THE MULTI-TIME PROPAGATORS AND THE CONSISTENCY CONDITION

SIWAPORN SUNGTED

A Thesis Submitted to Graduate School of Naresuan University
in Partial Fulfillment of the Requirements
for the Master of Science Degree in Theoretical Physics
April 2022

Thesis entitled “The multi-time propagators and the consistency condition”
by Siwaporn Sungted
has been approved by the Graduate School as partial fulfillment of the requirements for
the Master of Science in Theoretical Physics of
Naresuan University.

Oral Defense Committee

. . . . Chair
(Assistant Professor Monsit Tanasittikosol, Ph.D.)

. . . . Advisor
(Assistant Professor Sikarin Yoo-Kong, Ph.D.)

. . . . Co-Advisor
(Assistant Professor Pichet Vanichchapongjaroen, Ph.D.)

. . . . Internal Examiner
(Assistant Professor Seckson Sukhasena, Ph.D.)

Approved

. . . .
(Associate Professor Krongkarn Chootip, Ph.D.)
Dean of the Graduate School

ACKNOWLEDGMENTS

I would like to express the deepest appreciation to my advisor, Assistant Professor Sikarin Yoo-Kong, for giving me the opportunity to do this research and providing invaluable guidance. I am very grateful to you for teaching and explaining me useful knowledge about theoretical physics.

Futhermore, I would like to thank Assistant Professor Pichet Vanichchapongjaroen for the valuable discussion and suggestion to carry out the research.

I would like to thank all people at IF who always be friendly and kindly. Also I would like to thank my family for supporting and encouragement all the time in IF.

Finally, I am grateful for supporting by Development and Promotion of Science and Technology Talents Project (DPST).

Siwaporn Sungted

LIST OF CONTENTS

LIST OF CONTENTS (CONT.)

LIST OF TABLES

LIST OF FIGURES

Figure **Page**

Title	THE MULTI-TIME PROPAGATORS AND THE CONSISTENCY CONDITION
Author	Siwaporn Sungted
Advisor	Assistant Professor Sikarin Yoo-Kong, Ph.D.
Co-Advisor	Assistant Professor Pichet Vanichchapongjaroen, Ph.D.
Academic Paper	Thesis M.S. in Theoretical Physics, Naresuan University, 2022
Keywords	Multi-time system, Feynman's path integration, Consistency condition, Wilson line

ABSTRACT

For a non-relativistic quantum system of N particles, the wave function is a function of $3N$ spatial coordinates and one temporal coordinate. The relativistic generalisation of this wave function is a function of N time variables known as the multi-time wave function and its evolution is described by N Schrödinger equations, one for each time variable. To guarantee the existence of a non-trivial common solution to these N equations, the N Hamiltonians need to satisfy a compatible condition known as an integrability condition. In this work, the integrability condition will be expressed in terms of Lagrangians. The time evolution of a wave function with N time variables through the Feynman picture of quantum mechanics is derived. However, these evolutions will be compatible if and only if the N Lagrangians satisfy a certain relation called the consistency condition which could be expressed in terms of the Wilson line. As a consequence of this consistency condition, the evolution of the wave function gives rise to a key feature called the "path-independent" property on the space of time variables. This would suggest that one must consider all possible paths not only on the space of dependent variables

(spatial variables) but also on the space of independent variables (temporal variables). In the view of the geometry, this consistency condition can be considered as a zero curvature condition and the multi-time evolutions can be treated as compatible parallel transport processes on flat space of time variables.

CHAPTER I

INTRODUCTION

1.1 Background and motivation

In non-relativistic quantum mechanics, the wave function for N particles can be expressed as $\Psi(q_1, q_2, ..., q_N, t)$, where $q_k \in \mathbb{R}^d$ in d-dimensional space, $k = 1, 2, ..., N$. If one asks for the relativistic counterpart of this wave function we encounter with the difficulty as follows. Since there is only one time variable in the wave function, it is a bit puzzle to perform the Lorentz transformation. The argument of Ψ can be treated as a collection of N simultaneous space-time points $(t, q_1), ..., (t, q_N)$ which under the Lorentz transformation is changed to $(t'_1, q'_1), ..., (t'_N, q'_N)$, of course, in general, $t'_1 \neq t'_2 \neq ... \neq t'_N$. Then it is quite natural to introduce the multi-time structure into the wave function $\Phi(q_1, t_1, ..., q_N, t_N)$ to manifest the Lorentz transformation. This idea was first introduced by Dirac in 1932 [1]. To capture the Dirac's idea, we consider the system of N charged particles interacting with the electromagnetic field (EM-field). The evolution of the system can be described on the phase space with the coordinates $(p, q) = (p_1, p_2, ..., p_N, q_1, q_2, ..., q_N)$. The Schrödinger equation is given by

$$\left(H_{\text{EM}} + \sum_{j=1}^{N} H_j(q_j, p_j, a(q_j)) + \frac{\hbar}{i}\frac{\partial}{\partial t} \right) \Psi = 0 \ , \tag{1.1}$$

where H_{EM} is the Hamiltonian of the EM-field, H_j is the (time-independent) Hamiltonian of the j^{th}-particle and contains the coupling between the j^{th}-particle and the EM-field, i.e. $a(q_j)$. We now introduce the unitary operator

$$u = e^{-\frac{i}{\hbar} H_{\text{EM}} t} \tag{1.2}$$

such that the wave function Ψ will be transformed to

$$\Phi = u\Psi \ . \tag{1.3}$$

Then Equation (1.1) can be simplified as follows

$$\left(\sum_{j=1}^{N} u H_j u^{-1} + \frac{\hbar}{i} \frac{\partial}{\partial t} \right) \Phi \quad = 0 \; . \tag{1.4}$$

Next, we introduce the transformation of the coupling a as

$$\mathscr{U} = u a u^{-1} \; . \tag{1.5}$$

Now Equation (1.4) becomes

$$\left(\sum_{j=1}^{N} H_j(q_j, p_j, \mathscr{U}(q_j, t)) + \frac{\hbar}{i} \frac{\partial}{\partial t} \right) \Phi = 0 \; . \tag{1.6}$$

The thing is that since u depends on time and by introducing the transformation $\mathscr{U}$, the Hamiltonian H_j is now effectively time-dependent. Furthermore, in oder to decouple the temporal part of Equation (1.6), it is quite natural to introduce a set of time variables $t = (t_1, t_2, ..., t_N)$ such that

$$\sum_{j=1}^{N} \left(H_j(q_j, p_j, \mathscr{U}(q_j, t_j)) + \frac{\hbar}{i} \frac{\partial}{\partial t_j} \right) \Phi = 0 \; , \tag{1.7}$$

resulting in N separable time-dependent Schrödinger equations

$$\left(H_j(q_j, p_j, \mathscr{U}(q_j, t_j)) + \frac{\hbar}{i} \frac{\partial}{\partial t_j} \right) \Phi(q_1, t_1, q_2, t_2, ..., q_N, t_N) = 0 \; , \quad j = 1, 2, ..., N \; . \tag{1.8}$$

It seems to suggest that $\Phi(q_1, t_1, q_2, t_2, ..., q_N, t_N)$ must be treated as the multi-time wave function [2]. These multi-time systems will be compatible or a common non-trivial solution Φ exists if and only if the relation

$$\frac{\partial H_j}{\partial t_k} - \frac{\partial H_k}{\partial t_j} + i \left[H_j, H_k \right] = 0 \; , \quad \forall j \neq k \; , \tag{1.9}$$

holds. This is known as the consistency condition or integrability condition. The ordinary probability amplitude $\Phi(t)$ is retrieved by setting all time coordinates equal

$$\Phi(q_1, t, q_2, t, ..., q_N, t) = \Psi(q_1, q_2, ..., q_N, t) \; . \tag{1.10}$$

Here the single-time wave function satisfies the standard Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H \Psi \; , \tag{1.11}$$

where $H = \sum_{j=1}^{N} H_j$. Equation (1.10) and (1.11) suggests that the multi-time wave function coincides with the single-time wave function with respect to the Lorentz frame on configurations of N space-time points [3–8].

The idea of the multi-time wave function formalism could be possibly useful in many aspects. For example, Petrat and Tumulku [9] demonstrated that the relevant interacting quantum field theories can be reformulated in terms of multi-time wave functions and therefore, multi-time wave function, the Tomonaga-Schwinger and the Heisenberg approaches are equivalent and the consistency condition of the multi-time formulation explains why in nature the process that a fermion decays into two fermions cannot happen [10]. Lienert, Petrat and Tumulka [8] pointed out that multi-time wave function can be considered in discrete action principles and can be applied to study the cellular automata.

1.2 Objectives

To capture the multi-time consistency condition and inconsistency condition in terms of Lagrangians.

1.3 Frameworks

The main question of this work could be addressed as what is the Lagrangian analogue for the consistency condition? This kind of question is natural to be asked since normally in physics we could choose to work with either Hamiltonian or Lagrangian descriptions. Then in this work, the variational principle will play a central role in order to obtain the consistency condition or integrability condition and the quantum multi-time evolution will be captured through Feynman's path integration expressing in terms of the Wilson line.

1.4 Structure of the thesis

To make things flow smoothly, the remainder of this thesis is organised as follows. In chapter 2, a brief review of the both Lagrangian and Hamiltonian mechanics will be given and we will introduce the basic differential geometry and Wilson loop. In chapter 3, we will derive the multi-time system through Hamiltonian approach and Lagrangian approach will be explained by the Feynman path integration method. After that, the consistency condition of multi-time propagators will be constructed. The conclusion will be given in the last chapter.

CHAPTER II

THEORETICAL BACKGROUND

In this chapter, the background ingredients will be provided ranging from the classical mechanics to quantum mechanics. In the classical mechanics part, the Lagrangian and Hamiltonian formalism will be discussed. The variational principle will be mathematically explained. In the quantum mechanics part, the derivation of the Schrödinger equation will be presented through the connection with the Hamilton-Jacobi equation. The time evolution operator will be obtained as a map of the wave function from an initial time to a later time. The next part, the Feynman path integral description will be explained. The propagator, which is the transition probability amplitude of a particle from the initial point to the final point, will mathematically derived. The last part is devoted for some basic tools on differential geometry and the Wilson loop.

2.1 Lagrangian and Hamiltonian Mechanics

In this section, the Lagrangian and Hamiltonian descriptions in classical mechanics will be sufficiently explained. We shall first derive the Euler-Lagrange equation from the Newton equation known as the D'Alambert's principle and then the least action principle will also derived. The Hamiltonian will be obtained from the Lagrangian using Legendre transformation. The Hamilton's equations are also obtained which are equivalent to the Euler-Lagrange equation. The canonical transformations will be explained and a special case known as the Hamilton-Jacobi equation is also obtained.

2.1.1 Lagrangian Mechanics

In this section, we will derive the Euler-Lagrange equation from the Newton equation $\boldsymbol{F} = m\boldsymbol{a}$, where $\boldsymbol{F}$ is the resultant force, $\boldsymbol{a}$ is the acceleration and m is mass of a particle. Suppose the particle trajectory is on the rectangular space $\mathbb{R}^3$. The kinetic energy of the particle is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \; , \tag{2.1}$$

where $x = x(q_1, q_2, ..., q_n)$, $y = y(q_1, q_2, ..., q_n)$ and $z = z(q_1, q_2, ..., q_n)$. Here a set of $(q_1, q_2, ..., q_n)$ is the generalised coordinates. Then $\dot{x}$ can be expressed in terms of $\{q_k\}, k = 1, 2, ...n$ as

$$\begin{aligned} \dot{x} &= \frac{\partial x}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial x}{\partial q_2}\frac{\partial q_2}{\partial t} + ... + \frac{\partial x}{\partial q_n}\frac{\partial q_n}{\partial t} \\ &= \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\frac{\partial q_k}{\partial t} = \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\dot{q}_k = \dot{x}(q, \dot{q}) \; , \end{aligned} \tag{2.2}$$

where $\dot{q}_k$ is generalised velocities. In the same fashion, we also find that $\dot{y} = \dot{y}(q, \dot{q})$ and $\dot{z} = \dot{z}(q, \dot{q})$ and the kinetic energy becomes

$$T = \frac{1}{2}m\left(\dot{x}^2(q, \dot{q}) + \dot{y}^2(q, \dot{q}) + \dot{z}^2(q, \dot{q})\right) \; . \tag{2.3}$$

Next we take the derivative Equation (2.3) with respect to $\dot{q}$, resulting in

$$\frac{\partial T}{\partial \dot{q}_k} = m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_k} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{q}_k} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{q}_k}\right) \; . \tag{2.4}$$

Using $\frac{\partial \dot{x}}{\partial \dot{q}_k} = \frac{\partial x}{\partial q_k}$, we can write

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_k} &= m\left(\dot{x}\frac{\partial x}{\partial q_k} + \dot{y}\frac{\partial y}{\partial q_k} + \dot{z}\frac{\partial z}{\partial q_k}\right) \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) &= m\ddot{x}\frac{\partial x}{\partial q_k} + m\ddot{y}\frac{\partial y}{\partial q_k} + m\ddot{z}\frac{\partial z}{\partial q_k} + m\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) + m\dot{y}\frac{d}{dt}\left(\frac{\partial y}{\partial q_k}\right) \\ &\quad + m\dot{z}\frac{d}{dt}\left(\frac{\partial z}{\partial q_k}\right) \\ &= m\ddot{x}\frac{\partial x}{\partial q_k} + m\ddot{y}\frac{\partial y}{\partial q_k} + m\ddot{z}\frac{\partial z}{\partial q_k} + m\dot{x}\frac{\partial}{\partial q_k}\left(\frac{dx}{dt}\right) + m\dot{y}\frac{\partial}{\partial q_k}\left(\frac{dy}{dt}\right) \\ &\quad + m\dot{z}\frac{\partial}{\partial q_k}\left(\frac{dz}{dt}\right) \\ &= m\ddot{x}\frac{\partial x}{\partial q_k} + m\ddot{y}\frac{\partial y}{\partial q_k} + m\ddot{z}\frac{\partial z}{\partial q_k} + m\dot{x}\frac{\partial \dot{x}}{\partial q_k} + m\dot{y}\frac{\partial \dot{y}}{\partial q_k} + m\dot{z}\frac{\partial \dot{z}}{\partial q_k} \\ &= m\ddot{x}\frac{\partial x}{\partial q_k} + m\ddot{y}\frac{\partial y}{\partial q_k} + m\ddot{z}\frac{\partial z}{\partial q_k} + \frac{\partial}{\partial q_k}\left(\frac{1}{2}m\dot{x}^2\right) + \frac{\partial}{\partial q_k}\left(\frac{1}{2}m\dot{y}^2\right) \\ &\quad + \frac{\partial}{\partial q_k}\left(\frac{1}{2}m\dot{z}^2\right) \\ &= F_x\frac{\partial x}{\partial q_k} + F_y\frac{\partial y}{\partial q_k} + F_z\frac{\partial z}{\partial q_k} + \frac{\partial}{\partial q_k}\left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right) \; . \end{aligned} \tag{2.5}$$

We now assume that the particle is moving under the influence of the conservative force and then we write

$$F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k} + F_z \frac{\partial z}{\partial q_k} = -\left[\left(\frac{\partial V}{\partial x} \right) \frac{\partial x}{\partial q_k} + \left(\frac{\partial V}{\partial y} \right) \frac{\partial y}{\partial q_k} + \left(\frac{\partial V}{\partial z} \right) \frac{\partial z}{\partial q_k} \right]$$
$$= -\frac{\partial V}{\partial q_k} \ . \tag{2.6}$$

Substituting Equation (2.3) and (2.6) in Equation (2.5), one gets

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) = \frac{\partial T}{\partial q_k} - \frac{\partial V}{\partial q_k} \ . \tag{2.7}$$

Equation (2.7) can be rearranged in the form

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_k}(T - V) \right) = \frac{\partial}{\partial q_k}(T - V) \ , \tag{2.8}$$

since $V = V(q)$. Then we define

$$L(q, \dot{q}) = T(\dot{q}) - V(q) \ , \tag{2.9}$$

which is known as the Lagrangian. Now we arrive at

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \ , \quad k = 1, 2, 3, .., n \ . \tag{2.10}$$

This equation is known as **Euler-Lagrange's equations**. Of course, Equation (2.10) is equivalent to the Newton equation giving the second order differential equation. The difference between these two approaches is that the Newton equation comes with vector quantities, while the Euler-Lagrange's equation deals with scalar quantities, namely the kinetic and potential energies.

Now, we will start to derive the Euler-Lagrange's equation from another point of view namely the variational principle. An action functional is given by

$$S[q(t)] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} L(\dot{q}, q; t) dt \ , \tag{2.11}$$

where t_i is the initial time and t_f is the final time. According to the least action principle: **Of all possible paths of the system, the actual physical path is the one which**

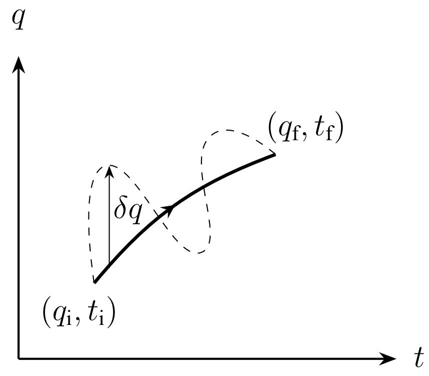


Figure 1 The classical particle will take only one path which minimizes the action that is the solid line.

minimizes the action [11, 12], see figure 1. Mathematically, the variation of the action functional vanishes

$$\delta S = \delta \int_{t_\mathrm{i}}^{t_\mathrm{f}} L(\dot{q}, q; t) dt = 0 \ . \tag{2.12}$$

The variation can be expressed as

$$\begin{aligned} 0 = \delta \int_{t_\mathrm{i}}^{t_\mathrm{f}} L(\dot{q}, q; t) dt &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) dt \\ &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} \delta q_k \right) dt \\ &= \sum_{k=1}^{n} \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_\mathrm{i}}^{t_\mathrm{f}} + \int_{t_\mathrm{i}}^{t_\mathrm{f}} \sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k dt \\ &= \int_{t_\mathrm{i}}^{t_\mathrm{f}} \sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k dt \ . \end{aligned} \tag{2.13}$$

Since $\delta q_k(t) \neq 0$, where $t_\mathrm{i} < t < t_\mathrm{f}$, the term inside the bracket must be zero. Of course, what we obtain from (2.13) is nothing but the Euler-Lagrange's equations.

2.1.2 Hamiltonian Mechanics

In this subsection, we will present an alternative approach to explain the dynamics of the system known as the Hamiltonian mechanics. We shall first define

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \tag{2.14}$$

as the canonical momentum variables then the Euler-Lagrange equations are expressed as

$$\dot{p_k} = \frac{\partial L}{\partial q_k} \ . \tag{2.15}$$

Next, we consider the total time derivative of the Lagrangian $L(q, \dot{q}; t)$

$$\frac{dL}{dt} = \sum_k \left(\frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q_k}}\frac{d\dot{q_k}}{dt} \right) + \frac{\partial L}{\partial t} \ . \tag{2.16}$$

Substituting Equation (2.14) and (2.15) into Equation (2.16), we obtain

$$\frac{dL}{dt} = \sum_k \left(\dot{p_k}\frac{dq_k}{dt} + p_k\frac{d\dot{q_k}}{dt} \right) + \frac{\partial L}{\partial t} \ . \tag{2.17}$$

If the Lagrangian does not depend explicitly on time $\frac{\partial L}{\partial t} = 0$, hence the total time derivative Equation (2.17) can be written as

$$\begin{aligned} \frac{dL}{dt} &= \sum_k \left(\frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial L}{\partial \dot{q_k}}\frac{\dot{q_k}}{dt} \right) \\ &= \sum_k \left(\frac{\partial L}{\partial q_k}\dot{q_k} + \frac{\partial L}{\partial \dot{q_k}}\ddot{q_k} \right) \\ &= \sum_k \left(\dot{q_k}\frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}} + \frac{\partial L}{\partial \dot{q_k}}\ddot{q_k} \right) \\ &= \sum_k \frac{d}{dt}\left(\dot{q_k}\frac{\partial L}{\partial \dot{q_k}} \right) \ , \end{aligned} \tag{2.18}$$

therefore,

$$\frac{d}{dt}\left(\sum_k \dot{q_k}\frac{\partial L}{\partial \dot{q_k}} - L \right) = 0 \ . \tag{2.19}$$

The terms inside the bracket must be constant with respect to time and shall be denoted as $H(q, p)$ called the Hamiltonian

$$H(q,p) = \sum_k \dot{q_k}\frac{\partial L}{\partial \dot{q_k}} - L = \sum_k \dot{q_k}p_k - L(q, \dot{q}) \ . \tag{2.20}$$

This equation is known as the Legendre transformation. We can also write

$$\begin{aligned} \frac{dH}{dt} &= \sum_k \left(\dot{q_k}\frac{dp_k}{dt} + p_k\frac{d\dot{q_k}}{dt} - \frac{\partial L}{\partial q_k}\frac{dq_k}{dt} - \frac{\partial L}{\partial \dot{q_k}}\frac{d\dot{q_k}}{dt} \right) + \frac{\partial H}{\partial t} \\ &= \sum_k \left(\dot{q_k}\frac{dp_k}{dt} - \dot{p_k}\frac{dq_k}{dt} \right) + \frac{\partial H}{\partial t} \ . \end{aligned} \tag{2.21}$$

The Hamiltonian is considered as a function of (q,p) by using Equation (2.14). Consequently, we can express $\dot{q_k}$ in terms of p_k. The total time derivative of H is therefore

$$\frac{dH}{dt} = \sum_k \left(\frac{\partial H}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial H}{\partial p_k}\frac{dp_k}{dt} \right) + \frac{\partial H}{\partial t} \ . \tag{2.22}$$

Comparing Equation (2.22) with Equation (2.21), we find

$$\dot{q_k} = \frac{\partial H}{\partial p_k} \ , \tag{2.23}$$

$$-\dot{p_k} = \frac{\partial H}{\partial q_k} \ , \tag{2.24}$$

Equation (2.23) and (2.24) are **Hamilton's equations**. Next, we consider the Lagrangian depends on explicitly on time $\frac{\partial L}{\partial t} \neq 0$, therefore the Hamiltonian is expressed as an explicit function of $(q,p;t)$. Thus, we have instead of Equation (2.22)

$$\frac{dH}{dt} = \sum_k \left(\frac{\partial H}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial H}{\partial p_k}\frac{dp_k}{dt} \right) + \frac{\partial H}{\partial t} \ , \tag{2.25}$$

and Equation (2.21) becomes

$$\begin{aligned} \frac{dH}{dt} &= \sum_k \left(\dot{q_k}\frac{dp_k}{dt} + p_k\frac{d\dot{q_k}}{dt} - \frac{\partial L}{\partial q_k}\frac{dq_k}{dt} - \frac{\partial L}{\partial \dot{q_k}}\frac{d\dot{q_k}}{dt} \right) - \frac{\partial L}{\partial t} \\ &= \sum_k \left(\dot{q_k}\frac{dp_k}{dt} - \dot{p_k}\frac{dq_k}{dt} \right) - \frac{\partial L}{\partial t} \ . \end{aligned} \tag{2.26}$$

Comparing Equation (2.25) with Equation (2.26), we find

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \ . \tag{2.27}$$

If we substitute Equation (2.23) and (2.24) into Equation (2.25), we obtain

$$\begin{aligned} \frac{dH}{dt} &= \sum_k \left(\frac{\partial H}{\partial q_k}\dot{q_k} + \frac{\partial H}{\partial p_k}\dot{p_k} \right) + \frac{\partial H}{\partial t} \\ &= \sum_k \left(\frac{\partial H}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial p_k}\frac{\partial H}{\partial q_k} \right) + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t} \ . \end{aligned} \tag{2.28}$$

This equation expresses the fact that the Hamiltonian is a constant if it does not contain the time explicitly $\frac{\partial H}{\partial t} = 0$ and this implies the conserved quantity. If the potential energy V is independent of the velocity $\dot{q_k}$, and using Equation (2.20), we find that

$$\begin{aligned} H &= \sum_k \dot{q_k} \frac{\partial L}{\partial \dot{q_k}} - L \\ &= \sum_k \dot{q_k} \frac{\partial (T-V)}{\partial \dot{q_k}} - (T-V) \\ &= \sum_k \dot{q_k} \frac{\partial T}{\partial \dot{q_k}} - (T-V) \\ &= 2T - (T-V) = T + V = E \ , \end{aligned} \tag{2.29}$$

the Hamiltonian is nothing but the total energy of the system.

One important feature of the Hamiltonian mechanics is that a set of coordinates (p, q) is not unique. This means that there are other set of coordinates that can be used to describe the dynamics of the system subject to invariance of the Hamilton's equations. Now let $(P_k(q,p,t), Q_k(q,p,t))$ be a new set of coordinates and $K(Q,P,t)$ be a new Hamiltonian such that the Hamilton's equations become

$$\dot{Q}_k = \frac{\partial K}{\partial P_k} \ , \tag{2.30}$$

$$-\dot{P}_k = \frac{\partial K}{\partial Q_k} \ . \tag{2.31}$$

According to the least action principle (2.11), we have

$$\delta S = \delta \int_{t_\mathrm{i}}^{t_\mathrm{f}} \left(p_k \dot{q}_k - H(q,p,t)\right) dt \tag{2.32}$$

in terms of the old set of coordinates (p, q). Similarly, we also have

$$\delta S = \delta \int_{t_\mathrm{i}}^{t_\mathrm{f}} \Big(P_k \dot{Q}_k - K(Q,P,t)\Big) \, dt \tag{2.33}$$

in terms of the old set of coordinates (P, Q). According to the non-uniqueness property of the Lagrangian, we have

$$P_k \dot{Q}_k - K(Q,P,t) = p_k \dot{q}_k - H(q,p,t) + \frac{d}{dt} F(q,Q,t) \ , \tag{2.34}$$

or

$$P_k\dot{Q}_k - K(Q,P,t) = p_k\dot{q}_k - H(q,p,t) + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial Q_k}\dot{Q}_k \ . \tag{2.35}$$

Therefore, comparing both sides of the equation, we obtain

$$p_k = \frac{\partial F}{\partial q_k} \ , \tag{2.36}$$

$$P_k = -\frac{\partial F}{\partial Q_k} \ , \tag{2.37}$$

and,

$$K = H + \frac{\partial F}{\partial t} \ . \tag{2.38}$$

The function $F = F_1$ is known as the first generating function providing the connection between the old Hamiltonian H and the new Hamiltonian K according to the transformation $(q,p) \to (Q,P)$. However, there are another three types of generators as shown in table 1. [13]

Table 1 Summary of four different basic generating functions.

Generating Function	Generating Function Derivatives
$F = F_1(q,Q,t)$	$p_k = \frac{\partial F_1}{\partial q_k}$ $\quad P_k = -\frac{\partial F_1}{\partial Q_k}$
$F = F_2(q,P,t) - Q_kP_k$	$p_k = \frac{\partial F_2}{\partial q_k}$ $\quad Q_k = \frac{\partial F_2}{\partial P_k}$
$F = F_3(Q,p,t) + q_kp_k$	$q_k = -\frac{\partial F_3}{\partial p_k}$ $\quad P_k = -\frac{\partial F_3}{\partial Q_k}$
$F = F_4(p,P,t) + q_kp_k - Q_kP_k$	$q_k = -\frac{\partial F_4}{\partial p_k}$ $\quad Q_k = \frac{\partial F_4}{\partial P_k}$

We now consider a special case of the transformation (q,p) to (Q,P) such that the Q and P are constant. Then the equation of motion of new Hamiltonian are

$$\frac{\partial K}{\partial P_k} = \dot{Q}_k = 0 \ , \tag{2.39}$$

$$\frac{\partial K}{\partial Q_k} = -\dot{P}_k = 0 \ . \tag{2.40}$$

It is trivial that Equation (2.39) and (2.40) have a common solution which is $K = 0$. Therefore, Equation (2.38) becomes

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0 \ . \tag{2.41}$$

Using the generating functions such as $F_2(q, P, t)$, Equation (2.41) can be written as

$$H\left(q, \frac{\partial F_2}{\partial q}, t\right) + \frac{\partial F_2}{\partial t} = 0 \ . \tag{2.42}$$

This equation is known as the Hamilton-Jacobi equation. Since the P and Q are constant, we can take the new constant momenta as $P_k = \alpha_k$ and the new constant coordinates $Q_k = \beta_k$. The generating function derivative relations are

$$p_k = \frac{\partial S(q, \alpha, t)}{\partial q_k} \ , \tag{2.43}$$

$$Q_k = \frac{\partial S(q, \alpha, t)}{\partial \alpha_k} \ , \tag{2.44}$$

where $S \equiv F_2$ which can be determined by computing its total time derivative

$$\begin{aligned} \frac{dS}{dt} &= \frac{\partial S}{\partial q_k}\dot{q}_k + \frac{\partial S}{\partial \alpha_k}\dot{\alpha}_k + \frac{\partial S}{\partial t} \\ &= \frac{\partial S}{\partial q_k}\dot{q}_k + \frac{\partial S}{\partial t} = p_k\dot{q}_k - H = L \ . \end{aligned} \tag{2.45}$$

This is indeed the action functional

$$S = \int L dt \ . \tag{2.46}$$

Now, the **Hamilton-Jacobi equation** can be written as

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \ . \tag{2.47}$$

where S is called the Hamilton's principal function.

2.2 The Schrödinger wave mechanics

In this section, we will derive the Schrödinger equation from the Hamilton-Jacobi equation for a particle of mass m with the potential V

$$\frac{1}{2m}\left\{\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2\right\}+V+\frac{\partial S}{\partial t}=0\ . \tag{2.48}$$

Now we take a transformation such that $\Psi = e^{\frac{i}{\hbar}S}$ or $S = -i\hbar\ln\Psi$. Since $S = S(x,y,z,t)$, we can compute

$$\begin{aligned}\frac{\partial S}{\partial x}&=-\frac{i\hbar}{\Psi}\frac{\partial\Psi}{\partial x}\\ \frac{\partial\Psi}{\partial x}&=-\frac{\Psi}{i\hbar}\frac{\partial S}{\partial x}=\frac{i\Psi}{\hbar}\frac{\partial S}{\partial x}\\ \frac{\partial^2\Psi}{\partial x^2}&=\frac{i}{\hbar}\frac{\partial\Psi}{\partial x}\frac{\partial S}{\partial x}+\frac{i\Psi}{\hbar}\frac{\partial^2 S}{\partial x^2}\ .\end{aligned} \tag{2.49}$$

Differentiating momentum equation (2.43),

$$\frac{\partial^2 S}{\partial x^2}=\frac{\partial p_x}{\partial x}=0\ . \tag{2.50}$$

Substituting this into Equation (2.49),

$$\begin{aligned}\frac{\partial^2\Psi}{\partial x^2}&=\frac{i}{\hbar}\frac{\partial S}{\partial x}\left(-\frac{\Psi}{i\hbar}\frac{\partial S}{\partial x}\right)=-\frac{\Psi}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2\\ \left(\frac{\partial S}{\partial x}\right)^2&=-\frac{\hbar^2}{\Psi}\frac{\partial^2\Psi}{\partial x^2}\ .\end{aligned} \tag{2.51}$$

Similarly, we also have

$$\left(\frac{\partial S}{\partial y}\right)^2=-\frac{\hbar^2}{\Psi}\frac{\partial^2\Psi}{\partial y^2}\ , \tag{2.52}$$

$$\left(\frac{\partial S}{\partial z}\right)^2=-\frac{\hbar^2}{\Psi}\frac{\partial^2\Psi}{\partial z^2}\ , \tag{2.53}$$

and

$$\frac{\partial S}{\partial t}=-\frac{i\hbar}{\Psi}\frac{\partial\Psi}{\partial t}\ . \tag{2.54}$$

Substituting Equation (2.51), (2.52), (2.53) and (2.54) into Equation (2.48), we obtain

$$\begin{aligned}\frac{1}{2m}\left\{-\frac{\hbar^2}{\Psi}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Psi\right\}+V-\frac{i\hbar}{\Psi}\frac{\partial\Psi}{\partial t}&=0\\ -\frac{\hbar^2}{2m}\nabla^2\Psi+V\Psi-i\hbar\frac{\partial\Psi}{\partial t}&=0\\ -\frac{\hbar^2}{2m}\nabla^2\Psi+V\Psi=i\hbar\frac{\partial\Psi}{\partial t}\ .\end{aligned} \tag{2.55}$$

This equation is the time-dependent **Schrödinger equation**. The solution which is indeed the wave function Ψ lives in the Hilbert space $\mathscr{H}$. The wave function $\Psi(x, y, z, t)$ itself is not measurable quantity. However, the $|\Psi(x, y, z, t)|^2 dV$ is the probability amplitude of finding a particle within the element volume dV and therefore

$$\int_{-\infty}^{+\infty} |\Psi(x, y, z, t)|^2 dV = 1\ , \tag{2.56}$$

which is called the normalisation condition.

Next, we will adopt the Dirac notation to represent the wave function. Let $|\Psi\rangle \in \mathscr{H}$ be a state vector in the space $\mathscr{H}$ and $\langle\Phi| \in \mathscr{H}^*$ be a dual state vector in the dual space $\mathscr{H}^*$. With this new way of presenting the wave function, we can write $\Psi(q, t) = \langle q|\Psi(t)\rangle$ and the normalisation condition (2.56) becomes $\langle\Psi|\Psi\rangle = 1$. The wave function can be mapped from the initial time t' to the final time t'' through the time-evolution operator $U(t', t)$ such that

$$|\Psi(t'')\rangle = U(t'', t')|\Psi(t')\rangle\ , \tag{2.57}$$

with the unitary property

$$U^\dagger(t'', t')U(t'', t') = 1\ . \tag{2.58}$$

The time-evolution operator can be decomposed as

$$U(t'', t') = U(t'', t)U(t, t')\ , \tag{2.59}$$

where $t'' > t > t'$. Next, let $t'' = t' + dt$, where dt is infinitesimal. Then we have

$$U(t' + dt, t) = U(t' + dt, t')U(t', t)\ . \tag{2.60}$$

We now write

$$U(t' + dt, t') = 1 - i\Omega dt \ , \tag{2.61}$$

where Ω is a Hermitian operator $\Omega^\dagger = \Omega$. It is not difficult to see that Equation (2.61) satisfies both composition and unitary property. We see that the dimension of Ω is that of the frequency. Therefore, using the Planck-Einstein relation $E = \hbar\Omega$, Equation (2.60) is written as

$$\begin{aligned} U(t' + dt, t) &= \left(1 - \frac{iHdt}{\hbar}\right) U(t', t) \\ U(t' + dt, t) - U(t', t) &= -i\left(\frac{H}{\hbar}\right) dt U(t', t) \ . \end{aligned} \tag{2.62}$$

Taking the partial derivative,

$$i\hbar\frac{\partial}{\partial t'}U(t', t) = HU(t', t) \ . \tag{2.63}$$

This equation is known as the Schrödinger equation for the time-evolution operator. Equation (2.63) yields the **Schrödinger equation** by applying the initial state ket as

$$\begin{aligned} i\hbar\frac{\partial}{\partial t'}U(t', t)|\Psi(t)\rangle &= HU(t', t)|\Psi(t)\rangle \\ i\hbar\frac{\partial}{\partial t'}|\Psi(t')\rangle &= H|\Psi(t')\rangle \ . \end{aligned} \tag{2.64}$$

Next, we will compute the explicit form of the time evolution operator which can be classified into three cases.

Case 1: The Hamiltonian operator does not depend on time. Let $\Psi(q', t')$ be the final wave function and $\Psi(q, t)$ be the initial wave function. We then have

$$\Psi(q', t') = U(t', t)\Psi(q, t) \ , \quad t' = t + \delta t \ . \tag{2.65}$$

If δt is infinitesimal, $\Psi(q', t')$ can be expanded as followed

$$\begin{aligned} \Psi(q', t') &= \Psi(q, t) + \delta t\frac{d}{dt}\Psi(q, t) + \frac{\delta t^2}{2}\frac{d^2}{dt^2}\Psi(q, t) + ... \\ &= \Psi(q, t) + \delta t\Big(\frac{-iH}{\hbar}\Big)\Psi(q, t) + \frac{\delta t^2}{2}\Big(\frac{-iH}{\hbar}\Big)^2\Psi(q, t) + ... \\ &= e^{-\frac{i}{\hbar}H\delta t}\Psi(q, t) \ . \end{aligned} \tag{2.66}$$

Comparing Equation (2.65) with Equation (2.66), we find that

$$U(t',t) = e^{-\frac{i}{\hbar}H\delta t} = e^{-\frac{i}{\hbar}H(t'-t)} \ . \tag{2.67}$$

This is an explicit formula of the time evolution operator for the case of time-independent Hamiltonian operator.

Case 2: The Hamiltonian operator is time dependent and they do commute at different time such that $[H(t), H(t')] = 0$. To derive the time evolution operator, we start dividing the time from t to t' into n equal intervals, i.e.,

$$\triangle t = \frac{t'-t}{n} \ . \tag{2.68}$$

Using Equation (2.59), we have

$$U(t',t) = \prod_{k=1}^{n} U(t+k\triangle t, t+(k-1)\triangle t) = \prod_{k=1}^{n} e^{-\frac{i\triangle t}{\hbar}H(t+(k-1)\triangle t)} \ . \tag{2.69}$$

Taking the limit in which $\triangle t \to 0$ and $n \to \infty$ simultaneously, we obtain

$$\begin{aligned} U(t',t) &= \lim_{\triangle t\to 0} \prod_{k=1}^{n} e^{-\frac{i\triangle t}{\hbar}H(t+(k-1)\triangle t)} \\ &= \lim_{\triangle t\to 0} e^{-\frac{i\triangle t}{\hbar}H(t)} e^{\frac{-i\triangle t}{\hbar}H(t+\triangle t)} ... e^{-\frac{i\triangle t}{\hbar}H(t'-\triangle t)} \ . \end{aligned} \tag{2.70}$$

Applying the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots} \ . \tag{2.71}$$

and using the fact that the Hamiltonian operators commute $[H(t_i), H(t_j)] = 0$. Equation (2.70) can be written as

$$U(t',t) = \lim_{\triangle t\to 0} e^{-\frac{i\triangle t}{\hbar}\sum_{k=0}^{N-1} H(t+k\triangle t)} = e^{-\frac{i}{\hbar}\int_t^{t'} H(t)dt} \ . \tag{2.72}$$

Here we obtain the explicit form of the time evolution operator for the case of the time-dependent Hamiltonian.

Case 3: This case can be considered as an extension of Case 2 to the situation that the Hamiltonian operators evaluated at different times do not commute, i.e.,

$[H(t), H(t')] \neq 0$. The time evolution can be considered from Equation (2.63), with the initial condition $U(t,t) = 1$,

$$U(t', t) = 1 - \frac{i}{\hbar} \int_t^{t'} H(t') U(t', t) dt' \ . \tag{2.73}$$

Doing the same process as above, but we need to concern the condition of commutation of each Hamiltonian, we obtain

$$\begin{aligned} U(t', t) &= 1 + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_t^{t'} dt_1 \int_t^{t_1} dt_2 ... \int_t^{t_{n-1}} dt_n H(t_1) H(t_2) ... H(t_n) \\ &= \mathsf{T} e^{-\frac{i}{\hbar} \int_t^{t'} dT H(T)} \ , \end{aligned} \tag{2.74}$$

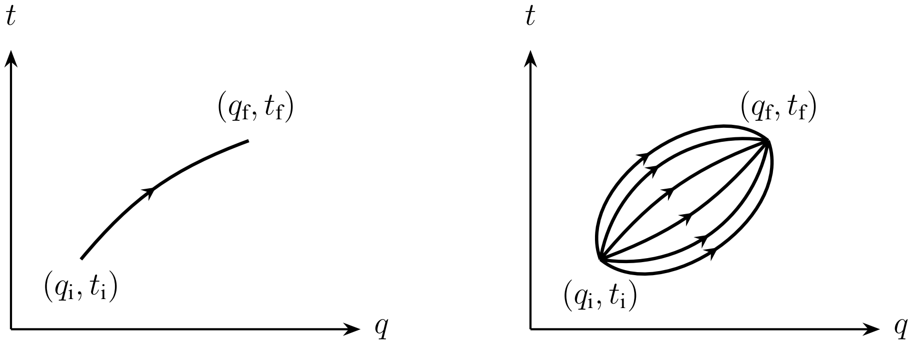
where T is the time ordering operator and this expansion (2.74) is sometimes known as Dyson series. [14–16]

2.3 The Propagator

In this section, we will give a brief tour on how to quantise the system with the Lagrangian description. We recall the least action principle: the path that the particle will follow is the one that the action functional is stationary. As a consequence of this principle, there is only one true path called the classical path, see figure 2a. Interestingly, in quantum context, the particle will take all possible paths simultaneously from the initial point to the final point, see figure 2b. The main mathematical object in this context is no longer the wave function, but the Feynman's propagator given by

$$K(q_{\mathrm{f}}, t_{\mathrm{f}}; q_{\mathrm{i}}, t_{\mathrm{i}}) = \langle q_{\mathrm{f}} | U(t_{\mathrm{f}} - t_{\mathrm{i}}) | q_{\mathrm{i}} \rangle \ . \tag{2.75}$$

The propagator provides the transition probability amplitude for a particle to travel from the initial point $(q_{\mathrm{i}}, t_{\mathrm{i}})$ to the final point $(q_{\mathrm{f}}, t_{\mathrm{f}})$. If we introduce the time t_1 such that



(a) The classical particle will take only one path.

(b) Quantum particle will take all possible paths.

Figure 2 The path connected between $(q_\mathrm{i}, t_\mathrm{i})$ and $(q_\mathrm{f}, t_\mathrm{f})$.

$t_\mathrm{f} > t_1 > t_\mathrm{i}$, the propagator can be factorised as follows

$$\begin{aligned} K(q_\mathrm{f}, t_\mathrm{f}; q_\mathrm{i}, t_\mathrm{i}) &= \Big\langle q_\mathrm{f} \Big| U(t_\mathrm{f} - t_1 + t_1 - t_\mathrm{i}) \Big| q_i \Big\rangle \\ &= \Big\langle q_\mathrm{f} \Big| U(t_\mathrm{f} - t_1) U(t_1 - t_\mathrm{i}) \Big| q_\mathrm{i} \Big\rangle \\ &= \Big\langle q_\mathrm{f} \Big| U(t_\mathrm{f} - t_1) \int dq_1 \Big| q_1 \Big\rangle \Big\langle q_1 \Big| U(t_1 - t_\mathrm{i}) \Big| q_\mathrm{i} \Big\rangle \\ &= \int dq_1 \Big\langle q_\mathrm{f} \Big| U(t_\mathrm{f} - t_1) \Big| q_1 \Big\rangle \Big\langle q_1 \Big| U(t_1 - t_\mathrm{i}) \Big| q_\mathrm{i} \Big\rangle \\ &= \int dq_1 K(q_\mathrm{f}, t_\mathrm{f}; q_1, t_1) K(q_1, t_1; q_\mathrm{i}, t_\mathrm{i}) \; . \end{aligned} \tag{2.76}$$

Equation (2.76) suggests that the transition amplitude of the quantum particle from the initial point to the final point must be taken into account of all possible points q_1 at time t_1. We could continue to make the time interval into n steps such that $t_n \equiv t_\mathrm{f} > t_{n-1} > t_{n-2} > ... > t_2 > t_1 > t_\mathrm{i} \equiv t_0$ (see figure 3) resulting in

$$K(q_n, t_n; q_0, t_0) = \left(\prod_{k=1}^{n-1} \int dq_k \right) \prod_{k=0}^{n-1} K(q_{k+1}, t_{k+1}; q_k, t_k) \; . \tag{2.77}$$

Using the fact that the Hamiltonian operator is $H(p, q) = T(p) + V(q)$ and applying the

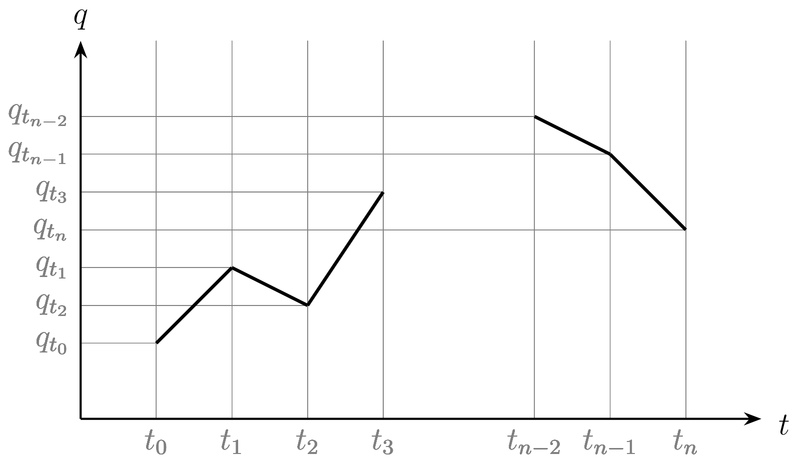


Figure 3 Time slicing with the interval $\Delta t = t_{j+1} - t_j$.

Baker-Campbell-Hausdorff formula, we can compute

$$\begin{aligned} K(q_{k+1}, t_{k+1}; q_k, t_k) &= \langle q_{k+1} \left| U(t_{k+1} - t_k) \right| q_k \rangle \\ &= \left\langle q_{k+1} \middle| e^{-\frac{i(t_{k+1}-t_k)}{\hbar} T(\hat{p})} e^{-\frac{i(t_{k+1}-t_k)}{\hbar} V(\hat{q})} \right. \\ &\quad \left. \times\, e^{-\frac{1}{2}\left(-\frac{i(t_{k+1}-t_k)}{\hbar}\right)^2 [T(\hat{p}), V(\hat{q})] + \dots} \middle| q_k \right\rangle . \end{aligned} \tag{2.78}$$

We take $t_{k+1} - t_k$ very small and therefore we can ignore higher order terms resulting in

$$\begin{aligned} K(q_{k+1}, t_{k+1}; q_k, t_k) &= \left\langle q_{k+1} \middle| e^{-\frac{i(t_{k+1}-t_k)}{\hbar} T(\hat{p})} e^{-\frac{i(t_{k+1}-t_k)}{\hbar} V(\hat{q})} \middle| q_k \right\rangle \\ &= \int dp_k \left\langle q_{k+1} \middle| p_k \right\rangle \left\langle p_k \middle| q_k \right\rangle e^{-\frac{i(t_{k+1}-t_k)}{\hbar} T(p_k)} e^{-\frac{i(t_{k+1}-t_k)}{\hbar} V(q_k)} \\ &= \left(\frac{1}{\sqrt{2\pi\hbar}} \right)^2 \int dp_k e^{\frac{i(t_{k+1}-t_k)}{\hbar} \left(\frac{p_k(q_{k+1}-q_k)}{(t_{k+1}-t_k)} - \frac{p_k^2}{2m} - V(q_k) \right)} \\ &= \sqrt{\frac{m}{2\pi i\hbar(t_{k+1}-t_k)}} e^{\frac{i(t_{k+1}-t_k)}{\hbar} L(q_k, q_{k+1})} , \end{aligned} \tag{2.79}$$

where

$$L(q_k, q_{k+1}) = \frac{m}{2} \left(\frac{q_{k+1} - q_k}{t_{k+1} - t_k} \right)^2 - V(q_k) . \tag{2.80}$$

Equation (2.79) is the discrete propagator and Equation (2.80) is indeed the discrete Lagrangian. Next, taking $t_{k+1} - t_k \equiv \Delta t \to 0$ and $n \to \infty$, the propagator (2.78) can be written as

$$K(q_n, t_n; q_0, t_0) = \lim_{\substack{n\to\infty \\ \Delta t \to 0}} \left(\frac{m}{2\pi i\hbar\Delta t} \right)^{n/2} \left(\prod_{k=1}^{n-1} \int dq_k \right) e^{\frac{i}{\hbar} \sum_{k=0}^{n-1} \Delta t L(q_k, q_{k+1})} . \tag{2.81}$$

Under the limits with $q_\mathrm{a} = q_\mathrm{i}$ and $q_\mathrm{b} = q_\mathrm{f}$

$$\sum_{k=0}^{n-1} \Delta t L(q_k, q_{k+1}) \Rightarrow \int_{t_\mathrm{a}}^{t_\mathrm{b}} dt L(q, \dot{q}) \ , \tag{2.82}$$

where

$$L(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - V(q)$$

is the standard Lagrangian. Now, the propagator can be written as

$$K(q_\mathrm{f}, t_\mathrm{f}; q_\mathrm{i}, t_\mathrm{i}) = \int_{q_\mathrm{i}}^{q_\mathrm{f}} \mathscr{D}[q(t)] e^{\frac{i}{\hbar}S[q(t)]} \ , \tag{2.83}$$

where

$$\int_{q_\mathrm{i}}^{q_\mathrm{f}} \mathscr{D}[q(t)] \equiv \lim_{\substack{n\to\infty \\ \Delta t\to 0}} \left(\sqrt{\frac{m}{2\pi i\hbar\Delta t}}\right)^{n/2} \left(\prod_{k=1}^{n-1} \int dq_k\right) \ ,$$

and

$$S[q(t)] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt L(\dot{q}, q; t) \ .$$

is the action functional.

Remark: The explicit form of the propagator can be obtained for the case of quadratic Lagrangian

$$K\left(q_\mathrm{f}, t_\mathrm{f}; q_\mathrm{i}, t_\mathrm{i}\right) = F(t_\mathrm{f} - t_\mathrm{i}) e^{\frac{i}{\hbar}S_\mathrm{c}} \ , \tag{2.84}$$

where S_c is the classical action and $F(t_\mathrm{f} - t_\mathrm{i}) = \sqrt{\frac{1}{2\pi i\hbar}\left|\frac{\partial^2 S_\mathrm{c}}{\partial q_\mathrm{i}\partial q_\mathrm{f}'}\right|}$ is the prefactor [17–19].

2.4 Basic Differential Geometry

In this section, we will provide some necessary ingredients on differential geometry. Conventionally, the discussion of geometry is concerned with properties of space such as curve, angle, moving of line segments, etc. The geometry, which could be expressed as differential equations, is important analytical technique to understand physical systems. A physical phenomenon usually takes place on the space which generally can be treated as the manifold [20–22].

A manifold is a set of points and each point has a neighborhood that is required to be one-to-one mapping on an open set of an n-dimensional Euclidean space. Every φ_i

from the open set U_i on a manifold M maps to $\mathbb{R}^n$: (U_i, φ_i) is called a chart and the set of chart (U_i, φ_i) is called an atlas. If $U_i \cap U_j \neq \varnothing$, then the map from $\varphi_j(U_j)$ to $\varphi_i(U_i)$ is $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$, see figure 4, [22–25].

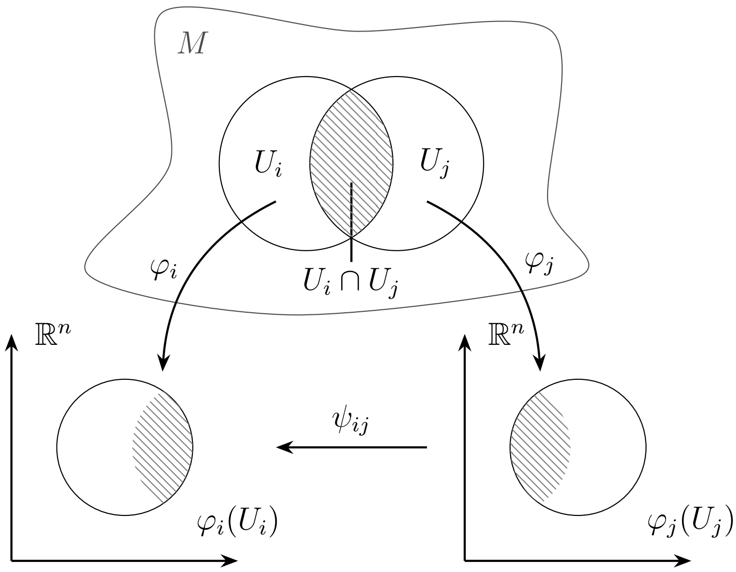


Figure 4 Relation between two charts, φ_i and φ_j, in the overlap region of neighborhoods U_i and U_j.

In order to compare the vector at different points, we need a necessary notion of parallelism. Imagine that there is a vector $\boldsymbol{V}$ at the north pole of the sphere, then vector $\boldsymbol{V}$ is transported along the curve ABC to the south pole, which is not rotated. This produces the vector $\boldsymbol{V}'$ at C, see the figure 5a. On the other hand, we transport $\boldsymbol{V}$ on the another path ADC shown in figure 5b. The vector $\boldsymbol{V}$ is perpendicular to ADC arising the vector $\boldsymbol{V}''$ at C. Both vectors at C, $\boldsymbol{V}'$ and $\boldsymbol{V}''$, are antiparallel. For this reason, the definition of parallel transport, moving along a curve without changing its direction with respect to the sphere's geometry, is important in geometry [22].

We can write the displacement vector of two dimensional coordinate system, see figure 6, as

$$d\boldsymbol{r} = dx^1\boldsymbol{e_1} + dx^2\boldsymbol{e_2} = \left(\frac{\partial \boldsymbol{r}}{\partial x^1}\right) dx^1 + \left(\frac{\partial \boldsymbol{r}}{\partial x^2}\right) dx^2 \ . \tag{2.85}$$

(a) Parallel transport of a vector V along the path ABC.

(b) Parallel transport of a vector V along the path ADC.

Figure 5 Parallel transport of a vector V on different paths.

The basis vectors vary with the position, which affect the basis vectors when applying

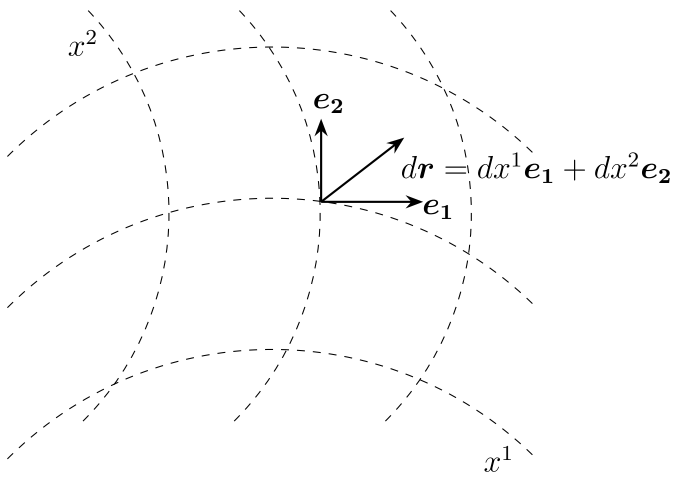


Figure 6 A two-dimensional coordinate system in curved space.

derivatives to vector on a curved space. The divergence of a vector $\boldsymbol{V} = V^i\boldsymbol{e_i}$ with $\boldsymbol{\nabla} \equiv \boldsymbol{e^i}\frac{\partial}{\partial x^i}$, becomes

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{V} &= \boldsymbol{e^j}\frac{\partial}{\partial x^j}\cdot\left(V^i\boldsymbol{e_i}\right) = \boldsymbol{e^j}\cdot\frac{\partial}{\partial x^j}\left(V^i\boldsymbol{e_i}\right)\\ &= \boldsymbol{e^j}\cdot\left[\left(\frac{\partial V^i}{\partial x^j}\right)\boldsymbol{e_i} + V^i\left(\frac{\partial \boldsymbol{e_i}}{\partial x^j}\right)\right]\ . \end{aligned} \tag{2.86}$$

The last term in the square brackets can be expressed as $\frac{\partial \boldsymbol{e_i}}{\partial x^j} = \Gamma^k_{ij}\boldsymbol{e_k}$, where Γ^k_{ij} is the

Christoffel symbol. We can calculate the Christoffel symbols by dot $\boldsymbol{e^l}$ on both sides

$$\boldsymbol{e^l} \cdot \frac{\partial \boldsymbol{e_i}}{\partial x^j} = \Gamma^k_{ij} \left(\boldsymbol{e^l} \cdot \boldsymbol{e_k} \right) \ , \tag{2.87}$$

yielding

$$\Gamma^l_{ij} = \boldsymbol{e^l} \cdot \frac{\partial \boldsymbol{e_i}}{\partial x^j} \ . \tag{2.88}$$

We know that the basis vectors can be written as $\boldsymbol{e_i} = \frac{\partial \boldsymbol{r}}{\partial x^i}$ and consequently the Christoffel symbol is rewritten

$$\Gamma^l_{ij} = \boldsymbol{e^l} \cdot \frac{\partial^2 \boldsymbol{r}}{\partial x^j \partial x^i} \ . \tag{2.89}$$

If we consider a torsionless manifold $\Gamma^k_{ij} = \Gamma^k_{ji}$ and $g_{ij} = \boldsymbol{e_i} \cdot \boldsymbol{e_j}$, we can define a Christoffel symbol of the first kind

$$\Gamma_{kij} \equiv g_{kl} \Gamma^l_{ij} \ . \tag{2.90}$$

Substituting Equation (2.88) into Equation (2.90),

$$\Gamma_{kij} = g_{kl} \boldsymbol{e^l} \cdot \frac{\partial \boldsymbol{e_i}}{\partial x^j} = \boldsymbol{e_k} \cdot \frac{\partial \boldsymbol{e_i}}{\partial x^j} \ . \tag{2.91}$$

Taking the derivative of the metric, we obtain

$$\frac{\partial g_{ij}}{\partial x^k} = \boldsymbol{e_i} \cdot \left(\frac{\partial \boldsymbol{e_j}}{\partial x^k} \right) + \left(\frac{\partial \boldsymbol{e_i}}{\partial x^k} \right) \cdot \boldsymbol{e_j} = \Gamma_{ijk} + \Gamma_{jik} \ . \tag{2.92}$$

Similarly, we also have

$$\frac{\partial g_{ik}}{\partial x^j} = \Gamma_{ikj} + \Gamma_{kij} \ , \tag{2.93}$$

and

$$\frac{\partial g_{jk}}{\partial x^i} = \Gamma_{jki} + \Gamma_{kji} \ , \tag{2.94}$$

Comparing Equation (2.92), (2.93) and (2.94), while using the quantities $\Gamma_{kij} = \Gamma_{kji}$, one gets

$$\Gamma_{kij} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \ . \tag{2.95}$$

An expression for the Christoffel symbol of the second kind is

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right) . \tag{2.96}$$

Equation (2.86) can now be written in term of the Christoffel symbol as

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = \boldsymbol{e^j} \cdot \left[\left(\frac{\partial V^i}{\partial x^j} \right) \boldsymbol{e_i} + V^i \Gamma^k_{ij} \boldsymbol{e_k} \right] . \tag{2.97}$$

The derivative in parentheses can also be defined a new type of derivative

$$\begin{aligned} \frac{\partial}{\partial x^j} \left(V^i \boldsymbol{e_i} \right) &= \frac{\partial V^i}{\partial x^j} \boldsymbol{e_i} + V^i \frac{\partial \boldsymbol{e_i}}{\partial x^j} \\ &= \frac{\partial V^i}{\partial x^j} \boldsymbol{e_i} + V^i \Gamma^k_{ij} \boldsymbol{e_k} \\ &= \frac{\partial V^i}{\partial x^j} \boldsymbol{e_i} + V^k \Gamma^i_{kj} \boldsymbol{e_i} \\ &= \left(\frac{\partial V^i}{\partial x^j} + V^k \Gamma^i_{kj} \right) \boldsymbol{e_i} \\ &= \left(\nabla_j V^i \right) \boldsymbol{e_i} , \end{aligned} \tag{2.98}$$

where

$$\nabla_j V^i \equiv \frac{\partial V^i}{\partial x^j} + \Gamma^i_{kj} V^k , \tag{2.99}$$

is called the **covariant derivative** of the contravariant component of a vector. On the other hand, the covariant derivative of the covariant components is

$$\nabla_j V_i \equiv \frac{\partial V_i}{\partial x^j} - \Gamma^k_{ij} V_k . \tag{2.100}$$

We can think that the covariant derivative is simply the partial derivative. If the covariant derivative of the vector $\boldsymbol{V}$ is zero, $\boldsymbol{V}$ is parallel transported along the curve [22, 26, 27].

Consider a vector $\boldsymbol{V}$ along the curve $\mathscr{C}$ parametrised by u.

$$\boldsymbol{V}(u) = V^i(u) \boldsymbol{e_i} . \tag{2.101}$$

Then, the derivative is given by

$$\begin{aligned}
\frac{d\boldsymbol{V}}{du} &= \frac{dV^i}{du}\boldsymbol{e_i} + V^i\frac{d\boldsymbol{e_i}}{du} \\
&= \frac{dV^i}{du}\boldsymbol{e_i} + V^i\frac{\partial \boldsymbol{e_i}}{\partial x^j}\frac{dx^j}{du} \\
&= \frac{dV^i}{du}\boldsymbol{e_i} + V^i\Gamma^k_{ij}\boldsymbol{e_k}\frac{dx^j}{du} \\
&= \frac{dV^i}{du}\boldsymbol{e_i} + V^k\Gamma^i_{kj}\boldsymbol{e_i}\frac{dx^j}{du} \\
&= \left(\frac{dV^i}{du} + V^k\Gamma^i_{kj}\frac{dx^j}{du}\right)\boldsymbol{e_i} \ . \qquad (2.102)
\end{aligned}$$

The term in the parentheses is defined as the intrinsic (or absolute) derivative and often denoted by

$$\frac{DV^i}{Du} \equiv \frac{dV^i}{du} + V^k\Gamma^i_{kj}\frac{dx^j}{du} \ . \qquad (2.103)$$

Since, we can write

$$\frac{dV^i}{du} = \frac{\partial V^i}{\partial x^j}\frac{dx^j}{du} \ , \qquad (2.104)$$

then Equation (2.103) becomes

$$\begin{aligned}
\frac{DV^i}{Du} &= \frac{\partial V^i}{\partial x^j}\frac{dx^j}{du} + V^k\Gamma^i_{kj}\frac{dx^j}{du} \\
&= \left(\frac{\partial V^i}{\partial x^j} + V^k\Gamma^i_{kj}\right)\frac{dx^j}{du} \\
&= \nabla_j V^i\frac{dx^j}{du} \ . \qquad (2.105)
\end{aligned}$$

The intrinsic derivative can be written in term of covariant derivative. The parallel transport of a vector along a curve is $\frac{DV^i}{Du} = 0$ [26].

The curvature is important property of the surface on manifold, which measures how much a manifold is curved. This is related to the path dependence of parallel transport. We can consider the parallel transport around a small closed loop, see figure 7. What we want to compute is the commutator of two covariant derivatives in order to define the curvature tensor or Riemann tensor. The second covariant differentiation is

$$\begin{aligned}\nabla_l \nabla_j V_i &= \partial_l \left(\nabla_j V_i\right) - \Gamma^m_{il} \nabla_j V_m - \Gamma^m_{jl} \nabla_m V_i \\ &= \partial_l \partial_j V_i - \left(\partial_l \Gamma^k_{ij}\right) V_k - \Gamma^k_{ij} \partial_l V_k - \Gamma^m_{il} \left(\partial_j V_m - \Gamma^k_{mj} V_k\right) \\ &\quad - \Gamma^m_{jl} \left(\partial_m V_i - \Gamma^k_{im} V_k\right) \ . \end{aligned} \tag{2.106}$$

The other one gives

$$\begin{aligned}\nabla_j \nabla_l V_i &= \partial_j \left(\nabla_l V_i\right) - \Gamma^m_{ij} \nabla_l V_m - \Gamma^m_{lj} \nabla_m V_i \\ &= \partial_j \partial_l V_i - \left(\partial_j \Gamma^k_{il}\right) V_k - \Gamma^k_{il} \partial_j V_k - \Gamma^m_{ij} \left(\partial_l V_m - \Gamma^k_{ml} V_k\right) \\ &\quad - \Gamma^m_{lj} \left(\partial_m V_i - \Gamma^k_{im} V_k\right) \ . \end{aligned} \tag{2.107}$$

Therefore, the commutator , $[\nabla_l, \nabla_j]V_i$, gives

$$\left(\nabla_l \nabla_j - \nabla_j \nabla_l\right) V_i = R^k_{ijl} V_k \ , \tag{2.108}$$

where the **curvature tensor** is defined by

$$R^k_{ijl} \equiv \partial_j \Gamma^k_{il} - \partial_l \Gamma^k_{ij} + \Gamma^m_{il} \Gamma^k_{mj} - \Gamma^m_{ij} \Gamma^k_{ml} \ . \tag{2.109}$$

The vanishing of the curvature tensor indicates that a manifold is flat that is the components of the metric g_{jl} are constant. [26, 28–30].

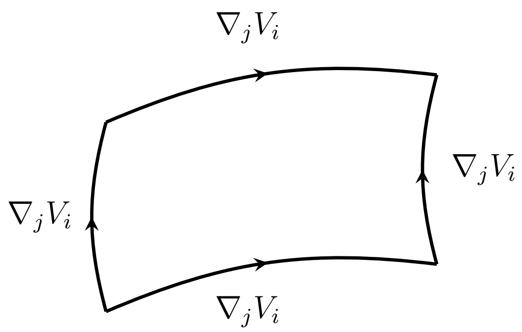


Figure 7 The parallel transport process on completing the parallelogram.

2.5 Heuristic introduction to the Wilson loop

A Wilson loop (or Wilson line) is an observable in gauge theory obtained from the holonomy of the gauge connection. It is usually discussed in the language of differential geometry involving the infinitesimal parallel transport. The **Wilson line** along path C is given by

$$W = \mathcal{P}e^{i\int_C A_\mu dx^\mu} , \tag{2.110}$$

where $\mathcal{P}$ is the path-ordering operator and A_μ are the components of the gauge connection. If one consider parallel transport along a closed loop C, we obtain

$$W = \mathcal{P}e^{i\oint_C A_\mu dx^\mu} , \tag{2.111}$$

which is called the **Wilson loop** [31, 32].

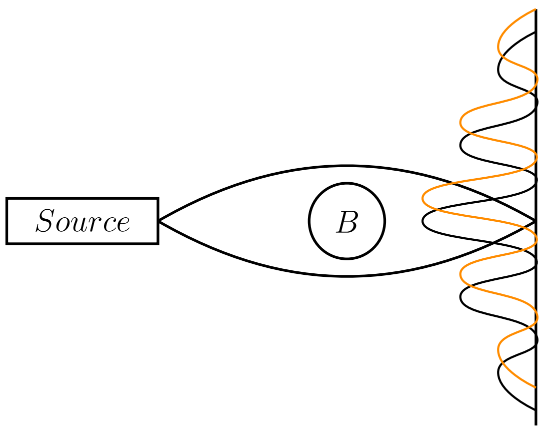


Figure 8 Aharonov-Bohm effect.

In order to see the Wilson loop in action, we shall consider the Aharonov-Bohm effect. The setup of the experiment is illustrated in figure 8. The electrons are emitted, one at a time, by a source and then a beam of electrons splits into two and each beam past infinite long solenoid on different sides [33]. Turning off and turning on the magnetic field inside the solenoid would affect the interference pattern of the electron, since the electron experiences the present of the vector potential $\boldsymbol{A}$.

To analyse this effect, we use the Feynman's path integral formalism. The Lagrangian of a free charged particle in the magnetic field is given by

$$L = \frac{m}{2}\left(\frac{d\boldsymbol{x}}{dt}\right)^2 + \frac{q}{c}\frac{d\boldsymbol{x}}{dt}\cdot\boldsymbol{A} = L_0 + \frac{q}{c}\frac{d\boldsymbol{x}}{dt}\cdot\boldsymbol{A}\ , \tag{2.112}$$

where L_0 is the free particle Lagrangian. Let define S_0 as the action of the free particle. Then the action of the system is

$$S = S_0 + \frac{q}{c}\int \boldsymbol{A}\cdot\frac{d\boldsymbol{x}}{dt}dt = S_0 + \frac{q}{c}\int \boldsymbol{A}\cdot d\boldsymbol{x}\ , \tag{2.113}$$

and let

$$S_{\text{above path}} = S_0 + \frac{q}{c}\int_{\text{above path}} \boldsymbol{A}\cdot d\boldsymbol{x}\ , \tag{2.114}$$

$$S_{\text{below path}} = S_0 + \frac{q}{c}\int_{\text{below path}} \boldsymbol{A}\cdot d\boldsymbol{x}\ , \tag{2.115}$$

be the actions of the path going above and the below the cylinder, respectively. From Equation (2.83), the propagator is proportional to function $e^{\frac{i}{\hbar}S}$. Then the propagator which obtains from two different paths

$$\begin{aligned} K &= \int_{\text{above path}} \mathscr{D}[x(t)]e^{\frac{i}{\hbar}S_{\text{above path}}} + \int_{\text{below path}} \mathscr{D}[x(t)]e^{\frac{i}{\hbar}S_{\text{below path}}} \\ &= \int_{\text{above path}} \mathscr{D}[x(t)]e^{\frac{i}{\hbar}\left[S_0+\frac{q}{c}\int_{\text{above path}}\boldsymbol{A}\cdot d\boldsymbol{x}\right]} + \int_{\text{below path}} \mathscr{D}[x(t)]e^{\frac{i}{\hbar}\left[S_0+\frac{q}{c}\int_{\text{below path}}\boldsymbol{A}\cdot d\boldsymbol{x}\right]} \\ &= \oint \mathscr{D}[x(t)]e^{\frac{i}{\hbar}S_{\text{above}}}\left(1+e^{\frac{i}{\hbar}\left[S_0+\frac{q}{c}\int_{\text{below}}\boldsymbol{A}\cdot d\boldsymbol{x}-S_{\text{above}}\right]}\right) \\ &= \oint \mathscr{D}[x(t)]e^{\frac{i}{\hbar}S_{\text{above}}}\left(1+e^{\frac{iq}{\hbar c}\left[\int_{\text{below}}\boldsymbol{A}\cdot d\boldsymbol{x}-\int_{\text{above}}\boldsymbol{A}\cdot d\boldsymbol{x}\right]}\right)\ . \end{aligned} \tag{2.116}$$

The exponential term inside the bracket can be expressed in a closed path (the trajectory goes around the solenoid and back to the origin)

$$e^{\frac{iq}{\hbar c}\left[\int_{\text{below path}}\boldsymbol{A}\cdot d\boldsymbol{x}-\int_{\text{above path}}\boldsymbol{A}\cdot d\boldsymbol{x}\right]} = e^{\frac{iq}{\hbar c}\left(\oint \boldsymbol{A}\cdot d\boldsymbol{x}\right)}\ . \tag{2.117}$$

The object in Equation (2.117) is nothing but Wilson loop representing the phase difference between the upper path and the lower path. Applying Stoke's theorem, we find that

$$\frac{iq}{\hbar c}\oint \boldsymbol{A}\cdot d\boldsymbol{x} = \frac{iq}{\hbar c}\int_{\text{enclosed}} \boldsymbol{B}\cdot d\boldsymbol{s} = \frac{iq}{\hbar c}\Phi_{\text{enclosed}}\ , \tag{2.118}$$

where Φ_{enclosed} is a magnetic flux inside the solenoid [16, 34]. Then the interference pattern will be proportional to magnetic field in the solenoid. In addition, Equation (2.117) implies that the upper path cannot be continuously deformed to the lower path because of the presenting magnetic field inside the solenoid. Topologically, we can think that there is a hole in the space (2 dimension plane) as an obstacle in the path deformation process. Then the upper path and the lower path are not homotopic to each other, see figure 9 [19].

(a) Turn off the magnetic field. **(b) Turn on the magnetic field.**

Figure 9 The topological point of view.

CHAPTER III

MULTI-TIME CLASSICAL AND QUANTUM MECHANICS

In this chapter, we shall give a main result of the work, namely the multi-time formalism. In the first section, the multi-time system will be discussed together with the consistency condition both in Hamiltonian and Lagrangian descriptions. Later, the quantum version for both descriptions will be derived.

3.1 Multi-time Hamiltonian

Suppose there is a set of Hamiltonians $\{H_1, H_2, ..., H_N\}$ and a multi-time Hamilton's principal function S associated with a set of time variables $t = (t_1, t_2, ..., t_N)$, where $t_j \in \mathbb{R}$. We then look for solutions for a set of the first order differential equations given by

$$\frac{\partial}{\partial t_j} S(q,t) + H_j \left(q, t, \frac{\partial}{\partial q} S(q,t) \right) = 0 \ , \tag{3.1}$$

where $q \in \mathbb{R}^d$ and $j = 1, 2, 3, ..., N$. It is well known that the set of equations in Equation (3.1) is the multi-time Hamilton-Jacobi equations and is overdetermined. Then, to get a nontrivial solution, one may need all Hamiltonians to commute in an appropriate way known as the Hamiltonian commuting flows. To obtain that particular consistency condition, we look at the compatible flows between t_i and t_j. What we have now are

$$\begin{aligned}
\frac{\partial S}{\partial t_j} &= -H_j \left(q, t, \frac{\partial S}{\partial q} \right) \\
\frac{\partial^2 S}{\partial t_i \partial t_j} &= -\frac{\partial}{\partial t_i} H_j \left(q, t, \frac{\partial S}{\partial q} \right) \\
&= -\frac{\partial H_j}{\partial t_i} - \frac{\partial H_j}{\partial \frac{\partial S}{\partial q_k}} \cdot \frac{\partial}{\partial t_i} \frac{\partial S}{\partial q_k} \\
&= -\frac{\partial H_j}{\partial t_i} - \frac{\partial H_j}{\partial \frac{\partial S}{\partial q_k}} \cdot \left(-\frac{\partial}{\partial q_k} H_i \left(q, t, \frac{\partial S}{\partial q} \right) \right)
\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 S}{\partial t_i \partial t_j} &= -\frac{\partial H_j}{\partial t_i} - \frac{\partial H_j}{\partial \frac{\partial S}{\partial q_k}} \cdot \left(-\frac{\partial H_i}{\partial q_k} - \frac{\partial H_i}{\partial \frac{\partial S}{\partial q_l}} \cdot \frac{\partial}{\partial q_k} \frac{\partial S}{\partial q_l} \right) \\ &= -\frac{\partial H_j}{\partial t_i} + \frac{\partial H_j}{\partial \frac{\partial S}{\partial q_k}} \cdot \left(\frac{\partial H_i}{\partial q_k} + \frac{\partial H_i}{\partial \frac{\partial S}{\partial q_l}} \cdot \frac{\partial^2 S}{\partial q_k \partial q_l} \right) \\ &= -\frac{\partial H_j}{\partial t_i} + \frac{\partial H_j}{\partial p_k} \cdot \left(\frac{\partial H_i}{\partial q_k} + \frac{\partial H_i}{\partial p_l} \cdot \frac{\partial^2 S}{\partial q_k \partial q_l} \right) \ , \end{aligned} \tag{3.2}$$

and

$$\frac{\partial^2 S}{\partial t_j \partial t_i} = -\frac{\partial H_i}{\partial t_j} + \frac{\partial H_i}{\partial p_k} \cdot \left(\frac{\partial H_j}{\partial q_k} + \frac{\partial H_j}{\partial p_l} \cdot \frac{\partial^2 S}{\partial q_k \partial q_l} \right) \ , \tag{3.3}$$

where $p_k = \frac{\partial S}{\partial q_k}$. The compatibility requires

$$\left(\frac{\partial^2}{\partial t_j \partial t_i} - \frac{\partial^2}{\partial t_i \partial t_j} \right) S = 0 \ , \tag{3.4}$$

leading to the condition [7, 35]

$$-\frac{\partial H_i}{\partial t_j} + \frac{\partial H_j}{\partial t_i} - \{H_i, H_j\} = 0 \ , \tag{3.5}$$

where $\{A, B\}$ is the standard Poisson bracket between A and B.

3.2 Multi-time Lagrangian

Next, we will express the consistency condition for multi-time evolution in terms of the Lagrangian of the classical system. We start to give the action functional along path Γ, see figure 10 in the case of two-time variables, defined on the space of time variables

$$S_\Gamma[t] = \int_\Gamma \sum_{i=1}^{N} L_i dt_i \ , \tag{3.6}$$

where $L_i = L_i(dq_i/dt_i, q_i; t)$ is the Lagrangian for i^{th} particle. We introduce a new variable $\sigma_0 \leqslant \sigma \leqslant \sigma_1$ such that $(t_1(\sigma), t_2(\sigma), ..., t_N(\sigma))$ and this new variable is actually a parametrisation of the path Γ. Then the action (3.6) becomes

$$S_\Gamma[t(\sigma)] = \int_{\sigma_0}^{\sigma_1} \mathcal{L} d\sigma \ , \quad \text{where} \ \ \mathcal{L} = \sum_{i=1}^{N} L_i \frac{dt_i}{d\sigma} \ . \tag{3.7}$$

In order to capture the consistency condition for multi-time evolution, we consider the time variation $t_i \to t_i + \delta t_i$ resulting in a new path Γ' with the action

$$S_{\Gamma'}\left[t(\sigma)+\delta t(\sigma)\right] = \int_{\sigma_0}^{\sigma_1} d\sigma \left(\sum_{i=1}^{N} L_i\left(t+\delta t\right)\frac{d\left(t_i+\delta t_i\right)}{d\sigma}\right) . \tag{3.8}$$

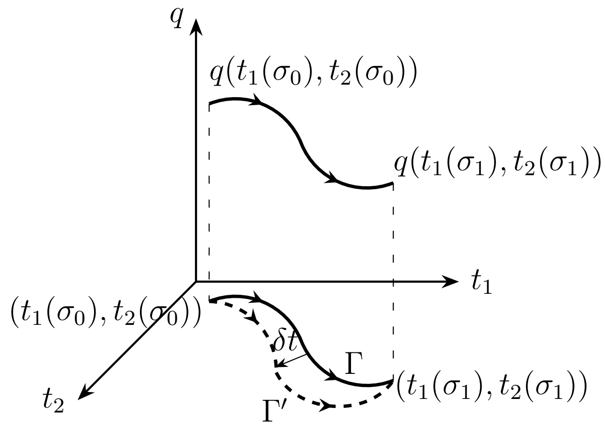


Figure 10 The variation of the path on the space of two-time variables.

Employing Taylor series to expand Lagrangians and ignoring the higher-order terms, each Lagrangian can be expressed as

$$L_i\left(t+\delta t\right) = L_i\left(t\right) + \sum_{j=1}^{N} \delta t_j \frac{\partial L_i}{\partial t_j} + \ldots , \quad i = 1, 2, \ldots, N . \tag{3.9}$$

The variation of the action is given by

$$\begin{aligned} S_{\Gamma'}\left[t(\sigma)+\delta t(\sigma)\right] - S_{\Gamma}[t] &\equiv \delta S \\ &\approx \int_{\sigma_0}^{\sigma_1} d\sigma \left\{\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\delta t_j \frac{\partial L_i}{\partial t_j}\right)\frac{dt_i}{d\sigma} + \sum_{i=1}^{N} L_i \frac{d\delta t_i}{d\sigma}\right\} . \end{aligned} \tag{3.10}$$

Using integration by parts, Equation (3.10) becomes

$$\delta S = \int_{\sigma_0}^{\sigma_1} d\sigma \left\{\sum_{i,j=1}^{N} \delta t_i \left(\frac{\partial L_j}{\partial t_i} - \frac{\partial L_i}{\partial t_j}\right)\frac{dt_j}{d\sigma}\right\} , \quad \forall i \neq j . \tag{3.11}$$

Imposing the condition of the principle of least action $\delta S = 0$, we obtain

$$\frac{\partial L_j}{\partial t_i} = \frac{\partial L_i}{\partial t_j} , \quad \forall i \neq j . \tag{3.12}$$

Equation (3.12)[1] is nothing but the consistency condition for the multi-time evolution in terms of the Lagrangian. Consequently, under condition (3.12) the action remains the same under the path variation on the space of time variables. There is nothing but the path-independent feature of the evolution on the space of time variables.

Remark: We shall point out that one can do the variation on the action with respect to the coordinate variables resulting in a set of Euler-Lagrange equations together with constraints [36].

From the geometric point of view, Equation (3.12) can also be obtained. Suppose that α is a differential (k-1)-form. The generalised Stokes' theorem states that *the integral of its exterior derivative over the surface of smooth oriented k-dimensional manifold Σ is equal to its integral of along the boundary $\partial\Sigma$ of the manifold Σ* [37]:

$$\int_{\partial\Sigma} \alpha = \iint_{\Sigma} d\alpha \ . \tag{3.13}$$

We now introduce an object dS from Equation (3.6) given by

$$dS = \sum_{i=1}^{N} L_i dt_i \ , \tag{3.14}$$

as a 1-form on the N-dimensional space of independent variables and, therefore, the action (3.6) becomes $S = \int_\Gamma dS$. Applying an exterior derivative to the smooth function coefficients which, in this case, is the Lagrangian $\alpha = \sum_{i=1}^{N} L_i dt_i$, the exterior derivative of α is

$$\begin{aligned} d\alpha &= \sum_i dL_i \wedge dt_i \\ &= \sum_{1\leq i<j\leq N}^{N} (dL_i \wedge dt_i + dL_j \wedge dt_j) \end{aligned}$$

[1] This equation was first derived in a different context, the integrable 1-dimensional many-body system [36], to capture also the consistency condition.

$$
\begin{aligned}
d\alpha &= \sum_{1\leq i<j\leq N}^{N} \left(\frac{\partial L_i}{\partial t_i}dt_i + \frac{\partial L_i}{\partial t_j}dt_j\right)\wedge dt_i + \left(\frac{\partial L_j}{\partial t_i}dt_i + \frac{\partial L_j}{\partial t_j}dt_j\right)\wedge dt_j \\
&= \sum_{1\leq i<j\leq N}^{N} \frac{\partial L_i}{\partial t_i}dt_i\wedge dt_i + \frac{\partial L_i}{\partial t_j}dt_j\wedge dt_i + \frac{\partial L_j}{\partial t_i}dt_i\wedge dt_j + \frac{\partial L_j}{\partial t_j}dt_j\wedge dt_j \\
&= \sum_{1\leq i<j\leq N}^{N} -\frac{\partial L_i}{\partial t_j}dt_i\wedge dt_j + \frac{\partial L_j}{\partial t_i}dt_i\wedge dt_j \\
&= \sum_{1\leq i<j\leq N}^{N} \left(\frac{\partial L_j}{\partial t_i} - \frac{\partial L_i}{\partial t_j}\right) dt_i\wedge dt_j \ . \qquad (3.15)
\end{aligned}
$$

Then, Equation (3.13) becomes

$$
\oint_{\partial\Sigma} \sum_{i=1}^{N} L_i dt_i = \iint_{\Sigma} \sum_{1\leq i<j\leq N}^{N} \left(\frac{\partial L_j}{\partial t_i} - \frac{\partial L_i}{\partial t_j}\right) dt_i\wedge dt_j \ . \qquad (3.16)
$$

The left-hand side of Equation (3.16) is equivalent to $\int_{\Gamma} dS - \int_{\Gamma'} dS$. The consistent system exists the path independent property and means that Lagrangian 1-form is closed-form. Thus, the right-hand side of Equation (3.16) vanishes, since the exterior derivative operating on the closed-form gives a vanishing result. Therefore, we obtain

$$
\frac{\partial L_j}{\partial t_i} - \frac{\partial L_i}{\partial t_j} = 0 \ , \quad i,j = 1,2,3,...,N \quad \text{and} \quad i\neq j \ , \qquad (3.17)
$$

which are the consistency conditions of the system that evolves in the N-dimensional space of independent variables. The main point is that Equation (3.16) is the Lagrangian version of parallel transport feature, which is explained in the next section, see Equation (3.55) if one defines

$$
F_{ij} = \frac{\partial L_j}{\partial t_i} - \frac{\partial L_i}{\partial t_j} \ , \quad i,j = 1,2,3,...,N \quad \text{and} \quad i\neq j \ , \qquad (3.18)
$$

and of course, consequently, the consistency condition (3.17) of multi-time evolution can be treated as the zero curvature condition in terms of the Lagrangians.

We find that the condition (3.12) violates if there is the interaction. To see this, we give a simple example as follows. Given $L_1 = \frac{m\dot{q_1}^2}{2} + kq_1q_2$ and $L_2 = \frac{m\dot{q_2}^2}{2}$, where

$q_1 = q_1(t_1)$, $q_2 = q_2(t_2)$, k is the constant and then

$$\frac{\partial L_1}{\partial t_2} = kq_1\frac{\partial q_2}{\partial t_2} \;, \tag{3.19}$$

$$\frac{\partial L_2}{\partial t_1} = 0 \;. \tag{3.20}$$

Therefore,

$$\frac{\partial L_1}{\partial t_2} \neq \frac{\partial L_2}{\partial t_1} \;. \tag{3.21}$$

Thus the presence of the interaction leads to inconsistency. We shall see later in the quantum case that the presence of the interaction gives also incompatible quantum evolution in terms of the propagators.

3.3 Multi-time Schrödinger wave mechanics

For multi-time quantum case, suppose there are N particles in the systems and $(q_1, q_2, .., q_N)$ is a set of coordinates. The single-time wave function is given by $\Psi(q_1, q_2, ..., q_N, t)$ and the relativistic version is $\Phi(q_1, t_1, q_2, t_2, ..., q_N, t_N)$ satisfying N separable time-dependent Schrödinger equations[2] [1,2]

$$\left(H_j + \frac{1}{i}\frac{\partial}{\partial t_j}\right)\Phi(q_1, t_1, q_2, t_2, ..., q_N, t_N) = 0 \;, \quad j = 1, 2, ..., N \;, \tag{3.22}$$

where H_j are the free Schrödinger Hamiltonians (or free Dirac Hamiltonians). The ordinary probability amplitude is retrieved by setting all-time coordinates equal

$$\Phi(q_1, t, q_2, t, ..., q_N, t) = \Psi(q_1, q_2, ..., q_N, t) \;. \tag{3.23}$$

Here the single-time wave function Ψ satisfies the standard Schrödinger equation (2.64) and $H = \sum_{j=1}^{N} H_j$. Equation (3.23) and (2.64) suggest that the multi-time wave function coincides with the single-time wave function with respect to the Lorentz frame on configurations of N space-time points [8].

---

[2] From now on, we set $\hbar = 1$.

Here comes to an interesting feature of the system of Equation (3.22). The multi-time evolution must satisfy a certain condition. Suppose the multi-time wave function evolves from the initial point $(0, 0)$ to the final point (t_1, t_2)[3]. In the case of time-independent Hamiltonians, we define $U_1(t_1) = e^{-iH_1t_1}$ as the unitary time operator in t_1 direction and $U_2(t_2) = e^{-iH_2t_2}$ as the unitary time operator in t_2 direction. There are two different ways to proceed the evolution map as follows

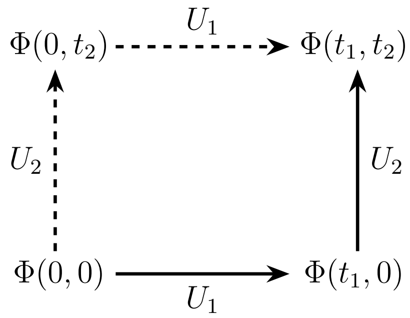


Figure 11 Two compatible maps of the wave function from the initial point $(0, 0)$ to the final point (t_1, t_2).

$$\Phi(t_1, t_2) = e^{-iH_2t_2}\Phi(t_1, 0) = e^{-iH_2t_2}e^{-iH_1t_1}\Phi(0, 0) = U_2U_1\Phi(0, 0) \ , \tag{3.24}$$

and

$$\Phi(t_1, t_2) = e^{-iH_1t_1}\Phi(0, t_2) = e^{-iH_1t_1}e^{-iH_2t_2}\Phi(0, 0) = U_1U_2\Phi(0, 0) \ . \tag{3.25}$$

From Equation (3.24) and (3.25), the evolution is compatible if and only if

$$[H_1, H_2] = 0 \ , \tag{3.26}$$

which is called the consistency condition or integrability criterion for the multi-time evolution, see figure 11. In the case of the time-dependent Hamiltonian, one could obtain **the consistency condition** [3]

$$\frac{\partial H_j}{\partial t_k} - \frac{\partial H_k}{\partial t_j} - i\,[H_j, H_k] = 0 \ , \ \ \forall j \neq k \ . \tag{3.27}$$

---

[3]For simplicity, we consider only two-time variables.

Here Equation (3.27) can be considered as the quantum analogue of Equation (3.5). We can derive this condition for simplicity only two time variables. First we introduce the time evolution of the wave function in t_1 direction from $(0,0)$ to $(t_1,0)$

$$\phi(t_1,0) = U(t_1,0)\phi(0,0) \ , \tag{3.28}$$

where $U(t_1,0) = \mathsf{T}e^{-i\int_0^{t1} H_1(T,t_2=0)dT}$ which can be expanded in the Dyson series as we show Equation (2.74) in the last chapter

$$U(t_1,0) = I + \sum_{n=1}^{\infty}(-i)^n \int_0^{t_1} dT_1 \int_0^{T_1} dT_2 .. \int_0^{T_{n-1}} dT_n H_1(T_1,t_2)..H_1(T_n,t_2) \ . \tag{3.29}$$

Next we proceed the evolution in the t_2 direction from $(t_1,0)$ to (t_1,t_2), we fix the first time variable t_1, resulting in

$$\phi(t_1,t_2) = U_2(t_1,t_2)\phi(t_1,0) = U_2(t_1,t_2)U(t_1,0)\phi(0,0) \ , \tag{3.30}$$

where $U_2(t_1,t_2) = \mathsf{T}e^{-i\int_0^{t_2} H_2(t_1,T)dT}$. On the other hand, one can go with t_2 and then t_1 resulting in

$$\phi(t_1,t_2) = U_1(t_1,t_2)\phi(0,t_2) = U_1(t_1,t_2)U(0,t_2)\phi(0,0) \ , \tag{3.31}$$

where $U(0,t_2) = \mathsf{T}e^{-i\int_0^{t_2} H_2(t_1=0,T)dT}$ and $U_1(t_1,t_2) = \mathsf{T}e^{-i\int_0^{t_1} H_1(T,t_2)dT}$. We take the paths shown in figure 12 to work with a small rectangle with side lengths Δt_1 and Δt_2 for each time direction in order to get same end points of each path. The evolution of each time direction for the lower corner path can be written as

$$\begin{aligned} U(t_1,0) = I &+ (-i)^1 \int_0^{t_1} dT_1 H_1(T_1,0) \\ &+ (-i)^2 \int_0^{t_1} dT_1 \int_0^{T_1} dT_2 H_1(T_1,0)H_1(T_2,0) + ... \ , \end{aligned} \tag{3.32}$$

$$\begin{aligned} U_2(t_1,t_2) = I &+ (-i)^1 \int_0^{t_2} dT_1 H_2(t_1,T_1) \\ &+ (-i)^2 \int_0^{t_2} dT_1 \int_0^{T_1} dT_2 H_2(t_1,T_1)H_2(t_1,T_2) + ... \ , \end{aligned} \tag{3.33}$$

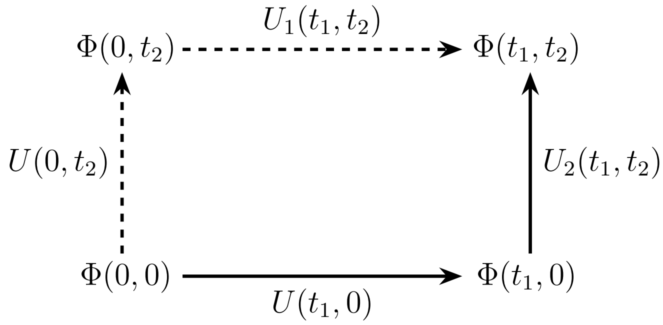


Figure 12 A small rectangle of the time evolution with side lengths Δt_1 and Δt_2 for each time direction from the initial point $(0,0)$ to the final point (t_1,t_2).

and for the upper corner path can be written as

$$\begin{aligned} U(0,t_2) &= I + (-i)^1 \int_0^{t_2} dT_1 H_2(0,T_1) \\ &\quad + (-i)^2 \int_0^{t_2} dT_1 \int_0^{T_1} dT_2 H_2(0,T_1) H_2(0,T_2) + \ldots \ , \end{aligned} \tag{3.34}$$

$$\begin{aligned} U_1(t_1,t_2) &= I + (-i)^1 \int_0^{t_1} dT_1 H_1(T_1,t_2) \\ &\quad + (-i)^2 \int_0^{t_1} dT_1 \int_0^{T_1} dT_2 H_1(T_1,t_2) H_1(T_2,t_2) + \ldots \ . \end{aligned} \tag{3.35}$$

Since the Hamiltonian operator depends on two time variables, then we find that

$$\begin{aligned} H_1(T_1,0) &= H_1(0,0) + (T_1-0)\frac{\partial H_1(0,0)}{\partial t_1} + (0-0)\frac{\partial H_1(0,0)}{\partial t_2} + \ldots \\ &= H_1(0,0) + T_1\frac{\partial H_1(0,0)}{\partial t_1} + \ldots \ , \end{aligned} \tag{3.36}$$

$$\begin{aligned} H_1(T_2,0) &= H_1(0,0) + (T_2-0)\frac{\partial H_1(0,0)}{\partial t_1} + (0-0)\frac{\partial H_1(0,0)}{\partial t_2} + \ldots \\ &= H_1(0,0) + T_2\frac{\partial H_1(0,0)}{\partial t_1} + \ldots \ , \end{aligned} \tag{3.37}$$

$$\begin{aligned} H_1(T_1,t_2) &= H_1(0,0) + (T_1-0)\frac{\partial H_1(0,0)}{\partial t_1} + (t_2-0)\frac{\partial H_1(0,0)}{\partial t_2} + \ldots \\ &= H_1(0,0) + T_1\frac{\partial H_1(0,0)}{\partial t_1} + \Delta t_2\frac{\partial H_1(0,0)}{\partial t_2} + \ldots \ , \end{aligned} \tag{3.38}$$

$$\begin{aligned} H_1(T_2,t_2) &= H_1(0,0) + (T_2-0)\frac{\partial H_1(0,0)}{\partial t_1} + (t_2-0)\frac{\partial H_1(0,0)}{\partial t_2} + \ldots \\ &= H_1(0,0) + T_2\frac{\partial H_1(0,0)}{\partial t_1} + \Delta t_2\frac{\partial H_1(0,0)}{\partial t_2} + \ldots \ , \end{aligned} \tag{3.39}$$

and

$$
\begin{aligned}
H_2(0,T_1) &= H_2(0,0) + (0-0)\frac{\partial H_2(0,0)}{\partial t_1} + (T_1-0)\frac{\partial H_2(0,0)}{\partial t_2} + \dots \\
&= H_2(0,0) + T_1\frac{\partial H_2(0,0)}{\partial t_2} + \dots \ , \qquad (3.40)\\
H_2(0,T_2) &= H_2(0,0) + (0-0)\frac{\partial H_2(0,0)}{\partial t_1} + (T_2-0)\frac{\partial H_2(0,0)}{\partial t_2} + \dots \\
&= H_2(0,0) + T_2\frac{\partial H_2(0,0)}{\partial t_2} + \dots \ , \qquad (3.41)\\
H_2(t_1,T_1) &= H_2(0,0) + (t_1-0)\frac{\partial H_2(0,0)}{\partial t_1} + (T_1-0)\frac{\partial H_2(0,0)}{\partial t_2} + \dots \\
&= H_2(0,0) + \Delta t_1\frac{\partial H_2(0,0)}{\partial t_1} + T_1\frac{\partial H_2(0,0)}{\partial t_2} + \dots \ , \qquad (3.42)\\
H_2(t_1,T_2) &= H_2(0,0) + (t_1-0)\frac{\partial H_2(0,0)}{\partial t_1} + T_2\frac{\partial H_2(0,0)}{\partial t_2} + \dots \\
&= H_2(0,0) + \Delta t_1\frac{\partial H_2(0,0)}{\partial t_1} + T_2\frac{\partial H_2(0,0)}{\partial t_2} + \dots \ . \qquad (3.43)
\end{aligned}
$$

Using Equation (3.36) - (3.43), the unitary operators can be expressed as

$$
\begin{aligned}
U(t_1,0) &= I - iH_1(0,0)\Delta t_1 - \frac{i}{2}\frac{\partial H_1(0,0)}{\partial t_1}(\Delta t_1)^2 \\
&\quad - \frac{1}{2}H_1^2(0,0)(\Delta t_1)^2 + \dots \ , \qquad (3.44)\\
U_2(t_1,t_2) &= I - iH_2(0,0)\Delta t_2 - i\frac{\partial H_2(0,0)}{\partial t_1}\Delta t_1\Delta t_2 - \frac{i}{2}\frac{\partial H_2(0,0)}{\partial t_2}(\Delta t_2)^2 \\
&\quad - \frac{1}{2}H_2^2(0,0)(\Delta t_2)^2 + \dots \ , \qquad (3.45)\\
U(0,t_2) &= I - iH_2(0,0)\Delta t_2 - \frac{i}{2}\frac{\partial H_2(0,0)}{\partial t_2}(\Delta t_2)^2 \\
&\quad - \frac{1}{2}H_2^2(0,0)(\Delta t_2)^2 + \dots \ , \qquad (3.46)\\
U_1(t_1,t_2) &= I - iH_1(0,0)\Delta t_1 - \frac{i}{2}\frac{\partial H_1(0,0)}{\partial t_1}(\Delta t_1)^2 - i\frac{\partial H_1(0,0)}{\partial t_2}\Delta t_2\Delta t_1 \\
&\quad - \frac{1}{2}H_1^2(0,0)(\Delta t_1)^2 + \dots \ . \qquad (3.47)
\end{aligned}
$$

For $\Delta t_1 \to 0$ and $\Delta t_2 \to 0$, we can ignore higher-order terms and then the evolutions (3.30) and (3.31) are compatible if and only if

$$
U_2(t_1,t_2)U(t_1,0) = U_1(t_1,t_2)U(0,t_2)
$$

leading to

$$\frac{\partial H_1}{\partial t_2} - \frac{\partial H_2}{\partial t_1} - i\,[H_1, H_2] = 0\ . \tag{3.48}$$

This proves the consistency condition (3.27) on the evolution of the multi-time system with the time-dependent Hamiltonians and the full derivation can be found in appendix B.

Remark: The wave function Φ is defined only on the space-like configurations and the system of multi-time equations with interaction automatically violates the consistency condition [3].

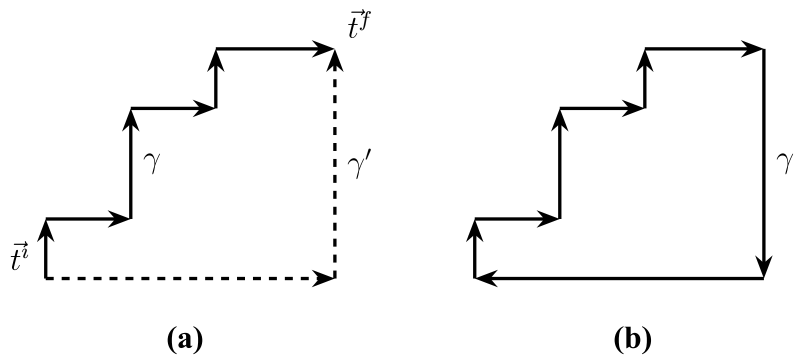


Figure 13 (a) Two different paths γ and γ' from the initial point $\vec{t^i}$ to the final point $\vec{t^f}$. (b) A loop γ.

The condition (3.27) implies the path-independent feature of the time evolution in the context of multi-time quantum theory. This can be seen by the following construction. If we consider the path which is parametised by γ, see figure 13a, where $\gamma\,:\,[0,1]$ from the initial point $\gamma(0) = \vec{t^i} = ({t_1}^i, {t_2}^i, ..., {t_N}^i)$ to the final point $\gamma(1) = \vec{t^f} = ({t_1}^f, {t_2}^f, ..., {t_N}^f)$, the time evolution operator along this particular path is given by

$$U_\gamma = \mathsf{T}e^{-i\int_\gamma \sum_j H_j dt_j}\ . \tag{3.49}$$

Another path parametised by γ', see also figure 13a, where $\gamma'\,:\,[0,1]$ from the initial point $\gamma'(0) = \vec{t^i} = ({t_1}^i, {t_2}^i, ..., {t_N}^i)$ to the final point $\gamma'(1) = \vec{t^f} = ({t_1}^f, {t_2}^f, ..., {t_N}^f)$, the time evolution operator along this path is given by

$$U_{\gamma'} = \mathsf{T}e^{-i\int_{\gamma'}\sum_j H_j dt_j} \ . \tag{3.50}$$

The path-independent feature requires the condition $U_\gamma = U_{\gamma'}$.

In the language of geometry, we can put the path-independent feature as the parallel transport process. To see this, we define the covariant derivative

$$\nabla_j = \partial_j - iA_j \ , \tag{3.51}$$

where $\partial_j = \partial/\partial_{t_j}$ and connection coefficient $A_j = -H_j$. Then U_γ can be treated as the parallel transport operator along the path γ known as the order path integral or Wilson line. For an arbitrary loop γ, see figure 13b, one can express the transport operator in the form

$$U_\gamma = \mathsf{T}e^{-i\oint_\gamma \sum_j H_j dt_j} \ , \tag{3.52}$$

which is known as the Wilson loop. Then the path-independent property is nothing but saying that all closed paths γ have trivial holonomy, i.e., $U_\gamma = I$. As a consequence, a gauge connection possesses trivial holonomies if and only if its **curvature** F is defined as

$$F_{jk} \equiv -\frac{\partial H_k}{\partial t_j} + \frac{\partial H_j}{\partial t_k} - i\big[H_j, H_k\big] \tag{3.53}$$

vanishes [3]:

$$F_{jk} = 0 \quad \forall j \neq k \ . \tag{3.54}$$

With the definition of the curvature, we can rewrite the argument of the exponential of the time evolution operator as

$$-i\oint_{\partial\Sigma}\sum_j H_j dt_j = -i\iint_\Sigma \sum_{ij} F_{ij} dt_i \wedge dt_j \ , \tag{3.55}$$

where Σ is a 2-dimensional surface whose boundary is $\partial\Sigma$. Obviously, condition (3.54) is identical to Equation (3.27) so we can consider the consistency condition in the viewpoint of curvature. We knew that curvature is the tool to test the difference of vector that parallel transported along a closed path. If the direction of the initial and the final vector is not different, there is no curvature of the surface, $F_{jk} = 0$, which means flat surface [3, 38, 39]. Consequently, the consistency condition (3.27) of the multi-time wave function can be treated as the zero curvature condition.

3.4 Multi-time Propagator

To capture the quantum version of the consistency condition in terms of the Lagrangian, the appropriate approach is the Feynman path integration method. We would like to discuss the multi-time evolutions in the Feynman picture.

Compatible evolutions: For simplicity, we consider the evolutions of the multi-time wave function from the initial point (t_1, t_2) to the final point (t'_1, t'_2), see figure 14, in two different paths in the context of Feynman path integration on the space of time variables.

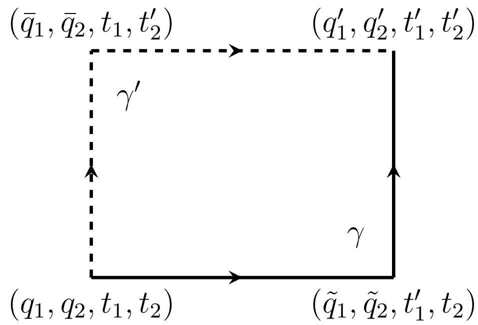


Figure 14 Two different paths γ and γ' from the initial point (t_1, t_2) to the final point (t'_1, t'_2).

The first path(solid line): The transition of the multi-time wave function from point (t_1, t_2) to (t'_1, t'_2) evolves from t_1 to t'_1 with the unitary operator U_1, then evolves from t_2 to t'_2 with the unitary operator U_2. The lower-half path can be captured in terms of the propagator as follows

$$\begin{aligned}
\left\langle q_1', q_2' \middle| \Phi_{\lrcorner}(t_1', t_2') \right\rangle &= \langle q_1', q_2' \left| U_2 U_1 \right| \Phi(t_1, t_2) \rangle \\
&= \iiiint d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \langle q_1', q_2' \left| U_2 \right| \tilde{q}_2, \tilde{q}_1 \rangle \langle \tilde{q}_2, \tilde{q}_1 \left| U_1 \right| q_1, q_2 \rangle \\
&\quad \times \langle q_2, q_1 | \Phi(t_1, t_2) \rangle \\
&= \iiiint d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \langle q_2' \left| U_2 \right| \tilde{q}_2 \rangle \langle q_1' | \tilde{q}_1 \rangle \langle \tilde{q}_1 \left| U_1 \right| q_1 \rangle \langle \tilde{q}_2 | q_2 \rangle \\
&\quad \times \Phi(q_1, q_2, t_1, t_2) \\
&= \iiiint d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \langle q_2' \left| U_2 \right| \tilde{q}_2 \rangle \langle \tilde{q}_1 \left| U_1 \right| q_1 \rangle \delta(q_1' - \tilde{q}_1) \delta(\tilde{q}_2 - q_2) \\
&\quad \times \Phi(q_1, q_2, t_1, t_2) \\
&= \iint dq_1 dq_2 \langle q_2' \left| U_2 \right| q_2 \rangle \langle q_1' \left| U_1 \right| q_1 \rangle \Phi(q_1, q_2, t_1, t_2) \\
&= \iint dq_1 dq_2 K_2(q_2', t_2'; q_2, t_2) K_1(q_1', t_1'; q_1, t_1) \\
&\quad \times \Phi(q_1, q_2, t_1, t_2) \; . \qquad (3.56)
\end{aligned}$$

The second path(dashed line): The transition of the multi-time wave function from point (t_1, t_2) to (t_1', t_2') through the upper-half path is given by

$$\begin{aligned}
\left\langle q_1', q_2' \middle| \Phi_{\ulcorner}(t_1', t_2') \right\rangle &= \langle q_1', q_2' \left| U_1 U_2 \right| \Phi(t_1, t_2) \rangle \\
&= \iiiint d\bar{q}_1 d\bar{q}_2 dq_1 dq_2 \langle q_1', q_2' \left| U_1 \right| \bar{q}_2, \bar{q}_1 \rangle \langle \bar{q}_2, \bar{q}_1 \left| U_2 \right| q_1, q_2 \rangle \\
&\quad \times \langle q_2, q_1 | \Phi(t_1, t_2) \rangle \\
&= \iiiint d\bar{q}_1 d\bar{q}_2 dq_1 dq_2 \langle q_1' \left| U_1 \right| \bar{q}_1 \rangle \langle q_2' | \bar{q}_2 \rangle \langle \bar{q}_2 \left| U_2 \right| q_2 \rangle \langle \bar{q}_1 | q_1 \rangle \\
&\quad \times \Phi(q_1, q_2, t_1, t_2) \\
&= \iiiint d\bar{q}_1 d\bar{q}_2 dq_1 dq_2 \langle q_1' \left| U_1 \right| \bar{q}_1 \rangle \langle \bar{q}_2 \left| U_2 \right| q_2 \rangle \delta(q_2' - \bar{q}_2) \delta(\bar{q}_1 - q_1) \\
&\quad \times \Phi(q_1, q_2, t_1, t_2) \\
&= \iint dq_1 dq_2 \langle q_1' \left| U_1 \right| q_1 \rangle \langle q_2' \left| U_2 \right| q_2 \rangle \Phi(q_1, q_2, t_1, t_2)
\end{aligned}$$

$$\Phi_{\ulcorner}\left(q_1', q_2', t_1', t_2'\right) = \iint dq_1 dq_2 K_1(q_1', t_1'; q_1, t_1) K_2(q_2', t_2'; q_2, t_2) \times \Phi(q_1, q_2, t_1, t_2) \ . \tag{3.57}$$

To make the both transitions compatible, one requires $\Phi_{\lrcorner}\left(q_1', q_2', t_1', t_2'\right) = \Phi_{\ulcorner}\left(q_1', q_2', t_1', t_2'\right)$, resulting in

$$\iint dq_1 dq_2 \left\{K_2 K_1 - K_1 K_2\right\} \Phi(q_1, q_2, t_1, t_2) = 0 \ . \tag{3.58}$$

If now we define $K_{\lrcorner}\left(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2\right) = K_2(q_2', t_2'; q_2, t_2) K_1(q_1', t_1'; q_1, t_1)$ as a lower-half propagator and $K_{\ulcorner}\left(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2\right) = K_1(q_1', t_1'; q_1, t_1) K_2(q_2', t_2'; q_2, t_2)$ as an upper-half propagator. Since $\Phi(q_1, q_2, t_1, t_2)$ cannot be zero, then Equation (3.58) gives us

$$K_{\lrcorner}\left(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2\right) = K_{\ulcorner}\left(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2\right) \ . \tag{3.59}$$

Here we obtain the consistency condition for the multi-time evolution in terms of the propagator. This equation is nothing but the commuting propagators: $[K_1, K_2] = 0$ reflecting the path-independent property of the propagator on the space of time variables.

One can treat these commuting propagators as the parallel transport operation in terms of Lagrangian. Now we may write the Wilson line associated with path γ as

$$\begin{aligned} &K_{\gamma}\left(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2\right) \\ &\quad = \int_{q_2}^{q_2'} \mathscr{D}[\tilde{q}_2(\tilde{t}_2)] \int_{q_1}^{q_1'} \mathscr{D}[\tilde{q}_1(\tilde{t}_1)] e^{i \int_{\gamma} L_1(\tilde{q}_1, \partial_{\tilde{t}_1}\tilde{q}_1) d\tilde{t}_1 + L_2(\tilde{q}_2, \partial_{\tilde{t}_2}\tilde{q}_2) d\tilde{t}_2} \ , \end{aligned} \tag{3.60}$$

and the Wilson line associated with path γ' as

$$\begin{aligned} &K_{\gamma'}\left(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2\right) \\ &\quad = \int_{q_1}^{q_1'} \mathscr{D}[\tilde{q}_1(\tilde{t}_1)] \int_{q_2}^{q_2'} \mathscr{D}[\tilde{q}_2(\tilde{t}_2)] e^{i \int_{\gamma'} L_1(\tilde{q}_1, \partial_{\tilde{t}_1}\tilde{q}_1) d\tilde{t}_1 + L_2(\tilde{q}_2, \partial_{\tilde{t}_2}\tilde{q}_2) d\tilde{t}_2} \ . \end{aligned} \tag{3.61}$$

The result in this section can be easily extended to the case of N time variables and the Wilson line γ in terms of the propagator is given by

$$\begin{aligned} &K_{\gamma}\left(q_1', t_1', q_2', t_2', ..., q_N', t_N'; q_1, t_1, q_2, t_2, ..., q_N, t_N\right) \\ &\quad = \mathsf{P} \prod_{i=1}^{N} \int_{q_i}^{q_i'} \mathscr{D}[\tilde{q}_i(\tilde{t}_i)] e^{i \int_{\gamma} \sum_{i=1}^{N} L_i(\tilde{q}_i, \partial_{\tilde{t}_i}\tilde{q}_i) d\tilde{t}_i} \ , \end{aligned} \tag{3.62}$$

where P stands for the permutation.

Time loops: We consider the other consistency condition in terms of the propagator that is the case of the loop evolution of the multi-time wave function. Before proceeding with the computation, we need to establish some useful relations. We start to consider the transition of the wave function from (q,t) to (q',t') given by

$$\Phi(q',t') = \int dqK(q',t';q,t)\Phi(q,t)\ . \tag{3.63}$$

Next, we consider the transition from (q',t') to $(\tilde{q},\tilde{t})$ given by

$$\Phi(\tilde{q},\tilde{t}) = \int dq'K(\tilde{q},\tilde{t};q',t')\Phi(q',t')\ . \tag{3.64}$$

Combining Equation (3.64) with Equation (3.63), we obtain

$$\Phi(\tilde{q},\tilde{t}) = \iint dq'dqK(\tilde{q},\tilde{t};q',t')K(q',t';q,t)\Phi(q,t)\ . \tag{3.65}$$

To change the transition (3.65) to the loop transition, we impose

$$\begin{aligned}\Phi(\tilde{q},\tilde{t}) &= \iint dq'dqK(\tilde{q},\tilde{t};q',t')K(q',t';q,t)\Phi(q,t)\\ &= \int dq\delta(\tilde{q}-q)\Phi(q,\tilde{t}) = \Phi(\tilde{q},\tilde{t})\ ,\end{aligned} \tag{3.66}$$

therefore, one requires

$$\delta(\tilde{q}-q) = \int dq'K(\tilde{q},\tilde{t};q',t')K(q',t';q,t) = K(\tilde{q},t;q,t)\ , \tag{3.67}$$

where $(\tilde{t}-t) = \delta t \to 0$. Equivalently, Equation (3.67) can be expressed in terms of Lagrangians as

$$\begin{aligned}\delta(\tilde{q}-q) &= \lim_{\delta t\to 0}\int dq'\left[\int_{q'}^{\tilde{q}}\mathscr{D}[\bar{q}(\bar{t})]e^{i\int_{t'}^{\tilde{t}}L(\bar{q},\partial_{\bar{t}}\bar{q})d\bar{t}}\right]\left[\int_{q}^{q'}\mathscr{D}[\bar{q}(\bar{t})]e^{i\int_{t}^{t'}L(\bar{q},\partial_{\bar{t}}\bar{q})d\bar{t}}\right]\\ &= \lim_{\delta t\to 0}\int dq'\int_{q'}^{\tilde{q}}\mathscr{D}[\bar{q}(\bar{t})]\int_{q}^{q'}\mathscr{D}[\bar{q}(\bar{t})]e^{i\left(\int_{t'}^{\tilde{t}}L(\bar{q},\partial_{\bar{t}}\bar{q})d\bar{t}+\int_{t}^{t'}L(\bar{q},\partial_{\bar{t}}\bar{q})d\bar{t}\right)}\\ &= \lim_{\delta t\to 0}\int_{q}^{\tilde{q}}\mathscr{D}[\bar{q}(\bar{t})]e^{i\int_{t}^{\tilde{t}}L(\bar{q},\partial_{\bar{t}}\bar{q})d\bar{t}} = \int_{q}^{\tilde{q}}\mathscr{D}[\bar{q}(\bar{t})]e^{i\oint L(\bar{q},\partial_{\bar{t}}\bar{q})d\bar{t}}\ .\end{aligned} \tag{3.68}$$

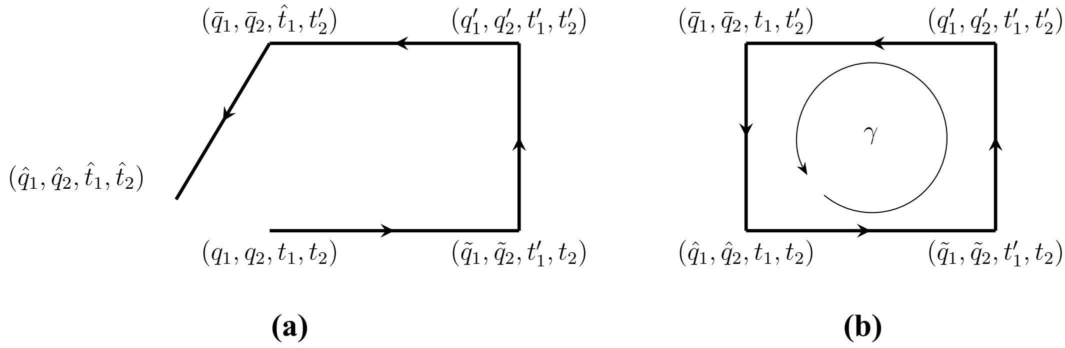


Figure 15 (a) The evolution from the initial point (q_1, q_2, t_1, t_2) to the final point $(\hat{q}_1, \hat{q}_2, \hat{t}_1, \hat{t}_2)$. (b) A loop evolution γ can be obtained by imposing $\hat{q}_i = q_i$ and $\hat{t}_i = t_i$, where $i = 1, 2$.

Now we are ready to consider the loop evolution. Let define $U_1(t'_1 - t_1)$ as the time evolution operator from t_1 to t'_1, $U_2(t'_2 - t_2)$ as the time evolution operator from t_2 to t'_2, $U'_1(\hat{t}_1 - t'_1)$ as the time evolution operator from t'_1 to $\hat{t}_1$ and $U'_2(\hat{t}_2 - t'_2)$ as the time evolution operator from t'_2 to $\hat{t}_2$. The transition map, shown in figure 15a, can be expressed as

$$\begin{aligned}
\left\langle \hat{q}_1, \hat{q}_2 \middle| \Phi(\hat{t}_1, \hat{t}_2) \right\rangle &= \langle \hat{q}_1, \hat{q}_2 \left| U'_2 U'_1 U_2 U_1 \right| \Phi(t_1, t_2) \rangle \\
\Phi\left(\hat{q}_1, \hat{q}_2, \hat{t}_1, \hat{t}_2\right) &= \iint dq_1 dq_2 \langle \hat{q}_1, \hat{q}_2 \left| U'_2 U'_1 U_2 U_1 \right| q_1, q_2 \rangle \left\langle q_2, q_1 \middle| \Phi(t_1, t_2) \right\rangle \\
&= \iint dq_1 dq_2 \int d\bar{q}_2 \left\langle \hat{q}_2 \left| U'_2(\hat{t}_2 - t'_2) \right| \bar{q}_2 \right\rangle \langle \bar{q}_2 \left| U_2(t'_2 - t_2) \right| q_2 \rangle \\
&\quad \times \int d\tilde{q}_1 \left\langle \hat{q}_1 \left| U'_1(\hat{t}_1 - t'_1) \right| \tilde{q}_1 \right\rangle \langle \tilde{q}_1 \left| U_1(t'_1 - t_1) \right| q_1 \rangle \, \Phi(q_1, q_2, t_1, t_2) \\
&= \iint dq_1 dq_2 \int d\bar{q}_2 K_2(\hat{q}_2, \hat{t}_2; \bar{q}_2, t'_2) K_2(\bar{q}_2, t'_2; q_2, t_2) \\
&\quad \times \int d\tilde{q}_1 K_1(\hat{q}_1, \hat{t}_1; \tilde{q}_1, t'_1) K_1(\tilde{q}_1, t'_1; q_1, t_1) \Phi(q_1, q_2, t_1, t_2) \ . \qquad (3.69)
\end{aligned}$$

The full derivation of Equation (3.69) can be found in appendix. Using the condition (3.67) where $(\hat{t}_1 - t_1) = \delta t_1 \to 0$ and $(\hat{t}_2 - t_2) = \delta t_2 \to 0$, we have

$$\delta(\hat{q}_2 - q_2) = \int d\bar{q}_2 K_2(\hat{q}_2, \hat{t}_2; \bar{q}_2, t'_2) K_2(\bar{q}_2, t'_2; q_2, t_2) = K_2(\hat{q}_2, t_2; q_2, t_2) \ , \qquad (3.70)$$

$$\delta(\hat{q}_1 - q_1) = \int d\tilde{q}_1 K_1(\hat{q}_1, \hat{t}_1; \tilde{q}_1, t'_1) K_1(\tilde{q}_1, t'_1; q_1, t_1) = K_1(\hat{q}_1, t_1; q_1, t_1) \ . \qquad (3.71)$$

Substituting Equation (3.70) and (3.71) into Equation (3.69), we find that

$$\begin{aligned}\Phi\left(\hat{q}_1, \hat{q}_2, t_1, t_2\right) &= \iint dq_1 dq_2 \delta(\hat{q}_2 - q_2)\delta(\hat{q}_1 - q_1)\Phi(q_1, q_2, t_1, t_2) \\ &= \Phi(\hat{q}_1, \hat{q}_2, t_1, t_2)\ ,\end{aligned} \tag{3.72}$$

which gives us the loop evolution shown in figure 15b.

Next, the condition for the propagator in Equation (3.69) can be expressed in terms of the Lagrangian as

$$\begin{aligned}&\delta\left(\hat{q}_2 - q_2\right)\delta\left(\hat{q}_1 - q_1\right) \\ &= \int d\bar{q}_2 K_2(\hat{q}_2, \hat{t}_2; \bar{q}_2, t_2')K_2(\bar{q}_2, t_2'; q_2, t_2) \int d\tilde{q}_1 K_1(\hat{q}_1, \hat{t}_1; \tilde{q}_1, t_1')K_1(\tilde{q}_1, t_1'; q_1, t_1) \\ &= \int d\bar{q}_2 \left[\int_{\bar{q}_2}^{\hat{q}_2} \mathscr{D}[\check{q}_2(\check{t}_2)] e^{i\int_{t_2'}^{\hat{t}_2} L_2(\check{q}_2, \partial_{\check{t}_2}\check{q}_2)d\check{t}_2}\right]\left[\int_{q_2}^{\bar{q}_2} \mathscr{D}[\check{q}_2(\check{t}_2)] e^{i\int_{t_2}^{t_2'} L_2(\check{q}_2, \partial_{\check{t}_2}\check{q}_2)d\check{t}_2}\right] \\ &\times \int d\tilde{q}_1 \left[\int_{\tilde{q}_1}^{\hat{q}_1} \mathscr{D}[\check{q}_1(\check{t}_1)] e^{i\int_{t_1'}^{\hat{t}_1} L_1(\check{q}_1, \partial_{\check{t}_1}\check{q}_1)d\check{t}_1}\right]\left[\int_{q_1}^{\tilde{q}_1} \mathscr{D}[\check{q}_1(\check{t}_1)] e^{i\int_{t_1}^{t_1'} L_1(\check{q}_1, \partial_{\check{t}_1}\check{q}_1)d\check{t}_1}\right] \\ &= \left[\int_{q_2}^{\hat{q}_2} \mathscr{D}[\check{q}_2(\check{t}_2)] e^{i\int_{t_2}^{\hat{t}_2} L_2(\check{q}_2, \partial_{\check{t}_2}\check{q}_2)d\check{t}_2}\right]\left[\int_{q_1}^{\hat{q}_1} \mathscr{D}[\check{q}_1(\check{t}_1)] e^{i\int_{t_1}^{\hat{t}_1} L_1(\check{q}_1, \partial_{\check{t}_1}\check{q}_1)d\check{t}_1}\right]\ .\end{aligned} \tag{3.73}$$

Taking $\delta t_1 \to 0$ and $\delta t_2 \to 0$, we obtain

$$\begin{aligned}&\delta\left(\hat{q}_2 - q_2\right)\delta\left(\hat{q}_1 - q_1\right) \\ &= \left[\lim_{\delta t_2 \to 0} \int_{q_2}^{\hat{q}_2} \mathscr{D}[\check{q}_2] e^{i\int_{t_2}^{\hat{t}_2} L_2(\check{q}_2, \partial_{\check{t}_2}\check{q}_2)d\check{t}_2}\right]\left[\lim_{\delta t_1 \to 0} \int_{q_1}^{\hat{q}_1} \mathscr{D}[\check{q}_1] e^{i\int_{t_1}^{\hat{t}_1} L_1(\check{q}_1, \partial_{\check{t}_1}\check{q}_1)d\check{t}_1}\right] \\ &= \left[\int_{q_2}^{\hat{q}_2} \mathscr{D}[\check{q}_2] e^{i\oint L_2(\check{q}_2, \partial_{\check{t}_2}\check{q}_2)d\check{t}_2}\right]\left[\int_{q_1}^{\hat{q}_1} \mathscr{D}[\check{q}_1] e^{i\oint L_1(\check{q}_1, \partial_{\check{t}_1}\check{q}_1)d\check{t}_1}\right]\ .\end{aligned} \tag{3.74}$$

The result in Equation (3.74) can be immediately extended to the case of N time variables resulting in

$$\prod_{k=1}^{N} \int_{q_k}^{\hat{q}_k} \mathscr{D}[\check{q}_k] e^{i\oint_\gamma L_k(\check{q}_k, \partial_{\check{t}_k}\check{q}_k)d\check{t}_k} = \prod_{k=1}^{N} \delta(\hat{q}_k - q_k)\ . \tag{3.75}$$

In the language of the Wilson line, we have the propagator for the loop γ, shown figure 15b for the two-time variables, as

$$\begin{aligned}K_\gamma(\hat{q}_1, t_1, \hat{q}_2, t_2, ..., \hat{q}_N, t_N; \hat{q}_1, t_1, \hat{q}_2, t_2, ..., \hat{q}_N, t_N) &= \prod_{k=1}^{N} \oint \mathscr{D}[\check{q}_k] e^{i\oint_\gamma L_k(\check{q}_k, \partial_{\check{t}_k}\check{q}_k)d\check{t}_k} \\ &= I\ .\end{aligned} \tag{3.76}$$

What we have from Equation (3.76) is the following. The quantum transition between two endpoints will be not contributed from the loops. In other words, the loops can be excluded from the whole evolution as shown in figure 16.

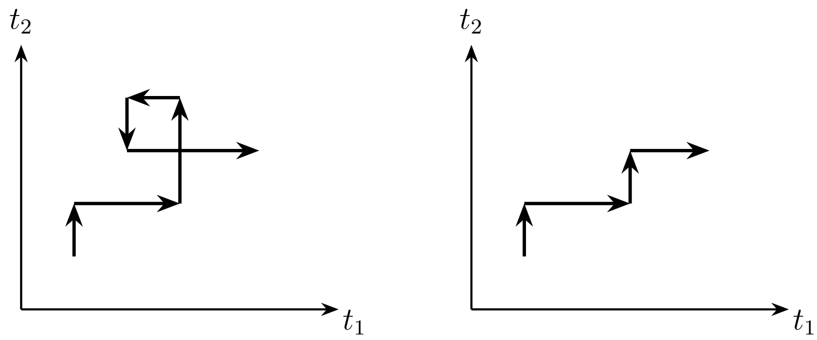


Figure 16 The close loop does not contribute to the evolution.

Example: Next, we will give an explicit computation to illustrate the path independent property, i.e., a loop evolution. Here, for simplicity, we choose a free particle to work with and the propagator is given by

$$K(q',t';q,t) = \sqrt{\frac{m}{2\pi i(t'-t)}}e^{\frac{im}{2}\frac{(q'-q)^2}{(t'-t)}} \ . \tag{3.77}$$

We now compute the propagator along the time variables t_i, where $i = 1, 2$

$$\begin{aligned}
&K_i(\hat{q}_i,\hat{t}_i;q_i,t_i)\\
&= \int d\bar{q}_i K_i(\hat{q}_i,\hat{t}_i;\bar{q}_i,t_i')K_i(\bar{q}_i,t_i';q_i,t_i)\\
&= \int d\bar{q}_i \sqrt{\frac{m}{2i\pi(\hat{t}_i-t_i')}}\sqrt{\frac{m}{2i\pi(t_i'-t_i)}}e^{\frac{im}{2}\frac{(\hat{q}_i-\bar{q}_i)^2}{(\hat{t}_i-t_i')}}e^{\frac{im}{2}\frac{(\bar{q}_i-q_i)^2}{(t_i'-t_i)}}\\
&= \int d\bar{q}_i \frac{m}{2i\pi}\sqrt{\frac{1}{(\hat{t}_i-t_i')(t_i'-t_i)}}e^{\frac{im}{2(\hat{t}_i-t_i')}(\hat{q}_i^2-2\hat{q}_i\bar{q}_i+\bar{q}_i^2)}e^{\frac{im}{2(t_i'-t_i)}(\bar{q}_i^2-2\bar{q}_iq_i+q_i^2)}\\
&= \int d\bar{q}_i \frac{m}{2i\pi}\sqrt{\frac{1}{(\hat{t}_i-t_i')(t_i'-t_i)}}e^{\hat{q}_i^2\left(\frac{im}{2(\hat{t}_i-t_i')}\right)}e^{q_i^2\left(\frac{im}{2(t_i'-t_i)}\right)}e^{\bar{q}_i^2\left(\frac{im}{2(\hat{t}_i-t_i')}+\frac{im}{2(t_i'-t_i)}\right)}\\
&\quad\times e^{\bar{q}_i\left(\frac{-im\hat{q}_i}{(\hat{t}_i-t_i')}+\frac{-imq_i}{(t_i'-t_i)}\right)}\\
&= \frac{m}{2i\pi}\sqrt{\frac{2\pi}{(-im)(\hat{t}_i-t_i)}}e^{\hat{q}_i^2\left(\frac{im}{2(\hat{t}_i-t_i')}\right)}e^{q_i^2\left(\frac{im}{2(t_i'-t_i)}\right)}\\
&\quad\times e^{\frac{(-im)^2}{-2im}\left(\frac{\hat{q}_i}{(\hat{t}_i-t_i')}+\frac{q_i}{(t_i'-t_i)}\right)^2\left(\frac{(\hat{t}_i-t_i')(t_i'-t_i)}{(\hat{t}_i-t_i)}\right)}
\end{aligned}$$

$$\begin{aligned} K_i(\hat{q}_i,\hat{t}_i;q_i,t_i) &= \sqrt{\frac{m}{2i\pi(\hat{t}_i-t_i)}}e^{\hat{q}_i^2\frac{im}{2(\hat{t}_i-t_i')}\left(1-\frac{(t_i'-t_i)}{(\hat{t}_i-t_i)}\right)}e^{q_i^2\frac{im}{2(t_i'-t_i)}\left(1-\frac{(\hat{t}_i-t_i')}{(\hat{t}_i-t_i)}\right)}e^{2\hat{q}_iq_i\left(\frac{-im}{2(\hat{t}_i-t_i)}\right)} \\ &= \sqrt{\frac{m}{2i\pi(\hat{t}_i-t_i)}}e^{\frac{im}{2(\hat{t}_i-t_i)}\left(\hat{q}_i^2-2\hat{q}_iq_i+q_i^2\right)} \\ &= \sqrt{\frac{m}{2i\pi(\hat{t}_i-t_i)}}e^{\frac{im}{2(\hat{t}_i-t_i)}(\hat{q}_i-q_i)^2} \; . \end{aligned} \tag{3.78}$$

Imposing $\hat{t}_i - t_i = \delta t_i$ and taking $\delta t_i \to 0$, we obtain [18]

$$K_i(\hat{q}_i,t_i;q_i,t_i) = \lim_{\delta t_i\to 0}\sqrt{\frac{m}{2i\pi\delta t_i}}e^{\frac{im}{2\delta t_i}(\hat{q}_i-q_i)^2} = \delta(\hat{q}_i-q_i) \; , \tag{3.79}$$

which are indeed Equation (3.70) for $i = 2$ and Equation (3.71) for $i = 1$.

Including interaction: The last point is that we will consider the system with the interaction. For simplicity, we work with the Hamiltonian for the system of two particles

$$H = H_1 + H_2 + V_{12} \; , \tag{3.80}$$

where V_{12} is a potential representing the interaction between the particles and H_i is the free Hamiltonian for the i^{th}-particle. What we are going to do is the same process as in figure 14.

Let us first define the unitary operators $U_1(t_1',t_1) = e^{-i(H_1+V_{12})(t_1'-t_1)}$ and $U_2(t_2',t_2)$ $= e^{-iH_2(t_2'-t_2)}$. Then the propagator for the lower corner path is given by

$$\begin{aligned} &K_{\lrcorner}(q_1',t_1',q_2',t_2';q_1,t_1,q_2,t_2) \\ &= \langle q_1',q_2'|U_2U_1|q_1,q_2\rangle \\ &= \iint d\tilde{q}_1d\tilde{q}_2\,\langle q_1',q_2'|U_2|\tilde{q}_2,\tilde{q}_1\rangle\,\langle\tilde{q}_2,\tilde{q}_1|U_1|q_1,q_2\rangle \\ &= \iint d\tilde{q}_1d\tilde{q}_2\,\langle q_1'|\tilde{q}_1\rangle\,\langle q_2'|U_2|\tilde{q}_2\rangle\,\langle\tilde{q}_2,\tilde{q}_1|U_1|q_1,q_2\rangle \\ &= \iint d\tilde{q}_1d\tilde{q}_2\delta(q_1'-\tilde{q}_1)K_2(q_2',t_2';\tilde{q}_2,t_2)\,\langle\tilde{q}_2,\tilde{q}_1|U_1|q_1,q_2\rangle \\ &= \int d\tilde{q}_2K_2(q_2',t_2';\tilde{q}_2,t_2)\left\langle\tilde{q}_2\right|\langle q_1'|U_1|q_1\rangle\left|q_2\right\rangle \\ &= \int d\tilde{q}_2K_2(q_2',t_2';\tilde{q}_2,t_2)\,\langle\tilde{q}_2|G(q_1',t_1';q_1,t_1;\hat{q}_2)|q_2\rangle \\ &= \int d\tilde{q}_2K_2(q_2',t_2';\tilde{q}_2,t_2)\,\langle\tilde{q}_2|q_2\rangle\,G(q_1',t_1';q_1,t_1;\tilde{q}_2,t_2;q_2,t_2) \end{aligned}$$

$$
\begin{aligned}
&K_{\lrcorner}\,(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2) \\
&\quad = \int d\tilde{q}_2 K_2(q_2', t_2'; \tilde{q}_2, t_2)\delta(\tilde{q}_2 - q_2)G(q_1', t_1'; q_1, t_1; \tilde{q}_2, t_2; q_2, t_2) \\
&\quad = K_2(q_2', t_2'; q_2, t_2)G(q_1', t_1'; q_1, t_1; q_2, t_2; q_2, t_2)\ ,
\end{aligned} \tag{3.81}
$$

and the propagator for the upper corner path is given by

$$
\begin{aligned}
&K_{\ulcorner}\,(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2) \\
&\quad = \langle q_1', q_2' \left| U_1 U_2 \right| q_1, q_2 \rangle \\
&\quad = \iint d\bar{q}_1 d\bar{q}_2 \left\langle q_1', q_2' \left| U_1 \right| \bar{q}_2, \bar{q}_1 \right\rangle \left\langle \bar{q}_2, \bar{q}_1 \left| U_2 \right| q_1, q_2 \right\rangle \\
&\quad = \iint d\bar{q}_1 d\bar{q}_2 \left\langle q_1', q_2' \left| U_1 \right| \bar{q}_2, \bar{q}_1 \right\rangle \left\langle \bar{q}_1 | q_1 \right\rangle \left\langle \bar{q}_2 \left| U_2 \right| q_2 \right\rangle \\
&\quad = \iint d\bar{q}_1 d\bar{q}_2 \left\langle q_1', q_2' \left| U_1 \right| \bar{q}_2, \bar{q}_1 \right\rangle \delta(\bar{q}_1 - q_1) K_2(\bar{q}_2, t_2'; q_2, t_2) \\
&\quad = \int d\bar{q}_2 \left\langle q_2' \left| \langle q_1' \left| U_1 \right| q_1 \rangle \right| \bar{q}_2 \right\rangle K_2(\bar{q}_2, t_2'; q_2, t_2) \\
&\quad = \int d\bar{q}_2 \left\langle q_2' \left| G'(q_1', t_1'; q_1, t_1; \hat{q}_2) \right| \bar{q}_2 \right\rangle K_2(\bar{q}_2, t_2'; q_2, t_2) \\
&\quad = \int d\bar{q}_2 \left\langle q_2' | \bar{q}_2 \right\rangle G'(q_1', t_1'; q_1, t_1; q_2', t_2'; \bar{q}_2, t_2') K_2(\bar{q}_2, t_2'; q_2, t_2) \\
&\quad = \int d\bar{q}_2 \delta(q_2' - \bar{q}_2) G'(q_1', t_1'; q_1, t_1; q_2', t_2'; \bar{q}_2, t_2') K_2(\bar{q}_2, t_2'; q_2, t_2) \\
&\quad = G'(q_1', t_1'; q_1, t_1; q_2', t_2'; q_2', t_2') K_2(q_2', t_2'; q_2, t_2)\ ,
\end{aligned} \tag{3.82}
$$

where $\hat{q}_2$ is a position operator for the 2^{nd} particle which is being fixed during the evolution of the 1^{st} particle. Then we find that the propagators for the upper and lower paths are not the same. This implies that the quantum evolution of the system with interaction is path-dependent. Of course, this path-dependent feature is a direct consequence of the violation of the consistency condition (3.12).

Example: Here we will show the explicit example. We choose to work with the following Lagrangians

$$
\begin{aligned}
L_1 &= \frac{m\dot{q}_1{}^2}{2} + kq_1q_2 = \frac{m\dot{q}_1{}^2}{2} + Fq_1 \qquad ; F = kq_2 \\
L_2 &= \frac{m\dot{q}_2{}^2}{2}\ .
\end{aligned} \tag{3.83}
$$

The propagator for a free particle with the constant force F with the Lagrangian $L = \frac{m\dot{q}^2}{2} + Fq$ is given by [17]

$$K^F(q',t';q,t) = \sqrt{\frac{m}{2\pi i(t'-t)}}e^{i\left\{\frac{m}{2}\frac{(q'-q)^2}{t'-t}+\frac{F}{2}(q'+q)(t'-t)-\frac{F^2}{24m}(t'-t)^3\right\}}. \tag{3.84}$$

We will process the same transition given in figure 14. Then the propagator of the lower path can be written as

$$\begin{aligned}
&K_{\lrcorner}(q_1',t_1',q_2',t_2';q_1,t_1,q_2,t_2)\\
&= \langle q_1',q_2'|U_2U_1|q_1,q_2\rangle\\
&= \sqrt{\frac{m}{2\pi i(t_2'-t_2)}}e^{\frac{im}{2}\frac{(q_2'-q_2)^2}{t_2'-t_2}}\\
&\quad\times\int d\tilde{q}_2\langle\tilde{q}_2|\sqrt{\frac{m}{2\pi i(t_1'-t_1)}}e^{i\left\{\frac{m}{2}\frac{(q_1'-q_1)^2}{t_1'-t_1}+\frac{F}{2}(q_1'+q_1)(t_1'-t_1)-\frac{F^2}{24m}(t_1'-t_1)^3\right\}}|q_2\rangle\\
&= \sqrt{\frac{m}{2\pi i(t_2'-t_2)}}e^{\frac{im}{2}\frac{(q_2'-q_2)^2}{t_2'-t_2}}\quad\sqrt{\frac{m}{2\pi i(t_1'-t_1)}}e^{\frac{im}{2}\frac{(q_1'-q_1)^2}{t_1'-t_1}}\\
&\quad\times\int d\tilde{q}_2\langle\tilde{q}_2|e^{i\left\{\frac{k\hat{q}_2}{2}(q_1'+q_1)(t_1'-t_1)-\frac{k^2\hat{q}_2^2}{24m}(t_1'-t_1)^3\right\}}|q_2\rangle\\
&= K_2(q_2',t_2';q_2,t_2)K_1(q_1',t_1';q_1,t_1)\int d\tilde{q}_2\,\langle\tilde{q}_2|q_2\rangle\,e^{i\left\{\frac{k\tilde{q}_2}{2}(q_1'+q_1)(t_1'-t_1)-\frac{k^2\tilde{q}_2{}^2}{24m}(t_1'-t_1)^3\right\}}\\
&= K_2(q_2',t_2';q_2,t_2)K_1(q_1',t_1';q_1,t_1)\\
&\quad\times\int d\tilde{q}_2\delta(\tilde{q}_2-q_2)e^{i\left\{\frac{k\tilde{q}_2}{2}(q_1'+q_1)(t_1'-t_1)-\frac{k^2\tilde{q}_2{}^2}{24m}(t_1'-t_1)^3\right\}}\\
&= K_2(q_2',t_2';q_2,t_2)K_1(q_1',t_1';q_1,t_1)e^{i\left\{\frac{kq_2}{2}(q_1'+q_1)(t_1'-t_1)-\frac{k^2q_2^2}{24m}(t_1'-t_1)^3\right\}}.
\end{aligned}\tag{3.85}$$

We proceed with the same computation for the upper path and we obtain

$$\begin{aligned}
&K_{\ulcorner}(q_1',t_1',q_2',t_2';q_1,t_1,q_2,t_2)\\
&= \langle q_1',q_2'|U_1U_2|q_1,q_2\rangle\\
&= \int d\bar{q}_2\langle q_2'|\sqrt{\frac{m}{2\pi i(t_1'-t_1)}}e^{i\left\{\frac{m}{2}\frac{(q_1'-q_1)^2}{t_1'-t_1}+\frac{F}{2}(q_1'+q_1)(t_1'-t_1)-\frac{F^2}{24m}(t_1'-t_1)^3\right\}}|\bar{q}_2\rangle\\
&\quad\times\sqrt{\frac{m}{2\pi i(t_2'-t_2)}}e^{\frac{im}{2}\frac{(q_2'-q_2)^2}{t_2'-t_2}}\\
&= \sqrt{\frac{m}{2\pi i(t_1'-t_1)}}e^{\frac{im}{2}\frac{(q_1'-q_1)^2}{t_1'-t_1}}\int d\bar{q}_2\langle q_2'|e^{i\left\{\frac{k\hat{q}_2}{2}(q_1'+q_1)(t_1'-t_1)-\frac{k^2\hat{q}_2^2}{24m}(t_1'-t_1)^3\right\}}|\bar{q}_2\rangle\\
&\quad\times\sqrt{\frac{m}{2\pi i(t_2'-t_2)}}e^{\frac{im}{2}\frac{(q_2'-q_2)^2}{t_2'-t_2}}
\end{aligned}$$

$$
\begin{aligned}
&K_{\ulcorner}(q_1',t_1',q_2',t_2';q_1,t_1,q_2,t_2)\\
&\quad = K_1(q_1',t_1';q_1,t_1)\int d\bar{q}_2\,\langle q_2'|\bar{q}_2\rangle\, e^{i\left\{\frac{k\bar{q}_2}{2}(q_1'+q_1)(t_1'-t_1)-\frac{k^2\bar{q}_2{}^2}{24m}(t_1'-t_1)^3\right\}}K_2(q_2',t_2';q_2,t_2)\\
&\quad = K_1(q_1',t_1';q_1,t_1)\int d\bar{q}_2\delta(q_2'-\bar{q}_2)e^{i\left\{\frac{k\bar{q}_2}{2}(q_1'+q_1)(t_1'-t_1)-\frac{k^2\bar{q}_2{}^2}{24m}(t_1'-t_1)^3\right\}}\\
&\qquad\times K_2(q_2',t_2';q_2,t_2)\\
&\quad = K_1(q_1',t_1';q_1,t_1)K_2(q_2',t_2';q_2,t_2)e^{i\left\{\frac{kq_2'}{2}(q_1'+q_1)(t_1'-t_1)-\frac{k^2q_2'^2}{24m}(t_1'-t_1)^3\right\}}\,. \qquad (3.86)
\end{aligned}
$$

This simple calculation shows that the interaction causes the violation of the relation (3.12) and consequently the commutation of the propagators. Of course, the path-independent is no longer applicable. In the geometry point of view, the present of the interaction can be viewed as a course of temporal space curvature and therefore, the parallel transport of different paths would give different results.

CHAPTER IV

SUMMARY

We study the multi-time formalism both in the Hamiltonian and Lagrangian descriptions. There is a necessary condition for the evolution of the system to be consistence. This condition is called the integrability condition. In the Hamiltonian mechanics, the system of Hamilton-Jacobi equations will possess an non-trivial common solution if the Poisson bracket of a pair of Hamiltonians vanishes known as the Hamiltonian commuting flows. In quantum level, there is a set of Schrödinger equations. These Schrödinger equations will have a common solution if all Hamiltonians are commute. With this commutation relation allows us to express the time evolution operator in terms of the Wilson loop and the Hamiltonian is nothing but the gauge variable. In geometrical view, the consistency condition can be viewed as the parallel transport completing the parallelogram.

We succeed to capture the consistency condition for the multi-time evolution in terms of the Lagrangian as the consequence of the variation of the action on the space of time variables. This consistency condition implies that the action is invariant under the local deformation, fixing end-points, of the path on the space of time variables. Actually, if we think that the continuous path is constituted from tiny discrete elements, then, path-independent property in the continuous-time case is a direct consequence of path-independent in the discrete-time case. Furthermore, with this property, there is a family of paths(homotopy), sharing the endpoints, that can be continuously transformed to each other in N-dimensional space of time variables. The consistency condition for the multi-time quantum evolution in terms of Feynman's path integrals is derived. The important point is the path-independent feature of the multi-time propagator which can be summarised as follows. In general, there are an infinite number of paths from the initial point to the final point on the space of time variables, see figure 17 in the case of

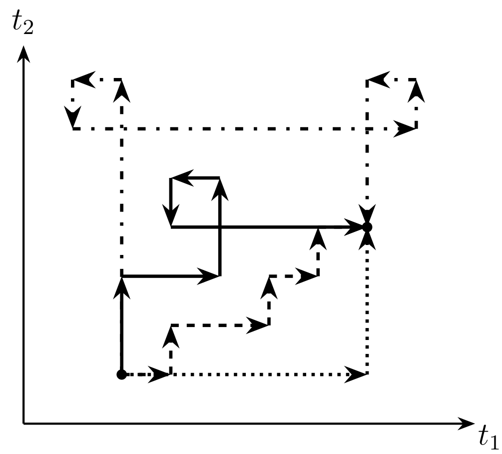


Figure 17 The possible paths, including shortest path, zigzag path and path with loops, $t_1 - t_2$ space from the initial point to the final point.

two-time variables. With a set of Lagrangians $\{L_1, L_2, ..., L_N\}$ satisfying the consistency condition, the propagator remains unchanged under the variation of the path on the space of time variables, and of course, this is nothing but the path-independent feature of the multi-time propagator. This would suggest that, apart from taking all possible paths in the configuration space as we normally do in the standard single-time path integration, one may need to take also the all possible paths in the space of time variables for the case of the multi-time path integration[4]. In the view of the geometry, the path-independent feature can be captured in terms of the parallel transport process on the flat space of time variables since the curvature vanishes. Then the consistency condition for a set of Lagrangians can be viewed as the zero curvature condition.

[4]This terminology arises also in the context of integrable systems [41]

REFERENCES

REFERENCES

[1] Dirac, P.A.M. (1932). Relativistic quantum mechanics. Proceedings of the Royal Society London A.

[2] Tomonaga, S. (1946). On a relativistically invariant formulation of the quantum theory of wave fields. Progress of Theoretical Physics.

[3] Petrat, S.,& Tumulka, R. (2014). Multi-time Schrödinger equations cannot contain interaction potentials. Journal of Mathematical Physics.

[4] Nickel, L. (2019). On the dynamics of multi-time systems. Ludwig-Maximilians-Universität München.

[5] Deckert, D.,& Nickel, L. (2018). Consistency of multi-time Dirac equations with general interaction potentials. Journal of Mathematical Physics.

[6] Petrat, S.P. (2010). Evolution equations for multi-time wave functions (master's thesis). The State University of New Jersey.

[7] Petrat, S.,& Tumulka, R. (2014). Multi-time equations, classical and quantum. Proceedings of the Royal Society A.

[8] Lienert, M., Petrat, S.,& Tumulka, R. (2017). Multi-time wave functions. Journal of Physics: Conference Series.

[9] Petrat, S.,& Tumulka, R. (2014). Multi-time wave functions for quantum field theory. Annals of Physics.

[10] Petrat, S.,& Tumulka, R. (2014). Multi-time formulation of pair creation. Journal of Physics A: Mathematical and Theoretical 47.

[11] José, J.V.,& Saletan, E. (1998). Classical dynamics: A contemporary approach. USA: Cambridge University Press.

[12] Thornton, S.T.,& Marion, J.B. (2004). Classical dynamics of particles and systems (5th ed.). USA: Thomson Learning.

[13] Goldstein, H., Poole, C.,& Safko, J. (2000). Classical mechanics (3rd ed.). Addison Wesley.

[14] Zwiebach, B. (2013). Quantum dynamics.

[15] Griffiths, D.J. (2005). Introduction to Quantum Mechanics (2nd ed.). USA: Pearson Education.

[16] Sakurai, J.J. (1994). Modern quantum mechanics (Rev. ed.). USA: Addison-Wesley.

[17] Feynman, R.,& Hibbs, A. (2010). Quantum mechanics and path integrals (Emended ed.). NY: McGraw-Hill.

[18] Walter, D.,& Martin R. (2020). Classical and quantum dynamics: From classical paths to path integrals (6th ed.). Springer Nature Switzerland.

[19] Yoo-Kong, S. (2018). Calculus of variations: With application in physics. BKK: Danex Intercorporation.

[20] Reid, M.,& Szendröi, B. (2005). Introduction to Quantum Mechanics (2nd ed.). NY: Cambridge University Press.

[21] Meyer, W. (2006). Geometry and its applications (2nd ed.). USA: Elsevier.

[22] Schutz, B.F. (1980). Geometrical methods of mathematical physics. USA: Cambridge University Press.

[23] Nakahara, M. (2003). Geometry, topology and physics (2nd ed.). IOP Publishing.

[24] Eschrig, H. (2011). Topology and geometry for physics. Springer-Verlag Berlin Heidelberg.

[25] Hicks, N.J. (1965). Notes on differential geometry. Litton Educational Publishing.

[26] Hobson, M.P., Efstathiou, G.P.,& Lasenby, A.N. (2006). General relativity: An introduction for physicists. Cambridge University Press.

[27] Kusse, B.R.,& Westwig, E.A. (2006). Mathematical physics: Applied mathematics for scientists and engineers (2nd ed.). WILEY-VCH Verlag GmbH & Co. KGaA.

[28] Wald, R.M. (1984). General relativity. USA: The University of Chicago Press.

[29] Fecko, M. (2006). Differential geometry and Lie groups for physicists. Cambridge University Press.

[30] Carroll, S.M. (1997). Lecture notes on general relativity. arXiv preprint gr-qc/9712019.

[31] Moore, G.D. (2003). Informal lectures on lattice gauge theory. Spring.

[32] Andosca, R. (2016). Time-dependent electromagnetic fields and 4-dimensional Stokes' theorem (master's thesis). California State University.

[33] Orasch, O. (2014). The Aharonov-Bohm-Effect (Bachelor Thesis). Karl-Franzens-Universität Graz.

[34] Wächter, S. (2018). The Aharonov-Bohm effect. ETH Zürich.

[35] Longhi, G., Lusanna, L.,& Pons, J.M. (1989). On the many-time formulation of classical particle dynamics. Journal of Mathematical Physics.

[36] Yoo-Kong, S., Lobb, S.,& Nijhoff, F. (2011). Discrete-time Calogero-Moser system and Lagrangian 1-form structure. Journal of Physics A: Mathematical and Theoretical.

[37] Fortney, J. P. (2018). A visual introduction to differential forms and calculus on manifolds. Birkhäuser.

[38] Broda, B. (1996). Non-abilian Stokes theorem. arXiv preprint hep-th/9511150v2.

[39] Aref'eva, I.Y. (1980). Non-abilian Stokes formula. Plenum Publishing.

[40] Lill, S., Lickel, L.,& Tumulke, R. (2014). Consistency proof for multi-time Schrodinger equations with particle creation and ultraviolet cut-off. Annales Henri Poincaré 22.

[41] King, S.D.,& Nijhoff, F.W. (2017). Quantum variational principle and quantum multiform structure: the case of quadratic Lagrangians. Nuclear Physics B.

APPENDIX

APPENDIX A THE DERIVATION OF THE TRANSITION MAP

Here we show the derivation of the transition that evolves from (t_1, t_2) to $(\hat{t}_1, \hat{t}_2)$, see figure 15a.

$$\begin{aligned}
\left\langle \hat{q}_1, \hat{q}_2 \middle| \Phi(\hat{t}_1, \hat{t}_2) \right\rangle &= \langle \hat{q}_1, \hat{q}_2 \left| U_2' U_1' U_2 U_1 \right| \Phi(t_1, t_2) \rangle \\
\Phi\left(\hat{q}_1, \hat{q}_2, \hat{t}_1, \hat{t}_2\right) &= \iint dq_1 dq_2 \langle \hat{q}_1, \hat{q}_2 \left| U_2' U_1' U_2 U_1 \right| q_1, q_2 \rangle \left\langle q_2, q_1 \middle| \Phi(t_1, t_2) \right\rangle \\
&= \iint dq_1 dq_2 \langle \hat{q}_1, \hat{q}_2 \left| U_2' U_1' U_2 U_1 \right| q_1, q_2 \rangle \Phi(q_1, q_2, t_1, t_2) \\
&= \iiiint d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \langle \hat{q}_1, \hat{q}_2 \left| U_2' U_1' U_2 \right| \tilde{q}_1, \tilde{q}_2 \rangle \langle \tilde{q}_2, \tilde{q}_1 \left| U_1 \right| q_1, q_2 \rangle \\
&\times \Phi(q_1, q_2, t_1, t_2) \\
&= \iiiint d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \langle \hat{q}_1, \hat{q}_2 \left| U_2' U_1' U_2 \right| \tilde{q}_1, \tilde{q}_2 \rangle \langle \tilde{q}_1 \left| U_1 \right| q_1 \rangle \left\langle \tilde{q}_2 \middle| q_2 \right\rangle \\
&\times \Phi(q_1, q_2, t_1, t_2) \\
&= \iiiint d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \langle \hat{q}_1, \hat{q}_2 \left| U_2' U_1' U_2 \right| \tilde{q}_1, \tilde{q}_2 \rangle \langle \tilde{q}_1 \left| U_1 \right| q_1 \rangle \delta(\tilde{q}_2 - q_2) \\
&\times \Phi(q_1, q_2, t_1, t_2) \\
&= \iiiiiint dq_1' dq_2' d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \langle \hat{q}_1, \hat{q}_2 \left| U_2' U_1' \right| q_1', q_2' \rangle \langle q_2', q_1' \left| U_2 \right| \tilde{q}_1, \tilde{q}_2 \rangle \\
&\times \langle \tilde{q}_1 \left| U_1 \right| q_1 \rangle \delta(\tilde{q}_2 - q_2) \Phi(q_1, q_2, t_1, t_2) \\
&= \iiiiiint dq_1' dq_2' d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \langle \hat{q}_1, \hat{q}_2 \left| U_2' U_1' \right| q_1', q_2' \rangle \langle q_2' \left| U_2 \right| \tilde{q}_2 \rangle \\
&\times \left\langle q_1' \middle| \tilde{q}_1 \right\rangle \langle \tilde{q}_1 \left| U_1 \right| q_1 \rangle \delta(\tilde{q}_2 - q_2) \Phi(q_1, q_2, t_1, t_2) \\
&= \iiiiiint dq_1' dq_2' d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \langle \hat{q}_1, \hat{q}_2 \left| U_2' U_1' \right| q_1', q_2' \rangle \langle q_2' \left| U_2 \right| \tilde{q}_2 \rangle \\
&\times \delta(q_1' - \tilde{q}_1) \langle \tilde{q}_1 \left| U_1 \right| q_1 \rangle \delta(\tilde{q}_2 - q_2) \Phi(q_1, q_2, t_1, t_2) \\
&= \iiiiiiiint d\bar{q}_1 d\bar{q}_2 dq_1' dq_2' d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \langle \hat{q}_1, \hat{q}_2 \left| U_2' \right| \bar{q}_1, \bar{q}_2 \rangle \\
&\times \langle \bar{q}_1, \bar{q}_2 \left| U_1' \right| q_1', q_2' \rangle \langle q_2' \left| U_2 \right| \tilde{q}_2 \rangle \delta(q_1' - \tilde{q}_1) \langle \tilde{q}_1 \left| U_1 \right| q_1 \rangle \delta(\tilde{q}_2 - q_2) \\
&\times \Phi(q_1, q_2, t_1, t_2)
\end{aligned}$$

$$
\begin{aligned}
\Phi\left(\hat{q}_1,\hat{q}_2,\hat{t}_1,\hat{t}_2\right) &= \iiiint\!\!\!\iiiint d\bar{q}_1 d\bar{q}_2 dq_1' dq_2' d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \left\langle \hat{q}_1,\hat{q}_2 \left| U_2' \right| \bar{q}_1,\bar{q}_2 \right\rangle \\
&\times \left\langle \bar{q}_1 \left| U_1' \right| q_1' \right\rangle \left\langle \bar{q}_2 \middle| q_2' \right\rangle \left\langle q_2' \left| U_2 \right| \tilde{q}_2 \right\rangle \delta(q_1' - \tilde{q}_1) \left\langle \tilde{q}_1 \left| U_1 \right| q_1 \right\rangle \delta(\tilde{q}_2 - q_2) \\
&\times \Phi(q_1,q_2,t_1,t_2) \\
&= \iiiint\!\!\!\iiiint d\bar{q}_1 d\bar{q}_2 dq_1' dq_2' d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \left\langle \hat{q}_1,\hat{q}_2 \left| U_2' \right| \bar{q}_1,\bar{q}_2 \right\rangle \\
&\times \left\langle \bar{q}_1 \left| U_1' \right| q_1' \right\rangle \delta(\bar{q}_2 - q_2') \left\langle q_2' \left| U_2 \right| \tilde{q}_2 \right\rangle \delta(q_1' - \tilde{q}_1) \left\langle \tilde{q}_1 \left| U_1 \right| q_1 \right\rangle \delta(\tilde{q}_2 - q_2) \\
&\times \Phi(q_1,q_2,t_1,t_2) \\
&= \iiiint\!\!\!\iiiint d\bar{q}_1 d\bar{q}_2 dq_1' dq_2' d\tilde{q}_1 d\tilde{q}_2 dq_1 dq_2 \left\langle \hat{q}_2 \left| U_2' \right| \bar{q}_2 \right\rangle \left\langle \hat{q}_1 \middle| \bar{q}_1 \right\rangle \\
&\times \left\langle \bar{q}_1 \left| U_1' \right| q_1' \right\rangle \ \delta(\bar{q}_2 - q_2') \left\langle q_2' \left| U_2 \right| \tilde{q}_2 \right\rangle \delta(q_1' - \tilde{q}_1) \left\langle \tilde{q}_1 \left| U_1 \right| q_1 \right\rangle \delta(\tilde{q}_2 - q_2) \\
&\times \Phi(q_1,q_2,t_1,t_2) \\
&= \iiiint d\bar{q}_2 d\tilde{q}_1 dq_1 dq_2 \left\langle \hat{q}_2 \left| U_2' \right| \bar{q}_2 \right\rangle \int d\bar{q}_1 \delta(\hat{q}_1 - \bar{q}_1) \left\langle \bar{q}_1 \left| U_1' \right| q_1' \right\rangle \\
&\times \int dq_2' \delta(\bar{q}_2 - q_2') \left\langle q_2' \left| U_2 \right| \tilde{q}_2 \right\rangle \int dq_1' \delta(q_1' - \tilde{q}_1) \left\langle \tilde{q}_1 \left| U_1 \right| q_1 \right\rangle \\
&\times \int d\tilde{q}_2 \delta(\tilde{q}_2 - q_2) \Phi(q_1,q_2,t_1,t_2) \\
&= \iiiint d\bar{q}_2 d\tilde{q}_1 dq_1 dq_2 \left\langle \hat{q}_2 \left| U_2' \right| \bar{q}_2 \right\rangle \left\langle \bar{q}_2 \left| U_2 \right| q_2 \right\rangle \left\langle \hat{q}_1 \left| U_1' \right| \tilde{q}_1 \right\rangle \left\langle \tilde{q}_1 \left| U_1 \right| q_1 \right\rangle \\
&\times \Phi(q_1,q_2,t_1,t_2) \\
&= \iint dq_1 dq_2 \int d\bar{q}_2 \left\langle \hat{q}_2 \left| U_2'(\hat{t}_2 - t_2') \right| \bar{q}_2 \right\rangle \left\langle \bar{q}_2 \left| U_2(t_2' - t_2) \right| q_2 \right\rangle \\
&\times \int d\tilde{q}_1 \left\langle \hat{q}_1 \left| U_1'(\hat{t}_1 - t_1') \right| \tilde{q}_1 \right\rangle \left\langle \tilde{q}_1 \left| U_1(t_1' - t_1) \right| q_1 \right\rangle \Phi(q_1,q_2,t_1,t_2) \\
&= \iint dq_1 dq_2 \int d\bar{q}_2 K_2(\hat{q}_2,\hat{t}_2;\bar{q}_2,t_2') K_2(\bar{q}_2,t_2';q_2,t_2) \\
&\times \int d\tilde{q}_1 K_1(\hat{q}_1,\hat{t}_1;\tilde{q}_1,t_1') K_1(\tilde{q}_1,t_1';q_1,t_1) \Phi(q_1,q_2,t_1,t_2)\ . \qquad \text{(A.1)}
\end{aligned}
$$

This proves the equation of transition (3.69) on the evolution-counter clockwise of two-time system.

APPENDIX B THE DERIVATION OF THE CONSISTENCY CONDITION FOR THE MULTI-TIME SYSTEM WITH THE TIME-DEPENDENT HAMILTONIANS

The detailed derivation of the consistency condition for the multi-time system with the time-dependent Hamiltonians is given in this section. We work with a small rectangle as shown in figure 12. The time evolution operators is given by substituting Equation (3.36) - (3.43) into Equation (3.32) - (3.35)

$$
\begin{aligned}
U(t_1,0) &= I - i\int_0^{t_1} dT_1 H_1(T_1,0) - \int_0^{t_1} dT_1 \int_0^{T_1} dT_2 H_1(T_1,0)H_1(T_2,0) + ... \\
&= I - i\int_0^{t_1} dT_1 \left(H_1(0,0) + T_1 \frac{\partial H_1(0,0)}{\partial t_1} \right) \\
&\quad - \int_0^{t_1} dT_1 \int_0^{T_1} dT_2 \left(H_1(0,0) + T_1 \frac{\partial H_1(0,0)}{\partial t_1} \right) \left(H_1(0,0) + T_2 \frac{\partial H_1(0,0)}{\partial t_1} \right) \\
&\quad + ... \\
&= I - i\left[H_1(0,0)T_1 + \frac{T_1^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} \right]_0^{t_1} \\
&\quad - \int_0^{t_1} dT_1 \int_0^{T_1} dT_2 \Bigg(H_1^2(0,0) + H_1(0,0)T_2 \frac{\partial H_1(0,0)}{\partial t_1} + H_1(0,0)T_1 \frac{\partial H_1(0,0)}{\partial t_1} \\
&\quad + T_1 T_2 \left(\frac{\partial H_1(0,0)}{\partial t_1} \right)^2 \Bigg) + ... \\
&= I - i\left(H_1(0,0)\Delta t_1 + \frac{(\Delta t_1)^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} \right) \\
&\quad - \int_0^{t_1} dT_1 \Bigg[H_1^2(0,0)T_2 + H_1(0,0)\frac{T_2^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} + H_1(0,0)T_1T_2 \frac{\partial H_1(0,0)}{\partial t_1} \\
&\quad + T_1 \frac{(T_2^2)}{2} \left(\frac{\partial H_1(0,0)}{\partial t_1} \right)^2 \Bigg]_0^{T_1} + ... \\
&= I - i\left(H_1(0,0)\Delta t_1 + \frac{(\Delta t_1)^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} \right) \\
&\quad - \int_0^{t_1} dT_1 \Bigg(H_1^2(0,0)T_1 + H_1(0,0)\frac{T_1^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} + H_1(0,0)T_1^2 \frac{\partial H_1(0,0)}{\partial t_1} \\
&\quad + \frac{T_1^3}{2} \left(\frac{\partial H_1(0,0)}{\partial t_1} \right)^2 \Bigg) + ...
\end{aligned}
$$

$$
\begin{aligned}
U(t_1,0) &= I - i\left(H_1(0,0)\Delta t_1 + \frac{(\Delta t_1)^2}{2}\frac{\partial H_1(0,0)}{\partial t_1}\right) \\
&\quad - \left[H_1^2(0,0)\frac{T_1^2}{2} + H_1(0,0)\frac{T_1^3}{6}\frac{\partial H_1(0,0)}{\partial t_1} + H_1(0,0)\frac{T_1^3}{3}\frac{\partial H_1(0,0)}{\partial t_1}\right. \\
&\quad \left. + \frac{T_1^4}{8}\left(\frac{\partial H_1(0,0)}{\partial t_1}\right)^2\right]_0^{t_1} + ... \\
&= I - i\left(H_1(0,0)\Delta t_1 + \frac{(\Delta t_1)^2}{2}\frac{\partial H_1(0,0)}{\partial t_1}\right) \\
&\quad - \left(H_1^2(0,0)\frac{(\Delta t_1)^2}{2} + H_1(0,0)\frac{(\Delta t_1)^3}{6}\frac{\partial H_1(0,0)}{\partial t_1}\right. \\
&\quad \left. + H_1(0,0)\frac{(\Delta t_1)^3}{3}\frac{\partial H_1(0,0)}{\partial t_1} + \frac{(\Delta t_1)^4}{8}\left(\frac{\partial H_1(0,0)}{\partial t_1}\right)^2\right) + ... \ , \qquad \text{(B.1)}
\end{aligned}
$$

$$
\begin{aligned}
U_2(t_1,t_2) &= I - i\int_0^{t_2} dT_1 H_2(t_1,T_1) - \int_0^{t_2} dT_1 \int_0^{T_1} dT_2 H_2(t_1,T_1)H_2(t_1,T_2) + ... \\
&= I - i\int_0^{t_2} dT_1\left(H_2(0,0) + \Delta t_1\frac{\partial H_2(0,0)}{\partial t_1} + T_1\frac{\partial H_2(0,0)}{\partial t_2}\right) \\
&\quad - \int_0^{t_2} dT_1\int_0^{T_1} dT_2\left(H_2(0,0) + \Delta t_1\frac{\partial H_2(0,0)}{\partial t_1} + T_1\frac{\partial H_2(0,0)}{\partial t_2}\right) \\
&\quad \times\left(H_2(0,0) + \Delta t_1\frac{\partial H_2(0,0)}{\partial t_1} + T_2\frac{\partial H_2(0,0)}{\partial t_2}\right) + ... \\
&= I - i\left[H_2(0,0)T_1 + \Delta t_1 T_1\frac{\partial H_2(0,0)}{\partial t_1} + \frac{T_1^2}{2}\frac{\partial H_2(0,0)}{\partial t_2}\right]_0^{t_2} \\
&\quad - \int_0^{t_2} dT_1\int_0^{T_1} dT_2\left(H_2^2(0,0) + H_2(0,0)\Delta t_1\frac{\partial H_2(0,0)}{\partial t_1}\right. \\
&\quad + H_2(0,0)T_2\frac{\partial H_2(0,0)}{\partial t_2} + H_2(0,0)\Delta t_1\frac{\partial H_2(0,0)}{\partial t_1} + (\Delta t_1)^2\left(\frac{\partial H_2(0,0)}{\partial t_1}\right)^2 \\
&\quad + \Delta t_1 T_2\frac{\partial H_2(0,0)}{\partial t_1}\frac{\partial H_2(0,0)}{\partial t_2} + H_2(0,0)T_1\frac{\partial H_2(0,0)}{\partial t_1} \\
&\quad \left. + \Delta t_1 T_1\frac{\partial H_2(0,0)}{\partial t_2}\frac{\partial H_2(0,0)}{\partial t_1} + T_1T_2\left(\frac{\partial H_2(0,0)}{\partial t_2}\right)^2\right) + ...
\end{aligned}
$$

$$
\begin{aligned}
U_2(t_1,t_2) = {} & I - i\left(H_2(0,0)\Delta t_2 + \Delta t_1 \Delta t_2 \frac{\partial H_2(0,0)}{\partial t_1} + \frac{(\Delta t_2)^2}{2}\frac{\partial H_2(0,0)}{\partial t_2}\right) \\
& - \int_0^{t_2} dT_1 \Bigg[H_2^2(0,0)T_2 + 2H_2(0,0)\Delta t_1 T_2 \frac{\partial H_2(0,0)}{\partial t_1} \\
& + H_2(0,0)\frac{T_2^2}{2}\frac{\partial H_2(0,0)}{\partial t_2} + (\Delta t_1)^2 T_2 \left(\frac{\partial H_2(0,0)}{\partial t_1}\right)^2 \\
& + \Delta t_1 \frac{T_2^2}{2}\frac{\partial H_2(0,0)}{\partial t_1}\frac{\partial H_2(0,0)}{\partial t_2} + H_2(0,0)T_1T_2\frac{\partial H_2(0,0)}{\partial t_1} \\
& + \Delta t_1 T_1 T_2 \frac{\partial H_2(0,0)}{\partial t_2}\frac{\partial H_2(0,0)}{\partial t_1} + T_1\frac{T_2^2}{2}\left(\frac{\partial H_2(0,0)}{\partial t_2}\right)^2 \Bigg]_0^{T_1} + ... \\
= {} & I - i\left(H_2(0,0)\Delta t_2 + \Delta t_1 \Delta t_2 \frac{\partial H_2(0,0)}{\partial t_1} + \frac{(\Delta t_2)^2}{2}\frac{\partial H_2(0,0)}{\partial t_2}\right) \\
& - \int_0^{t_2} dT_1 \Bigg(H_2^2(0,0)T_1 + 2H_2(0,0)\Delta t_1 T_1 \frac{\partial H_2(0,0)}{\partial t_1} \\
& + H_2(0,0)\frac{T_1^2}{2}\frac{\partial H_2(0,0)}{\partial t_2} + (\Delta t_1)^2 T_1 \left(\frac{\partial H_2(0,0)}{\partial t_1}\right)^2 \\
& + \Delta t_1 \frac{T_1^2}{2}\frac{\partial H_2(0,0)}{\partial t_1}\frac{\partial H_2(0,0)}{\partial t_2} + H_2(0,0)T_1^2\frac{\partial H_2(0,0)}{\partial t_1} \\
& + \Delta t_1 T_1^2 \frac{\partial H_2(0,0)}{\partial t_2}\frac{\partial H_2(0,0)}{\partial t_1} + \frac{T_1^3}{2}\left(\frac{\partial H_2(0,0)}{\partial t_2}\right)^2 \Bigg) + ... \\
= {} & I - i\left(H_2(0,0)\Delta t_2 + \Delta t_1 \Delta t_2 \frac{\partial H_2(0,0)}{\partial t_1} + \frac{(\Delta t_2)^2}{2}\frac{\partial H_2(0,0)}{\partial t_2}\right) \\
& - \Bigg[H_2^2(0,0)\frac{T_1^2}{2} + 2H_2(0,0)\Delta t_1 \frac{T_1^2}{2}\frac{\partial H_2(0,0)}{\partial t_1} + H_2(0,0)\frac{T_1^3}{6}\frac{\partial H_2(0,0)}{\partial t_2} \\
& + (\Delta t_1)^2 \frac{T_1^2}{2}\left(\frac{\partial H_2(0,0)}{\partial t_1}\right)^2 + \Delta t_1 \frac{T_1^3}{6}\frac{\partial H_2(0,0)}{\partial t_1}\frac{\partial H_2(0,0)}{\partial t_2} \\
& + H_2(0,0)\frac{T_1^3}{3}\frac{\partial H_2(0,0)}{\partial t_1} + \Delta t_1 \frac{T_1^3}{3}\frac{\partial H_2(0,0)}{\partial t_2}\frac{\partial H_2(0,0)}{\partial t_1} \\
& + \frac{T_1^4}{8}\left(\frac{\partial H_2(0,0)}{\partial t_2}\right)^2 \Bigg]_0^{t_2} + ... \\
= {} & I - i\left(H_2(0,0)\Delta t_2 + \Delta t_1 \Delta t_2 \frac{\partial H_2(0,0)}{\partial t_1} + \frac{(\Delta t_2)^2}{2}\frac{\partial H_2(0,0)}{\partial t_2}\right) \\
& - \Bigg(H_2^2(0,0)\frac{(\Delta t_2)^2}{2} + H_2(0,0)\Delta t_1 (\Delta t_2)^2 \frac{\partial H_2(0,0)}{\partial t_1} \\
& + H_2(0,0)\frac{(\Delta t_2)^3}{6}\frac{\partial H_2(0,0)}{\partial t_2} + (\Delta t_1)^2 \frac{(\Delta t_2)^2}{2}\left(\frac{\partial H_2(0,0)}{\partial t_1}\right)^2 \\
& + \Delta t_1 \frac{(\Delta t_2)^3}{6}\frac{\partial H_2(0,0)}{\partial t_1}\frac{\partial H_2(0,0)}{\partial t_2} + H_2(0,0)\frac{(\Delta t_2)^3}{3}\frac{\partial H_2(0,0)}{\partial t_1} \\
& + \Delta t_1 \frac{(\Delta t_2)^3}{3}\frac{\partial H_2(0,0)}{\partial t_2}\frac{\partial H_2(0,0)}{\partial t_1} + \frac{(\Delta t_2)^4}{8}\left(\frac{\partial H_2(0,0)}{\partial t_2}\right)^2 \Bigg) + ., \qquad \text{(B.2)}
\end{aligned}
$$

$$
\begin{aligned}
U(0,t_2) &= I - i\int_0^{t_2} dT_1 H_2(0,T_1) - \int_0^{t_2} dT_1 \int_0^{T_1} dT_2 H_2(0,T_1)H_2(0,T_2) + ... \\
&= I - i\int_0^{t_2} dT_1 \left(H_2(0,0) + T_1 \frac{\partial H_2(0,0)}{\partial t_2} \right) \\
&\quad - \int_0^{t_2} dT_1 \int_0^{T_1} dT_2 \left(H_2(0,0) + T_1 \frac{\partial H_2(0,0)}{\partial t_2} \right) \left(H_2(0,0) + T_2 \frac{\partial H_2(0,0)}{\partial t_2} \right) \\
&\quad + ... \\
&= I - i\left[H_2(0,0)T_1 + \frac{T_1^2}{2}\frac{\partial H_2(0,0)}{\partial t_2} \right]_0^{t_2} \\
&\quad - \int_0^{t_2} dT_1 \int_0^{T_1} dT_2 \Bigg(H_2^2(0,0) + H_2(0,0)T_2 \frac{\partial H_2(0,0)}{\partial t_2} + H_2(0,0)T_1 \frac{\partial H_2(0,0)}{\partial t_2} \\
&\quad + T_1T_2 \left(\frac{\partial H_2(0,0)}{\partial t_2} \right)^2 \Bigg) + ... \\
&= I - i\left(H_2(0,0)\Delta t_2 + \frac{(\Delta t_2)^2}{2}\frac{\partial H_2(0,0)}{\partial t_2} \right) \\
&\quad - \int_0^{t_2} dT_1 \Bigg[H_2^2(0,0)T_2 + H_2(0,0)\frac{T_2^2}{2}\frac{\partial H_2(0,0)}{\partial t_2} + H_2(0,0)T_1T_2 \frac{\partial H_2(0,0)}{\partial t_2} \\
&\quad + T_1 \frac{(T_2^2)}{2} \left(\frac{\partial H_2(0,0)}{\partial t_2} \right)^2 \Bigg]_0^{T_1} + ... \\
&= I - i\left(H_2(0,0)\Delta t_2 + \frac{(\Delta t_2)^2}{2}\frac{\partial H_2(0,0)}{\partial t_2} \right) \\
&\quad - \int_0^{t_2} dT_1 \Bigg(H_2^2(0,0)T_1 + H_2(0,0)\frac{T_1^2}{2}\frac{\partial H_2(0,0)}{\partial t_2} + H_2(0,0)T_1^2 \frac{\partial H_2(0,0)}{\partial t_2} \\
&\quad + \frac{T_1^3}{2} \left(\frac{\partial H_2(0,0)}{\partial t_2} \right)^2 \Bigg) + ... \\
&= I - i\left(H_2(0,0)\Delta t_2 + \frac{(\Delta t_2)^2}{2}\frac{\partial H_2(0,0)}{\partial t_2} \right) \\
&\quad - \Bigg[H_2^2(0,0)\frac{T_1^2}{2} + H_2(0,0)\frac{T_1^3}{6}\frac{\partial H_2(0,0)}{\partial t_2} + H_2(0,0)\frac{T_1^3}{3}\frac{\partial H_2(0,0)}{\partial t_2} \\
&\quad + \frac{T_1^4}{8} \left(\frac{\partial H_2(0,0)}{\partial t_2} \right)^2 \Bigg]_0^{t_2} + ... \\
&= I - i\left(H_2(0,0)\Delta t_2 + \frac{(\Delta t_2)^2}{2}\frac{\partial H_2(0,0)}{\partial t_2} \right) \\
&\quad - \Bigg(H_2^2(0,0)\frac{(\Delta t_2)^2}{2} + H_2(0,0)\frac{(\Delta t_2)^3}{6}\frac{\partial H_2(0,0)}{\partial t_2} \\
&\quad + H_2(0,0)\frac{(\Delta t_2)^3}{3}\frac{\partial H_2(0,0)}{\partial t_2} + \frac{(\Delta t_2)^4}{8} \left(\frac{\partial H_2(0,0)}{\partial t_2} \right)^2 \Bigg) + ... \ ,
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
U_1(t_1,t_2) &= I - i\int_0^{t_1} dT_1 H_1(T_1,t_2) - \int_0^{t_1} dT_1 \int_0^{T_1} dT_2 H_1(T_1,t_2)H_1(T_2,t_2) + ... \\
&= I - i\int_0^{t_1} dT_1 \left(H_1(0,0) + T_1\frac{\partial H_1(0,0)}{\partial t_1} + \Delta t_2 \frac{\partial H_1(0,0)}{\partial t_2}\right) \\
&- \int_0^{t_1} dT_1 \int_0^{T_1} dT_2 \left(H_1(0,0) + T_1\frac{\partial H_1(0,0)}{\partial t_1} + \Delta t_2 \frac{\partial H_1(0,0)}{\partial t_2}\right) \\
&\times \left(H_1(0,0) + T_2\frac{\partial H_1(0,0)}{\partial t_1} + \Delta t_2 \frac{\partial H_1(0,0)}{\partial t_2}\right) + ... \\
&= I - i\left[H_1(0,0)T_1 + \frac{T_1^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} + \Delta t_2 T_1 \frac{\partial H_1(0,0)}{\partial t_2}\right]_0^{t_1} \\
&- \int_0^{t_1} dT_1 \int_0^{T_1} dT_2 \Bigg(H_1^2(0,0) + H_1(0,0)T_2\frac{\partial H_1(0,0)}{\partial t_1} \\
&+ H_1(0,0)\Delta t_2 \frac{\partial H_1(0,0)}{\partial t_2} + H_1(0,0)T_1\frac{\partial H_1(0,0)}{\partial t_1} + T_1T_2\left(\frac{\partial H_1(0,0)}{\partial t_1}\right)^2 \\
&+ T_1\Delta t_2 \frac{\partial H_1(0,0)}{\partial t_1}\frac{\partial H_1(0,0)}{\partial t_2} + H_1(0,0)\Delta t_2 \frac{\partial H_1(0,0)}{\partial t_2} \\
&+ T_2\Delta t_2 \frac{\partial H_1(0,0)}{\partial t_2}\frac{\partial H_1(0,0)}{\partial t_1} + (\Delta t_2)^2\left(\frac{\partial H_1(0,0)}{\partial t_2}\right)^2 \Bigg) + ... \\
&= I - i\left(H_1(0,0)\Delta t_1 + \frac{(\Delta t_1)^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} + \Delta t_2 \Delta t_1 \frac{\partial H_1(0,0)}{\partial t_2}\right) \\
&- \int_0^{t_1} dT_1 \Bigg[H_1^2(0,0)T_2 + H_1(0,0)\frac{T_2^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} \\
&+ 2H_1(0,0)\Delta t_2 T_2 \frac{\partial H_1(0,0)}{\partial t_2} + H_1(0,0)T_1T_2\frac{\partial H_1(0,0)}{\partial t_1} \\
&+ T_1\frac{T_2^2}{2}\left(\frac{\partial H_1(0,0)}{\partial t_1}\right)^2 + T_1\Delta t_2 T_2 \frac{\partial H_1(0,0)}{\partial t_1}\frac{\partial H_1(0,0)}{\partial t_2} \\
&+ \frac{T_2^2}{2}\Delta t_2 \frac{\partial H_1(0,0)}{\partial t_2}\frac{\partial H_1(0,0)}{\partial t_1} + (\Delta t_2)^2 T_2 \left(\frac{\partial H_1(0,0)}{\partial t_2}\right)^2 \Bigg]_0^{T_1} + ... \\
&= I - i\left(H_1(0,0)\Delta t_1 + \frac{(\Delta t_1)^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} + \Delta t_2 \Delta t_1 \frac{\partial H_1(0,0)}{\partial t_2}\right) \\
&- \int_0^{t_1} dT_1 \Bigg(H_1^2(0,0)T_1 + H_1(0,0)\frac{T_1^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} \\
&+ 2H_1(0,0)\Delta t_2 T_1 \frac{\partial H_1(0,0)}{\partial t_2} + H_1(0,0)T_1^2\frac{\partial H_1(0,0)}{\partial t_1} + \frac{T_1^3}{2}\left(\frac{\partial H_1(0,0)}{\partial t_1}\right)^2 \\
&+ \Delta t_2 T_1^2 \frac{\partial H_1(0,0)}{\partial t_1}\frac{\partial H_1(0,0)}{\partial t_2} + \frac{T_1^2}{2}\Delta t_2 \frac{\partial H_1(0,0)}{\partial t_2}\frac{\partial H_1(0,0)}{\partial t_1} \\
&+ (\Delta t_2)^2 T_1 \left(\frac{\partial H_1(0,0)}{\partial t_2}\right)^2 \Bigg) + ...
\end{aligned}
$$

$$
\begin{aligned}
U_1(t_1,t_2) = I - i & \left(H_1(0,0)\Delta t_1 + \frac{(\Delta t_1)^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} + \Delta t_2 \Delta t_1 \frac{\partial H_1(0,0)}{\partial t_2} \right) \\
& - \left[H_1^2(0,0)\frac{T_1^2}{2} + H_1(0,0)\frac{T_1^3}{6}\frac{\partial H_1(0,0)}{\partial t_1} + 2H_1(0,0)\Delta t_2 \frac{T_1^2}{2}\frac{\partial H_1(0,0)}{\partial t_2} \right. \\
& + H_1(0,0)\frac{T_1^3}{3}\frac{\partial H_1(0,0)}{\partial t_1} + \frac{T_1^4}{8}\left(\frac{\partial H_1(0,0)}{\partial t_1}\right)^2 \\
& + \Delta t_2 \frac{T_1^3}{3}\frac{\partial H_1(0,0)}{\partial t_1}\frac{\partial H_1(0,0)}{\partial t_2} + \frac{T_1^3}{6}\Delta t_2 \frac{\partial H_1(0,0)}{\partial t_2}\frac{\partial H_1(0,0)}{\partial t_1} \\
& \left. + (\Delta t_2)^2 \frac{T_1^2}{2}\left(\frac{\partial H_1(0,0)}{\partial t_2}\right)^2 \right]_0^{t_1} + \ldots \\
= I - i & \left(H_1(0,0)\Delta t_1 + \frac{(\Delta t_1)^2}{2}\frac{\partial H_1(0,0)}{\partial t_1} + \Delta t_2 \Delta t_1 \frac{\partial H_1(0,0)}{\partial t_2} \right) \\
& - \left(H_1^2(0,0)\frac{(\Delta t_1)^2}{2} + H_1(0,0)\frac{(\Delta t_1)^3}{6}\frac{\partial H_1(0,0)}{\partial t_1} \right. \\
& + H_1(0,0)\Delta t_2 (\Delta t_1)^2 \frac{\partial H_1(0,0)}{\partial t_2} + H_1(0,0)\frac{(\Delta t_1)^3}{3}\frac{\partial H_1(0,0)}{\partial t_1} \\
& + \frac{(\Delta t_1)^4}{8}\left(\frac{\partial H_1(0,0)}{\partial t_1}\right)^2 + \Delta t_2 \frac{(\Delta t_1)^3}{3}\frac{\partial H_1(0,0)}{\partial t_1}\frac{\partial H_1(0,0)}{\partial t_2} \\
& + \frac{(\Delta t_1)^3}{6}\Delta t_2 \frac{\partial H_1(0,0)}{\partial t_2}\frac{\partial H_1(0,0)}{\partial t_1} \\
& \left. + (\Delta t_2)^2 \frac{(\Delta t_1)^2}{2}\left(\frac{\partial H_1(0,0)}{\partial t_2}\right)^2 \right) + \ldots \ . \qquad \text{(B.4)}
\end{aligned}
$$

Ignoring higher-order terms, we obtain

$$
U(t_1,0) = I - iH_1(0,0)\Delta t_1 - \frac{i}{2}\frac{\partial H_1(0,0)}{\partial t_1}(\Delta t_1)^2 - \frac{1}{2}H_1^2(0,0)(\Delta t_1)^2 \ , \qquad \text{(B.5)}
$$

$$
\begin{aligned}
U_2(t_1,t_2) = I - iH_2(0,0)\Delta t_2 - i\frac{\partial H_2(0,0)}{\partial t_1}\Delta t_1 \Delta t_2 - \frac{i}{2}\frac{\partial H_2(0,0)}{\partial t_2}(\Delta t_2)^2 \\
- \frac{1}{2}H_2^2(0,0)(\Delta t_2)^2 \ , \qquad \text{(B.6)}
\end{aligned}
$$

$$
U(0,t_2) = I - iH_2(0,0)\Delta t_2 - \frac{i}{2}\frac{\partial H_2(0,0)}{\partial t_2}(\Delta t_2)^2 - \frac{1}{2}H_2^2(0,0)(\Delta t_2)^2 \ , \qquad \text{(B.7)}
$$

$$
\begin{aligned}
U_1(t_1,t_2) = I - iH_1(0,0)\Delta t_1 - \frac{i}{2}\frac{\partial H_1(0,0)}{\partial t_1}(\Delta t_1)^2 - i\frac{\partial H_1(0,0)}{\partial t_2}\Delta t_2 \Delta t_1 \\
- \frac{1}{2}H_1^2(0,0)(\Delta t_1)^2 \ . \qquad \text{(B.8)}
\end{aligned}
$$

Next, we consider the evolution for the lower corner path given by

$$\begin{aligned} U_2(t_1,t_2)U(t_1,0) = &\Bigg(I - iH_2(0,0)\Delta t_2 - i\frac{\partial H_2(0,0)}{\partial t_1}\Delta t_1\Delta t_2 - \frac{i}{2}\frac{\partial H_2(0,0)}{\partial t_2}(\Delta t_2)^2 \\ &- \frac{1}{2}H_2^2(0,0)(\Delta t_2)^2\Bigg)\Bigg(I - iH_1(0,0)\Delta t_1 - \frac{i}{2}\frac{\partial H_1(0,0)}{\partial t_1}(\Delta t_1)^2 \\ &- \frac{1}{2}H_1^2(0,0)(\Delta t_1)^2\Bigg) \\ = &\, I - iH_1(0,0)\Delta t_1 - \frac{i}{2}\frac{\partial H_1(0,0)}{\partial t_1}(\Delta t_1)^2 - \frac{1}{2}H_1^2(0,0)(\Delta t_1)^2 \\ &- iH_2(0,0)\Delta t_2 - H_2H_1\Delta t_2\Delta t_1 - i\frac{\partial H_2(0,0)}{\partial t_1}\Delta t_1\Delta t_2 \ , \end{aligned} \tag{B.9}$$

and the evolution for the upper path given by

$$\begin{aligned} U_1(t_1,t_2)U(0,t_2) = &\Bigg(I - iH_1(0,0)\Delta t_1 - \frac{i}{2}\frac{\partial H_1(0,0)}{\partial t_1}(\Delta t_1)^2 - i\frac{\partial H_1(0,0)}{\partial t_2}\Delta t_2\Delta t_1 \\ &- \frac{1}{2}H_1^2(0,0)(\Delta t_1)^2\Bigg)\Bigg(I - iH_2(0,0)\Delta t_2 - \frac{i}{2}\frac{\partial H_2(0,0)}{\partial t_2}(\Delta t_2)^2 \\ &- \frac{1}{2}H_2^2(0,0)(\Delta t_2)^2\Bigg) \\ = &\, I - iH_2(0,0)\Delta t_2 - \frac{i}{2}\frac{\partial H_2(0,0)}{\partial t_2}(\Delta t_2)^2 - \frac{1}{2}H_2^2(0,0)(\Delta t_2)^2 \\ &- iH_1(0,0)\Delta t_1 - H_1H_2\Delta t_1\Delta t_2 - i\frac{\partial H_1(0,0)}{\partial t_2}\Delta t_2\Delta t_1 \ , \end{aligned} \tag{B.10}$$

where we neglect higher-order terms of Δt_1 and Δt_2. To make the both evolutions compatible, one requires $U_2(t_1,t_2)U(t_1,0) - U_1(t_1,t_2)U(0,t_2) = 0$, resulting in

$$\begin{aligned} 0 = &-H_1(0,0)H_2(0,0)\Delta t_1\Delta t_2 - i\frac{\partial H_1(0,0)}{\partial t_2}\Delta t_2\Delta t_1 + H_2(0,0)H_1(0,0)\Delta t_2\Delta t_1 \\ &+ i\frac{\partial H_2(0,0)}{\partial t_1}\Delta t_1\Delta t_2 \\ = &\left(-H_1(0,0)H_2(0,0) + H_2(0,0)H_1(0,0) - i\frac{\partial H_1(0,0)}{\partial t_2} + i\frac{\partial H_2(0,0)}{\partial t_1}\right)\Delta t_1\Delta t_2 \\ = &\left(\frac{\partial H_1}{\partial t_2} - \frac{\partial H_2}{\partial t_1} - i\left[H_1,H_2\right]\right)\Delta t_1\Delta t_2 \ . \end{aligned} \tag{B.11}$$

Therefore, the terms inside the bracket must vanish. This proves the consistency condition (3.27) on the evolution of the multi-time system with the time-dependent Hamiltonians.

BIOGRAPHY

BIOGRAPHY

Name-Surname	Siwaporn Sungted
Date of Birth	April 9, 1997
Place of Birth	Phetchabun Province, Thailand
Address	4 Moo 6 Tambon Palao, Amphoe Muang, Phetchabun Province, Thailand 76000

Education Background

2019	B.S. (Physics), Naresuan University, Phitsanulok, Thailand