi c}+w_{\phi c}\Omega_{\phi c}+\Omega_{\phi c}+\lambda_3\right)\,. \qquad (5.69)
\end{aligned}$$

In the above eigenvalues, we have expressed for λ and λ_1 in terms of $w_{\phi c}$ and $\Omega_{\phi c}$ using Eqs. (5.51) and (5.52). We obtain the fixed point x_{c+} and x_{c-} in Eq. (5.53) leading to the same μ_2 and μ_3 but different μ_1. The first eigenvalue μ_1 can be negative when

$$\lambda_3 < \pm\frac{\lambda_2\sqrt{3\Omega_{\phi c}(1+w_{\phi c})}}{3(1+w_{\phi c}\Omega_{\phi c})}\,. \qquad (5.70)$$

The eigenvalues μ_2 and μ_3 in Eq. (5.67) can be infinite if $r_b = 0$ occurring when

$$\lambda_3 = \lambda_{3b} = -\frac{\left(w_{\phi c}+1\right)\Omega_{\phi c}}{2w_{\phi c}\Omega_{\phi c}+\Omega_{\phi c}+1}\,. \qquad (5.71)$$

If the ratio $r_a/r_b < 0$,the real parts of both μ_2 and μ_3 can be ensured to be negative. To estimate the sign of this ratio, λ_3 at which $r_a = 0$ is computed. We can show that $r_a = 0$ when

$$\begin{aligned}
\lambda_3 &= \lambda_{3a} \\
&= -\frac{\Omega_{\phi c}\left(w_{\phi c}^2\left(w_{\phi c}+1\right)\Omega_{\phi c}^2-2\left(w_{\phi c}^2+5w_{\phi c}-4\right)\Omega_{\phi c}+9w_{\phi c}-7\right)}{w_{\phi c}^2\left(2w_{\phi c}+1\right)\Omega_{\phi c}^3+\left(-3w_{\phi c}^2-18w_{\phi c}+16\right)\Omega_{\phi c}^2+\left(16w_{\phi c}-15\right)\Omega_{\phi c}+1}\,. \qquad (5.72)
\end{aligned}$$

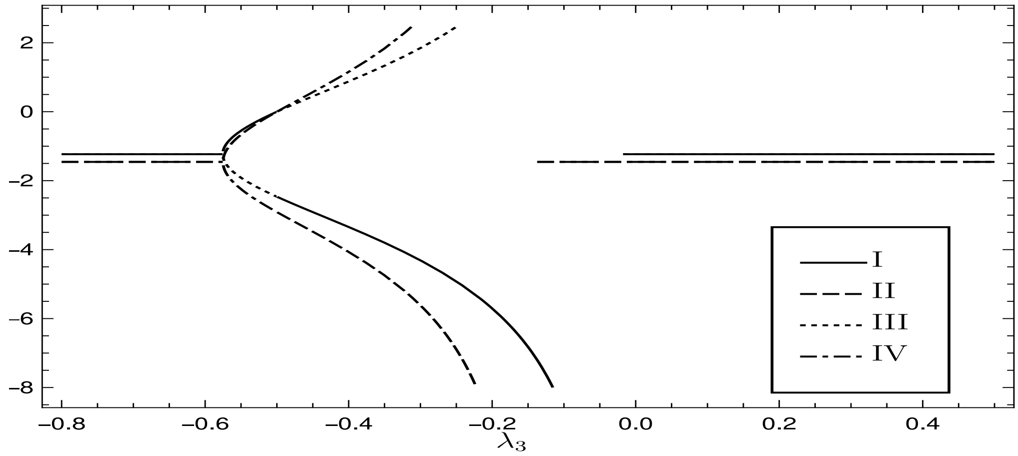

Figure 4 Plots of the real parts of μ_2 and μ_3 for scaling fixed point. In the plots, $w_{\phi c} = -0.99$ and $\lambda_2 = 1$. The lines I and II represent the real part of μ_2 while the lines III and IV represent the real part of μ_3. The lines I and III show the cases of $\Omega_{\phi c} = 0.65$ while the lines II and IV show the cases of $\Omega_{\phi c} = 0.95$.

For the case that $\Omega_{\phi c} > 0.6$ and $w_{\phi c} \gtrsim -1$, the coefficient of λ_3 in Eq. (5.68) is negative while that in Eq. (5.69) is positive. Hence, $r_b < 0$ when $\lambda_3 < \lambda_{3b}$ while $r_a < 0$ when $\lambda_3 > \lambda_{3a}$. Since $\lambda_{3a} < \lambda_{3b}$, the ratio r_a/r_b is negative when $\lambda_3 < \lambda_{3a}$ or $\lambda_3 > \lambda_{3b}$. As a result, the scaling point is stable when $\lambda_3 < \lambda_{3a}$ or $\lambda_3 > \lambda_{3b}$ for suitable choice of λ_2 according to Eq. (5.70). For the case $\lambda_3 \in (\lambda_{3a}, \lambda_{3b})$, we have to evaluate μ_2 and μ_3 numerically. The real parts of μ_2 and μ_3 for some choices of $\Omega_{\phi c}$ are plotted in Fig. 4. In this figure, the real parts of the eigenvalues weakly depend on λ_2.

(c) Kinetic Dominated Point and ϕMDE Point

Let us first consider the kinetic dominated points where $x_c = \pm 1$. For these points, the eigenvalues of the Jacobian matrix are

$$\mu_1 = \frac{3\,(\lambda_3+1)}{2\lambda_3+1} \pm \frac{\sqrt{6}\,(\lambda_1+\lambda_2)}{4\lambda_3+2}\,, \quad \mu_2 = 6\lambda_3 \mp \sqrt{6}\lambda_2\,, \quad \text{and} \quad \mu_3 = 6 \pm \sqrt{6}\lambda\,. \quad (5.73)$$

Depending on the values of λ_2 and λ_3, the second eigenvalue μ_2 can be either positive or negative. This means that these points can be saddle point, so that they could be reached for some ranges of λ_2, λ_3 and some choices of initial conditions. However, we focus on the cases where the cosmic evolution satisfies observational data, so that we will not discuss these points in more detail.

We next consider the ϕMDE points given by Eq. (5.59). Since the eigenvalues for these fixed points are complicated and their values include many possible cases according to the range of λ_1, λ_2 and λ_3, it is hard for discussion. However, if we are interested in the case where the ϕMDE is followed by accelerating epoch described by scaling points, we have to demand that λ_1 from Eq. (5.52) is equal to that from Eq. (5.61). Therefore we match these two equations. We then obtain the relation between $\Omega_{\phi c}^{(\phi)}$ and $\Omega_{\phi c}$ as

$$\Omega_{\phi c\mp}^{(\phi)} = \frac{A \mp |\lambda_3 (3w_{\phi c}\Omega_{\phi c} - 1) + 2w_{\phi c}\Omega_{\phi c}| \sqrt{B}}{(3\lambda_3 + 2)^2 (w_{\phi c} + 1)\, \Omega_{\phi c}}, \tag{5.74}$$

where

$$\begin{aligned} A &= \lambda_3^2 \left(9w_{\phi c}^2 \Omega_{\phi c}^2 - 3(w_{\phi c} - 1)\,\Omega_{\phi c} + 1\right) \\ &\quad + 2\lambda_3 \Omega_{\phi c} \left(6w_{\phi c}^2 \Omega_{\phi c} - w_{\phi c} + 1\right) + 4w_{\phi c}^2 \Omega_{\phi c}^2, \end{aligned} \tag{5.75}$$

$$B = \lambda_3^2 \left(9w_{\phi c}^2 \Omega_{\phi c}^2 + 6\Omega_{\phi c} + 1\right) + 4\lambda_3 \Omega_{\phi c} \left(3w_{\phi c}^2 \Omega_{\phi c} + 1\right) + 4w_{\phi c}^2 \Omega_{\phi c}^2. \tag{5.76}$$

If λ_3 is equal to $-2/3$, the right-hand side of Eq. (5.74) could be infinite. Nevertheless, if we take the limit $\lambda_3 \to -2/3$, Eq. (5.74) gives

$$\Omega_{\phi c-}^{(\phi)} = \frac{1}{2}(w_{\phi c} + 1)\,\Omega_{\phi c} \quad , \quad \Omega_{\phi c+}^{(\phi)} = \infty. \tag{5.77}$$

Hence, in the following consideration, we concentrate only $\Omega_{\phi c-}^{(\phi)}$ which will be denoted by $\Omega_{\phi c}^{(\phi)}$. From Eq. (5.74), we see that $\Omega_{\phi c}^{(\phi)}$ can have an imaginary part if B is negative occurring when

$$\frac{2\left(-3w_{\phi c}^2 \Omega_{\phi c}^2 - \sqrt{\Omega_{\phi c}^2 - w_{\phi c}^2 \Omega_{\phi c}^2} - \Omega_{\phi c}\right)}{9w_{\phi c}^2 \Omega_{\phi c}^2 + 6\Omega_{\phi c} + 1} < \lambda_3 < \frac{2\left(-3w_{\phi c}^2 \Omega_{\phi c}^2 + \sqrt{\Omega_{\phi c}^2 - w_{\phi c}^2 \Omega_{\phi c}^2} - \Omega_{\phi c}\right)}{9w_{\phi c}^2 \Omega_{\phi c}^2 + 6\Omega_{\phi c} + 1}. \tag{5.78}$$

For $w_{\phi c} = -0.99$, the above condition becomes $-0.45 < \lambda_3 < -0.41$ and $-0.51 < \lambda_3 < -0.47$ corresponding to $\Omega_{\phi c} = 0.65$ and $\Omega_{\phi c} = 0.95$, respectively. To ensure that the scaling points are stable, we choose λ_3 in the ranges $\lambda_3 < \lambda_{3a}$ or $\lambda_3 > \lambda_{3b}$. For $w_{\phi c} = -0.99$, from Eqs. (5.72) and (5.71) we have $\lambda_{3a} \approx -0.57$ for both $\Omega_{\phi c} = 0.65$ and

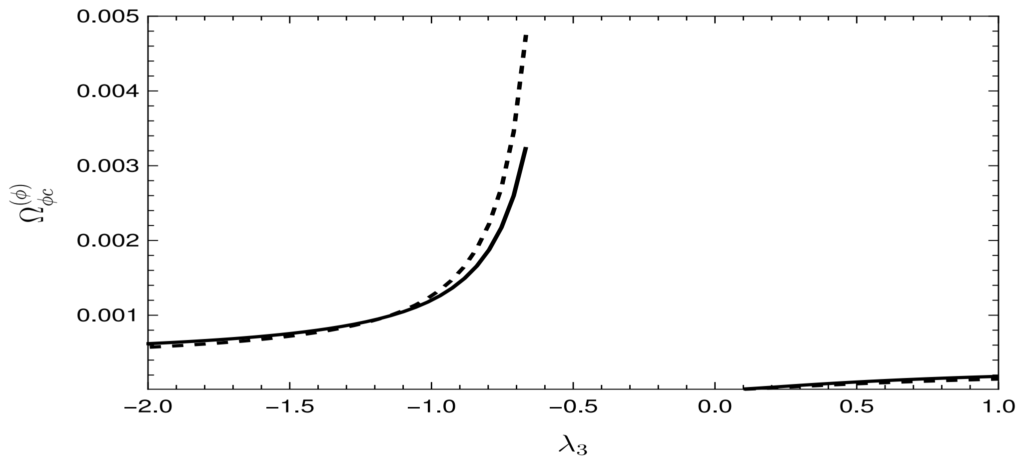

Figure 5 Plots of $\Omega_{\phi c}^{(\phi)}$ as a function of λ_3. The solid line shows the case $\Omega_{\phi c} = 0.65$, while the dashed line shows the case $\Omega_{\phi c} = 0.95$. In the plots, $w_{\phi c} = -0.99$, $\lambda_2 = 1$ and λ_3 lies within the range $\lambda_3 \leq -2/3$ and $0 < \lambda_3 \leq 1$.

$\Omega_{\phi c} = 0.95$. Moreover, we have $\lambda_{3b} \approx -0.01$ and -0.13 for $\Omega_{\phi c} = 0.65$ and $\Omega_{\phi c} = 0.95$ respectively. Hence, for $\lambda_3 < \lambda_{3a}$ or $\lambda_3 > \lambda_{3b}$, $\Omega_{\phi c}^{(\phi)}$ is real. In the case where $w_{\phi c} \gtrsim -1$ and $\Omega_{\phi c} > 0.65$, Eq. (5.74) gives $\Omega_{\phi c}^{(\phi)} \lesssim 10^{-3}$. From the numerical values of λ_{3a} and λ_{3b}, we have to set λ_3 within the ranges $\lambda_3 \leq -2/3$ and $0 < \lambda_3 \leq 1$ in the following analysis. For the upper bound $\lambda_3 \leq 1$, we impose to avoid divergence of x' and Ω_m' which can occur when λ_3 satisfies Eq. (5.37).

The quantity $\Omega_{\phi c}^{(\phi)}$ is the value of Ω_ϕ at the ϕMDE point. We plot this quantity as a function of λ_3 in Fig. 5. In the plots, $\Omega_{\phi c}$ are not sensitive to λ_2. We note that λ_1 in Eqs. (5.52) and (5.61) can be matched only for suitable conditions for λ_3. For example, we obtain the same expression for $\Omega_{\phi c}^{(\phi)}$ when we solve for it from the equations which are constructed by matching λ_{1+} from Eq. (5.52) with either $\lambda_{1+}^{(\phi)}$ or $\lambda_{1-}^{(\phi)}$ from Eq. (5.61). However, if we compute the numerical value of $\Omega_{\phi c}^{(\phi)}$ from Eq. (5.74) for given values of $\Omega_{\phi c}, w_{\phi c}, \lambda_2$ and λ_3, and insert the result back into Eq. (5.61), the numerical value of λ_{1+} will be equal to $\lambda_{1-}^{(\phi)}$ when $\lambda_3 \leq -2/3$ while it will be equal to $\lambda_{1+}^{(\phi)}$ when $\lambda_3 > 0$. Moreover, $|x_{c-}^{(\phi)}| < 1$ and $|x_{c+}^{(\phi)}| > 1$ for the former case while $|x_{c-}^{(\phi)}| > 1$ and $|x_{c+}^{(\phi)}| < 1$ for the latter case. The case where $|x_c^{(\phi)}| > 1$ is not physically relevant case. We summarize

Table 2 Matching of λ_1 from Eqs. (5.52) and (5.61) and the required conditions on λ_3. The fourth column shows the magnitude of $x_c^{(\phi)}$. The fifth and the sixth columns present the signs of λ_1 and λ computed from Eqs. (5.52) and (5.51). The main conclusions from the table do not change if $|\lambda_2| \sim \mathcal{O}(1)$, $w_{\phi c} \gtrsim -1$ and $\Omega_{\phi c} > 0.65$.

Matching Cases	Scaling = ϕMDE	λ_3	$x_c^{(\phi)}$	λ_1	λ
I	$\lambda_{1+} = \lambda_{1-}^{(\phi)}$	$\lambda_3 \leq -2/3$	$\lvert x_{c-}^{(\phi)}\rvert < 1$ and $\lvert x_{c+}^{(\phi)}\rvert > 1$	< 0	< 0
II	$\lambda_{1+} = \lambda_{1+}^{(\phi)}$	$\lambda_3 > 0$	$\lvert x_{c-}^{(\phi)}\rvert > 1$ and $\lvert x_{c+}^{(\phi)}\rvert < 1$	> 0	< 0
III	$\lambda_{1-} = \lambda_{1+}^{(\phi)}$	$\lambda_3 \leq -2/3$	$\lvert x_{c-}^{(\phi)}\rvert > 1$ and $\lvert x_{c+}^{(\phi)}\rvert < 1$	> 0	> 0
IV	$\lambda_{1-} = \lambda_{1-}^{(\phi)}$	$\lambda_3 > 0$	$\lvert x_{c-}^{(\phi)}\rvert < 1$ and $\lvert x_{c+}^{(\phi)}\rvert > 1$	< 0	> 0

Table 3 The first eigenvalues for all possible matching cases.

First eigenvalue	Cases I and II	Cases III and IV
μ_1	$\lambda\sqrt{6\Omega_{\phi c}^{(\phi)}} + 3(\Omega_{\phi c}^{(\phi)} + 1)$	$-\lambda\sqrt{6\Omega_{\phi c}^{(\phi)}} + 3(\Omega_{\phi c}^{(\phi)} + 1)$

the matching of λ_1 and $\lambda_1^{(\phi)}$ and the conditions on λ_3 in Tab. 2. Based on the choices of parameters in Tab. 2, We now investigate the eigenvalues of the ϕMDE points . The first eigenvalues of all cases are simple and are shown in Tab. 3. From the table, we see that the eigenvalues could be negative depending on the sign of λ. Nevertheless, the terms λ are multiplied by $\sqrt{\Omega_{\phi c}^{(\phi)}}$ which is in order of 10^{-2}, so that these terms have no sufficient contribution to make the eigenvalues negative. For these ϕMDE points, the polynomial for the eigenvalues is complicated. Fortunately, the form the first eigenvalue is simple, therefore the order of the polynomial can be reduced by dividing the polynomial with $(\mu_1 - \mu)$. It yields the resulting polynomial which can be written in the form

$$\mu^2 + a_1\mu + a_2 = 0\,, \tag{5.79}$$

where a_1 and a_2 are complicated functions of the parameters and $\Omega_{\phi c}^{(\phi)}$. Since $\Omega_{\phi c}^{(\phi)} \lesssim 10^{-3}$, we expand a_1 and a_2 around $\Omega_{\phi c}^{(\phi)} = 0$ up to $\Omega_{\phi c}^{(\phi)}$ as shown in Eqs. (5.80) - (5.83).

cases I and II:

$$a_1 = \frac{3}{2} - \lambda_1 \sqrt{6\Omega_{\phi c}^{(\phi)}} + \left(-24\lambda_3 + \frac{6}{\lambda_3} - \frac{3}{2}\right)\Omega_{\phi c}^{(\phi)} + \dots , \quad (5.80)$$

$$\begin{aligned} a_2 &= -\frac{9}{2} + \frac{3\sqrt{\frac{3}{2}}[\lambda_1\lambda_3(\lambda_3-5)+3\lambda_2(\lambda_3+1)]}{\lambda_3(\lambda_3+1)}\sqrt{\Omega_{\phi c}^{(\phi)}} \\ &+ \frac{3\left[\lambda_1^2(-2\lambda_3^2+5\lambda_3+1) - \lambda_1\lambda_2 a_{2b} - 2\left(\lambda_2^2(\lambda_3+1) - 3\lambda_3 a_{2c}\right)\right]}{\lambda_3^2(\lambda_3+1)}\Omega_{\phi c}^{(\phi)} + \dots , \end{aligned} \quad (5.81)$$

cases III and IV:

$$a_1 = \frac{3}{2} + \lambda_1 \sqrt{6\Omega_{\phi c}^{(\phi)}} + \left(-24\lambda_3 + \frac{6}{\lambda_3} - \frac{3}{2}\right)\Omega_{\phi c}^{(\phi)} + \dots , \quad (5.82)$$

$$\begin{aligned} a_2 &= -\frac{9}{2} - \frac{3\sqrt{\frac{3}{2}}[\lambda_1(\lambda_3-5)\lambda_3+3\lambda_2(\lambda_3+1)]}{\lambda_3(\lambda_3+1)}\sqrt{\Omega_{\phi c}^{(\phi)}} \\ &- \frac{3\left[\lambda_1^2(2\lambda_3^2-5\lambda_3-1) + \lambda_1\lambda_2 a_{2b} + 2\left(\lambda_2^2(\lambda_3+1) - 3\lambda_3 a_{2c}\right)\right]}{\lambda_3^2(\lambda_3+1)}\Omega_{\phi c}^{(\phi)} + \dots , \end{aligned} \quad (5.83)$$

where $a_{2b} = 2\lambda_3^2 - 3\lambda_3 + 1$ and $a_{2c} = 2\lambda_3^3 - 2\lambda_3^2 + 3\lambda_3 + 1$. The solutions of Eq. (5.79) are

$$\mu_{\pm} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} . \quad (5.84)$$

From these solutions we see that the real part of at least one solution is negative if $a_1 > 0$. If $a_1 < 0$, the real part of one solution is negative when $a_2 < 0$. According to Eqs. (5.80) and (5.82) and the sign of λ_1 in Tab. 2, the main contributions to a_1 for the cases I and III are positive. As a result, the real part of at least one eigenvalue for each case is negative. For the cases II and IV, it follows from Eqs. (5.81) and (5.83) together with the sign of λ_1 and the range of λ_3 in Tab. 2 that the main contributions to a_2 can be negative. However, to ensure that a_2 is negative, we suppose that $|\lambda_2| < |\lambda_1|$ and impose the additional condition $\lambda_3 \leq 1$ which is required to avoid divergence of x' and Ω_m'.

This suggests that the real part of one eigenvalue for each case is negative. From the above discussion, we conclude that the ϕMDE point can be saddle for λ_3 given in the table, $\lambda_3 \leq 1$, $|\lambda_2| \sim \mathcal{O}(1)$ and for $w_{\phi c}$, $\Omega_{\phi c}$ satisfying the observational bound, e.g., $w_{\phi c} = -0.99$ and $\Omega_{\phi c} > 0.65$.

5.2.4 Evolution from the ϕMDE Point to Scaling point

From the fixed points which we have already discussed in the previous sections, we now numerically study the evolution of the background universe. The evolution equations used in the numerical integration are obtained by substituting Eq. (5.36) into Eqs. (5.28)-(5.31). To explain some results in the previous sections, The evolutions of Ω_ϕ for various values of λ_3 is plotted in Fig. 6. In the figure, we fix $\lambda_2 = 1$, $\Omega_b = 0$ and specify λ and λ_1 by setting $\Omega_{\phi c} = 0.7$ and $w_{\phi c} = -0.99$. From the figure, we see that the fixed point $\Omega_\phi = \Omega_{\phi c} = 0.7$ can be reached at late time. From the numerical investigation, the all evolution of Ω_ϕ weakly depends on λ_2, and the late-time evolution is robust under the change of initial conditions. We then set $\Omega_b \simeq 0.022$ at present for adding the contribution from the baryon energy density Ω_b into the numerical integration. Now the evolutions of Ω_r, Ω_m and Ω_ϕ for $\lambda_3 = -3/2$ are plotted in Fig. 7. In these plots, we fix $\lambda_2 = 1$ and also specify the parameters λ and λ_1 by setting $\Omega_{\phi c} = 0.95$ and $w_{\phi c} = -0.99$. Since this scaling point can be reached in the future when $\Omega_b \sim 0$, we set $\Omega_{\phi c}$ to be larger than the observational bound for the present value of Ω_ϕ. From the figure we see that the universe evolves from the radiation domination to ϕMDE point and then evolves towards the scaling point at late time with $\Omega_\phi \to 0.95$ and $\Omega_b \to 0$. This pattern of the evolution is achieved for wide ranges of λ_2 and initial conditions. Before moving to the late-time attractor, the cosmic evolution can pass the point $\Omega_\phi \simeq 0.68, \Omega_m \simeq 0.3$ and $\Omega_b \simeq 0.022$ at present as required by observational data. We note that Ω_c in Fig. 7 is Ω_m.

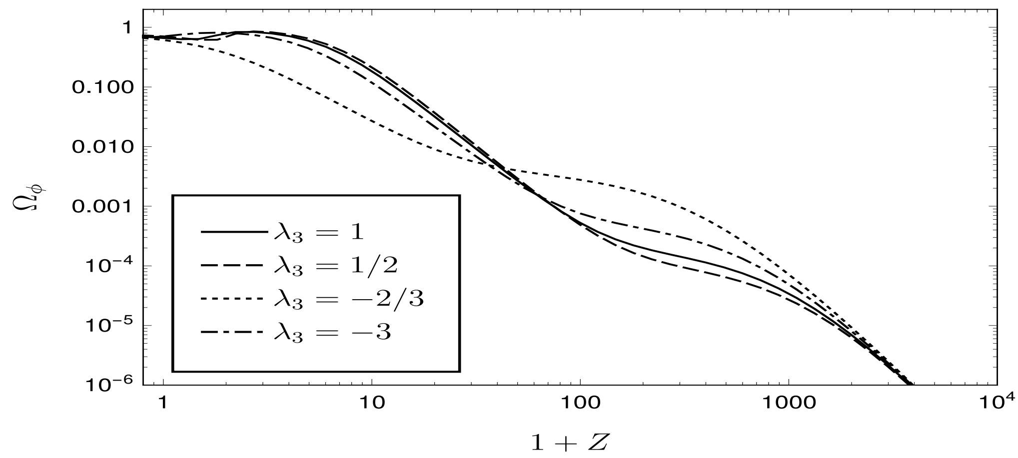

Figure 6 Evolutions of Ω_ϕ for various values of λ_3. In the plots, $1 + Z = 1/a$.

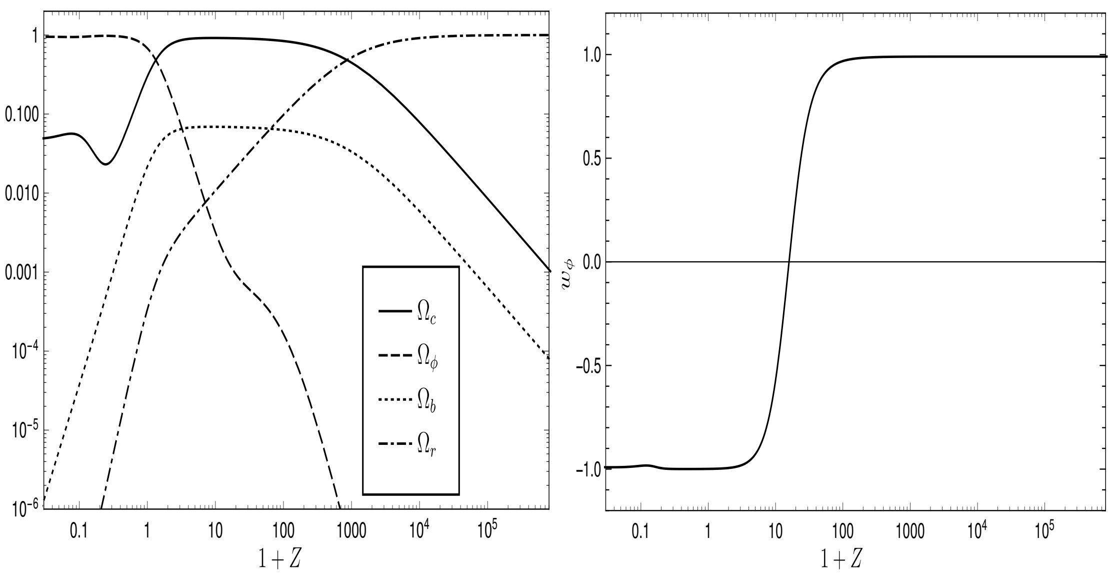

Figure 7 The left panel shows the evolutions of $\Omega_r, \Omega_b, \Omega_c$ and Ω_ϕ, while the right panel shows the evolution of w_ϕ. The ϕMDE takes place around $1 + Z \sim 20$.

CHAPTER VI

CONCLUSIONS

According to the observational data, the expansion of the universe is accelerating at late time. There are numerous attempts to describe the accelerated expansion of universe by introducing dark energy or assuming that physics of gravity on large scale obeys the modified theory of gravity. However, one question arises because of coincidence problem which is a puzzle why energy density of dark energy and matter that independently evolve with time have the same order of magnitude at the present. To solve such problem, we demand that the energy density of dark energy is proportional to the energy density of matter during some period of time. A solution of the evolution equations for the background universe that leads to the constant ratio of the matter and dark energy densities is a scaling solution. Since ratio ρ_d/ρ_m is constant, ρ_m is no longer scale as a^{-3} during the scaling regime but the effective equation of state parameter of matter is negative as the dark energy. To realize such property of ρ_m, one assumes that there is an interaction between the matter and the dark energy.

The scaling solution lead to the existence of ϕMDE point in which there is a small fraction of dark energy during matter domination. The coincidence problem could be alleviated if the universe can evolve from radiation domination through ϕMDE toward acceleration epoch at late time.

In this thesis, the scaling solutions in two cosmological models are studied. Firstly, the scaling solutions in the modified theory of gravity are investigated. The modified theory of gravity used in our study is DHOST theory which satisfies the gravitational wave constraints and has the scaling solutions . To get a suitable attractor at late time, the coupling between scalar degree of freedom and dark matter is assumed. The coupling for this model is inspired from conformal transformation in which the coefficient of the conformal transformation depends only on scalar field. Then, the scaling solutions in the coupled dark energy model constructed from the general conformal transformation are studied. The general conformal transformation is the conformal transformation in which the coefficient of the conformal transformation depends on both the scalar field and its kinetic term. For the first part of our analysis, the coupling between the dark components is the same as literatures, but the different gravity is used for the second part of our analysis, we use more general coupling term in Einstein gravity.

6.1 DHOST Theory with Scaling Solution

For the analysis of DHOST theory, we concentrate on the model parameters which the expression of z_r is given by Eq. (C.7). We have found that the scaling fixed point corresponding to the comic acceleration at late time, is stable when n_2 and n_6 are not negative for $n_4 = -1$ and -2. The stabilities of this scaling fixed point also depend on the parameter x_λ which corresponds to the expansion rate of the universe at late time. There are ranges of parameters in which the scaling fixed point and the field dominated point are simultaneously stable. If x_r during the matter domination is positive, the cosmic evolution will reach the scaling fixed point at late time. If x_r during the matter domination is negative, the cosmic evolution will reach the field dominated point.

The density parameter of the matter can be larger than unity during the cosmic evolution if c_4 is large enough and the initial value of y_r during the matter domination is significantly different from its value at scaling fixed points. Here, the deviation from the Einstein gravity is parametrized by c_4. In our consideration, the allowed value of c_4 depends on the initial conditions for x_r and y_r during the matter domination.

Even though the gravity is described by the different theories, the eigenvalues for the field dominated and ϕMDE points in the model considered here are similar to those for coupled dark energy models presented in [23]. However, for DHOST theory, the expressions for the eigenvalues corresponding to the scaling points are complicated, and consequently stability of the fixed points has to evaluate numerically. In our numerical investigation, the universe can only evolve from the ϕMDE to the field dominated point. Since the evolution from ϕMDE toward the scaling point corresponding to the cosmic acceleration at late time cannot be achieved in this model, the coincidence problem cannot be alleviated.

6.2 Coupled Dark Energy Model with General Conformal Coupling

In the analysis of coupled dark energy model, the ϕMDE point can be a saddle point, while the solution for the cosmic acceleration at late time can be scaling attractor. The cosmic evolution that starts from the radiation dominated epoch can move towards the ϕMDE and then reaches the cosmic acceleration epoch at late time. This sequence of the evolution can be achieved for the cosmological parameters which satisfy the observational bounds. This suggests that the coincidence problem can be alleviated.

We conclude that if coupling term is inspired from conformal transformation in which the coefficient of the conformal transformation depends only on scalar field, the coincidence problem cannot be alleviated in both DHOST theory and coupled dark energy model. If coupling term is inspired from general conformal transformation in which the coefficient of the conformal transformation depends on both scalar field and its kinetic term, the coincidence problem can be alleviated in coupled dark energy model.

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APPENDIX

APPENDIX A SOME DETAILS OF CALCULATION FOR GALILEON THEORIES

In the following, we present some details of several calculations for Galileon theories. Considering the first term in Eq. (3.7), we obtain

$$
\begin{aligned}
&\partial_\mu\partial_\nu \frac{\partial\left(\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\right)}{\partial\phi_{\mu\nu}} \\
= \;& \partial_\mu\partial_\nu\left[\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}\frac{\partial\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\right)}{\partial\phi_{\mu\nu}}\right] , \\
= \;& \partial_\mu\partial_\nu\left[\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}\left(\phi_{\alpha_1\beta_1}\frac{\partial\phi_{\alpha_2\beta_2}}{\partial\phi_{\mu\nu}}+\phi_{\alpha_2\beta_2}\frac{\partial\phi_{\alpha_1\beta_1}}{\partial\phi_{\mu\nu}}\right)\right] , \\
= \;& \partial_\mu\partial_\nu\left[\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}\left(\phi_{\alpha_1\beta_1}\delta^\mu_{\alpha_2}\delta^\nu_{\beta_2}+\phi_{\alpha_2\beta_2}\delta^\mu_{\alpha_1}\delta^\nu_{\beta_1}\right)\right] , \\
= \;& \partial_{\alpha_2}\partial_{\beta_2}\left(\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}\phi_{\alpha_1\beta_1}\right)+\underbrace{\partial_{\alpha_1}\partial_{\beta_1}\left(\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}\phi_{\alpha_2\beta_2}\right)}_{\alpha_1\leftrightarrow\alpha_2\,,\,\beta_1\leftrightarrow\beta_2} , \\
= \;& 2\partial_{\alpha_2}\partial_{\beta_2}\left(\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}\phi_{\alpha_1\beta_1}\right) , \\
= \;& 2\phi_{\alpha_1\beta_1}\left(\partial_{\alpha_2}\partial_{\beta_2}\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}\right) , \\
= \;& 2\phi_{\alpha_1\beta_1}\partial_{\alpha_2}\left(\frac{\partial\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial\phi_\mu}\partial_{\beta_2}\phi_\beta\right) , \\
= \;& 2\phi_{\alpha_1\beta_1}\left[\partial_{\beta_2}\phi_\mu\partial_{\alpha_2}\left(\frac{\partial\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial\phi_\mu}\right)+\frac{\partial\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial\phi_\mu}\left(\partial_{\alpha_2}\partial_{\beta_2}\phi_\mu\right)\right] , \\
= \;& 2\phi_{\alpha_1\beta_1}\left[\partial_{\beta_2}\phi_\mu\left(\frac{\partial^2\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial\phi_\mu\partial\phi_\nu}\partial_{\alpha_2}\phi_\nu\right)+\frac{\partial\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial\phi_\mu}\left(\partial_{\alpha_2}\partial_{\beta_2}\phi_\mu\right)\right] . \qquad \text{(A.1)}
\end{aligned}
$$

For the second term in Eq. (3.7), it yields

$$
\begin{aligned}
&\partial_\mu\frac{\partial\left(\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\right)}{\partial\phi_\mu} \\
= \;& \partial_\mu\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\frac{\partial\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial\phi_\mu}\right) , \\
= \;& \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\partial_\mu\left(\frac{\partial\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial\phi_\mu}\right)+\frac{\partial\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial\phi_\mu}\partial_\mu\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\right) , \\
= \;& \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\left(\frac{\partial^2\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial\phi_\mu\partial\phi_\nu}\partial_\mu\phi_\nu\right)+2\phi_{\alpha_1\beta_1}\frac{\partial\mathcal{T}^{\alpha_1\alpha_2\beta_1\beta_2}}{\partial\phi_\mu}\left(\partial_\mu\phi_{\alpha_2\beta_2}\right) . \qquad \text{(A.2)}
\end{aligned}
$$

Inserting Eq. (3.9) into Eq. (3.8), we obtain

$$\begin{aligned}
0 &= 2\phi_{\mu\beta_2}\phi_{\nu\alpha_2}\phi_{\alpha_1\beta_1}\frac{\partial}{\partial\phi_\mu}\left[\frac{\partial\left(\mathcal{A}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}\phi_{\alpha_3}\phi_{\beta_3}\right)}{\partial\phi_\nu}\right] \\
&\quad -\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\mu\nu}\frac{\partial}{\partial\phi_\mu}\left[\frac{\partial\left(\mathcal{A}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}\phi_{\alpha_3}\phi_{\beta_3}\right)}{\partial\phi_\nu}\right], \\
&= 2\phi_{\mu\beta_2}\phi_{\nu\alpha_2}\phi_{\alpha_1\beta_1}\left(\delta^\mu_{\beta_3}\delta^\nu_{\alpha_3}+\delta^\mu_{\alpha_3}\delta^\nu_{\beta_3}\right)\mathcal{A}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3} \\
&\quad -\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\mu\nu}\left(\delta^\mu_{\beta_3}\delta^\nu_{\alpha_3}+\delta^\mu_{\alpha_3}\delta^\nu_{\beta_3}\right)\mathcal{A}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}, \\
&= \underbrace{2\phi_{\alpha_3\beta_2}\phi_{\beta_3\alpha_2}\phi_{\alpha_1\beta_1}\mathcal{A}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}}_{\beta_2\leftrightarrow\beta_3}-2\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\mathcal{A}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}, \\
&= -4\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\mathcal{A}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}. \qquad \text{(A.3)}
\end{aligned}$$

Inserting Lagrangian in Eq. (3.11) into Eq. (3.5), we obtain

$$\begin{aligned}
0 &= -\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\partial_\mu\left[\frac{\partial}{\partial\phi_\mu}\left(\phi_{\alpha_{n+1}}\phi_{\beta_{n+1}}\right)\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right] \\
&\quad +\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\partial_\mu\partial_\nu\left[\frac{\partial}{\partial\phi_{\mu\nu}}\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\phi_{\alpha_{n+1}}\phi_{\beta_{n+1}}\right], \\
&= -\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\partial_\mu\left[\left(\phi_{\alpha_{n+1}}\delta^\mu_{\beta_{n+1}}+\phi_{\beta_{n+1}}\delta^\mu_{\alpha_{n+1}}\right)\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right] \\
&\quad +\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\partial_\mu\partial_\nu\left[\frac{\partial}{\partial\phi_{\mu\nu}}\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\phi_{\alpha_{n+1}}\phi_{\beta_{n+1}}\right], \\
&= -2\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}\beta_{n+1}} \\
&\quad +\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\partial_\mu\partial_\nu\left[n\frac{\partial}{\partial\phi_{\mu\nu}}\left(\phi_{\alpha_1\beta_1}\right)\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}}\phi_{\beta_{n+1}}\right], \\
&= -2\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}\beta_{n+1}} \\
&\quad +n\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\partial_\mu\partial_\nu\left[\left(\delta^\mu_{\alpha_1}\delta^\nu_{\beta_1}\right)\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}}\phi_{\beta_{n+1}}\right], \\
&= -2\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}\beta_{n+1}} \\
&\quad +n\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\partial_{\alpha_1}\partial_{\beta_1}\left(\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}}\phi_{\beta_{n+1}}\right), \\
&= -2\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}\beta_{n+1}} \\
&\quad +n\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\underbrace{\phi_{\beta_1\alpha_{n+1}}\phi_{\alpha_1\beta_{n+1}}}_{\beta_1\leftrightarrow\beta_{n+1}}, \\
&= -2\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}\beta_{n+1}} \\
&\quad -n\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}\beta_{n+1}}, \\
&= -\left(2+n\right)\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_{n+1}\beta_1\beta_2\ldots\beta_{n+1}}_{(2n+2)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\phi_{\alpha_{n+1}\beta_{n+1}}. \qquad \text{(A.4)}
\end{aligned}$$

Showing the calculation of the first and second term in Eq. (3.19), we obtain Eq. (A.6) and Eq. (A.7) respectively

$$\begin{aligned}
&\partial_\mu\partial_\nu \frac{\partial\left[\left(\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_1}\phi_\lambda\phi^\lambda_{\beta_1}\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right]}{\partial\phi_{\mu\nu}} \\
=\;& \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\mu\partial_\nu\left[\phi_{\alpha_1}\phi^\lambda\frac{\partial\left(\phi_{\lambda\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right)}{\partial\phi_{\mu\nu}}\right], \\
=\;& \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\mu\partial_\nu\left[\phi_{\alpha_1}\phi_\lambda\phi^\lambda_{\beta_1}\frac{\partial\left(\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right)}{\partial\phi_{\mu\nu}}\right] \\
&+\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\mu\partial_\nu\left[\phi_{\alpha_1}\phi^\lambda\frac{\partial\left(\phi_{\lambda\beta_1}\right)}{\partial\phi_{\mu\nu}}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right], \\
=\;& \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\mu\partial_\nu\left[(n-1)\phi_{\alpha_1}\phi_\lambda\phi^\lambda_{\beta_1}\frac{\partial\left(\phi_{\alpha_2\beta_2}\right)}{\partial\phi_{\mu\nu}}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right] \\
&+\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\mu\partial_\nu\left[\phi_{\alpha_1}\phi^\lambda\frac{\partial\left(\phi_{\lambda\beta_1}\right)}{\partial\phi_{\mu\nu}}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right], \\
=\;& \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\mu\partial_\nu\left[(n-1)\phi_{\alpha_1}\phi_\lambda\phi^\lambda_{\beta_1}(\delta^\mu_{\alpha_2}\delta^\nu_{\beta_2})\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right] \\
&+\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\mu\partial_\nu\left[\phi_{\alpha_1}\phi^\lambda(\delta^\mu_\lambda\delta^\nu_{\beta_1})\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right], \\
=\;& (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_{\alpha_2}\partial_{\beta_2}\left(\phi_{\alpha_1}\phi_\lambda\phi^\lambda_{\beta_1}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right) \\
&+\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\lambda\partial_{\beta_1}\left(\phi_{\alpha_1}\phi^\lambda\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right), \\
=\;& (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_{\alpha_2}\left[\phi^\lambda_{\beta_1}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_{\beta_2}\left(\phi_{\alpha_1}\phi_\lambda\right)\right] \\
&+\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\lambda\left[\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_{\beta_1}\left(\phi_{\alpha_1}\phi^\lambda\right)\right], \\
=\;& (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_{\alpha_2}\left[\phi^\lambda_{\beta_1}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\left(\phi_\lambda\phi_{\alpha_1\beta_2}+\phi_{\alpha_1}\phi_{\lambda\beta_2}\right)\right] \\
&+\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\lambda\left[\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\left(\phi^\lambda\phi_{\alpha_1\beta_1}+\phi_{\alpha_1}\phi^\lambda_{\beta_1}\right)\right], \qquad \text{(A.5)}
\end{aligned}$$

$$
\begin{aligned}
& \partial_\mu \partial_\nu \frac{\partial\left[\left(\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_1}\phi_\lambda\phi^\lambda_{\beta_1}\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right]}{\partial\phi_{\mu\nu}} \\
= \; & (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_{\alpha_2}\left[\phi^\lambda_{\beta_1}\left(\phi_\lambda\phi_{\alpha_1\beta_2}+\phi_{\alpha_1}\phi_{\lambda\beta_2}\right)\right] \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_\lambda\left(\phi^\lambda\phi_{\alpha_1\beta_1}+\phi_{\alpha_1}\phi^\lambda_{\beta_1}\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\left(\phi^\lambda\phi_{\alpha_1\beta_1}+\phi_{\alpha_1}\phi^\lambda_{\beta_1}\right)\partial_\lambda\left(\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right)\,, \\
= \; & (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi^\lambda_{\beta_1}\partial_{\alpha_2}\left(\phi_\lambda\phi_{\alpha_1\beta_2}+\phi_{\alpha_1}\phi_{\lambda\beta_2}\right) \\
& +(n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\underbrace{\phi^\lambda_{\beta_1\alpha_2}\left(\phi_\lambda\phi_{\alpha_1\beta_2}+\phi_{\alpha_1}\phi_{\lambda\beta_2}\right)}_{\beta_1\leftrightarrow\beta_2} \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_\lambda\left(\phi^\lambda\phi_{\alpha_1\beta_1}+\phi_{\alpha_1}\phi^\lambda_{\beta_1}\right) \\
& +(n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\left(\phi^\lambda\phi_{\alpha_1\beta_1}+\phi_{\alpha_1}\phi^\lambda_{\beta_1}\right)\phi_{\alpha_2\beta_2\lambda}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,, \\
= \; & (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi^\lambda_{\beta_1}\left(\phi_{\lambda\alpha_2}\phi_{\alpha_1\beta_2}+\phi_{\alpha_1}\phi_{\lambda\beta_2\alpha_2}\right) \\
& -(n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi^\lambda_{\beta_2\alpha_2}\left(\phi_\lambda\phi_{\alpha_1\beta_1}+\phi_{\alpha_1}\phi_{\lambda\beta_1}\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_\lambda\left(\phi^\lambda\phi_{\alpha_1\beta_1}+\phi_{\alpha_1}\phi^\lambda_{\beta_1}\right) \\
& +(n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\left(\phi^\lambda\phi_{\alpha_1\beta_1}+\phi_{\alpha_1}\phi^\lambda_{\beta_1}\right)\phi_{\lambda\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,, \\
= \; & (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi^\lambda_{\beta_1}\left(\phi_{\lambda\alpha_2}\phi_{\alpha_1\beta_2}+\phi_{\alpha_1}\phi_{\lambda\beta_2\alpha_2}\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_\lambda\left(\phi^\lambda\phi_{\alpha_1\beta_1}+\phi_{\alpha_1}\phi^\lambda_{\beta_1}\right)\,, \\
= \; & (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi^\lambda_{\beta_1}\underbrace{\left(\phi_{\lambda\alpha_2}\phi_{\alpha_1\beta_2}\right)}_{\alpha_1\leftrightarrow\alpha_2} \\
& +(n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi^\lambda_{\beta_1}\left(\phi_{\alpha_1}\phi_{\lambda\beta_2\alpha_2}\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\left(\phi^\lambda_\lambda\phi_{\alpha_1\beta_1}+\phi^\lambda\phi_{\alpha_1\beta_1\lambda}\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\left(\phi_{\alpha_1\lambda}\phi^\lambda_{\beta_1}+\phi_{\alpha_1}\phi^\lambda_{\beta_1\lambda}\right)\,, \\
= \; & -(n-2)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi^\lambda_{\beta_1}\left(\phi_{\lambda\alpha_1}\phi_{\alpha_2\beta_2}\right) \\
& +(n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi^\lambda_{\beta_1}\left(\phi_{\alpha_1}\phi_{\lambda\beta_2\alpha_2}\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi^\lambda_\lambda\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\left(\phi_{\alpha_1}\phi^\lambda_{\beta_1\lambda}+\phi^\lambda\phi_{\alpha_1\beta_1\lambda}\right)\,,
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
& \partial_\mu \frac{\partial\left[\left(\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_1}\phi_\lambda\phi_{\beta_1}^\lambda\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right]}{\partial\phi_\mu} \\
= \;& \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\mu\left[\frac{\partial\left(\phi_{\alpha_1}\phi_\lambda\right)}{\partial\phi_\mu}\phi_{\beta_1}^\lambda\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right], \\
= \;& \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\mu\left[\left(\phi_{\alpha_1}\delta_\lambda^\mu+\phi_\lambda\delta_{\alpha_1}^\mu\right)\phi_{\beta_1}^\lambda\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right], \\
= \;& \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_\lambda\left(\phi_{\alpha_1}\phi_{\beta_1}^\lambda\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\partial_{\alpha_1}\left(\phi_\lambda\phi_{\beta_1}^\lambda\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right), \\
= \;& \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_1}\phi_{\beta_1}^\lambda\partial_\lambda\left(\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_\lambda\left(\phi_{\alpha_1}\phi_{\beta_1}^\lambda\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_{\alpha_1}\left(\phi_\lambda\phi_{\beta_1}^\lambda\right), \\
= \;& (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi_{\beta_1}^\lambda\left(\phi_{\alpha_1}\phi_{\alpha_2\beta_2\lambda}\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_\lambda\left(\phi_{\alpha_1}\phi_{\beta_1}^\lambda\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_{\alpha_1}\left(\phi_\lambda\phi_{\beta_1}^\lambda\right), \\
= \;& (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi_{\beta_1}^\lambda\left(\phi_{\alpha_1}\phi_{\alpha_2\beta_2\lambda}\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\left(\phi_{\alpha_1\lambda}\phi_{\beta_1}^\lambda+\phi_{\alpha_1}\phi_{\beta_1\lambda}^\lambda\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\left(\phi_{\lambda\alpha_1}\phi_{\beta_1}^\lambda+\phi_\lambda\phi_{\beta_1\alpha_1}^\lambda\right), \\
= \;& (n-1)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\phi_{\beta_1}^\lambda\left(\phi_{\alpha_1}\phi_{\alpha_2\beta_2\lambda}\right) \\
& +2\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\left(\phi_{\alpha_1\lambda}\phi_{\beta_1}^\lambda\right) \\
& +\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\left(\phi_{\alpha_1}\phi_{\beta_1\lambda}^\lambda+\phi_\lambda\phi_{\beta_1\alpha_1}^\lambda\right).
\end{aligned}
\tag{A.7}
$$

Inserting Lagrangian in Eq. (3.18) into Eq. (3.5), we obtain

$$
\begin{aligned}
0 &= \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_\mu} , \\
&= \partial_\mu \partial_\nu \frac{\partial \left[\left(\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi_\lambda \phi^\lambda \right) \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right]}{\partial \phi_{\mu\nu}} \\
&\quad -\partial_\mu \frac{\partial \left[\left(\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi_\lambda \phi^\lambda \right) \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right]}{\partial \phi_\mu} , \\
&= \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_\mu \partial_\nu \left[\phi_\lambda \phi^\lambda \frac{\partial \left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right)}{\partial \phi_{\mu\nu}} \right] \\
&\quad -\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_\mu \left[\frac{\partial \left(\phi_\lambda \phi^\lambda \right)}{\partial \phi_\mu} \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right] , \\
&= \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_\mu \partial_\nu \left[n\phi_\lambda \phi^\lambda \frac{\partial \left(\phi_{\alpha_1\beta_1} \right)}{\partial \phi_{\mu\nu}} \phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right] \\
&\quad -\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_\mu \left[\frac{\partial \left(\phi_\lambda \phi^\lambda \right)}{\partial \phi_\mu} \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right] , \\
&= \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_\mu \partial_\nu \left[n\phi_\lambda \phi^\lambda \left(\delta^\mu_{\alpha_1} \delta^\nu_{\beta_1} \right) \phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right] \\
&\quad -\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_\mu \left[\left(2\phi^\lambda \delta^\mu_\lambda \right) \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right] , \\
&= n\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_{\alpha_1} \partial_{\beta_1} \left(\phi_\lambda \phi^\lambda \phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right) \\
&\quad -2\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_\lambda \left(\phi^\lambda \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right) , \\
&= n\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_{\alpha_1} \left[\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \partial_{\beta_1} \left(\phi_\lambda \phi^\lambda \right) \right] \\
&\quad -2\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi^\lambda \partial_\lambda \left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right) , \\
&\quad -2\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_\lambda \left(\phi^\lambda \right) \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} , \\
&= n\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_{\alpha_1} \left[\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \left(2\phi_{\lambda\beta_1} \phi^\lambda \right) \right] \\
&\quad -2n\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi^\lambda \phi_{\alpha_1\beta_1\lambda}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \\
&\quad -2\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi^\lambda_\lambda \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} , \\
&= 2n\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \partial_{\alpha_1} \left(\phi^\lambda \phi_{\lambda\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \right) \\
&\quad -2n\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi^\lambda \phi_{\alpha_1\beta_1\lambda}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} \\
&\quad -2\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi^\lambda_\lambda \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n} , \qquad \text{(A.8)}
\end{aligned}
$$

$$\begin{aligned}
0 &= 2n\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\partial_{\alpha_1}\left(\phi^\lambda\phi_{\lambda\beta_1}\right)\\
&\quad -2n\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi^\lambda\phi_{\alpha_1\beta_1\lambda}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\\
&\quad -2\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi^\lambda_\lambda\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,,\\
&= 2n\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\left(\phi^\lambda_{\alpha_1}\phi_{\lambda\beta_1}+\phi^\lambda\phi_{\lambda\beta_1\alpha_1}\right)\\
&\quad -2n\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi^\lambda\phi_{\alpha_1\beta_1\lambda}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\\
&\quad -2\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi^\lambda_\lambda\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,,\\
&= 2n\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi^\lambda_{\alpha_1}\phi_{\lambda\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\\
&\quad -2\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi^\lambda_\lambda\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,,\\
&= n\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi^\lambda_{\alpha_1}\phi_{\lambda\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\\
&\quad -\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi^\lambda_\lambda\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,. \qquad \text{(A.9)}
\end{aligned}$$

From Eq. (3.22), one can show that

$$\begin{aligned}
\partial_\alpha J^\alpha_N &= \partial_\alpha\left(X\mathcal{A}^{\alpha\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right)\,,\\
&= \partial_\alpha\left(\mathcal{A}^{\alpha\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_\mu\phi^\mu\phi_{\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right)\,,\\
&= \mathcal{A}^{\alpha\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\partial_\alpha\left(\phi_\mu\phi^\mu\phi_{\beta_1}\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,,\\
&= \mathcal{A}^{\alpha\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\left(2\phi_\mu\phi^\mu_\alpha\phi_{\beta_1}+\phi_\mu\phi^\mu\phi_{\alpha\beta_1}\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,,\\
&= 2\mathcal{A}^{\alpha\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\underbrace{\left(\phi_\mu\phi^\mu_\alpha\phi_{\beta_1}\right)}_{\alpha\leftrightarrow\beta_1}\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\\
&\quad +\mathcal{A}^{\alpha\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\left(\phi_\mu\phi^\mu\phi_{\alpha\beta_1}\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\,,\\
&= 2\,\underbrace{\mathcal{A}^{\alpha\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\left(\phi_\alpha\phi_\mu\phi^\mu_{\beta_1}\phi_\alpha\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}}_{\mathcal{L}^{\text{Gal},2}_N}\\
&\quad +\underbrace{\mathcal{A}^{\alpha\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\left(\phi_\mu\phi^\mu\phi_{\alpha\beta_1}\right)\phi_{\alpha_2\beta_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}}_{\mathcal{L}^{\text{Gal},3}_N}\,,\\
&= 2\mathcal{L}^{\text{Gal},2}_N+\mathcal{L}^{\text{Gal},3}_N\,. \qquad \text{(A.10)}
\end{aligned}$$

Considering the first term on the right-hand side of Eq. (3.27), we can obtain

$$
\begin{aligned}
-\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_2\beta_3\ldots\beta_{n+1}}\phi_{\alpha_1}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}} &= -\delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_2\beta_3\ldots\beta_{n+1}}\underbrace{\phi_{\alpha_1}\phi^{\alpha_1}}_{\alpha 1\to\alpha}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}\,, \\
&= -\underbrace{\underbrace{\delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_2\beta_3\ldots\beta_{n+1}}\phi_{\alpha}\phi^{\alpha}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}}_{\beta_2\beta_3\ldots\beta_{n+1}\to\beta_1\beta 2\ldots\beta_n}}_{\alpha_2\alpha_3\ldots\alpha_{n+1}\to\alpha_1\alpha_2\ldots\alpha_n}\,, \\
&= -\delta^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_n}\phi_{\alpha}\phi^{\alpha}\phi^{\beta_1}_{\alpha_1}\ldots\phi^{\beta_n}_{\alpha_n}\,, \\
&= \mathcal{A}_{(2n)}{}^{\alpha_1\alpha_2\ldots\alpha_n}{}_{\beta_1\beta_2\ldots\beta_n}\phi_{\alpha}\phi^{\alpha}\phi^{\beta_1}_{\alpha_1}\ldots\phi^{\beta_n}_{\alpha_n}\,, \\
&= X\mathcal{A}_{(2n)}{}^{\alpha_1\alpha_2\ldots\alpha_n}{}_{\beta_1\beta_2\ldots\beta_n}\phi^{\beta_1}_{\alpha_1}\ldots\phi^{\beta_n}_{\alpha_n}\,, \\
&= \mathcal{L}^{\text{Gal},3}_{N}\,.
\end{aligned}
\tag{A.11}
$$

Considering the second term on the right-hand side of Eq. (3.27), starting at $i = 2$ we now obtain

$$
\begin{aligned}
\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_1\beta_3\ldots\beta_{n+1}}\phi_{\alpha_1}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}} &= \delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_1\beta_3\ldots\beta_{n+1}}\underbrace{\phi_{\beta_2}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}}_{\beta_2\to\lambda}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}\,, \\
&= \underbrace{\underbrace{\delta^{\alpha_2\alpha_3\ldots\alpha_{n+1}}_{\beta_1\beta_3\ldots\beta_{n+1}}\phi_{\lambda}\phi^{\beta_1}\phi^{\lambda}_{\alpha_2}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}}_{\beta_1\beta_3\ldots\beta_{n+1}\to\beta_1\beta_2\ldots\beta_n}}_{\alpha_2\alpha_3\ldots\alpha_{n+1}\to\alpha_1\alpha_2\ldots\alpha_n}\,, \\
&= \delta^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_n}\phi_{\lambda}\phi^{\beta_1}\phi^{\lambda}_{\alpha_1}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_n}_{\alpha_n}\,, \\
&= -\mathcal{A}_{(2n)}{}^{\alpha_1\alpha_2\ldots\alpha_n}{}_{\beta_1\beta_2\ldots\beta_n}\phi_{\lambda}\phi^{\beta_1}\phi^{\lambda}_{\alpha_1}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_n}_{\alpha_n}\,, \\
&= -\mathcal{L}^{\text{Gal},2}_{N}\,.
\end{aligned}
\tag{A.12}
$$

At $i = 3$, it yields

$$
\begin{aligned}
-\delta^{\alpha_1}_{\beta_3}\delta^{\alpha_2\alpha_3\alpha_4\ldots\alpha_{n+1}}_{\beta_1\beta_2\beta_4\ldots\beta_{n+1}}\phi_{\alpha_1}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\phi^{\beta_3}_{\alpha_3}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}
&= -\,\delta^{\alpha_2\alpha_3\alpha_4\ldots\alpha_{n+1}}_{\beta_1\beta_2\beta_4\ldots\beta_{n+1}}\underbrace{\phi_{\beta_3}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\phi^{\beta_3}_{\alpha_3}}_{\beta_3\to\lambda}\phi^{\beta_4}_{\alpha_4}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}\,,\\
&= -\,\delta^{\alpha_2\alpha_3\alpha_4\ldots\alpha_{n+1}}_{\beta_1\beta_2\beta_4\ldots\beta_{n+1}}\underbrace{\phi_{\lambda}\phi^{\beta_1}\phi^{\beta_2}_{\alpha_2}\phi^{\lambda}_{\alpha_3}}_{\alpha_2\leftrightarrow\alpha_3}\phi^{\beta_4}_{\alpha_4}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}\,,\\
&= \delta^{\alpha_2\alpha_3\alpha_4\ldots\alpha_{n+1}}_{\beta_1\beta_2\beta_4\ldots\beta_{n+1}}\phi_{\lambda}\phi^{\beta_1}\phi^{\lambda}_{\alpha_2}\phi^{\beta_2}_{\alpha_3}\phi^{\beta_4}_{\alpha_4}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}\,,\\
&= \underbrace{\underbrace{\delta^{\alpha_2\alpha_3\alpha_4\ldots\alpha_{n+1}}_{\beta_1\beta_2\beta_4\ldots\beta_{n+1}}\phi_{\lambda}\phi^{\beta_1}\phi^{\lambda}_{\alpha_2}\phi^{\beta_2}_{\alpha_3}\phi^{\beta_4}_{\alpha_4}\ldots\phi^{\beta_{n+1}}_{\alpha_{n+1}}}_{\beta_1\beta_2\beta_4\ldots\beta_{n+1}\to\beta_1\beta_2\beta_3\ldots\beta_n}}_{\alpha_2\alpha_3\alpha_4\ldots\alpha_{n+1}\to\alpha_1\alpha_2\alpha_3\ldots\alpha_n}\,,\\
&= \delta^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_n}\phi_{\lambda}\phi^{\beta_1}\phi^{\lambda}_{\alpha_1}\phi^{\beta_2}_{\alpha_2}\phi^{\beta_3}_{\alpha_3}\ldots\phi^{\beta_n}_{\alpha_n}\,,\\
&= -\,\mathcal{A}_{(2n)}{}^{\alpha_1\alpha_2\ldots\alpha_n}{}_{\beta_1\beta_2\ldots\beta_n}\phi_{\lambda}\phi^{\beta_1}\phi^{\lambda}_{\alpha_1}\phi^{\beta_2}_{\alpha_2}\ldots\phi^{\beta_n}_{\alpha_n}\,,\\
&= -\,\mathcal{L}^{\text{Gal,2}}_{N}\,.
\end{aligned}
\tag{A.13}
$$

APPENDIX B CALCULATION FOR THE CORRECTTION TERMS

To search for the suitable correction terms, let us start with the generalized Lagrangian Eq. (3.44) that is changed all partial derivatives as covariant derivatives.

$$\begin{aligned}\mathcal{L}_n\{f\} &= f(\phi,X)\times\mathcal{L}^{\text{Gal},3}_{N=n+2}\\ &= f(\phi,X)\left(X\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\right)\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\,. \end{aligned}\tag{B.1}$$

This is the covariant generalized Galileons. If the coefficients f do not depend on ϕ, this is the covariant Galileons or the extended Galileons. Considering the variation with respect to ϕ, it is expressed as

$$\begin{aligned}\delta\mathcal{L}_n\{f\} &= \delta\left[Xf(\phi,X)\right]\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\\ &\quad +Xf(\phi,X)\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\delta\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\,. \end{aligned}\tag{B.2}$$

Considering the first term in this equation, we obtain

$$\begin{aligned}&\delta\left[Xf(\phi,X)\right]\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\\ =\;& f\delta\left(\phi_\lambda\phi^\lambda\right)\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\\ &+\phi_\lambda\phi^\lambda\left(\delta f\right)\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\,,\\ =\;& 2f\left(\phi^\lambda\nabla_\lambda\delta\phi\right)\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\\ &+\phi_\lambda\phi^\lambda\left(f_{,\phi}\delta\phi+f_{,X}\delta X\right)\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\,,\\ =\;& 2f\left(\phi^\lambda\nabla_\lambda\delta\phi\right)\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\\ &+\phi_\lambda\phi^\lambda f_{,\phi}\delta\phi\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\\ &+2\phi_\lambda\phi^\lambda f_{,X}\left(\phi^\rho\nabla_\rho\delta\phi\right)\mathcal{A}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}_{(2n)}\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\,. \end{aligned}\tag{B.3}$$

After performing integration by parts and paying attention only on the dangerous terms, we obtain

$$
\begin{aligned}
& \delta\left[X f(\phi, X)\right] \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n} \\
\sim\;& -2\nabla_\lambda\left(\phi^\lambda f \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\delta\phi \\
& -2\nabla_\rho\left(\phi_\lambda\phi^\lambda f_{,X}\phi^\rho \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\delta\phi\,, \\
\sim\;& -2\phi^\lambda f \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \nabla_\lambda\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\delta\phi \\
& -2\,\underbrace{\phi_\lambda\phi^\lambda f_{,X}\phi^\rho \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \nabla_\rho\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\delta\phi}_{\rho\leftrightarrow\lambda}\,, \\
\sim\;& -2\left(f + X f_{,X}\right)\phi^\lambda \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \nabla_\lambda\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\delta\phi\,, \\
\sim\;& -2n\left(f + X f_{,X}\right)\phi^\lambda \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \nabla_\lambda\left(\phi_{\alpha_1\beta_1}\right)\ \phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,.
\end{aligned} \tag{B.4}
$$

For the second term in Eq. (B.2), we obtain

$$
\begin{aligned}
& X f(\phi, X) \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \delta\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right) \\
=\;& n X f(\phi, X) \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \delta\left(\nabla_{\beta_1}\nabla_{\alpha_1}\delta\phi\right)\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\,.
\end{aligned} \tag{B.5}
$$

On performing twice integration by parts, the above expression can be given by

$$
\begin{aligned}
& X f(\phi, X) \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \delta\left(\phi_{\alpha_1\beta_1}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right) \\
=\;& n\nabla_{\alpha_1}\nabla_{\beta_1}\left(X f \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\delta\phi\,, \\
=\;& n\nabla_{\alpha_1}\nabla_{\beta_1}\left(X f \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\delta\phi\,, \\
=\;& \underbrace{2n\nabla_{\beta_1}\left(X f\right) \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \nabla_{\alpha_1}\left(\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\delta\phi}_{A} \\
& +\underbrace{n\nabla_{\alpha_1}\nabla_{\beta_1}\left(X f\right) \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\delta\phi}_{B} \\
& +\underbrace{n X f \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \nabla_{\alpha_1}\nabla_{\beta_1}\left(\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\right)\delta\phi}_{C}\,.
\end{aligned} \tag{B.6}
$$

Let us first consider the term A

$$\begin{aligned} A &= 2n\,(n-1)\,\nabla_{\beta_1}\,(Xf)\,\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\nabla_{\alpha_1}\,(\phi_{\alpha_2\beta_2})\ldots\phi_{\alpha_n\beta_n}\delta\phi\,,\\ &\sim \nabla_{\alpha_1}\,(\phi_{\alpha_2\beta_2})\;,\\ &\sim \nabla_{\alpha_1}\,(\nabla_{\beta_2}\partial_{\alpha_2}\phi)\;,\\ &\sim \partial_{\alpha_1}\left(\partial_{\beta_2}\partial_{\alpha_2}\phi+\Gamma^{\lambda}_{\ \alpha_2\beta_2}\partial_\lambda\phi\right)\;,\\ &\sim \partial_{\alpha_1}\partial_{\beta_2}\partial_{\alpha_2}\phi\,. \end{aligned} \tag{B.7}$$

The above term, $\partial_{\alpha_1}\partial_{\beta_2}\partial_{\alpha_2}\phi$, is eliminated by $\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}$. Therefore, there are no higher order derivatives in the term A. We now consider the term B in Eq. (B.6)

$$\begin{aligned} B &= n\nabla_{\alpha_1}\nabla_{\beta_1}\,(Xf)\,\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,,\\ &= n\nabla_{\alpha_1}\left[(f+Xf_{,X})\,\nabla_{\beta_1}X+Xf_{,\phi}\nabla_{\beta_1}\phi\right]\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,,\\ &\sim n\,(f+Xf_{,X})\,\nabla_{\alpha_1}\nabla_{\beta_1}\left(\phi_\lambda\phi^\lambda\right)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,,\\ &\sim 2n\,(f+Xf_{,X})\,\nabla_{\alpha_1}\left(\phi^\lambda\phi_{\lambda\beta_1}\right)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,,\\ &\sim 2n\,(f+Xf_{,X})\left(\phi^\lambda\phi_{\lambda\beta_1\alpha_1}\right)\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,. \end{aligned} \tag{B.8}$$

This term is canceled by Eq. (B.4). Therefore, the dangerous terms arising from Eq. (B.2), remain only the term C coming from Eq. (B.6). So far Eq. (B.2) yields

$$\begin{aligned} \delta\mathcal{L}_n\{f\} &\sim nXf\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\nabla_{\alpha_1}\nabla_{\beta_1}\,(\phi_{\alpha_2\beta_2}\ldots\phi_{\alpha_n\beta_n})\,\delta\phi\,,\\ &\sim n\,(n-1)\,Xf\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\nabla_{\alpha_1}\left[(\nabla_{\beta_1}\phi_{\alpha_2\beta_2})\,\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\right]\delta\phi\,,\\ &\sim n\,(n-1)\,Xf\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\,(\nabla_{\beta_1}\phi_{\alpha_2\beta_2})\,\nabla_{\alpha_1}\,(\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n})\,\delta\phi\\ &\quad+n\,(n-1)\,Xf\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\,(\nabla_{\alpha_1}\nabla_{\beta_1}\phi_{\alpha_2\beta_2})\,\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,,\\ &\sim n\,(n-1)\,Xf\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\,(\nabla_{\beta_1}\phi_{\alpha_2\beta_2})\,\nabla_{\alpha_1}\,(\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n})\,\delta\phi\\ &\quad+n\,(n-1)\,Xf\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\,(\nabla_{\alpha_1}\nabla_{\beta_1}\phi_{\alpha_2\beta_2})\,\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,. \end{aligned} \tag{B.9}$$

Based on the similar consideration, the first term in above equation includes the terms, i.e., $\nabla_{\beta_1}\phi_{\alpha_2\beta_2}$ that is the same as Eq. (B.7). Thus there are no third order derivatives in the first term. We then get

$$\begin{aligned}\delta\mathcal{L}_n\{f\} &\sim n\,(n-1)\,Xf\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\left(\nabla_{\alpha_1}\nabla_{\beta_1}\phi_{\alpha_2\beta_2}\right)\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,,\\ &\sim n\,(n-1)\,Xf\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\left(\nabla_{\alpha_1}\nabla_{\beta_1}\nabla_{\beta_2}\nabla_{\alpha_2}\phi\right)\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,.\end{aligned} \tag{B.10}$$

Since a generic tensor can be written in terms of symmetric and antisymmetric tensor as

$$\mathcal{T}_{\beta_1\beta_2} = \mathcal{T}_{(\beta_1\beta_2)} + \mathcal{T}_{[\beta_1\beta_2]}\,, \tag{B.11}$$

one can write

$$\begin{aligned}\nabla_{\beta_1}\nabla_{\beta_2}\phi &= \phi_{(\beta_1\beta_2)} + \phi_{[\beta_1\beta_2]}\,,\\ &= \frac{1}{2}\left(\nabla_{\beta_2}\nabla_{\beta_1} + \nabla_{\beta_1}\nabla_{\beta_2}\right)\phi + \frac{1}{2}\left(\nabla_{\beta_2}\nabla_{\beta_1}\phi - \nabla_{\beta_1}\nabla_{\beta_2}\right)\phi\,.\end{aligned} \tag{B.12}$$

Replacing the above relation into Eq. (B.10), we obtain

$$\delta\mathcal{L}_n\{f\} \sim \frac{1}{2}n\,(n-1)\,Xf\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\nabla_{\alpha_1}\left[\nabla_{\beta_1}\nabla_{\beta_2}\right]\phi_{\alpha_2}\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,. \tag{B.13}$$

We know that the Riemann tensor can be defined via the relation

$$\left[\nabla_\alpha, \nabla_\beta\right]\phi_\lambda = -R^{\sigma}{}_{\lambda\alpha\beta}\phi_\sigma = -R_{\sigma\lambda\alpha\beta}\phi^\sigma = R_{\lambda\sigma\alpha\beta}\phi^\sigma\,. \tag{B.14}$$

Then Eq. (B.13) takes the form

$$\delta\mathcal{L}_n\{f\} \sim \frac{1}{2}n\,(n-1)\,Xf\mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n}\nabla_{\alpha_1}\left(R_{\alpha_2\lambda\beta_1\beta_2}\right)\phi^\lambda\phi_{\alpha_3\beta_3}\ldots\phi_{\alpha_n\beta_n}\delta\phi\,. \tag{B.15}$$

Applying the Bianchi identity, we have

$$\nabla_\lambda R_{\alpha\beta\rho\sigma} + \nabla_\beta R_{\lambda\alpha\rho\sigma} + \nabla_\alpha R_{\beta\lambda\rho\sigma} = 0\,. \tag{B.16}$$

The term, $\nabla_{\alpha_1} R_{\alpha_2\lambda\beta_1\beta_2}$, in Eq. (B.15) can be written as

$$\begin{aligned}
\nabla_{\alpha_1} R_{\alpha_2\lambda\beta_1\beta_2} &= -\nabla_{\lambda} R_{\alpha_1\alpha_2\beta_1\beta_2} - \underbrace{\nabla_{\alpha_2} R_{\lambda\alpha_1\beta_1\beta_2}}_{\alpha_2\leftrightarrow\alpha_1} , \\
&= -\nabla_{\lambda} R_{\alpha_1\alpha_2\beta_1\beta_2} + \nabla_{\alpha_1} R_{\lambda\alpha_2\beta_1\beta_2} , \\
&= -\nabla_{\lambda} R_{\alpha_1\alpha_2\beta_1\beta_2} - \nabla_{\alpha_1} R_{\alpha_2\lambda\beta_1\beta_2} , \\
\nabla_{\lambda} R_{\alpha_1\alpha_2\beta_1\beta_2} &= -\frac{1}{2}\nabla_{\alpha_1} R_{\alpha_2\lambda\beta_1\beta_2} . \qquad \text{(B.17)}
\end{aligned}$$

Inserting this relation into Eq. (B.15), we obtain

$$\delta\mathcal{L}_n\{f\} \sim -\frac{1}{4} n\,(n-1)\, X f \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \nabla_{\lambda} R_{\alpha_1\alpha_2\beta_1\beta_2} \phi^{\lambda} \phi_{\alpha_3\beta_3} \ldots \phi_{\alpha_n\beta_n} \delta\phi . \qquad \text{(B.18)}$$

Starting with the above equation, one can add the correction term, $\mathcal{C}_r$, that its variation with respect to ϕ gives the term that cancels out Eq.(B.15). Then we choose

$$\mathcal{C}_r = \left[-\frac{1}{8} n\,(n-1) \int_{X_0}^{X} f(\phi, X_1) X_1 dX_1\right] \mathcal{A}_{(2n)}^{\alpha_1\ldots\alpha_n\beta_1\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \phi_{\alpha_3\beta_3} \ldots \phi_{\alpha_n\beta_n} . \qquad \text{(B.19)}$$

where X_0 is a constant.

The variation of $\mathcal{C}_r$ with respect to ϕ is

$$\begin{aligned}
\delta\mathcal{C}_r = &\underbrace{-\frac{1}{8} n\,(n-1)\, f(\phi, X) X\, (\delta X)\, \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \phi_{\alpha_3\beta_3} \ldots \phi_{\alpha_n\beta_n}}_{C_1} \\
&\underbrace{-\frac{1}{8} n\,(n-1) \int_{X_0}^{X} f(\phi, X_1) X_1 dX_1 \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \delta\left(\phi_{\alpha_3\beta_3} \ldots \phi_{\alpha_n\beta_n}\right)}_{C_2} .
\end{aligned} \qquad \text{(B.20)}$$

Considering the C_1 term, it reads

$$\begin{aligned}
C_1 &= -\frac{1}{8} n\,(n-1)\, f X \left(2\phi^{\lambda}\nabla_{\lambda}\delta\phi\right) \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \phi_{\alpha_3\beta_3} \ldots \phi_{\alpha_n\beta_n} , \\
&= -\frac{1}{4} n\,(n-1)\, f X \left(\phi^{\lambda}\nabla_{\lambda}\delta\phi\right) \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \phi_{\alpha_3\beta_3} \ldots \phi_{\alpha_n\beta_n} .
\end{aligned} \qquad \text{(B.21)}$$

After integration by parts, it leads to

$$
\begin{aligned}
C_1 &= \frac{1}{4} n\left(n-1\right) \nabla_\lambda \left(f X \phi^\lambda \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \phi_{\alpha_3\beta_3} \ldots \phi_{\alpha_n\beta_n} \right) \delta\phi \,, \\
&\sim \frac{1}{4} n\left(n-1\right) f X \phi^\lambda \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} \nabla_\lambda \left(R_{\alpha_1\alpha_2\beta_1\beta_2} \right) \phi_{\alpha_3\beta_3} \ldots \phi_{\alpha_n\beta_n} \delta\phi \\
&\quad + \frac{1}{4} n\left(n-1\right) f X \phi^\lambda \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \nabla_\lambda \left(\phi_{\alpha_3\beta_3} \ldots \phi_{\alpha_n\beta_n} \right) \delta\phi \,.
\end{aligned}
\tag{B.22}
$$

We can see that the second term in above expression cancels with Eq. (B.18). However, there is still the third order derivative of ϕ. To eliminate this third order derivative, we write the term C_2 in Eq. (B.20) as

$$
C_2 = \int_{X_0}^{X} \mathcal{D}_n dX_1 \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \left(\nabla_{\beta_3} \nabla_{\alpha_3} \delta\phi \right) \phi_{\alpha_4\beta_4} \ldots \phi_{\alpha_n\beta_n} \,,
\tag{B.23}
$$

where $\mathcal{D}_n \equiv -\frac{1}{8} n\left(n-1\right)\left(n-2\right) f(\phi, X_1) X_1$.

After twice integration by parts, it yields

$$
\begin{aligned}
C_2 &= \nabla_{\alpha_3} \nabla_{\beta_3} \left[\int_{X_0}^{X} \mathcal{D}_n dX_1 \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \phi_{\alpha_4\beta_4} \ldots \phi_{\alpha_n\beta_n} \right] \delta\phi \,, \\
&\sim \mathcal{P}_n \nabla_{\alpha_3} \nabla_{\beta_3} \left(\phi_\lambda \phi^\lambda \right) \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \phi_{\alpha_4\beta_4} \ldots \phi_{\alpha_n\beta_n} \delta\phi \\
&\quad + \int_{X_0}^{X} \mathcal{D}_n dX_1 \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \nabla_{\alpha_3} \nabla_{\beta_3} \left(\phi_{\alpha_4\beta_4} \ldots \phi_{\alpha_n\beta_n} \right) \delta\phi \,, \\
&\sim \tilde{\mathcal{P}}_n \phi^\lambda \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \nabla_\lambda \left(\phi_{\alpha_4\beta_4} \ldots \phi_{\alpha_n\beta_n} \right) \delta\phi \\
&\quad + \int_{X_0}^{X} \mathcal{D}_n dX_1 \mathcal{A}_{(2n)}^{\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots\beta_n} R_{\alpha_1\alpha_2\beta_1\beta_2} \nabla_{\alpha_3} \nabla_{\beta_3} \left(\phi_{\alpha_4\beta_4} \ldots \phi_{\alpha_n\beta_n} \right) \delta\phi \,,
\end{aligned}
\tag{B.24}
$$

where $\mathcal{P}_n \equiv -\frac{1}{8} n\left(n-1\right)\left(n-2\right) f(\phi, X) X$ and $\tilde{\mathcal{P}}_n \equiv -\frac{1}{4} n\left(n-1\right) f(\phi, X) X$.

From the above result, the first term is canceled with the second term in C_1. However, the fourth order derivative still appears from the second term of the above result. In order to eliminate the fourth order derivative terms, we have to add other correction term. Fortunately, the fourth order terms do not appear for $n = 2$ and $n = 3$ which correspond to quartic and quintic Lagrangians of Galileon models. Therefore, the additional correction terms are unneeded. In order to construct the covariantized Galileons for the quartic

Lagrangian, we have to combine Eq. (3.44) and Eq. (B.19) as

$$\begin{aligned}
\mathcal{L}_4 &= \mathcal{L}_{n=2}\{f\} + \mathcal{C}_r \,, \\
&= \left[-\frac{1}{4}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \mathcal{A}_{(2n=4)}^{\alpha_1\alpha_2\beta_1\beta_2} R_{\alpha_1\alpha_2\beta_1\beta_2} \\
&\quad - f(\phi, X)X\left(\Box\phi^2 - \phi_{\alpha\beta}\phi^{\alpha\beta}\right) \,, \\
&= \left[-\frac{1}{4}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \mathcal{A}_{(2n=4)}{}^{\alpha_1\alpha_2}{}_{\beta_1\beta_2} R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \\
&\quad - f(\phi, X)X\left(\Box\phi^2 - \phi_{\alpha\beta}\phi^{\alpha\beta}\right) \,, \\
&= \left[\frac{1}{4}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \delta^{\alpha_1\alpha_2}_{\beta_1\beta_2} R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \\
&\quad - f(\phi, X)X\left(\Box\phi^2 - \phi_{\alpha\beta}\phi^{\alpha\beta}\right) \,, \\
&= \left[\frac{1}{4}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \left(\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} - \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\right) R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} \\
&\quad - f(\phi, X)X\left(\Box\phi^2 - \phi_{\alpha\beta}\phi^{\alpha\beta}\right) \,, \\
&= \left[\frac{1}{2}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] R - f(\phi, X)X\left(\Box\phi^2 - \phi_{\alpha\beta}\phi^{\alpha\beta}\right) \,. \qquad \text{(B.25)}
\end{aligned}$$

We suppose that

$$G_4(\phi, X) \equiv \frac{1}{2}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1 \,, \qquad \text{(B.26)}$$

so that

$$G_{4,X} \equiv \frac{1}{2} f(\phi, X)X \,. \qquad \text{(B.27)}$$

According to the above two equations, Eq. (B.25) can be written as

$$\mathcal{L}_4^H = G_4(\phi, X)R - 2G_{4,X}\left(\Box\phi^2 - \phi_{\alpha\beta}\phi^{\alpha\beta}\right) \,. \qquad \text{(B.28)}$$

This is the quartic Horndeski Lagrangian. Similarly, for the quintic Lagrangian, we now obtain

$$
\begin{aligned}
\mathcal{L}_5 &= \mathcal{L}_{n=3}\{f\} + \mathcal{C}_r \,, \\
&= \left[-\frac{3}{4}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \mathcal{A}_{(2n=6)}^{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3} R_{\alpha_1\alpha_2\beta_1\beta_2}\phi_{\alpha_3\beta_3} \\
&\quad -f(\phi, X)X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right) \,, \\
&= \left[-\frac{3}{4}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \mathcal{A}_{(2n=6)}{}^{\alpha_1\alpha_2\alpha_3}{}_{\beta_1\beta_2\beta_3} R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\phi^{\beta_3}_{\alpha_3} \\
&\quad -f(\phi, X)X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right) \,, \\
&= \left[\frac{3}{4}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\phi^{\beta_3}_{\alpha_3} \\
&\quad -f(\phi, X)X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right) \,, \\
&= \left[\frac{3}{4}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \left(\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\delta^{\alpha_3}_{\beta_3} - \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_3}\delta^{\alpha_3}_{\beta_2} + \delta^{\alpha_1}_{\beta_3}\delta^{\alpha_2}_{\beta_1}\delta^{\alpha_3}_{\beta_2}\right. \\
&\quad \left. -\delta^{\alpha_1}_{\beta_3}\delta^{\alpha_2}_{\beta_2}\delta^{\alpha_3}_{\beta_1} + \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_3}\delta^{\alpha_3}_{\beta_1} - \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\delta^{\alpha_3}_{\beta_3}\right) R_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\phi^{\beta_3}_{\alpha_3} \\
&\quad -f(\phi, X)X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right) \,, \\
&= \left[\frac{3}{4}\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \left(2R\Box\phi - 4R_{\alpha\beta}\phi^{\alpha\beta}\right) \\
&\quad -f(\phi, X)X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right) \\
&\quad -f(\phi, X)X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right) \,, \\
&= \left[3\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \left(\frac{1}{2}R\Box\phi - R_{\alpha\beta}\phi^{\alpha\beta}\right) \\
&\quad -f(\phi, X)X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right) \,, \\
&= -\left[3\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] \left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)\phi^{\alpha\beta} \\
&\quad -f(\phi, X)X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right) \,, \\
&= -\left[3\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1\right] G_{\alpha\beta}\phi^{\alpha\beta} \\
&\quad -f(\phi, X)X\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right) \,,
\end{aligned} \tag{B.29}
$$

where $G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is Einstein tensor. Again, we suppose that

$$G_5(\phi, X) \equiv -3\int_{X_0}^{X} f(\phi, X_1)X_1 dX_1 \,, \tag{B.30}$$

so that

$$G_{5,X} \equiv -3f(\phi, X)X \,. \tag{B.31}$$

From the above two equations, Eq. (B.29) can be written as

$$\mathcal{L}_5^H = G_5(\phi, X)G_{\alpha\beta}\phi^{\alpha\beta} + \frac{1}{3}G_{5,X}\left(\Box\phi^3 - 3\Box\phi\phi_{\alpha\beta}\phi^{\alpha\beta} + 2\phi_{\lambda\alpha}\phi^{\alpha\beta}\phi^{\lambda}_{\beta}\right) . \tag{B.32}$$

APPENDIX C CONSTRAINT EQUATIONS IN TERMS OF DIMENSIONLESS VARIABLES

In terms of the dimensionless variables, we can write Eq. (4.49) as

$$
\begin{aligned}
0 \;=\; & \frac{1}{2c_4+v_r^{n_4}}\Bigg[v_r^{-n_4}\Bigg(v_r^{-n_2-n_6}\left(2c_4+v_r^{n_4}\right)\Bigg(2c_4n_4v_r^{n_2+n_6}\Bigg(z_rx_\lambda\Bigg(-\frac{\dot{H}}{H^2} \\
& +z_rx_\lambda-2\Bigg)-x_\lambda z_r'+n_4x_\lambda^2\Bigg)-x_r^2v_r^{n_4}\left(c_6v_r^{n_2}-c_2v_r^{n_6}\right)\Bigg) \\
& +\left(2c_4+v_r^{n_4}\right)\left(c_4(4\frac{\dot{H}}{H^2}+6)+\left(2\frac{\dot{H}}{H^2}+\Omega_\gamma+3\right)v_r^{n_4}\right) \\
& +c_4n_4z_r^2x_\lambda^2\left(c_4\left(n_4-4\right)+2\left(n_4-1\right)v_r^{n_4}\right)-4c_4n_4^2z_rx_\lambda^2\left(2c_4+v_r^{n_4}\right) \\
& -c_4n_4x_\lambda\left(2c_4+v_r^{n_4}\right)\left(z_rx_\lambda-4\right)\Bigg)\Bigg]\;, \qquad \text{(C.1)}
\end{aligned}
$$

where $v_r \equiv y_r/x_r^2$.

Eq. (4.50) can be written in terms of the dimensionless variables as

$$
\begin{aligned}
0 \;=\; & \frac{1}{\left(2c_4+v_r^{n_4}\right)^2}\Big[v_r^{-n_2-n_4-n_6}\Big(-c_4^2v_r^{n_4}\Big(-4c_6\left(3n_4-2n_6-1\right)x_r^2v_r^{n_2} \\
& -4c_2\left(2n_2-3n_4+1\right)x_r^2v_r^{n_6}+3v_r^{n_2+n_6}\Big(4n_4^3x_r^2x_\lambda^2+4n_4\left(2x_rx_\lambda\right. \\
& \left.-2z_rx_\lambda+\Omega_\gamma-2\right)+n_4^2x_\lambda\left(x_r\left(8-2z_rx_\lambda\right)+z_r\left(z_rx_\lambda+4\right)\right)+12\Big)\Big) \\
& -2c_4v_r^{2n_4}\Big(-c_6\left(3n_4-4n_6-2\right)x_r^2v_r^{n_2}-c_2\left(4n_2-3n_4+2\right)x_r^2v_r^{n_6} \\
& +3v_r^{n_2+n_6}\left(n_4\left(x_rx_\lambda-z_rx_\lambda+\Omega_\gamma-1\right)+3\right)\Big)-6c_4^3v_r^{n_2+n_6}\Big(n_4^3x_\lambda^2\left(-4x_rz_r\right. \\
& \left.+4x_r^2-z_r^2\right)+n_4^2x_\lambda\left(x_r\left(8-2z_rx_\lambda\right)+z_r\left(z_rx_\lambda+4\right)\right) \\
& +4n_4\left(x_\lambda\left(x_r-z_r\right)-1\right)+4\Big)+v_r^{3n_4}\Big(-c_6\left(2n_6+1\right)x_r^2v_r^{n_2} \\
& +c_2\left(2n_2+1\right)x_r^2v_r^{n_6}-3v_r^{n_2+n_6}\Big)\Big)\Big]+3\left(\Omega_m+\Omega_\gamma\right)\;. \qquad \text{(C.2)}
\end{aligned}
$$

This equation can be used to express Ω_m in terms of the other dimensionless variables. Eq. (4.55) can be written in terms of the dimensionless variables as

$$0 = \tilde{E}_1 + \tilde{E}_2\,. \qquad \text{(C.3)}$$

where $\tilde{E}_1$ and $\tilde{E}_2$ are respectively written as

$$
\begin{aligned}
\tilde{E}_1 = v_r^{-n_2-4n_4-n_6} & \left(c_4^3\left(-6n_4^4\left(8x_r^3-18z_rx_r^2+9z_r^2x_r+z_r^3\right)x_\lambda^3v_r^{n_2+n_6}\right.\right.\\
&+3n_4^3x_\lambda^2\left(8x_\lambda x_r^3-96x_r^2-3z_r\left(z_rx_\lambda-24\right)x_r+z_r^2\left(z_rx_\lambda+12\right)\right)v_r^{n_2+n_6}\\
&+12n_4\left(c_6x_r^2\left(\left(2n_6-1\right)x_rx_\lambda-\left(2n_6z_r+z_r\right)x_\lambda+8\right)v_r^{n_2}\right.\\
&-c_2x_r^2\left(\left(2n_2-1\right)x_rx_\lambda-\left(2n_2z_r+z_r\right)x_\lambda+8\right)v_r^{n_6}\\
&\left.+\left(x_r-z_r\right)\left(\Omega_\gamma-3\right)x_\lambda v_r^{n_2+n_6}\right)-12n_4^2x_\lambda\left(c_6x_r^2\left(2x_r+z_r\right)v_r^{n_2}\right.\\
&-c_2x_r^2\left(2x_r+z_r\right)v_r^{n_6}+\left(x_\lambda\left(z_rx_\lambda-4\right)x_r^2-2\left(\Omega_\gamma+2z_rx_\lambda-7\right)x_r\right.\\
&\left.\left.+z_r\left(-\Omega_\gamma+2z_rx_\lambda+4\right)\right)v_r^{n_2+n_6}\right)+8x_r\left(3v_r^{n_2+n_6}\left(\Omega_m+\Omega_\gamma\right)Q_\lambda x_\lambda\right.\\
&-x_r\left(c_6v_r^{n_2}\left(2\left(x_r-z_r\right)x_\lambda n_6^2+\left(x_rx_\lambda-3z_rx_\lambda+6\right)n_6-z_rx_\lambda+6\right)\right.\\
&\left.\left.\left.-c_2v_r^{n_6}\left(2\left(x_r-z_r\right)x_\lambda n_2^2+\left(x_rx_\lambda-3z_rx_\lambda+6\right)n_2-z_rx_\lambda+6\right)\right)\right)\right)v_r^{n_4}\\
&-3c_4^2\left(8n_4^4x_r^2\left(x_r-z_r\right)x_\lambda^3v_r^{n_2+n_6}-2n_4^3x_\lambda^2\left(x_\lambda x_r^3+\left(z_rx_\lambda-16\right)x_r^2+8z_rx_r\right.\right.\\
&\left.+2z_r^2\right)v_r^{n_2+n_6}+n_4\left(-4c_6x_r^2\left(\left(2n_6-1\right)x_rx_\lambda-\left(2n_6z_r+z_r\right)x_\lambda+8\right)v_r^{n_2}\right.\\
&\left.+4c_2x_r^2\left(\left(2n_2-1\right)x_rx_\lambda-\left(2n_2z_r+z_r\right)x_\lambda+8\right)v_r^{n_6}-2\left(x_r-z_r\right)\left(2\Omega_\gamma-3\right)x_\lambda v_r^{n_2+n_6}\right)\\
&+n_4^2x_\lambda\left(6c_6x_r^2z_rv_r^{n_2}-6c_2x_r^2z_rv_r^{n_6}+\left(x_\lambda\left(z_rx_\lambda-4\right)x_r^2+\left(8-4z_rx_\lambda\right)x_r\right.\right.\\
&\left.\left.+2z_r\left(-3\Omega_\gamma+z_rx_\lambda+5\right)\right)v_r^{n_2+n_6}\right)-4x_r\left(3v_r^{n_2+n_6}\left(\Omega_m+\Omega_\gamma\right)Q_\lambda x_\lambda\right.\\
&-x_r\left(c_6v_r^{n_2}\left(2\left(x_r-z_r\right)x_\lambda n_6^2+\left(x_rx_\lambda-3z_rx_\lambda+6\right)n_6-z_rx_\lambda+6\right)\right.\\
&\left.\left.\left.\left.-c_2v_r^{n_6}\left(2\left(x_r-z_r\right)x_\lambda n_2^2+\left(x_rx_\lambda-3z_rx_\lambda+6\right)n_2-z_rx_\lambda+6\right)\right)\right)\right)v_r^{2n_4}\\
&-3c_4\left(2\left(x_r-z_r\right)\left(-c_6x_r^2v_r^{n_2}+c_2x_r^2v_r^{n_6}+\left(\Omega_\gamma-1\right)v_r^{n_2+n_6}\right)x_\lambda n_4^2\right.\\
&+\left(-c_6x_r^2\left(\left(2n_6-1\right)x_rx_\lambda-\left(2n_6z_r+z_r\right)x_\lambda+8\right)v_r^{n_2}+c_2x_r^2\left(\left(2n_2-1\right)x_rx_\lambda\right.\right.,
\end{aligned}
\tag{C.4}
$$

$$\tilde{E}_2 = -\left(2n_2z_r + z_r\right)x_\lambda + 8\right)v_r^{n_6} - \left(x_r - z_r\right)\left(\Omega_\gamma - 1\right)x_\lambda v_r^{n_2+n_6}\Big)\, n_4$$
$$- 2x_r\,\Big(3v_r^{n_2+n_6}\left(\Omega_m + \Omega_\gamma\right)Q_\lambda x_\lambda - x_r\,\Big(c_6v_r^{n_2}\,\Big(2\left(x_r - z_r\right)x_\lambda n_6^2$$
$$+ \left(x_rx_\lambda - 3z_rx_\lambda + 6\right)n_6 - z_rx_\lambda + 6\Big) - c_2v_r^{n_6}\,\Big(2\left(x_r - z_r\right)x_\lambda n_2^2$$
$$+ \left(x_rx_\lambda - 3z_rx_\lambda + 6\right)n_2 - z_rx_\lambda + 6\Big)\Big)\Big)\Big)\, v_r^{3n_4} + x_r\,\Big(3v_r^{n_2+n_6}\left(\Omega_m + \Omega_\gamma\right)Q_\lambda x_\lambda$$
$$-x_r\,\Big(c_6v_r^{n_2}\,\Big(2\left(x_r - z_r\right)x_\lambda n_6^2 + \left(x_rx_\lambda - 3z_rx_\lambda + 6\right)n_6 - z_rx_\lambda + 6\Big)$$
$$-c_2v_r^{n_6}\,\Big(2\left(x_r - z_r\right)x_\lambda n_2^2 + \left(x_rx_\lambda - 3z_rx_\lambda + 6\right)n_2 - z_rx_\lambda + 6\Big)\Big)\Big)\, v_r^{4n_4}$$
$$+6c_4^4n_4x_\lambda\,\Big(4n_4^2x_\lambda^2x_r^3 + 2n_4x_\lambda\,\Big(2z_rx_\lambda n_4^2 - 2\left(z_rx_\lambda + 8\right)n_4 - z_rx_\lambda + 4\Big)\, x_r^2$$
$$-\,\Big(12z_r^2x_\lambda^2n_4^3 + z_rx_\lambda\left(3z_rx_\lambda - 40\right)n_4^2 - 8\left(z_rx_\lambda - 4\right)n_4 + 4\Big)\, x_r$$
$$-\left(n_4 - 1\right)z_r\left(n_4z_rx_\lambda - 2\right)^2\Big)\, v_r^{n_2+n_6}\Big)\ . \tag{C.5}$$

To compute the equation for z_r, we substitute Ω_m solved from Eq. (C.2) into the above equation. the resulting equation can be written in the form

$$b_3z_r^3 + b_2z_r^2 + b1z_r + b_0 = 0\,, \tag{C.6}$$

where b_0, b_1, b_2 and b_3 are complicated funtions of the dimensionless variables of Ω_γ, x_r, y_r and x_λ. Using Eq. (C.6), we can compute the expression for z_r in the form

$$z_{r\,1} = -\frac{\sqrt[3]{2}\left(3b_1b_3 - b_2^2\right)}{3b_3\sqrt[3]{\Delta}} + \frac{\sqrt[3]{\Delta}}{3\sqrt[3]{2}b_3} - \frac{b_2}{3b_3}\,, \tag{C.7}$$

$$z_{r\,2} = \frac{\left(1 + i\sqrt{3}\right)\left(3b_1b_3 - b_2^2\right)}{3(2^{2/3}b_3\sqrt[3]{\Delta})} - \frac{\left(1 - i\sqrt{3}\right)\sqrt[3]{\Delta}}{6\sqrt[3]{2}b_3} - \frac{b_2}{3b_3}\,, \tag{C.8}$$

$$z_{r\,3} = \frac{\left(1 - i\sqrt{3}\right)\left(3b_1b_3 - b_2^2\right)}{3(2^{2/3}b_3\sqrt[3]{\Delta})} - \frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{\Delta}}{6\sqrt[3]{2}b_3} - \frac{b_2}{3b_3}\,, \tag{C.9}$$

where $\Delta = -2b_2^3 + 9b_1b_3b_2 - 27b_0b_3^2 + \sqrt{4\left(3b_1b_3 - b_2^2\right)^3 + \left(-2b_2^3 + 9b_1b_3b_2 - 27b_0b_3^2\right)^2}$. The physically relevant solution is selected from the above solutions by the requirement that z_r becomes unity when $x_r = y_r = 1, \Omega_\gamma = 0$ and c_2 as well as c_6 are given by Eqs. (4.66) and (4.67).

BIOGRAPHY

BIOGRAPHY

Name-Surname	Wittaya Thipaksorn
Date of Birth	May 12, 1980
Place of Birth	Trang Province, Thailand
Address	8/3 Moo 4 Tambon Na Chum Hed, Amphoe Yan Ta Khao, Trang Province, Thailand 92140
Work Experiences	
2010	Lecturer, Thaksin University, Phatthalung Campus, Phatthalung, Thailand 93110
2009	Lecturer, Phetchaburi Rajabhat University, Phetchaburi, Thailand 76000
Education Background	
2015	H.G.D. (Quantum fields, Gravitation and Cosmology), Naresuan University, Phitsanulok, Thailand 65000
2007	M.Sc. (Physics), Thaksin University, Songkhla, Thailand 90000
2003	B.Sc. (Physics), Thaksin University, Songkhla, Thailand 90000