# THERMODYNAMICS OF BLACK STRING FROM RÉNYI ENTROPY IN DE RHAM-GABADAZE-TOLLEY MASSIVE GRAVITY THEORY

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# ABSTRACT

The de Rham-Gabadadze-Tolley (dRGT) black string solution is a cylindrically symmetric and static solution of the Einstein field equation with graviton mass term. For the asymptotically de Sitter (dS) solution, it is possible to obtain the black string with two event horizons corresponding to two thermodynamic systems. However, one found that the dRGT black string entropy is a non-extensive quantity. This indicates that the dRGT black string is the non-extensive system. The Rényi entropy is one of the entropic forms which is suitable to deal with the non-extensive properties of the black string. In this work, we investigated the possibility to obtain a stable black string by using the Rényi entropy in both separated and effective approaches. We found that the non-extensivity provides the thermodynamically stable black string with moderate size in both approaches. The transition from the hot gas phase to the moderate-sized stable black string in the separated/effective description is a first-order/zeroth-order phase transition. The significant ways to distinguish the black string from both approaches are discussed.

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# CHAPTER I

# INTRODUCTION

## 1.1 Background and Motivation

Massive gravity is a theory that modifies general relativity (GR) at the large scale by adding consistent interaction terms interpreted as a graviton mass into the Einstein-Hilbert action. This theory can be used to describe the accelerating expansion of the universe without introducing a cosmological constant, while at local scales its predictions are the same as GR does. The field theory for massive graviton was firstly proposed by Fierz and Pauli in 1993 [1] by added the interaction terms at the linearized level of GR. However, it was later found that there is a discontinuity when taking the massless limit, pointed out by van Dam, Veltman, and Zakharov in 1970 [2, 3] called van Dam-Veltman-Zakharov (vDVZ) discontinuity. This discontinuity invoked further studies on the non-linear generalization of Fierz-Pauli massive gravity. However, in 1972 Boulware and Deser found that additional mass terms in messive theory usually generate the ghost instability, later called Boulware-Deser (BD) ghost [4]. Eventually, the theory of massive gravity without ghost instability was proposed by de Rham, Gabadadze and Tolley (dRGT) in 2010 [5, 6]. Although the dRGT massive gravity theory is constructed by adding suitable mass term into GR, but it is difficult to find the exact solutions in this theory. Nevertheless, spherically symmetric solutions in dRGT massive gravity are also found [7, 8, 9, 10, 11, 12, 13]. The black hole solutions have been intensively investigated, for example, thermodynamic properties [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28], greybody factor [29, 30, 31, 32], quasinormal modes [33, 34, 35], critical heat engine [36] and phase transition in Ruppeiner geometry [37, 38].

It is interesting to ask whether a black hole can have cylindrical symmetry.

Many investigations have led us to know that the gravitational collapse of massive stars cannot form in cylindrical symmetry. This restriction comes from the Hoop conjecture which states that the horizon can be formed when the mass of an object gets compacted into a region whose circumference is less than its Schwarzschild circumference, $4\pi MG$ in every direction [39]. However, the Hoop conjecture was given for a spacetime with zero cosmological constants. This suggests that the Hoop conjecture may be violated in asymptotically dS/AdS spacetime. Indeed, Lemos shown that the cylindrical solutions can exist in the GR with the existence of the cosmological constant [40, 41, 42]. With cylindrical symmetry, the horizons usually are circular, and then such corresponding object is commonly known as the black string. Since the dRGT solution is one of the solutions with asymptotically dS/AdS spacetime, it is possible to obtain the black string solution. As a result, the dRGT black string solution including their thermodynamical properties has been investigated in [43]. The solution for the charged black string as well as rotating black string are explored [44, 45] and the greybody factor [46] of the dRGT black string also have been investigated.

It is important to note that the investigation on the black string is mostly performed in the asymptotically AdS spacetime. This comes from the fact that dS black string has no horizon, then it does not correspond to the thermodynamic system. For the dRGT black string with asymptotically dS spacetime, even though there exist horizons, the corresponding thermodynamic systems are found to be unstable since there is a negative heat capacity when its thermodynamic quantities are defined based on the Gibbs-Boltzmann statistics. This may be a consequence of the fact that the entropy of the black hole including the black string is proportional to its area and then corresponds to the non-extensive thermodynamic system. Therefore, the Bekenstein-Hawking entropy based on the Gibbs-Boltzmann statistics is not suitable to use to investigate the non-extensive system. Tsallis entropy is a candidate for studying non-extensive systems [47, 48]. However, Tsallis entropy has the problem of the incompatibility with the zeroth law of thermodynamics [49]. In order to investigate thermodynamic properties we have to map the Tsallis entropy to formal logarithm form, which is known as the Rényi entropy [50, 51]. It is possible to obtain the stable black hole by considering the Rényi entropy instead of the usual Gibbs-Boltzmann (GB) entropy [52, 53, 54, 55, 56, 57].

Due to the existence of multi horizons, the temperatures of the system evaluated at each horizon are generically different. This means that the system is not in the thermal equilibrium state. In order to study thermodynamics for such a system, we can separate our consideration into two approaches. First, the thermodynamic system of each horizon can be defined separately [58]. The systems are treated to be in the quasi-equilibrium state, in which the timescale of the heat transfer between each system is much longer than the timescale of the thermodynamics process. Second, one can treat the whole system as a single system called effective system described by the effective thermodynamic quantities [59, 60, 61, 62, 63, 64, 65, 66]. In this work, we investigate the thermodynamic properties of the black string from Rényi entropy in dRGT massive gravity with asymptotically dS spacetime. For the separated system approach, we analyze the local stability of black string by considering the temperature profile and heat capacity. By analyzing the Gibbs free energy, we found that it is possible to obtain the globally stable black string. For the effective system approach, we use the suitable definition of the effective quantities in [57] to perform the stability analysis in the same fashion as done in separated system approach. For convenience, in this thesis, we now dealing with the natural unit $c = \hbar = G = k_B = 1$.

## 1.2 Objectives

- To study thermodynamic stability of dRGT black string in the separated system approach by using Rényi entropy.
- To study thermodynamic stability of dRGT black string in the effective system approach by using Rényi entropy.

# CHAPTER II

# A BRIEF REVIEW OF GENERAL RELATIVITY

General relativity (GR) is the theory on the relationship between spacetime and gravity, which was proposed by Albert Einstein in 1915. In particular, we can say that gravity is the curvature of spacetime. Hence, we can explain this theory by using the concept of differential geometry.

## 2.1 Manifold

To discuss the geometry of spacetime, we will build up the general continuous object in which mathematical objects (e.g., vector field, tensor, or differential operator) can live on. This is known as manifold. In mathematics, the manifold can be defined as any set of points that are parameterized along with a set of neighborhoods for each point. In a very small area, the manifold is locally flat. In other words, a manifold is the object constructed by smoothly sewing many of the local regions together as shown in Fig.1. The concept of the manifold is quite abstract and complicated, one can see in [67, 68] for more details.

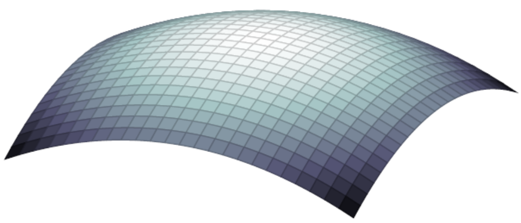

**Figure 1 An example of manifold.**

## 2.2 Vectors, dual vector and tensor

In this section, we will introduce the kinds of mathematical objects on a manifold. We begin with scalar fields on a manifold. The scalar or scalar field $\phi(x^\mu)$ is the function of spacetime which is independent of the choice of coordinates. Consider a curve $\gamma$ parameterized by $\lambda$ on a manifold and described in arbitrary coordinate by $x^\mu(\lambda)$. One can calculate the rate of change of a scalar function along this curve by,

$$\frac{d\phi}{d\lambda} = \frac{\partial \phi}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \partial_\mu \phi u^\mu. \tag{2.1}$$

This step allows us to introduce two types of objects on the manifold as follows: $u^\mu$ is a vector which is tangent to everywhere on a curve $\gamma$ and $\partial_\mu \phi$ is a dual vector interpreted as the gradient of the scalar function. Under the general coordinate transformation, $x^\mu \to x'^\mu$, one found that these objects transform as follows

$$\partial_{\mu'}\phi = \frac{\partial \phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu \phi, \quad \text{and} \quad u^{\mu'} = \frac{dx^{\mu'}}{d\lambda} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{\partial x^{\mu'}}{\partial x^\mu}u^\mu. \tag{2.2}$$

This suggest that the scalar function is invariant under the general coordinate transformation,

$$\phi'(x'^\mu) = \phi(x^\mu). \tag{2.3}$$

In curved spacetime, the vector fields are expressed as,

$$V = V^\mu \partial_\mu, \tag{2.4}$$

where $V^\mu$ is contravariant component of the vector and $\partial_\mu = e_\mu$ is the basis vector. One can see that under the coordinate transformation, we have

$$\begin{aligned} V^\mu \partial_\mu &= V^{\mu'}\partial_{\mu'}, \\ &= V^{\mu'}\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu. \end{aligned} \tag{2.5}$$

And then, the component of contravariant vector transforms as

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}. \tag{2.6}$$

On the other hand, the component of covariant vector $X_\mu$ transforms as

$$X_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu,}} X_{\mu}. \tag{2.7}$$

The dual vector can be interpreted as a map of vector to scalar. Both vector and dual vector can be thought of tensors of the rank $(1,0)$ and $(0,1)$ respectively. Then, we can generalize the transformation to the tensor of higher rank $(k,l)$ as

$$T^{\mu'_1 \ldots \mu'_k}{}_{\nu'_1 \ldots \nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \tag{2.8}$$

The tangent vector at each point along the curve on a manifold is defined in a tangent plane on the manifold at that point as illustrated in Fig.2, this plane is called the tangent space, $T_P$.

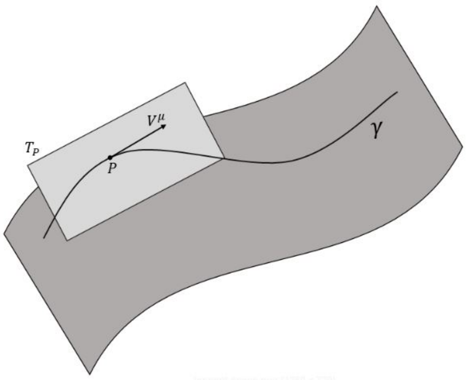


**Figure 2 A tangent vector, $V^\mu$ at the point $P$ along the curve $\gamma$ in tangent space, $T_P$ on the manifold.**

### 2.3 The metric tensor

A very important tensor in GR is the metric tensor $g_{\mu\nu}$, which is used to define the inner product between two vectors, $V^{\mu}X_{\mu} = g_{\mu\nu}V^{\nu}X^{\mu}$, and also represents the dynamical field in this theory. The metric tensor is a symmetric tensor rank $(0,2)$. We usually assume that the metric is non degenerate, the inverse metric $g^{\mu\nu}$ can be defined by

$$g^{\mu\nu}g_{\nu\sigma} = \delta^{\mu}_{\sigma}. \tag{2.9}$$

The symmetry of $g_{\mu\nu}$ implies that $g^{\mu\nu}$ is also symmetric. Generally, the metric $g_{\mu\nu}$ and its inverse can be used to raise and lower indices on vectors, one-forms, and tensors. For example, $V^{\mu} = g^{\mu\nu}V_{\nu}$ and $X_{\mu} = g_{\mu\nu}X^{\nu}$. Moreover, the metric tensor is invariant under the coordinate transformation,

$$g_{\mu\nu} \rightarrow g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}g_{\mu\nu} \tag{2.10}$$

In order to understand the behavior of a particle in spacetime, we will describe curved spacetime by determining the infinitesimal distance (or often call interval), $ds$ between two separated points, $x^{\mu}$ and $x^{\mu} + dx^{\mu}$ on the manifold. The interval can be written in terms of the line element as

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}, \tag{2.11}$$

In the context of GR, the metric tensor is required to be one negative and three positive eigenvalues with signature $(-,+,+,+)$. One found that the sign of $ds^2$ provides the characteristic feature of spacetime. When $ds^2 < 0$ the interval is timelike spacetime, while $ds^2 > 0$ the interval is spacelike spacetime and $ds^2 = 0$ correspond to lightlike spacetime.

### 2.4 Covariant derivative

By definition, the vector evaluated at different points on manifold lives in the different tangent spaces. To define the derivative of the vector field, one must compare these vectors to the limit that the distance between them tends to zero. In flat space, the partial derivative operator allows vectors at different points to be compared. However, in curved spacetime, the partial derivative is not a good operator because it depends on the coordinate system we used. Therefore, we will introduce an additional object, known as the affine connection for mapping one tangent space into the other on the manifold and then the derivative can be generalized to be the operator which is independent of choices of coordinates, known as the covariant derivative. In order to find the covariant derivative, let consider the basis vector $e_\mu$ at two nearby points in curved spacetime $P$ and $Q$ with lie on coordinates $x^\mu$ and $x^\mu + dx^\mu$ respectively. Generally, the basis vectors at $Q$ will

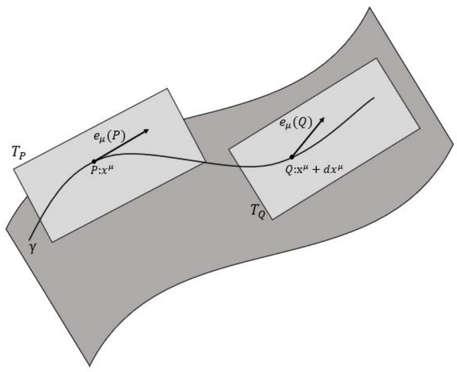


**Figure 3 A tangent vector, $V^\mu$ at the point $P$ along the curve $\gamma$ in tangent space, $T_P$ on the manifold**

infinitesimal different from the one at $P$ [68], so that

$$e_\mu(Q) = e_\mu(P) + \delta e_\mu. \tag{2.12}$$

The standard partial derivative of the basis vector is given by $\delta e_\mu/\delta x^\nu$ in the limit $\delta x^\nu \to 0$. However, the resulting vector will not lie on the tangent space at point $P$.

Hence, we can define the derivative in the manifold of the basis vector by projecting into the tangent space at point $P$ as,

$$\frac{\partial e_\mu}{\partial x^\nu} \equiv \left( \lim_{\delta x^\nu \to 0} \frac{\delta e_\mu}{\delta x^\nu} \right)_{||T_P} . \tag{2.13}$$

We can expand the above derivative in terms of the basis vectors $e_\mu(P)$ at point $P$ as,

$$\frac{\partial e_\mu}{\partial x^\nu} = \Gamma^\rho_{\mu\nu} e_\rho, \tag{2.14}$$

where the symbol $\Gamma^\rho_{\mu\nu}$ denote the affine connection. Then, suppose that a vector field, $V$ is defined on some region of manifold. The derivative of this vector field can be obtained by

$$\frac{\partial V}{\partial x^\nu} = \frac{\partial V^\mu}{\partial x^\nu} e_\mu + V^\mu \frac{\partial e_\mu}{\partial x^\nu}. \tag{2.15}$$

By using the Eq.(2.14), one may write (2.15) as

$$\begin{aligned}
\frac{\partial V}{\partial x^\nu} &= \frac{\partial V^\mu}{\partial x^\nu} e_\mu + V^\mu \Gamma^\rho_{\mu\nu} e_\rho, \\
&= \frac{\partial V^\mu}{\partial x^\nu} e_\mu + V^\rho \Gamma^\mu_{\rho\nu} e_\mu, \\
&= \left( \frac{\partial V^\mu}{\partial x^\nu} + V^\rho \Gamma^\mu_{\rho\nu} \right) e_\mu, \\
&= \nabla_\nu V^\mu e_\mu,
\end{aligned} \tag{2.16}$$

where

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + V^\rho \Gamma^\mu_{\rho\nu}, \tag{2.17}$$

is a covariant derivative of the vector. In the same way, the corresponding result for the covariant component of the vector, $V = V_\mu e^\mu$ can be expressed as

$$\nabla_\nu V_\mu = \partial_\nu V_\mu - V_\rho \Gamma^\rho_{\mu\nu}, \tag{2.18}$$

This kind of derivative does not change the form under a general coordinate transformation. This operator, $\nabla_\mu$ can reduce to the partial derivative, $\partial_\mu$ in locally flat spacetime. In addition, we can extend to the tensor rank $(k, l)$ as follows,

$$\nabla_\rho T_{\mu\nu} = \partial_\rho T_{\mu\nu} - T_{\lambda\nu}\Gamma^\lambda_{\rho\mu} - T_{\mu\lambda}\Gamma^\lambda_{\rho\nu}, \tag{2.19}$$

$$\nabla_\rho T^{\mu\nu} = \partial_\rho T^{\mu\nu} + T_{\lambda\nu}\Gamma^\lambda_{\rho\mu} + T^{\mu\lambda}\Gamma^\lambda_{\rho\nu}, \tag{2.20}$$

$$\nabla_\rho T^\mu_\nu = \partial_\rho T^\mu_\nu + T^\lambda_\nu\Gamma^\mu_{\rho\lambda} - T^\mu_\lambda\Gamma^\lambda_{\rho\nu}, \tag{2.21}$$

$$\vdots$$

$$\begin{aligned}\nabla_\rho T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} &= \partial_\rho T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \\ &\quad + \Gamma^{\mu_1}_{\rho\lambda}T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_2}_{\rho\lambda}T^{\mu_1\lambda\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \ldots \\ &\quad - \Gamma^\lambda_{\rho\nu_1}T^{\mu_1\mu_2\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_l} - \Gamma^\lambda_{\rho\nu_2}T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\lambda\ldots\nu_l} - ..\end{aligned} \tag{2.22}$$

One can say that the connection can be used to map one vector space into the other. It is important to note that the connections are not tensor. It does not transform as the components of a tensor, but the combination in Eq.(2.17) and (2.18) can be transformed as a tensor. We can find the relationship of a connection and metric tensor by introducing two additional assumptions as follows,

1. Torsion free: $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$.
2. Metric compatibility: $\nabla_\rho g_{\mu\nu} = 0$.

We then expand the equation for metric compatibility into three different permutations of the indices as follows,

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - g_{\alpha\nu}\Gamma^\alpha_{\rho\mu} - g_{\mu\alpha}\Gamma^\alpha_{\rho\nu}, \tag{2.23}$$

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - g_{\alpha\rho}\Gamma^\alpha_{\mu\nu} - g_{\nu\alpha}\Gamma^\alpha_{\mu\rho}, \tag{2.24}$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - g_{\alpha\mu}\Gamma^\alpha_{\nu\rho} - g_{\rho\alpha}\Gamma^\alpha_{\mu\nu}. \tag{2.25}$$

By using [Eq.(2.23)-Eq.(2.24)]-Eq.(2.25) and the first assumption; torsion free, we than obtain

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2g_{\rho\alpha}\Gamma^\alpha_{\mu\nu} = 0. \tag{2.26}$$

Contracting with $g^{\rho\lambda}$, we obtain

$$
\begin{aligned}
2(g^{\rho\lambda}g_{\rho\alpha})\Gamma^{\alpha}_{\mu\nu} &= g^{\rho\lambda}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}), \\
2(\delta^{\lambda}_{\alpha})\Gamma^{\alpha}_{\mu\nu} &= g^{\rho\lambda}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}), \\
\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \qquad (2.27)
\end{aligned}
$$

This unique connection is so called Christoffel symbol. The study of manifolds with the metric tensor, $g^{\mu\nu}$ and their associated connections is called Riemannian geometry.

### 2.5 Parallel transport

To understand the description of spacetime curvature, we have introduced the connection for mapping a vector from one tangent space to other on the manifold and defined the covariant derivative to compare one vector and its neighbor. But the question is, how to move a vector from one point to another? The answer is to move a vector from one point to another by paralleling the path and keeping the angle constant. This concept is known as parallel transport. Notice that, the idea of parallel transport can be also used in flat spacetime. The crucial difference between flat and curved spaces is that, in a curve space, the resulting from parallel transport will depend on the path taken between the points while in flat space is not [67].

Before going to parallel transport, it is worthwhile to understand the two different kinds of curvature: extrinsic and intrinsic curvature. Considering a surface that rounds up a cylinder, we immediately know that it is curved. On the other hand, a cylinder can be made by rolling a flat paper, so that the geometry of a cylinder corresponds to the original flat paper. This means that the distance between any points on the surface of the cylinder is the same as in the flat paper. This kind of curvature so called extrinsic curvature must be measured by an observer in higher-dimension. The intrinsic curvature is the actual curvature of

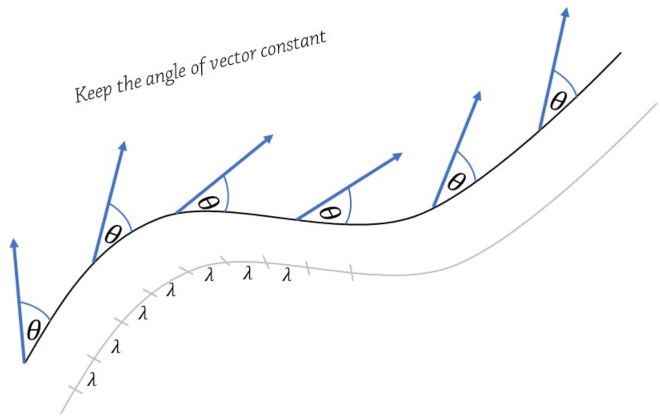


**Figure 4 Parallel transport of a vector along the curve parameterized by $\lambda$.**

spacetime without using the notion of a higher-dimensional spacetime, while the extrinsic curvature is based on the notion of a higher-dimensional spacetime [69]. Hence, when we talk about curved spacetime, meaning the intrinsic curvature of spacetime.

To construct the parallel transport equation of vector $V^\nu$ on a curve $x^\mu(\lambda)$ parameterized by $\lambda$ with fixing the angle and norm, we have to define the directional covariant derivative, given by

$$\frac{D}{d\lambda} = \frac{dx^\mu(\lambda)}{d\lambda}\nabla_\mu. \tag{2.28}$$

The vector $V^\nu$ must be constant along the curve. Hence, the directional covariant derivative of $V^\nu$ which is used to define parallel transport can be expressed as

$$
\begin{aligned}
\frac{D}{d\lambda}V^{\nu} &= 0, \\
\frac{dx^{\mu}}{d\lambda}\nabla_{\mu}V^{\nu} &= 0, \\
\frac{dx^{\mu}}{d\lambda}\left(\partial_{\mu}V^{\nu} + \Gamma^{\nu}_{\rho\mu}V^{\rho}\right) &= 0, \\
\frac{dV^{\nu}}{d\lambda} + \Gamma^{\nu}_{\rho\mu}\frac{dx^{\mu}}{d\lambda}V^{\rho} &= 0.
\end{aligned}
\tag{2.29}
$$

We then define the parallel transport equation for the tensor rank $(k, l)$ along the path $x^{\mu}(\lambda)$, via

$$
\left(\frac{D}{d\lambda}T\right)^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} \equiv \frac{dx^{\sigma}}{d\lambda}\nabla_{\sigma}T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} = 0 \tag{2.30}
$$

## 2.6 Geodesic equation

To study particle motion in general relativity, we have to discuss the shortest distance between two points in the Riemannian manifold, as known geodesics. The geodesic equation can be defined by taking parallel transport to the tangent vector. In other words, a geodesic is a curve along which the tangent vector is parallel transported. Hence, the geodesic equation can be obtained by

$$
\begin{aligned}
\frac{D}{d\lambda}\left(\frac{dx^{\nu}}{d\lambda}\right) &= 0, \\
\frac{dx^{\mu}}{d\lambda}\nabla_{\mu}\left(\frac{dx^{\nu}}{d\lambda}\right) &= 0, \\
\frac{dx^{\mu}}{d\lambda}\left(\partial_{\mu}\left(\frac{dx^{\nu}}{d\lambda}\right) + \Gamma^{\nu}_{\rho\mu}\left(\frac{dx^{\rho}}{d\lambda}\right)\right) &= 0, \\
\frac{dx^{\mu}}{d\lambda}\partial_{\mu}\left(\frac{dx^{\nu}}{d\lambda}\right) + \Gamma^{\nu}_{\rho\mu}\frac{dx^{\mu}}{d\lambda}\left(\frac{dx^{\rho}}{d\lambda}\right) &= 0, \\
\frac{d^2x^{\nu}}{d\lambda^2} + \Gamma^{\nu}_{\rho\mu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\rho}}{d\lambda} &= 0.
\end{aligned}
\tag{2.31}
$$

Note that one can be derived the geodesic equation by using the variation principle of as action for a particle moving in spacetime.

### 2.7 Curvature Tensor

The curvature of the spacetime can be examined by using a difference of the vector field resulting from the parallel transport around a closed loop. Mathematically, the difference of vector field is equivalent to the commutator of two covariant derivatives which is proportional to the vector field. As a result, this commutator can be written as,

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\rho &= \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho, \\
&= (\partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}(\nabla_\lambda V^\rho) - (\partial_\nu(\nabla_\mu V^\rho) - \Gamma^\rho_{\mu\sigma}(\nabla_\nu V^\sigma))), \\
&= \left[\partial_\mu(\partial_\nu V^\rho + \Gamma^\rho_{\nu\sigma}V^\sigma) - \Gamma^\lambda_{\mu\nu}(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma}V^\sigma)\right] \\
&\quad - \left[\partial_\nu(\partial_\mu V^\rho + \Gamma^\rho_{\mu\sigma}V^\sigma) - \Gamma^\rho_{\mu\sigma}(\partial_\nu V^\sigma + \Gamma^\sigma_{\mu\lambda}V^\lambda)\right], \\
&= R^\rho_{\sigma\mu\nu}V^\sigma,
\end{aligned} \tag{2.32}$$

where $R^\rho_{\sigma\mu\nu}$ is Riemann tensor which is a measure of the spacetime curvature and can be expressed as

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{2.33}$$

One can see that in flat spacetime the connection is zero, $\Gamma^\rho_{\mu\nu} = 0$ then $\partial_\sigma \Gamma^\rho_{\mu\nu} = 0$, thus Riemann tensor will be zero at every point. Therefore, the non zero of the Riemann tensor implies the existence of spacetime curvature.

It is easy to see properties of Riemann tensor, $R^\rho_{\sigma\mu\nu}$ because the indices $\mu$ and $\nu$ are related to covariant derivatives $\nabla_\mu$ and $\nabla_\nu$. Therefore, we will see that the Riemann tensor is anti-symmetric for swapping the order of $\mu$ and $\nu$,

$$R^\rho_{\sigma\mu\nu} = -R^\rho_{\sigma\nu\mu}. \tag{2.34}$$

The further properties for Riemann tensor can be found by lowering an index and swapping the order through the component (2.33) as follows as,

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \tag{2.35}$$

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}, \tag{2.36}$$

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}, \tag{2.37}$$

where $R_{\rho\sigma\mu\nu} = g_{\rho\lambda}R^{\lambda}_{\sigma\mu\nu}$. Moreover, one can write the sum of cyclic identity as

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0. \tag{2.38}$$

The Riemann tensor also obeys the Bianchi identity, given by

$$\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = 0, \tag{2.39}$$

where the square brackets denote totally anti-symmetric over the indices inside. In addition, we can construct the curvature tensor rank $(0, 2)$ by considering the contractions of the Riemann tensor. The resulting is called Ricci tensor, which is defined by

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}. \tag{2.40}$$

We can see that the symmetric of Ricci tensor is inherited from the Riemann tensor,

$$g^{\sigma\rho}R_{\sigma\mu\rho\nu} = g^{\rho\sigma}R_{\rho\nu\sigma\mu}, \tag{2.41}$$

giving us,

$$R_{\mu\nu} = R_{\nu\mu}. \tag{2.42}$$

As a consequence of the symmetry for Ricci tensor, one can see that the trace of Ricci tensor is giving the curvature scalar which is known as Ricci scalar, is then defined by

$$R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}. \tag{2.43}$$

## 2.8 Einstein field equation

We have discussed curvature in the previous section. In this section, we will construct the equation to relate matter/energy and gravity. One side of the equation would describe the spacetime curvature constructed from $R$, $R_{\mu\nu}$ or $R^{\rho}_{\mu\nu\sigma}$ and the other side should be proportional to the energy momentum tensor, $T^{\mu\nu}$ since it contains all information of the matter/energy. Therefore, the relation can be written as

$$f(R) \propto T_{\mu\nu}, \tag{2.44}$$

where $f(R)$ is any function of curvature tensors. One may assume that the equation might be written as

$$R_{\mu\nu} = kT_{\mu\nu}, \tag{2.45}$$

where $k$ is a proportional constant. However, the Eq.(2.45) is not possible, because the energy momentum tensor satisfies the conservation equation, $\nabla_\mu T^{\mu\nu}$ but the Ricci tensor does not in general. Hence, we have to find other quantities to satisfy this condition. Fortunately, Bianchi identity (2.39) provides us the conservation quantity, which can be expressed as

$$\nabla_\lambda R_{\mu\nu\rho\sigma} - \nabla_\lambda R_{\nu\mu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma} - \nabla_\mu R_{\lambda\nu\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma} - \nabla_\nu R_{\mu\lambda\rho\sigma} = 0, \tag{2.46}$$

$$2\left[\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma}\right] = 0. \tag{2.47}$$

Contracting with $g^{\lambda\rho}$ to the Eq. (2.47), we obtain

$$\begin{aligned} 0 &= 2g^{\lambda\rho}\left[\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma}\right], \\ &= 2\left[\nabla^\rho R_{\mu\nu\rho\sigma} - \nabla_\mu R_{\nu\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma}\right]. \end{aligned} \tag{2.48}$$

Contracting with $g^{\mu\sigma}$ to the Eq.(2.48), we then obtain

$$
\begin{aligned}
0 &= 2g^{\mu\sigma}\left[\nabla^{\rho}R_{\mu\nu\rho\sigma} - \nabla_{\mu}R_{\nu\sigma} + \nabla_{\nu}R_{\lambda\mu\rho\sigma}\right], \\
&= 2\left[-\nabla^{\rho}R_{\nu\rho} - \nabla^{\sigma}R_{\sigma\nu} + \nabla_{\nu}R\right], \\
&= 2\left[-2\nabla^{\rho}R_{\nu\rho} + \nabla_{\nu}R\right], \\
&= 2[-2\nabla^{\rho}R_{\nu\rho} + g_{\rho\nu}\nabla^{\rho}R], \\
&= -4\nabla^{\rho}(R_{\nu\rho} - \frac{1}{2}g_{\nu\rho}R).
\end{aligned}
\tag{2.49}
$$

The terms in a bracket from the Eq.(2.49) is called the Einstein tensor,

$$
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{2.50}
$$

Therefore, we will see that the Bianchi identity in the Eq.(2.39) is equivalent to,

$$
\nabla^{\mu}G_{\mu\nu} = 0. \tag{2.51}
$$

Hence, the relationship between gravity and matter/energy can be written as,

$$
G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.52}
$$

where the constant $k = 8\pi$, obtained by using the solutions in the Newtonian limit, see the detail in the appendix A . The Eq.(2.52) is called Einstein's field equations, it is very important in GR. It helps us to understand how the gravitational field responds to matter. Furthermore, this equation has been used to describe the expansion of the universe, the behavior of black holes, the propagation of the gravitational wave, etc.

### 2.9 The Einstein-Hilbert action

Most theories in physics can be described by the variational principle. Einstein's field equation is also obtained this way. Hear, we will show the action corresponding to Einstein's field equation in the vacuum, and the energy momentum tensor is also obtained by using the variational principle. We begin with the vacuum case, the action for spacetime curvature part can be written as,

$$S = \int d^4x\sqrt{-g}L, \tag{2.53}$$

where $L$ is Lagrangian density and $g = \det(g_{\mu\nu})$.

According to GR, the Lagrangian density $L$ must be covariant scalar quantities since there is a general coordinate transformation. By the restriction, one can write the action as,

$$S_{EH} = \int d^4x\sqrt{-g}R. \tag{2.54}$$

This is Einstein-Hilbert action. In fact, this action was firstly constructed by David Hilbert, using the variational principle in 1915 [70] the same year that GR was proposed while Einstein himself derived the equations independently. Therefore, this action was later called The Einstein-Hilbert action.

Using $R = g^{\mu\nu}R_{\mu\nu}$, and varying this action with respect to the metric tensor $g_{\mu\nu}$, we then obtain

$$\begin{aligned}\delta S_{EH} &= \int d^4x\delta(\sqrt{-g}g^{\mu\nu}R_{\mu\nu}),\\ &= \int d^4x\big[(\delta\sqrt{-g})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}(\delta R_{\mu\nu})\big].\end{aligned} \tag{2.55}$$

Using the identity $\ln(\det M) = \mathrm{Tr}(\ln M)$, where $\det M \neq 0$, we can write

$$\ln(g) = \mathrm{Tr}(\ln g_{\mu\nu}), \tag{2.56}$$

varying the Eq.(2.56), we then obtain

$$\begin{aligned} \delta\ln(g) &= \delta\mathrm{Tr}(\ln g_{\mu\nu}), \\ \frac{1}{g}\delta g &= g^{\mu\nu}\delta g_{\mu\nu}, \\ \delta g &= g g^{\mu\nu}\delta g_{\mu\nu}. \end{aligned} \tag{2.57}$$

It is found that,

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\delta(-g) = -\frac{1}{2\sqrt{-g}} g g^{\mu\nu}\delta g_{\mu\nu}, \tag{2.58}$$

$$= \frac{1}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu}. \tag{2.59}$$

Varying the inverse metric tensor can be obtained by,

$$g_{\mu\rho}g^{\nu\rho} = \delta^{\nu}_{\mu}, \Rightarrow (\delta g_{\mu\rho})g^{\rho\nu} + g_{\mu\rho}\delta g^{\rho\nu} = 0, \tag{2.60}$$

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}. \tag{2.61}$$

Then, the Eq.(2.59) can be written as

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}. \tag{2.62}$$

Substituting the Eq.(2.62) to the Eq.(2.55), we obtain

$$\begin{aligned} \delta S_{EH} &= \int d^4x \left[ -\frac{1}{2}\sqrt{-g} g_{\rho\sigma}\delta g^{\rho\sigma} g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{-g} g^{\mu\nu}(\delta R_{\mu\nu}) \right], \\ &= \int d^4x \sqrt{-g}\left( R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \right)\delta g^{\mu\nu} + \int d^4x \sqrt{-g} g^{\mu\nu}(\delta R_{\mu\nu}). \end{aligned} \tag{2.63}$$

Let's consider the second term which is proportional to $\delta R_{\mu\nu}$. The varying for Ricci tensor can be written as,

$$\begin{aligned} \delta R_{\mu\nu} = \delta R^{\rho}_{\mu\rho\nu} &= \partial_\rho \delta\Gamma^{\rho}_{\mu\nu} - \partial_\mu \delta\Gamma^{\rho}_{\nu\rho} + \delta\Gamma^{\rho}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\rho}_{\rho\sigma}\delta\Gamma^{\sigma}_{\mu\nu} - \delta\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\rho}_{\mu\sigma}\delta\Gamma^{\sigma}_{\nu\rho}, \\ &= \partial_\rho \delta\Gamma^{\rho}_{\mu\nu} + \delta\Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\rho\sigma} - \delta\Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} - \delta\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} \\ &\quad - (\partial_\mu \delta\Gamma^{\rho}_{\nu\rho} + \delta\Gamma^{\sigma}_{\nu\rho}\Gamma^{\rho}_{\mu\sigma} - \delta\Gamma^{\rho}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} - \delta\Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho}), \\ &= \nabla_\rho(\delta\Gamma^{\rho}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\mu\rho}). \end{aligned} \tag{2.64}$$

Therefore, the second term in the Eq.(2.63) can be expressed as

$$\int d^4x\sqrt{-g}g^{\mu\nu}(\delta R_{\mu\nu}) = \int d^4x\sqrt{-g}\big(\nabla_\rho(\delta(g^{\mu\nu}\Gamma^\rho_{\mu\nu})) - \nabla_\nu(\delta(g^{\mu\nu}\Gamma^\rho_{\mu\rho}))\big). \tag{2.65}$$

One can see that the Eq.(2.65) is an integration of the total derivative. It is considered as the boundary term, which can be set to zero to make the variation vanish at infinity. Hence, the variation of the action in the Eq.(2.63) can be written as,

$$\delta S_{EH} = \int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}. \tag{2.66}$$

The equation of motion can be obtain by solving $\delta S_{EH} = 0$, and then we have

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \qquad \rightarrow \qquad G_{\mu\nu} = 0. \tag{2.67}$$

This is Einstein's field equation in the vacuum. Next, we consider Einstein's field equation with sources. We thus include the action for matter field into our consideration. Therefore, the action corresponding to the Einstein's field equation with matter can be written in the form as,

$$S = S_{EH} + S_m = \int d^4x\sqrt{-g}\big(\frac{1}{2k}R + L_m\big), \tag{2.68}$$

where $S_m$ and $L_m$ are action and Lagrangian density for matter respectively. Then, varying this action with respect to the metric tensor $g^{\mu\nu}$, we obtain

$$\begin{aligned}\delta S &= \int d^4x\big(\sqrt{-g}\frac{1}{2k}G_{\mu\nu}\delta g^{\mu\nu} + \delta\mathcal{L}_m\big),\\ &= \int d^4x\sqrt{-g}\frac{1}{2k}\big(G_{\mu\nu} + \frac{2\kappa}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}\big)\delta g^{\mu\nu}.\end{aligned} \tag{2.69}$$

Requiring $\delta S = 0$, the Einstein's field with matter can be written as

$$G_{\mu\nu} + \frac{2k}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} = 0. \tag{2.70}$$

Comparing to the Einstein's field equations in the Eq.(2.52), we then have

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}, \tag{2.71}$$

$$= \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}, \tag{2.72}$$

$$= \frac{-2}{\sqrt{-g}}\left(\sqrt{-g}\frac{\delta L_m}{\delta g^{\mu\nu}} + L_m\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right), \tag{2.73}$$

$$= -\frac{2\delta L_m}{\delta g^{\mu\nu}} + L_m g_{\mu\nu} \tag{2.74}$$

where $k = 8\pi$.

In many situations in GR, the source of the gravitational field can be taken to be the suitable matter which is called the perfect fluid. Generally, the perfect fluid is defined as a fluid that has no heat transfer and no viscosity. The energy momentum tensor for the perfect fluid is written as [69],

$$T_{\mu\nu} = (P+\rho)U_\mu U_\nu + Pg_{\mu\nu}, \tag{2.75}$$

where $P$, $\rho$ and $U_\mu$ are pressure, energy density and four velocity of the fluid respectively. When the fluid is at the rest, $U_\mu = (1,0,0,0)$ so the energy momentum tensor can be written as,

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{2.76}$$

# CHAPTER III

# BLACK HOLE/STRING SOLUTION AND BLACK HOLE THERMODYNAMICS

## 3.1 The first look of black hole solution

The Einstein's field equation is a complicated non-linear differential equation. It is quite difficult to solve it analytically. However, the first exact solution for Einstein's field equation was found by Karl Schwarzschild [71]. From the fact that the shape of the astronomical object is almost a sphere, it is useful to assume that the spacetime geometry around the object is spherically symmetric. For more simplicity, the spacetime is assumed to be static meaning that the components of the metric tensor, $g_{\mu\nu}$ are independent of $x^0$ and the line element, $ds^2$ is invariant under timereversal symmetry, $x^0 \to -x^0$. By imposing two conditions to Einstein's field equation one can say that the Schwarzschild solution represents the static spherically symmetric spacetime surrounding some massive spherical object without matter or energy. In general, the line element for static and spherical symmetric spacetime can be expressed as,

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)r^2 d\Omega^2 \tag{3.1}$$

where $A(r)$, $B(r)$ and $C(r)$ are arbitrary function of $r$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on a unit 2-sphere. Since GR is invariant under general coordinate transformation, one can transform the line element in the Eq.(3.1) to other types of coordinate system without losing generality. For convenience, let's us define the new radial parameter, $\tilde{r} = r\sqrt{C(r)}$. Therefore, the line element in the Eq.(3.1) can be written in the form of $\tilde{r}$ as

$$ds^2 = -A(\tilde{r})dt^2 + B(\tilde{r})d\tilde{r}^2 + \tilde{r}^2 d\Omega^2. \tag{3.2}$$

Defining new function $e^{2\alpha(\tilde{r})}$ and $e^{2\beta(\tilde{r})}$ as $A(\tilde{r})$ and $B(\tilde{r})$ respectively and then removing the tilde, one obtains the line element as

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2. \tag{3.3}$$

The spherical metric can be written in the form as,

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix}. \tag{3.4}$$

The inverse of this metric is obtained by $g^{\mu\rho}g_{\rho\nu} = \delta^{\mu}_{\nu} = \mathbb{I}$, and then we obtain

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2\alpha(r)} & 0 & 0 & 0 \\ 0 & e^{-2\beta(r)} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} sin^{-2}\theta \end{pmatrix}. \tag{3.5}$$

As mentioned, we are interested in spacetime outside of a spherical mass object. Let us start by considering the Einstein's field equation for empty spacetime,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{3.6}$$

Contracting with $g^{\mu\nu}$, we obtain

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = 0, \tag{3.7}$$

$$R - \frac{1}{2}(4)R = 0, \tag{3.8}$$

$$\rightarrow R = 0. \tag{3.9}$$

Substituting it to the Eq.(3.6), one obtains

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(0) = 0,$$

$$\rightarrow R_{\mu\nu} = 0. \tag{3.10}$$

Notice that for $R_{\mu\nu} = 0$ and $R = 0$, it doesn't mean that the spacetime is not curved since the Riemann tensor, $R^{\rho}_{\sigma\mu\nu}$ does not necessarily equal to zero.

Let us label coordinates $x^{\mu} = (x^0, x^1, x^2, x^3)$ as $(t, r, \theta, \phi)$ for computing curvature quantities. By requiring the conditions for spherical symmetry, $\partial_{\phi} g_{\mu\nu} = 0$ and static spacetime, $\partial_0 g_{\mu\nu} = 0$, the non vanishing independent components of the Christoffel symbol can be written as

$$\Gamma^0_{01} = \frac{1}{2} g^{00}(\partial_0 g_{01} + \partial_1 g_{00} - \partial_0 g_{01}) = \frac{1}{2} g^{00}(\partial_1 g_{00}) = \alpha', \tag{3.11}$$

$$\Gamma^1_{00} = \frac{1}{2} g^{11}(\partial_0 g_{10} + \partial_0 g_{10} - \partial_1 g_{00}) = -\frac{1}{2} g^{11} \partial_1 g_{00} = e^{2(\alpha-\beta)} \alpha', \tag{3.12}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = -\frac{1}{2} g^{11} \partial_1 g_{11} = \beta', \tag{3.13}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) = -\frac{1}{2} g^{11} \partial_1 g_{22} = -r e^{-2\beta}, \tag{3.14}$$

$$\Gamma^1_{33} = \frac{1}{2} g^{11}(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33}) = -\frac{1}{2} g^{11} \partial_1 g_{33} = -r e^{-2\beta} \sin^2\theta, \tag{3.15}$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22}(\partial_1 g_{22} + \partial_2 g_{21} - \partial_2 g_{12}) = \frac{1}{2} g^{22} \partial_1 g_{22} = \frac{1}{r}, \tag{3.16}$$

$$\Gamma^2_{33} = \frac{1}{2} g^{22}(\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}) = -\frac{1}{2} g^{22} \partial_2 g_{33} = -\sin\theta\cos\theta, \tag{3.17}$$

$$\Gamma^3_{13} = \frac{1}{2} g^{33}(\partial_1 g_{33} + \partial_3 g_{31} - \partial_3 g_{13}) = \frac{1}{2} g^{33} \partial_1 g_{33} = \frac{1}{r}, \tag{3.18}$$

$$\Gamma^3_{23} = \frac{1}{2} g^{33}(\partial_2 g_{33} + \partial_3 g_{32} - \partial_3 g_{23}) = \frac{1}{2} g^{33} \partial_2 g_{33}, = \cot\theta, \tag{3.19}$$

where prime denotes the derivative with respect to $r$. By using the results of Christoffel symbols, the non vanishing independent components of the Ricci tensor can be written as

$$
\begin{aligned}
R_{00} &= R^{\rho}_{0\rho 0} = \partial_\rho \Gamma^{\rho}_{00} - \partial_0 \Gamma^{\rho}_{\rho 0} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{00} - \Gamma^{\rho}_{0\lambda}\Gamma^{\lambda}_{\rho 0}, \\
&= \partial_1 \Gamma^{1}_{00} + \Gamma^{1}_{11}\Gamma^{1}_{00} + \Gamma^{2}_{21}\Gamma^{1}_{00} + \Gamma^{3}_{31}\Gamma^{1}_{00} - \Gamma^{0}_{01}\Gamma^{1}_{00}, \\
&= \frac{e^{2(\alpha-\beta)}}{r}\left[r\alpha'' + r\alpha'^2 - r\alpha'\beta' + 2\alpha'\right] \qquad (3.20) \\
R_{11} &= R^{\rho}_{1\rho 1} = \partial_\rho \Gamma^{\rho}_{11} - \partial_1 \Gamma^{\rho}_{\rho 1} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{11} - \Gamma^{\rho}_{1\lambda}\Gamma^{\lambda}_{\rho 1}, \\
&= \partial_1 \Gamma^{1}_{11} - \partial_1 \Gamma^{0}_{01} - \partial_1 \Gamma^{1}_{11} - \partial_1 \Gamma^{2}_{21} - \partial_1 \Gamma^{3}_{31} + \Gamma^{0}_{01}\Gamma^{1}_{11} + \Gamma^{1}_{11}\Gamma^{1}_{11} \\
&\quad + \Gamma^{2}_{21}\Gamma^{1}_{11} + \Gamma^{3}_{31}\Gamma^{1}_{11} - \Gamma^{0}_{10}\Gamma^{0}_{01} - \Gamma^{1}_{11}\Gamma^{1}_{11} - \Gamma^{2}_{12}\Gamma^{2}_{21} - \Gamma^{3}_{13}\Gamma^{3}_{31}, \\
&= -\frac{1}{r}\left[r\alpha'' + r\alpha'^2 - r\beta'\alpha' - 2\beta'\right] \qquad (3.21) \\
R_{22} &= R^{\rho}_{2\rho 2} = \partial_\rho \Gamma^{\rho}_{22} - \partial_2 \Gamma^{\rho}_{\rho 2} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{22} - \Gamma^{\rho}_{2\lambda}\Gamma^{\lambda}_{\rho 2}, \\
&= \partial_1 \Gamma^{1}_{22} - \partial_2 \Gamma^{3}_{32} + \Gamma^{0}_{01}\Gamma^{1}_{22} + \Gamma^{1}_{11}\Gamma^{1}_{22} + \Gamma^{3}_{31}\Gamma^{1}_{22} - \Gamma^{2}_{21}\Gamma^{1}_{22} - \Gamma^{3}_{23}\Gamma^{3}_{32}, \\
&= e^{-2\beta}\left[r(\beta' - \alpha') - 1\right] + 1 \qquad (3.22) \\
R_{33} &= \sin^2\theta R_{22}. \qquad (3.23)
\end{aligned}
$$

Each component of Eq.(3.10) leads to three equations as follows

$$\frac{e^{2(\alpha-\beta)}}{r}\left[r\alpha'' + r\alpha'^2 - r\alpha'\beta' + 2\alpha'\right] = 0, \qquad (3.24)$$

$$-\frac{1}{r}\left[r\alpha'' + r\alpha'^2 - r\beta'\alpha' - 2\beta'\right] = 0, \qquad (3.25)$$

$$e^{-2\beta}\left[r(\beta' - \alpha') - 1\right] + 1 = 0. \qquad (3.26)$$

Multiplying Eq.(3.25) by $e^{2(\alpha-\beta)}$ and then combining with Eq (3.24), we obtain

$$2(\alpha' + \beta') = 0. \qquad (3.27)$$

As a consequent result in the Eq.(3.27) one find that,

$$(\alpha+\beta)' = 0, \quad \rightarrow \quad \alpha + \beta = C, \quad \rightarrow \quad \alpha = -\beta + C, \qquad (3.28)$$

where $C$ is an integration constant. The line element from Eq.(3.3) becomes

$$ds^2 = -e^{-2\beta + C}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega^2. \qquad (3.29)$$

To find the constant $C$, let us use the fact that the metric is asymptotically flat. In other words, the spacetime must be approached flat at very far from this massive object, $r \to \infty$. One can write the line element as

$$ds^2_{\text{flat}} = -dt^2 + dr^2 + r^2 d\Omega^2. \tag{3.30}$$

In this limit we immediately have the condition $\beta(r) = 0$ so that the constant $C$ must be zero. The line element thus becomes

$$ds^2 = -e^{-2\beta} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2. \tag{3.31}$$

Next, we will find the expression of $\beta$. From Eq.(3.28), we then have

$$\alpha = -\beta. \tag{3.32}$$

Substituting it to Eq.(3.26), we obtain

$$\begin{aligned} e^{-2\beta}\left[r(\beta' - \alpha') - 1\right] + 1 &= 0, \\ e^{-2\beta}\left[r(2\beta') - 1\right] + 1 &= 0, \\ re^{-2\beta} 2\beta'(r) - e^{-2\beta} + 1 &= 0, \\ e^{-2\beta} - re^{-2\beta} 2\beta' &= 1, \\ (re^{-2\beta})' &= 1. \end{aligned} \tag{3.33}$$

Integrating this, one obtains

$$e^{-2\beta} = 1 + \frac{D}{r}, \tag{3.34}$$

where $D$ is the integration constant. Eventually, we now have the line element in the form,

$$ds^2 = -\left(1 + \frac{D}{r}\right) dt^2 + \left(1 + \frac{D}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3.35}$$

In order to interpret the constant D, we have to use the fact that GR should be reduced to Newton's theory of gravity (or weak gravitational) in a limit called Newtonian limit. We have 3 assumptions as follows

1. The particle moves slowly with respect to the speed of light.

2. The gravitational field is static (unchanging with time).

3. The gravitational field is weak so it can be considered as a perturbation of flat space.

Let us consider the trajectory of particle described by the geodesic equation

$$\frac{d^2x^\rho}{d\lambda^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{3.36}$$

For the time-like particle, the affine parameter, $\lambda$ can be used as the proper time $\tau$. By the first assumption; the particle moving slowly means that,

$$\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau}. \tag{3.37}$$

The geodesic equation can be approximated as

$$\begin{aligned} 0 &= \frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \\ &= \frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \underbrace{2\Gamma^\rho_{0i}\frac{dx^0}{d\tau}\frac{dx^i}{d\tau} + \Gamma^\rho_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}_{\text{can be ignored due to Eq.(3.37)}}, \\ &\approx \frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}. \end{aligned} \tag{3.38}$$

For static spacetime, the metric does not change with time $\partial_0 g_{\mu\nu} = 0$. As a result the Christoffel symbol $\Gamma^\rho_{00}$ becomes.

$$\Gamma^\rho_{00} = \frac{1}{2}g^{\rho\sigma}(\partial_0 g_{\sigma 0} + \partial_0 g_{\sigma 0} - \partial_\sigma g_{00}) = -\frac{1}{2}g^{\rho\sigma}\partial_\sigma g_{00}. \tag{3.39}$$

By the last assumption; the weakness of gravitational field allows us to decompose the metric into the flat spacetime plus a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \mid h_{\mu\nu} \mid \ll 1, \tag{3.40}$$

where $h_{\mu\nu}$ is the perturbed metric. From the definition of the inverse metric,

$g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$ and consider at first order of $h$, one find that

$$\begin{aligned}
\delta^\mu_\nu &= g^{\mu\rho}g_{\rho\nu}, \\
&= (\eta^{\mu\rho} + \delta g^{\mu\rho})(\eta_{\rho\nu} + h_{\rho\nu}), \\
&= \eta^{\mu\rho}\eta_{\rho\nu} + \eta^{\mu\rho}h_{\rho\nu} + \delta g^{\mu\rho}\eta_{\rho\nu} + \cancelto{\approx 0}{\delta g^{\mu\rho}h_{\rho\nu}}, \\
&= \delta^\mu_\nu + \underbrace{(\eta^{\mu\rho}h_{\rho\nu} + \delta g^{\mu\rho}\eta_{\rho\nu})}_{=0}.
\end{aligned} \tag{3.41}$$

Then, we obtain

$$\begin{aligned}
\delta g^{\mu\rho}\eta_{\rho\nu} &= -\eta^{\mu\rho}h_{\rho\nu}, \\
\delta g^{\mu\rho} &= -\eta^{\rho\nu}\eta^{\mu\rho}h_{\rho\nu} = -h^{\mu\rho}.
\end{aligned} \tag{3.42}$$

As a result, the inverse metric can be written as

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{3.43}$$

Substituting $g_{\mu\nu}$ and $g^{\mu\nu}$ into the Eq.(3.39), one obtains

$$\begin{aligned}
\Gamma^\rho_{00} &= -\frac{1}{2}(\eta^{\rho\sigma} - h^{\rho\sigma})\partial_\sigma(\eta_{00} + h_{00}), \\
&= -\frac{1}{2}\left[\eta^{\rho\sigma}\cancelto{0}{\partial_\sigma\eta_{00}} + \eta^{\rho\sigma}\partial_\sigma h_{00} - h^{\rho\sigma}\cancelto{0}{\partial_\sigma\eta_{00}} - \cancelto{}{h^{\rho\sigma}\partial_\sigma h_{00}}\right]^{\approx 0}, \\
&= -\frac{1}{2}\eta^{\rho\sigma}\partial_\sigma h_{00}.
\end{aligned} \tag{3.44}$$

The geodesic equation in Eq.(3.38) for the Newtonian limit is then expressed as

$$0 = \frac{d^2x^\rho}{d\tau^2} - \frac{1}{2}\eta^{\rho\sigma}\partial_\sigma h_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau}. \tag{3.45}$$

Since $h_{00}$ is static, $\partial_0 h_{00} = 0$ it implies that the zero component the above equation can be written as,

$$\frac{d^2x^0}{d\tau^2} = 0, \quad \rightarrow \frac{d}{d\tau}(\frac{dx^0}{d\tau}) = 0. \tag{3.46}$$

This means that $dx^0/d\tau$ is constant. Using the relation

$$\frac{d}{d\tau} = \frac{dx^0}{d\tau}\frac{d}{dx^0}, \tag{3.47}$$

we therefore obtain

$$\frac{d^2x^\rho}{d(x^0)^2} = \frac{1}{2}\eta^{\rho\sigma}\partial_\sigma h_{00}. \tag{3.48}$$

Since $h_{00}$ does not depend on $x^0$, and the flat spacetime metric is diagonal (the component $\eta^{i0}$ vanishes), the Eq. (3.48) can be written as

$$\frac{d^2x^i}{d(x^0)^2} = \frac{1}{2}\eta^{ij}\partial_j h_{00} = \frac{1}{2}\partial^i h_{00}. \tag{3.49}$$

The result in Eq.(3.49) is also written in the familiar form as

$$\vec{a} = \frac{1}{2}\vec{\nabla}h_{00}. \tag{3.50}$$

where $\vec{\nabla}$ denotes the gradient of $h_{00}$ and $a$ is the acceleration corresponding to the gravitational force acts on a spherically symmetric mass $M$ in Newton's theory, which is obtained by

$$\vec{F} = -\frac{Mm}{r^2}\vec{r}, \qquad \to \qquad \vec{a} = -\frac{M}{r^2}\vec{r} = \partial_r\left(\frac{M}{r}\right)\hat{r}. \tag{3.51}$$

By compare the Eq.(3.50) and (3.51), we obtain

$$h_{00} = \frac{2M}{r}. \tag{3.52}$$

Hence, we now obtained

$$g_{00} = \eta_{00} + h_{00} = -1 + \frac{2M}{r} = -\left(1 - \frac{2M}{r}\right). \tag{3.53}$$

By comparing this result to the line element in Eq.(3.35), the integration constant can be written as, $D = -2M$. Therefore, we now obtain the Schwarzschild solution as

$$ds^2 = -f dt^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2, \qquad f \equiv 1 - \frac{2M}{r}. \tag{3.54}$$

It is important to note that $M$ is a parameter that has a mass dimension to make curved spacetime, which can be interpreted as the Newtonian mass at

$r \to \infty$. Note also that in the limit $M \to 0$ and/or $r \to \infty$, we retrieve the flat space. The spacetime for the Schwarzschild solution is not well defined at every point. It is obvious to see that the metric components are infinite at $r = 0$ and $r = 2M$. These points is called singularity. However, the components of a metric depends on the choice of coordinate. The singularity may disappear if we choose the other suitable coordinates. In order to find the real singular of spacetime, we have to find the scalar quantity which is not dependent on the choice of coordinates. For example, the Kretschmann scalar can be written as

$$K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48M^2}{r^6}. \tag{3.55}$$

One can see that the Kretschmann scalar is diverge at only $r = 0$. Therefore, the spacetime for the Schwarzschild solution indeed diverges at $r = 0$. This point is called real singularity. It is impossible to eliminate by coordinate transformation while at $r = 2M$ is a coordinate singularity.

## 3.2 Event horizon

As we mentioned before, the event horizon is a surface on which even the light cannot escape. Hence, one can say that the event horizon is a null surface where its normal vector is a null vector at every point. Consider an arbitrary surface such that,

$$\Phi(x^\mu) = \text{constant} = 0. \tag{3.56}$$

The normal vector of this surface can be written as

$$\partial_\mu \Phi = n_\mu. \tag{3.57}$$

If the normal vector is a null vector, this implies that it is a tangent vector to the surface. Therefore, the condition of the null surface in which the normal vector is a null vector can be written as.

$$n_\mu n^\mu = 0. \tag{3.58}$$

Let us consider the Schwarzschild case in which $\Phi = f(r)$. The condition for null surface in the Eq.(3.58) can be written as,

$$g^{\mu\nu} n_\mu n_\nu = g^{\mu\nu} \partial_\mu f \partial_\nu f = g^{11} (\partial_1 f)^2 = 0 \tag{3.59}$$

Therefore, the event horizon for Schwarzschild metric is

$$g^{11} = 0 \quad \Rightarrow \quad 1 - \frac{2M}{r} = 0 \quad \Rightarrow r = 2M. \tag{3.60}$$

The radius at $r = 2M$ is known as the Schwarzschild radius $r_s$. There is a dramatic thing that happened when we cross inside the event horizon, time and space would be reversed. The metric signature change form $(-,+,+,+)$ to $(+,-,+,+)$. The light-cone will be flipped in which $x$ and $t$ exchange at $r = r_s$. Hence, for the region $r < r_s$, we have to move forward in the direction of decreasing $r$ to $r = 0$. The objects with mass enough to collapse to a size smaller than the Schwarzschild radius will continue to collapse until it becomes a certain point in spacetime with infinite curvature. The result of this collapse is known as a Schwarzschild black hole. Anything that enters a black hole must inevitably slam into the singularity at $r = 0$. Therefore one can say that there is no particles or even electromagnetic waves can come out from the black hole. Note also that, an observer outside a black hole will never be able to perceive what is happening in spacetime within a black hole since the signals close to the event horizon get infinitely red-shifted.

The concept of null surface can be used to define the Killing horizon, which is used to resolve the conflict of the definition of surface gravity as we will discuss in the Sec 3.6.

### 3.3 Reissner-Nordström black hole

In this subsection, we will consider spherically symmetric and static spacetime with charge. The Einstein–Hilbert action with the electromagnetic field can be written as,

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2k}R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \tag{3.61}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the electromagnetic field strength tensor and $A_\mu$ is component of gauge field. The energy momentum tensor of the charged object can obtain by

$$T_{\mu\nu} = \frac{1}{4\pi}\Big(F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big), \tag{3.62}$$

where $k = 8\pi$. In flat spacetime, components of the field strength tensor can be written in terms of the electric field $E$ and magnetic field $B$ as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{3.63}$$

To obtain the energy momentum tensor satisfying spherical symmetry, the electric components $E_\theta, E_\phi$ and magnetic components $B_\theta, B_\phi$ must vanish, and $E_r, B_r$ must depend only on $r$. Therefore, the field strength tensor can be written the form as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & -r^2\sin\theta B(r) \\ 0 & 0 & r^2\sin\theta B(r) & 0 \end{pmatrix}. \tag{3.64}$$

The equations of motion can be described by Maxwell's equation,

$$\nabla_\mu F^{\mu\nu} = 0, \tag{3.65}$$

$$\nabla_\rho F_{\mu\nu} + \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} = 0. \tag{3.66}$$

Since $F^{\mu\nu}$ is antisymmetric tensor, one can write the covariant derivative in the Eq.(3.65) as,

$$\begin{aligned}
\nabla_\mu F^{\mu\nu} &= \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\rho} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho}, \\
&= \partial_\mu F^{\mu\nu} + \frac{1}{\sqrt{-g}} \partial_\rho \sqrt{-g} F^{\rho\nu} + \Gamma^\nu_{\mu\rho} \frac{1}{2} \cancelto{0}{(F^{\mu\rho} - F^{\rho\mu})}, \\
&= \frac{1}{\sqrt{-g}} \partial_\rho (\sqrt{-g} F^{\rho\nu}) = 0.
\end{aligned} \tag{3.67}$$

For the spherical coordinates we have $g = -e^{2(\alpha+\beta)} r^4 \sin^2\theta$, substituting $g$ into the Eq.(3.67), we obtain

$$\frac{1}{e^{(\alpha+\beta)} r^2 \sin\theta} \partial_\mu (e^{(\alpha+\beta)} r^2 \sin\theta F^{\mu\nu}) = 0. \tag{3.68}$$

Considering $\nu = 0$, the above equation becomes

$$\begin{aligned}
0 &= \frac{1}{e^{(\alpha+\beta)} r^2 \sin\theta} \partial_\mu (e^{(\alpha+\beta)} r^2 \sin\theta F^{\mu 0}), \\
&= \frac{1}{e^{(\alpha+\beta)} r^2 \sin\theta} \partial_1 (e^{(\alpha+\beta)} r^2 \sin\theta F^{10}), \\
&= \frac{1}{e^{(\alpha+\beta)} r^2 \sin\theta} \partial_1 (e^{(\alpha+\beta)} r^2 \sin\theta g^{11} g^{00} F_{10}), \\
&= \frac{1}{e^{(\alpha+\beta)} r^2 \sin\theta} \partial_r (e^{(\alpha+\beta)} r^2 \sin\theta (e^{-2\beta})(-e^{-2\alpha}) E), \\
&= -\frac{1}{e^{(\alpha+\beta)} r^2} \partial_r (e^{-(\alpha+\beta)} r^2 E).
\end{aligned} \tag{3.69}$$

As a result, we obtain the electric field as

$$E = \frac{C}{e^{-(\alpha+\beta)} r^2}, \tag{3.70}$$

where $C$ is integration constant. To interpret the constant $C$, we have to use the fact that the spacetime must be flat at vary large $r$ when $\alpha \approx \beta \approx 0$, so that the function $E$ must reduce to the electric field of a point charge as follows

$$E_{r\to\infty} = \frac{q}{r^2}, \quad \Rightarrow \quad C = q, \tag{3.71}$$

where $q$ is electric charge. Therefore, the function $E$ representing the electric field of a charged object can be written as

$$E = q \frac{e^{(\alpha+\beta)}}{r^2}. \tag{3.72}$$

In order to find $B$, we have to use the component $(\rho\mu\nu) = (r\theta\phi)$ for the Eq.(3.66). Therefore, we now obtain

$$\partial_r F_{\theta\phi} = -\partial_r (r^2 \sin\theta B) = 0, \tag{3.73}$$

which implies that

$$B = \frac{C}{r^2}, \tag{3.74}$$

where $C$ is a constant. However, for the large $r$, the constant $C$ can be interpreted as the magnetic charge, p. Therefore, the function $B$ representing the magnetic field of a point magnetic monopole can be written as

$$B = \frac{p}{r^2}. \tag{3.75}$$

Next, computing the non vanishing components of the energy-momentum tensor,

$$\begin{aligned}
T_{00} &= \frac{1}{4\pi}\Big(F_{0\rho}F_0^\rho - \frac{1}{4}g_{00}F_{\rho\sigma}F^{\rho\sigma}\Big), \\
&= \frac{1}{4\pi}\Big(F_{0\rho}g^{\rho\nu}F_{0\nu} - \frac{1}{4}g_{00}g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta}\Big), \\
&= \frac{1}{4\pi}\Big(F_{01}g^{11}F_{01} - \frac{1}{4}g_{00}[g^{00}g^{11}F_{01}F_{01} + g^{11}g^{00}F_{10}F_{10} + g^{22}g^{33}F_{23}F_{23} \\
&\quad + g^{33}g^{22}F_{32}F_{32}]\Big), \\
&= \frac{1}{4\pi}\Big(F_{01}g^{11}F_{01} - \frac{1}{4}g_{00}2(g^{00}g^{11}F_{01}F_{01} + g^{22}g^{33}F_{23}F_{23})\Big), \\
&= \frac{1}{4\pi}\Big(e^{-2\beta}E^2 + \frac{1}{2}e^{2\alpha}(-e^{-2(\alpha+\beta)}E^2 + \frac{1}{r^2}\frac{1}{r^2 sin^2\theta}r^4 sin^2\theta B^2)\Big), \\
&= \frac{1}{4\pi}\Big(e^{-2\beta}E^2 + \frac{1}{2}e^{2\alpha}(-e^{-2(\alpha+\beta)}E^2 + B^2)\Big), \\
&= \frac{1}{8\pi}e^{2\alpha}\Big(e^{-2(\alpha+\beta)}E^2 + B^2\Big),
\end{aligned} \tag{3.76}$$

$$\begin{aligned}
T_{11} &= \frac{1}{4\pi}\Big(F_{1\rho}F_1^\rho - \frac{1}{2}g_{11}(-e^{-2(\alpha+\beta)}E^2 + B^2)\Big), \\
&= \frac{1}{4\pi}\Big(F_{10}g^{00}F_{10} - \frac{1}{2}g_{11}(-e^{-2(\alpha+\beta)}E^2 + B^2)\Big), \\
&= \frac{1}{4\pi}\Big(-e^{-2\alpha}E^2 - \frac{1}{2}e^{2\beta}(-e^{-2(\alpha+\beta)}E^2 + B^2)\Big), \\
&= -\frac{1}{8\pi}e^{2\beta}\Big(e^{-2(\alpha+\beta)}E^2 + B^2\Big),
\end{aligned} \tag{3.77}$$

$$
\begin{aligned}
T_{22} &= \frac{1}{4\pi}\Big(F_{23}g^{33}F_{23} - \frac{1}{2}g_{22}(-e^{-2(\alpha+\beta)}E^2 + B^2)\Big), \\
&= \frac{1}{4\pi}\Big(r^{-2}\sin^{-2}\theta(r^4\sin^2\theta B^2) - \frac{1}{2}r^2(-e^{-2(\alpha+\beta)}E^2 + B^2)\Big), \\
&= \frac{1}{4\pi}\Big(r^2B^2 - \frac{1}{2}r^2(-e^{-2(\alpha+\beta)}E^2 + B^2)\Big),
\end{aligned} \tag{3.78}
$$

$$
\begin{aligned}
T_{33} &= \frac{1}{4\pi}\Big(F_{32}g^{22}F_{32} - \frac{1}{2}g_{33}(-e^{-2(\alpha+\beta)}E^2 + B^2)\Big), \\
&= \frac{1}{4\pi}\Big(r^2\sin^2\theta B^2 - \frac{1}{2}r^2\sin^2\theta(-e^{-2(\alpha+\beta)}E^2 + B^2)\Big).
\end{aligned} \tag{3.79}
$$

Therefore,

$$T^0_0 = g^{00}T_{00} = -\frac{1}{8\pi}\Big(e^{-2(\alpha+\beta)}E^2 + B^2\Big). \tag{3.80}$$

$$T^1_1 = g^{11}T_{11} = -\frac{1}{8\pi}\Big(e^{-2(\alpha+\beta)}E^2 + B^2\Big). \tag{3.81}$$

$$T^2_2 = g^{22}T_{22} = \frac{1}{8\pi}\Big(e^{-2(\alpha+\beta)}E^2 + B^2\Big). \tag{3.82}$$

$$T^3_3 = g^{33}T_{33} = \frac{1}{8\pi}\Big(e^{-2(\alpha+\beta)}E^2 + B^2\Big). \tag{3.83}$$

Substituting these results to the Einstein's field equation in Eq.(2.52), we obtain

$$-\frac{e^{-2\beta}}{r^2}\Big(e^{2\beta} + 2r\beta' - 1\Big) = -\Big(e^{-2(\alpha+\beta)}E^2 + B^2\Big), \tag{3.84}$$

$$\frac{e^{-2\beta}}{r^2}\Big(-e^{2\beta} + 2r\alpha' + 1\Big) = -\Big(e^{-2(\alpha+\beta)}E^2 + B^2\Big), \tag{3.85}$$

$$\frac{e^{-2\beta}}{r}\Big(\alpha' - \beta' + r(\alpha'' - \alpha'\beta' + \alpha'^2)\Big) = \Big(e^{-2(\alpha+\beta)}E^2 + B^2\Big). \tag{3.86}$$

The Eq.(3.84) - Eq.(3.85), one obtains

$$\alpha' + \beta' = 0 \Rightarrow \alpha = -\beta + C, \tag{3.87}$$

where $C$ is a integration constant. By using the same fashion as same as in Schwarzschild case, one can obtain $C = 0$. From the Eq.(3.84), one can solve for $\beta$ as

$$
\begin{aligned}
e^{-2\beta}\Big(e^{2\beta}+2r\beta'-1\Big) &= r^2\Big(e^{-2(-\beta+\beta)}E^2+B^2\Big),\\
1+2r\beta' e^{-2\beta}-e^{-2\beta} &= r^2(E^2+B^2),\\
1-\frac{d}{dr}(re^{-2\beta}) &= \Big(\frac{q^2}{r^2}+\frac{p^2}{r^2}\Big),\\
\frac{d}{dr}(re^{-2\beta}) &= 1-\Big(\frac{q^2}{r^2}+\frac{p^2}{r^2}\Big),\\
re^{-2\beta} &= r+\frac{1}{r}\Big(q^2+p^2\Big)+D,\\
e^{-2\beta} &= 1+\frac{1}{r^2}\Big(q^2+p^2\Big)+\frac{D}{r},\\
e^{-2\beta} &= 1-\frac{2M}{r}+\frac{Q^2}{r^2},
\end{aligned}
\tag{3.88}
$$

where the integration constant $D=-2M$ obtained by using the Newtonian limit and $Q^2=q^2+p^2$. Therefore, the solution can be expressed as

$$
ds^2=-\Delta dt^2+\Delta^{-1}dr^2+r^2d\Omega^2, \quad \Delta\equiv\Big(1-\frac{2M}{r}+\frac{Q^2}{r^2}\Big). \tag{3.89}
$$

This is known as the Reissner-Nordström metric, which describes spacetime with spherical symmetry and charges inside. Note that, in limit $Q\to 0$, we retrieve the Schwarzschild solution. From this metric, it can be seen that the singularity occurs at $\Delta=0$ and $r=0$. However, from the Kretschmann scalar

$$
K=R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}=\frac{12(Mr-Q^2)^2}{r^8}+\frac{2Q^4}{r^8}, \tag{3.90}
$$

one can see that the real singularity is occurring only at $r=0$ while the singularity locates at $\Delta=0$ is just the coordinate singularity. The two roots of $\Delta$ are given by

$$
r_\pm=M\pm\sqrt{M^2-Q^2}. \tag{3.91}
$$

The singularity at $r=r_\pm$ will be the location of the event horizon. Moreover, one can write the the Reissner-Nordström metric in terms of $r_\pm$ as,

$$
ds^2=-fdt^2+f^{-1}dr^2+r^2d\Omega^2, \quad f\equiv\frac{(r-r_+)(r-r_-)}{r^2}. \tag{3.92}
$$

However, there are three possible physical interpretations of $r_\pm$ depending on $M$ and $Q$ as follows;

1. $M^2 < Q^2$: In this case, the spacetime has an only real singularity at $r = 0$. There is no event horizon for $\Delta = 0$. The observer outside the black hole can perceive the information about a singularity. This is a naked singularity. However, the naked singularity is not considered as reasonable situation in physics since in this case there is a problem with negative energy and also contradicts the cosmic censorship conjecture.

2. $M^2 > Q^2$: For this case, there are two horizons located at $r = r_\pm$, where the horizon at $r_-$ and $r_+$ are inner and outer horizon respectively. Therefore, one can obtain the Reissner-Nordström black hole or charge black hole. The metric signature of Reissner-Nordström solution are given by

- $r > r_+$: $(-, +, +, +)$
- $r_- < r < r_+$: $(+, -, +, +)$
- $r < r_-$: $(-, +, +, +)$.

From the metric signatures, one can see that geodesic for the observers inside this black hole does not necessarily end at $r = 0$ since the observer are forced to move in the direction of decreasing $r$ only for $r_- < r < r_+$.

3. $M^2 = Q^2$: This case is known as the extremal Reissner-Nordström black hole solution. The event horizon at $r_-$ and $r_+$ will be merge as a single horizon at $r = M$. The extremal black holes are extremely unstable, when matter falls into this black hole, the mass of the black hole will be increased to more than its charge, $M^2 > Q^2$, and then becomes to the normal Reissner-Nordström black hole.

### 3.4 Kerr black hole

Previously, we have studied the Einstein's field equation with spherically symmetric and static spacetime. In this subsection, we will consider the stationary non zero angular momentum black hole with axial symmetry known as the rotating black hole. The solution was found by Roy P. Kerr in 1963 [72]. Since the black hole that we are considering is a rotating object, the Kerr solution is therefore invariant under the combination of angular and time reversed transformation,

$$\phi \to -\phi, \quad t \to -t. \tag{3.93}$$

With the Boyer-Lindquist coordinate, the Kerr metric is given by

$$\begin{aligned} ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \\ \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2, \end{aligned} \tag{3.94}$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = (r^2 + a^2\cos^2\theta) \tag{3.95}$$

and $a$ is the angular momentum per unit mass,

$$a = \frac{J}{M}. \tag{3.96}$$

In the limit, $a \to 0$ one can see that $\rho \to r$ and then this gives us the Schwarzschild solution. The real singularity for this metric can be obtain by evaluating the Kretschmann scalar,

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{6M^2}{\rho^2}\left[2r^2(r^2 - 3a^2\cos^2\theta)^2 - a^2\cos^2\theta(3r^2 - a^2\cos^2\theta)^2\right]. \tag{3.97}$$

As a result, the Kretschmann scalar is diverge at $\rho^2 = (r^2 + a^2\cos^2\theta) = 0$. Therefore, the real singularity arise at

$$r = 0, \quad \theta = \frac{\pi}{2}. \tag{3.98}$$

In the Boyer-Lindquist coordinate, this corresponds to

$$z = 0, \quad x^2 + y^2 = a^2. \tag{3.99}$$

Hence, one can see that the real singularity for Kerr black hole is not a certain point in spacetime, but the circular ring with radius $a$. The events horizons are located at the points at which $g_{11} \to \infty$, it can be found by,

$$\Delta = r^2 - 2Mr + a^2 = 0. \tag{3.100}$$

By solving the above equation, there are two horizon located at

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}. \tag{3.101}$$

It is also found that there are three possible physical interpretations of $r_{\pm}$ similar to the Reissner-Nordström case.

1. $M^2 < a^2$: In this case, there exists the naked singularity.

2. $M^2 > a^2$: For this case, we obtain the Kerr black hole. There are exist the inner and outer horizon two horizons located at at $r_-$ and $r_+$ respectively which play the same role as in Reissner-Nordström case. The metric signature is $(+, -, +, +)$ for $r_- < r < r_+$, and $(-, +, +, +)$ for $r > r_+$ and $r < r_-$.

3. $M^2 = a^2$: In this case, we obtain the extreme Kerr black hole.

There is something spacial for the Kerr black hole. To see this, we will consider the behavior of the stationary limit surfaces. The condition for these surface can be written as,

$$g_{00} = -\frac{1}{\rho^2}(r^2 - 2M + a^2 \cos^2\theta) = 0. \tag{3.102}$$

Then, we obtain the stationary limit surfaces are located at

$$r = M \pm \sqrt{M^2 - a^2 \cos^2\theta}. \tag{3.103}$$

It can be seen that, the smaller root in the Eq.(3.103) is inside the inner horizon, $r_-$, while the larger root is outside the outer horizon, $r_+$ except at $\theta = 0, \pi$ these surfaces coincide with the event horizon. Moreover, the inner surface at $\theta = \pi/2$ also coincides with the real singularity. The region between the outer horizon and the larger stationary limit surfaces is known as the ergosphere as illustrated in Fig.5. The outer boundary of the ergosphere is called the ergosurface as ill. This region does not exist in both Reissner-Nordström and Schwarzschild case. The observer or even the light inside the ergosphere will be move in the direction of the rotation of the black hole. Here is an example of the dragging of inertial frame. This phenomena provides that the rotation can be made the spacetime curvature. However, the ergosphere is outside the outer horizon so that the particle in this region can be escape from this black hole.

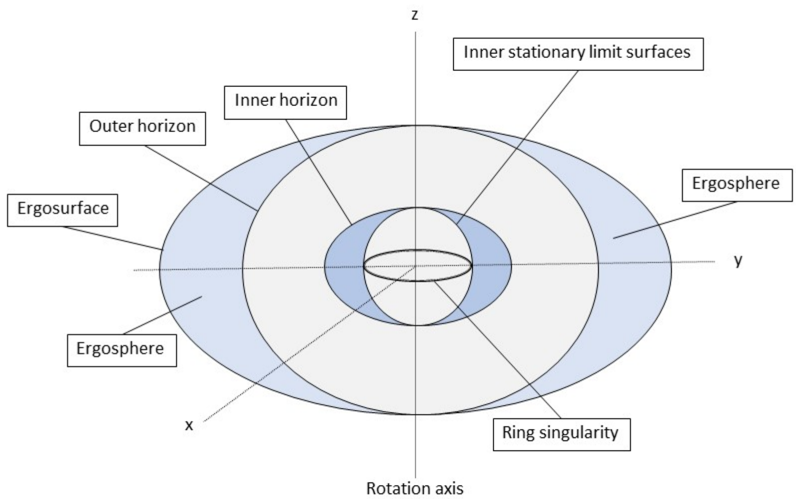


**Figure 5 Horizon structure around the Kerr Black hole.**

We can generalize the Kerr solution to the most general solution by including the electric charge $q$ and magnetic charge $p$ and replacing $2Mr$ with $2Mr-(q^2+p^2)$.

As a result, the resulting metric is known as the Kerr-Newman metric, given by

$$ds^2 = -\Big(1 - \frac{2Mr - Q^2}{\rho^2}\Big)dt^2 - \frac{2(2Mr - Q^2)a\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \Big(r^2 + a^2 + \frac{(2Mr - Q^2)a^2\sin^2\theta}{\rho^2}\Big)\sin^2\theta d\phi^2, \tag{3.104}$$

where $Q^2 = q^2 + p^2$, $\Delta = r^2 - (2Mr - Q^2) + a^2$. Note that, this metric reduces to the Schwarzschild metric for $a = 0$ and $Q = 0$, and also reduces to the Reissner-Nordström metric if $a = 0$ but $Q \neq 0$.

### 3.5 Schwarzschild de Sitter black hole

Nowadays, there have been several observations that suggest that our universe is expanding with acceleration. Although we study the universe based on GR, it may be only valid at the local gravity scale, whereas the large scale GR cannot be used to describe the accelerating expansion of the universe without introducing an exotic matter. Hence, a modification of GR called modified gravity theory is another possible way to explain this phenomenon. The familiar one is the cosmological constant model. The key idea of this model is that the accelerating expansion of the universe is driven by something called the cosmological constant $\Lambda$, so the black hole solution should involk to the cosmological constant. In this subsection, we will solve the static and spherical symmetric solution for GR with cosmological constant. The action for this modified gravity theory can be written as

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big(R - 2\Lambda\Big). \tag{3.105}$$

This solution was found by Tangherlini [73]. Note that the asymptotic behavior of spacetime will depend on the sign of cosmological constant, they can be asymptotically flat ($\Lambda = 0$), de Sitter ($\Lambda > 0$) or Anti-de Sitter ($\Lambda < 0$). Then, varying this action with respect to the metric tensor $g^{\mu\nu}$, we have

$$\begin{aligned}\delta S &= \frac{1}{16\pi}\int d^4x\Big(\delta(\sqrt{-g}R) - 2\delta\sqrt{-g}\Lambda\Big),\\ &= \frac{1}{16\pi}\int d^4x\Big(\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\Lambda\Big),\\ &= \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big(G_{\mu\nu} + g_{\mu\nu}\Lambda\Big)\delta g^{\mu\nu}. \end{aligned} \tag{3.106}$$

The Einstein's field equation is then obtained as

$$G_{\mu\nu} = -g_{\mu\nu}\Lambda, \tag{3.107}$$

$$\rightarrow \quad G^{\mu}_{\nu} = -\delta^{\mu}_{\nu}\Lambda. \tag{3.108}$$

Substituting the result in the Eq.(3.108) into Einstein's field equation, we then obtained the non vanishing components of the Einstein tensor as,

$$G^0_0 = -\frac{e^{-2\beta}}{r^2}\Big(e^{2\beta} + 2r\beta' - 1\Big) = -\Lambda, \tag{3.109}$$

$$G^1_1 = \frac{e^{-2\beta}}{r^2}\Big(-e^{2\beta} + 2r\alpha' + 1\Big) = -\Lambda, \tag{3.110}$$

$$G^2_2 = G^3_3 = \frac{e^{-2\beta}}{r}\Big(\alpha' - \beta' + r(\alpha'' - \alpha'\beta' + \alpha'^2)\Big) = -\Lambda. \tag{3.111}$$

Combining Eq.(3.109) and Eq.(3.110), we obtained

$$2r(\alpha' + \beta') = 0, \quad \Rightarrow \alpha = -\beta. \tag{3.112}$$

Hence, The line element for this solution becomes

$$ds^2 = -e^{-2\beta}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega^2. \tag{3.113}$$

One can solve the solution for $\beta$ by using the Eq.(3.109) as follows

$$-\frac{e^{-2\beta}}{r^2}\Big(e^{2\beta} + 2r\beta' - 1\Big) = -\Lambda,$$

$$1 + 2re^{-2\beta}\beta' - e^{-2\beta} = r^2\Lambda \tag{3.114}$$

$$\frac{d}{dr}\left(re^{-2\beta}\right) = 1 - r^2\Lambda. \tag{3.115}$$

Integrating this equation, we obtain

$$e^{-2\beta} = 1 + \frac{D}{r} - \frac{\Lambda}{3}r^2. \tag{3.116}$$

The integration constant $D$ can be obtained by using the Newtonian limit as $D = -2M$. Therefore, we obtain

$$e^{-2\beta} = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2. \tag{3.117}$$

Substituting back into the line element in Eq.(3.113), we obtain the static and spherical symmetric solution for GR with the cosmological constant as follow

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2 d\Omega^2, \qquad f \equiv 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2. \tag{3.118}$$

Generally, for the asymptotically dS case, there exist two event horizons: the smaller one is a black hole horizon, $r_b$ and the larger one is a cosmic horizon $r_c$, where the observer is in the middle between the two event horizons.

### 3.6 Surface gravity

The surface gravity, $\kappa$ is one of the important properties for the study of black hole thermodynamics. It is measured in the dimension of acceleration. In Newtonian theory, the surface gravity on the Earth surface can be written as

$$\kappa_{\text{Earth}} = \frac{M}{R^2_{\text{Earth}}}, \tag{3.119}$$

which corresponds to the acceleration of the particle on the Earth's surface. However, for the black hole case, one cannot define surface gravity as the acceleration experienced by a test particle at the surface of the object, because there is no real surface. Therefore, the surface gravity for the black hole may be defined analogously to the acceleration corresponding to the force acting on the test mass at the horizon, as measured by the observer at infinity. In order to define the surface gravity of a simple static black hole with the metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{3.120}$$

let us consider a 4-velocity of the particle stay at rest,

$$u^\mu = \frac{dx^\mu}{d\tau} = (\frac{dt}{d\tau}, 0, 0, 0), \tag{3.121}$$

where $\tau$ is the proper time of the particle and $t$ is a time of the observer at far away. By definition, the proper time interval between two events along a timelike path $P$ is defined as,

$$d\tau = \int_P \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \int_P \sqrt{-g_{00}}dx^0, \quad \rightarrow \quad d\tau = f^{1/2}dt. \tag{3.122}$$

Therefore, we now obtain,

$$u^\mu = (f^{-1/2}, 0, 0, 0). \tag{3.123}$$

The 4-acceleration can be defined as

$$a^\mu = u^\nu \nabla_\nu u^\mu = u^t \nabla_t u^\mu. \tag{3.124}$$

It is found that the non zero component of $a^\mu$ is determined by

$$a^1 = u^0(\partial_0 u^1 + \Gamma^1_{00} u^0) = f^{-1}\Gamma^1_{00} = \frac{1}{2}f', \tag{3.125}$$

the proper acceleration can be defined as

$$\begin{aligned} a = \sqrt{a^\mu a_\mu} &= \sqrt{g_{11}}a^1, \\ &= f^{-1/2}\frac{f'}{2}. \end{aligned} \tag{3.126}$$

The proper acceleration diverges at the event horizon. However, the surface gravity corresponds to the acceleration used to hold the particle at the horizon from the observer at infinity. One could imagine the particle connected to infinity by a very long massless string. If the observer at infinity moves the string with a very small distance $\delta l$, the amount of work that the observer has received is written as

$$\delta W_\infty = m_0 a_\infty \delta l, \tag{3.127}$$

where $m_0$ is considered as a mass of the particle. At the position of the particle nearby the horizon, the amount of work is

$$\delta W(r) = m_0 a(r) \delta l. \tag{3.128}$$

These amounts of work are different, but we can imagine that the amount of work at position $r$ can be converted into radiation which is then radiates to the observer at infinity. In this process, the energy of radiation is factor by $\sqrt{-g_{00}}$ due to the red-shift effect,

$$\delta E_\infty = \sqrt{-g_{00}} m_0 a(r) \delta l. \tag{3.129}$$

By the energy conservation, $\delta E_\infty = \delta W_\infty$, we thus obtain

$$a_\infty = a(r)\sqrt{-g_{00}} = \frac{1}{2} f'. \tag{3.130}$$

Therefore, the surface gravity can be properly defined by taking a particle at the horizon as

$$\kappa = a_\infty(r_h) = \frac{1}{2} f'(r_h). \tag{3.131}$$

For the general case, we can mathematically define the surface gravity by the idea that there is a surface in which the normal vector is the Killing vector, a vector field that preserves the metric tensor. And then, the surface gravity $\kappa$ can be defined via

$$k_\mu \kappa = -\frac{1}{2} \nabla_\mu (k^\nu k_\nu), \tag{3.132}$$

where $k_\mu$ is the Killing vector. Note that, if the component of the metric is independent of one of the coordinates, then the spacetime must automatically have a Killing vector in a direction independent of that coordinate. For static and spherically symmetric black holes which are represented by the Eq.(3.120) the metric is independent of time. In other words, there is a time translation invariant. Hence, it is suitable to use the timelike vector as a Killing vector, $k^\mu_{(t)} = (1, 0, 0, 0)$. Therefore, the surface gravity can be written as

$$\begin{aligned} k_{\mu(t)} \kappa &= -\frac{1}{2} \partial_\mu (g_{\rho\nu} k^\rho_{(t)} k^\nu_{(t)}), \\ &= -\frac{1}{2} \partial_\mu (g_{00} k^0 k^0), \\ &= -\frac{1}{2} \partial_\mu (g_{00}). \end{aligned} \tag{3.133}$$

It is obvious to see that $g_{00}$ is the only function of $r$, which shows that $\partial_\mu(g_{00})$ is in the direction of $r$ while we expect it to be in the direction of $t$. This conflict can be resolved by noting that surface gravity is evaluated at the Killing horizon, a surface for a suitably chosen Killing vector become normal and null, such as $k^\mu_{(t)} = (k^0, k^1, 0, 0)$ one can see in [67, 74, 75] for more details about Killing horizon and Killing vector. Therefore, we obtain the condition as,

$$g_{\mu\nu} k^\nu_{(t)} k^\mu_{(t)} = g_{00}(k^0)^2 + g_{11}(k^1)^2 = 0 \tag{3.134}$$

and then one obtains,

$$k^1 = \sqrt{\frac{-g_{00}}{g_{11}}} k^0 = f k^0. \tag{3.135}$$

We eventually obtain,

$$\begin{aligned} g_{11} k^1 \kappa = g_{11}(f k^0)\kappa = -\frac{1}{2}\partial_1(g_{00}), & \\ \kappa = \frac{1}{2}\partial_r(f(r)) = \frac{|f'(r)|}{2}|_{r=r_h}. & \end{aligned} \tag{3.136}$$

As a result, one found that the surface gravity for other kinds of black hole in general relativity as follows;

- Schwarzschild black hole:

$$\kappa = \frac{M}{r_h^2}, \text{ where } r_h = 2M. \tag{3.137}$$

- Reissner-Nordström black hole:

$$\kappa = \frac{\sqrt{M^2 - Q^2}}{r_h^2}, \text{ where } r_h = M + \sqrt{M^2 - Q^2}. \tag{3.138}$$

- Schwarzschild de Sitter black hole:

$$\kappa = \frac{M}{r_h^2} - \frac{\Lambda}{3} r_h, \text{where } r_h \quad = \frac{2}{\Lambda^{1/2}} \cos\left(\frac{1}{3}\cos^{-1}\left(-3M\Lambda^{1/2}\right) - \frac{2\pi k}{3}\right), \tag{3.139}$$

with $k = 0, 1, 2$ for the three distinguished solutions. In order to obtain two horizons, the cosmological constant must lie on the range $0 < \Lambda \leq 1/9M^2$. As a result, for $k = 2$, $r_h$ is negative, and for $k = 1$ and $k = 0$, the solutions can be respectively approximated as

$$r_h|_{k=1} \approx 2M, \quad r_h|_{k=0} \approx \sqrt{\frac{3}{\Lambda}} - M. \tag{3.140}$$

It is important to note that, the definition of surface gravity in the Eq.(3.133) was originally formulated from the component $(0,0)$ of the line element, hence the formulation $\kappa = \frac{|f'(r)|}{2}|_{r=r_h}$ can be used to calculate the surface gravity in other static solutions, i.e., the static and cylindrical symmetric black hole solution.

For the Kerr black hole the Killing vector which is normal vector is defined as the linear combination of the time-translation and rotational Killing vector, given by

$$k^\mu = K^\mu_{(t)} + \Omega_H K^\mu_{(\phi)}, \tag{3.141}$$

where $\Omega_H$ interpret as the angular velocity of the black hole, $K^\mu_{(t)} = \partial_t = (1,0,0,0)$ and $K^\mu_{(\phi)} = \partial_\phi = (0,0,0,1)$, the surface gravity can be written as,

$$\kappa = \frac{\sqrt{M^2 - a^2}}{r_h^2 + a^2}, \tag{3.142}$$

where $r_h = M + \sqrt{M^2 - a^2}$ and $\Omega_H = \frac{a}{a^2 + r_+^2}$. One can extend the surface gravity to the case of Kerr–Newman black hole as

$$\kappa = \frac{\sqrt{M^2 - a^2 - Q^2}}{r_h^2 + a^2}, \text{ where } r_h = M + \sqrt{M^2 - a^2 - Q^2}. \tag{3.143}$$

### 3.7 Black hole thermodynamics

The study of black hole thermodynamics began in 1971, Stephen Hawking found that the surface area of the black hole cannot decrease in any physical process [76]. This means that the area of a resulting black hole from the merging of two black holes is always greater than the sum of the areas of the original ones. Therefore, the area law of the black hole can be formulated as

$$\delta A \geq 0. \tag{3.144}$$

In 1973, Jacob Bekenstein conjectured that the surface area of a black hole can play the role of its entropy [77]. However, if a black hole carries entropy, the black hole would have temperature and must emit thermal radiation. Nevertheless, according to GR, nothing can come out form the black hole. Therefore, the temperature of a black hole will be equal to zero.

The contradiction of this entropy was fixed by Hawking in 1974 [78]. He claimed that black holes could be interpret as thermal objects. And if we study the properties of a black hole by using GR and taking the quantum effect into the account, black holes can emit thermal radiation (Hawking radiation) corresponding to a certain temperature (Hawking temperature) given by

$$T_H = \frac{\kappa}{2\pi} = \frac{|f'(r_h)|}{4\pi}. \tag{3.145}$$

As a consequent result, the black hole can carries entropy. The entropy of black hole was defined by the area of the black hole as

$$S_{BH} = \frac{A}{4}, \tag{3.146}$$

where $S_{BH}$ is Bekenstein–Hawking entropy. According to the area law, one can see that the entropy of black hole can never decrease in any process,

$$\Delta S_{BH} \geq 0. \tag{3.147}$$

From this analogy, one can see that the Eq.(3.147) is similar to the second law of thermodynamics. Therefore, there is a deep connection between black hole mechanics and thermodynamics. One can formulate the law of black hole mechanics equivalently to the law of thermodynamics as follow [79],

- **The zeroth law of black hole thermodynamics**

In a stationary state of black hole the surface gravity $\kappa$ is constant over an event horizon. It is analogous to the zeroth law of thermodynamics, which states that the temperature is constant for the system in thermal equilibrium.

- **The first law of black hole thermodynamics**

The $1^{st}$ is originally coming from the rotating black hole, it provides us how the energy $M$ (proportional to mass) of the black hole change with its area $A$, charge $Q$, and angular momentum $J$. The $1^{st}$ law of black hole thermodynamics states that the changes in mass, angular momentum, and surface area are related by [75],

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q, \tag{3.148}$$

where $\Omega_H$ and $\Phi_H$ are angular velocity and electric potential at the horizon respectively. One can see that the expression in Eq.(3.148) is analogous to the first law of thermodynamics,

$$dE = TdS - PdV. \tag{3.149}$$

It is obvious to see that the first term in RHS of Eq.(3.148), $\frac{\kappa}{8\pi G}\delta A$ can be interpreted as heat term appears in the first law of thermodynamics and the term $\Omega_H \delta J + \Phi_H \delta Q$ corresponds to the work terms of the first law. Moreover, one was found that the state variables in the $1^{st}$ law are also satisfied the relation

$$M = 2TS + 2\Omega_H J + 2\Phi_H Q, \tag{3.150}$$

which is known as the Smarr formula. The way to derive the Smarr formula can be seen in [80].

- **The second law of black hole thermodynamics**

As previously discussed, the surface area of the black hole cannot decrease in any physical process,

$$\delta A \geq 0. \tag{3.151}$$

This law is similar to the second law of thermodynamic, which state that in any physically allowed process the total entropy of the universe cannot decrease, $\delta S \geq 0$

- **The third law of black hole thermodynamics**

For the black hole, the third law may be stated that it is impossible to obtain a black hole with $\kappa = 0$. However, it is obvious to see that this law is violated since $\kappa$ always be zero for the extreme black hole. In this case, the area of the black hole does not need to be zero as $\kappa = 0$.

We also analyze the local stability of the black hole solutions by studying its heat capacity,

$$C = \frac{dM}{dT}, \tag{3.152}$$

it can be seen that the system with locally stable requires a positive heat capacity. The system with negative heat capacity will radiate thermal energy. Then the system gets hotter and eventually vanishes.

The global stability can be studied by considering the Gibbs free energy,

$$G = M - TS. \tag{3.153}$$

Thermodynamically stable systems prefer the lower free energy, which means that the system with lower Gibbs free energy at a given temperature prefers to exist compared to those with higher free energy. For example, if the free energy of the system without a black hole is zero, thus black hole can be formed by the condition, $G < 0$.

### 3.8 Separated and Effective System

In the case of Schwarzschild-de Sitter black hole, there generally exist two horizons; black hole horizon $r_b$ and cosmic horizon $r_c$. As a result, there are two distinct temperatures, given by

$$T(r_b) = \frac{f'(r_b)}{4\pi} \quad \text{and} \quad T(r_c) = -\frac{f'(r_c)}{4\pi}. \tag{3.154}$$

This suggests that the systems are not in thermal equilibrium. This is one of the obstructions to investigate thermodynamics of Schwarzschild-de Sitter black hole. To fix such a problem we can separate our consideration into two approaches as follows;

1. The thermodynamic system of each horizon can be defined separately and independently. The systems are treated to be in the quasi-equilibrium state, in which the timescale of the heat transfer between each system is much longer than the timescale of the thermodynamics process. Therefore, each system can be characterized by its thermodynamics behaviour. In order to find the first law of this system, let us treat the mass parameter $M$ as the homogeneous function with degree 1/2 as $M = M(S, P^{-1})$ where $S = \pi r^2$ is the entropy of the system and the cosmological constant is interpreted as the pressure of the system given by $\Lambda = -8\pi P$. By using Euler's theorem, it is obvious to see that $M$ can be expressed as

$$\frac{1}{2}M = S\frac{\partial M}{\partial S} + P^{-1}\frac{\partial M}{\partial P^{-1}} = S\frac{\partial M}{\partial S} - P\frac{\partial M}{\partial P} \tag{3.155}$$

By using the Smarr formula of the Schwarzschild-de Sitter black hole the mass parameter $M$ can be expressed as

$$M = \frac{r_h}{2}\left(1 - \frac{\Lambda}{3}r_h^2\right) = \pm 2TS - 2PV. \tag{3.156}$$

As a result, the temperature and volume can be defined as

$$T = \pm\left(\frac{\partial M}{\partial S}\right)_P, \quad V = \left(\frac{\partial M}{\partial P}\right)_S. \tag{3.157}$$

Therefore, the first law of black hole thermodynamic evaluated at each horizon can be written as

$$dM(r_b) = T(r_b)dS(r_b) + V(r_b)dP, \tag{3.158}$$

$$dM(r_c) = -T(r_c)dS(r_c) + V(r_c)dP, \tag{3.159}$$

where $M$ plays the role of enthalpy of the system. In order to study the thermodynamical stability of the system, the heat capacity and Gibbs free energy are defined as,

$$C(r_{b,c}) = \left(\frac{\partial M}{\partial T}\right)_P = T(r_{b,c})\left(\frac{\partial S(r_{b,c})}{\partial T(r_{b,c})}\right)_P, \tag{3.160}$$

and

$$G(r_{b,c}) = M(r_{b,c}) - T(r_{b,c})S(r_{b,c}). \tag{3.161}$$

2. The systems can be considered as a single system called an effective system. In this approach, one can think that an observer is located in the region between a black horizon and a cosmic horizon. Therefore, the thermodynamic system can be characterized by effective thermodynamics quantities. For this case, the entropy of the effective system can be written as the sum of entropies of each system,

$$S = S(r_b) + S(r_b). \tag{3.162}$$

The effective system is satisfied the following first law,

$$dM = T_{\text{eff}}dS + A_{\text{eff}}dB, \tag{3.163}$$

where $T_{\text{eff}}$ is the effective temperature and $A_{\text{eff}}$ is some effective quantity. One can see that $S$ and $B$ are entropy and another state function, which depends on two horizons $r_b$ and $r_c$,

$$S = S(r_b, r_c), \qquad B = B(r_b, r_c). \tag{3.164}$$

The total derivative of $S$ and $B$ can be written as

$$dS = \left(\frac{\partial S}{\partial r_b}\right)_{r_c} dr_b - \left(\frac{\partial S}{\partial r_c}\right)_{r_b} dr_c. \tag{3.165}$$

$$dB = \left(\frac{\partial B}{\partial r_b}\right)_{r_c} dr_b + \left(\frac{\partial B}{\partial r_c}\right)_{r_b} dr_c. \tag{3.166}$$

The minus sign in the second term on the right hand side of the Eq.(3.165) comes from the idea that we live in the region between the black hole horizon and cosmic horizon. When we identify the direction of heat flow, one found that it will be an opposite sign of each other as seen in the heat term from the Eq.(3.158) and (3.159). The consequent entropy of the system will be increased for the black hole horizon and will be increased for the cosmic horizon which is in the opposite direction to the black hole horizon one. One can find the expression of $dr_b$ by subtracting $\left(\frac{\partial B}{\partial r_c}\right)_{r_b}$ Eq.(3.165)+$\left(\frac{\partial S}{\partial r_c}\right)_{r_b}$ Eq.(3.166),

$$dr_b = \frac{\left(\frac{\partial B}{\partial r_c}\right)_{r_b} dS + \left(\frac{\partial S}{\partial r_c}\right)_{r_b} dB}{\left[\left(\frac{\partial B}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} + \left(\frac{\partial S}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c}\right]}. \tag{3.167}$$

The expression of $dr_c$ can be obtained by subtracting $\left(\frac{\partial B}{\partial r_b}\right)_{r_c}$ Eq.(3.165)-$\left(\frac{\partial S}{\partial r_b}\right)_{r_c}$ Eq.(3.166),

$$dr_c = -\frac{\left(\frac{\partial B}{\partial r_b}\right)_{r_c} dS + \left(\frac{\partial S}{\partial r_b}\right)_{r_c} dB}{\left[\left(\frac{\partial B}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} + \left(\frac{\partial S}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c}\right]}. \tag{3.168}$$

Since $M$ is also a function of $r_b$ and $r_c$, Hence we can write the total derivative of $M$ as

$$dM = \left(\frac{\partial M}{\partial r_b}\right)_{r_c} dr_b + \left(\frac{\partial M}{\partial r_c}\right)_{r_b} dr_c. \tag{3.169}$$

Substituting Eq.(3.167) and Eq.(3.168) into the total derivative of $M$ we then obtain

$$dM = \frac{\left(\frac{\partial M}{\partial r_b}\right)_{r_c}\left(\frac{\partial B}{\partial r_c}\right)_{r_b} dS + \left(\frac{\partial M}{\partial r_b}\right)_{r_c}\left(\frac{\partial S}{\partial r_c}\right)_{r_b} dB}{\left[\left(\frac{\partial B}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} + \left(\frac{\partial S}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c}\right]}$$

$$- \frac{\left(\frac{\partial M}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c} dS + \left(\frac{\partial M}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} dB}{\left[\left(\frac{\partial B}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} + \left(\frac{\partial S}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c}\right]} \tag{3.170}$$

$$= \left(\frac{\left(\frac{\partial M}{\partial r_b}\right)_{r_c}\left(\frac{\partial B}{\partial r_c}\right)_{r_b} - \left(\frac{\partial M}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c}}{\left(\frac{\partial B}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} + \left(\frac{\partial S}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c}}\right) dS$$

$$+ \left(-\frac{\left(\frac{\partial M}{\partial r_b}\right)_{r_c}\left(\frac{\partial S}{\partial r_c}\right)_{r_b} - \left(\frac{\partial M}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c}}{\left(\frac{\partial B}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} + \left(\frac{\partial S}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c}}\right) dB, \tag{3.171}$$

Comparing to Eq.(3.163), we can see

$$T_{\text{eff}} = \frac{\left(\frac{\partial M}{\partial r_b}\right)_{r_c}\left(\frac{\partial B}{\partial r_c}\right)_{r_b} - \left(\frac{\partial M}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c}}{\left(\frac{\partial B}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} + \left(\frac{\partial S}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c}}, \tag{3.172}$$

$$A_{\text{eff}} = -\frac{\left(\frac{\partial M}{\partial r_b}\right)_{r_c}\left(\frac{\partial S}{\partial r_c}\right)_{r_b} - \left(\frac{\partial M}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c}}{\left(\frac{\partial B}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} + \left(\frac{\partial S}{\partial r_c}\right)_{r_b}\left(\frac{\partial B}{\partial r_b}\right)_{r_c}}. \tag{3.173}$$

For the case of $M$ play role of the enthalpy of system, the first law becomes

$$dM = T_{\text{eff}} dS + V_{\text{eff}} dP, \tag{3.174}$$

The effective temperature are define in terms of $r_b$ and $r_c$ as

$$T_{\text{eff}} = \left(\frac{\partial M}{\partial S}\right)_P = \frac{\left(\frac{\partial M}{\partial r_b}\right)_{r_c}\left(\frac{\partial P}{\partial r_c}\right)_{r_b} - \left(\frac{\partial M}{\partial r_c}\right)_{r_b}\left(\frac{\partial P}{\partial r_b}\right)_{r_c}}{\left(\frac{\partial P}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} + \left(\frac{\partial S}{\partial r_c}\right)_{r_b}\left(\frac{\partial P}{\partial r_b}\right)_{r_c}}. \tag{3.175}$$

It is found that the effective temperature is related to the temperatures at the black hole and cosmic horizons as follows

$$T_{\text{eff}} = \left(\frac{1}{T(r_b)} + \frac{1}{T(r_c)}\right)^{-1}. \tag{3.176}$$

The effective volume of the system can be obtained by

$$V_{\text{eff}} = \left(\frac{\partial M}{\partial P}\right)_S = -\frac{\left(\frac{\partial M}{\partial r_b}\right)_{r_c}\left(\frac{\partial S}{\partial r_c}\right)_{r_b} - \left(\frac{\partial M}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c}}{\left(\frac{\partial P}{\partial r_c}\right)_{r_b}\left(\frac{\partial S}{\partial r_b}\right)_{r_c} + \left(\frac{\partial S}{\partial r_c}\right)_{r_b}\left(\frac{\partial P}{\partial r_b}\right)_{r_c}}. \tag{3.177}$$

It is also found that the effective volume related to the volume and temperature evaluated at the black hole and cosmic horizons can be written as

$$V_{\text{eff}} = T_{\text{eff}}\left(\frac{V(r_b)}{T(r_b)} + \frac{V(r_c)}{T(r_c)}\right). \tag{3.178}$$

The heat capacity of the effective system is defined as

$$C = \left(\frac{\partial M}{\partial T_{\text{eff}}}\right)_P. \tag{3.179}$$

The Gibbs free energy of the effective system is defined as

$$G = M - T_{\text{eff}}S. \tag{3.180}$$

The existence of the positive cosmological constant provides the system with negative heat capacity when the thermodynamic quantities are defined based on the Gibbs-Boltzmann statistics. This may come from the fact that the black hole is not an extensive system since its entropy is proportional to its area. By using the Rényi entropy instead of the Gibbs-Boltzmann one, one found that, it is possible to have the positive slope for temperature of the system which is directly related to the positive heat capacity. As a result, the system in our consideration is local stable in some certain range. The thermodynamics of black hole with the cosmological constant by using Rényi entropy is also investigated [54, 57].

### 3.9 Non-extensive system and Reýi entropy

Since the entropy of black hole is proportional to its area. This suggests that black hole is not an extensive system. Hence, we can not write the Bekenstein-Hawking entropy in the extensive entropic forms (i.e. $S(X,Y) = S(X) + S(Y)$). This may be a consequence from the fact that the traditional entropy is scaled by its volume while the Bekenstein–Hawking entropy is scaled by the area of black hole. Hence, the thermodynamics behaviors based on the Gibbs-Boltzmann statistics maybe not be suitable for the case of black hole. For more understand, let's consider the Bekenstein–Hawking entropy for the Schwarzschild black hole,

$$S_{BH} = \frac{4\pi r^2}{4} = \pi(2M)^2, \tag{3.181}$$

$$M = \sqrt{\frac{S_{BH}}{4\pi}}, \tag{3.182}$$

where the mass $M$ is an extensive quantity. Hence, the total mass of the system can be written as

$$M_{12} = M_1 + M_2 \tag{3.183}$$

where $M_1$ and $M_2$ are mass of subsystem. Therefore, the total entropy corresponding to this mass can be obtained by

$$M_{12}^2 = M_1^2 + 2M_1M_2 + M_2^2, \tag{3.184}$$

$$S_{BH_{12}} = S_{BH_1} + 2\sqrt{S_{BH_1}S_{BH_2}} + S_{BH_2}. \tag{3.185}$$

From the Eq.(3.185), it can be seen that the Bekenstein–Hawking entropy is the non-extensive quantity. It is worthwhile to study the entropy for black holes by using a more general form which can be applied to the non-extensive system. The Tsallis entropy is one of such non-extensive entropy, defined as

$$S_T = \sum_i p_i \ln_q\left(\frac{1}{p_i}\right) = \frac{1}{1-q}\left(\sum_i p_i^q - 1\right), \tag{3.186}$$

where $p_i$ is the probability of finding the microstate $i^{th}$ and $0 < q < \infty$ is the non-extensive parameter. Notice that the Tsallis entropy can be reduced to Gibbs-Boltzmann one (the entropy formula: $S_{GB} = -\sum_i p_i \ln p_i$) in the limit $q \to 1$,

$$S_T|_{q\to 1} = \sum_i p_i \ln\left(\frac{1}{p_i}\right) = -\sum_i p_i \ln p_i = S_{GB}. \tag{3.187}$$

However, as we seen in the Eq.(3.185), the Bekenstein–Hawking entropy also has the non-additive composition rule. In order to study non-extensive thermodynamics system, we have to relax one of the Shannon-Khinchin axioms of the entropy function called the strong additivity to a weaker non-additive composition rule. The general form of the non-additive composition rule of the entropic function can be expressed as,

$$S(X, Y) = S(X) + S(Y) + \lambda S(X)S(Y), \tag{3.188}$$

where $\lambda \in \mathbb{R}$ is non-extensive parameter. The composition rule for the Tsallis entropy from Eq.(3.186) can be written as

$$S_{T_{12}} = S_{T_1} + S_{T_2} + (1-q)S_{T_1}S_{T_2}. \tag{3.189}$$

By comparing to the Eq.(3.188), we obtain $\lambda = 1 - q$. The applications of the Tsallis entropy can be seen in [81, 82, 83]. Nevertheless, the Tsallis entropy is incompatible with the zeroth law of thermodynamics, it is not easy to define the empirical temperature for any system with non-extensive entropy (with the non-additive composition rule). To fix such a problem, one can use the formal logarithm map proposed by Biró and Ván to transform the entropy from the non-additive to the additive one [49]. We thus obtain the entropy for the non-extensive system with additive composition rule as,

$$L(S_T) = \frac{1}{1-q}\ln\left[1 + (1-q)S_T\right] = \frac{1}{1-q}\ln\left(\sum_i p_i^q\right), \tag{3.190}$$

where $L(S_T)$ is the formal logarithms of Tsallis entropy. It is found that this form of entropy looks similar to the form of the Rényi entropy proposed by Rényi in 1959

[50]. In other words, the expression in the Eq.(3.190) is indeed the Rényi entropy, $L(S_T) = S_R$. Notice that, when taking limit $q \to 1$ and using L'hopital's rule the Rényi entropy can be reduced to the Gibbs-Boltzmann one,

$$\begin{aligned}\lim_{q\to 1} \frac{1}{1-q} \ln\left(\sum_i p_i^q\right) &= -\lim_{q\to 1} \frac{d}{dq} \ln\left(\sum_i p_i^q\right),\\ &= -\lim_{q\to 1} \frac{1}{\sum_i p_i^q} \sum_i p_i^q \ln p_i,\\ &= -\sum_i p_i \ln p_i. \end{aligned} \tag{3.191}$$

By assuming that the Bekenstein–Hawking entropy of the black hole is regarded as the Tsallis statistics entropy. Therefore, it is possible to write the Rényi entropy in terms of Bekenstein–Hawking entropy as

$$S_R = \frac{1}{\lambda} \ln(1 + \lambda S_{BH}), \tag{3.192}$$

with $\lambda$ valid in the range $-\infty < \lambda < 1$. Note also that, by taking limit $\lambda \to 0$ the Rényi entropy can be reduced to the Gibbs-Boltzmann one, this limit is called Gibbs-Boltzmann limit.

### 3.10 Black string solution

In GR, the axial symmetry has two important particular cases, one is spherical symmetry which is also widely investigated. The another is cylindrical symmetry. In an astrophysical context, cylindrical symmetry has been applied to the study of cosmic strings which in the role of conical singularities. However, many investigations have led us to know that in order to form a black hole, the mass of an object must be radially compacted into a region whose circumference is in all directions. This implies that the gravitational collapse of massive stars cannot form in cylindrical symmetry. This restriction leads to the formulation of the Hoop conjecture which states that the horizon can be formed if and only if the mass of an object gets compacted into a region whose circumference is less than its Schwarzschild circumference, $4\pi M$ in every direction. However, the Hoop conjecture was formulated for a spacetime with zero cosmological constants. This suggests that the Hoop conjecture may be violated in asymptotically dS/AdS spacetime.

Indeed, in 1995 Lemos have been shown that there are charged and rotating black hole solutions in cylindrical symmetry with AdS spacetime [40, 42]. This cylindrically static black hole solutions in an anti-de Sitter spacetime are called as black strings. Subsequently, the pioneering works on the black string are also investigated [84, 85, 86]. With cylindrical symmetry, the horizons are usually circular, and then such corresponding object is commonly known as the black string. In this section, we will show how to obtain the cylindrical symmetry solutions of Einstein's field equations with a negative cosmological constant. The Einstein-Hilbert action with cosmological constant, $\Lambda$ is given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big( R - 2\Lambda \Big). \tag{3.193}$$

Assume that the spacetime is static and cylindrically symmetric, the metric can be written as in the from,

$$ds^2 = -A(r)d\bar{t}^2 + B(r)dr^2 + C(r)r^2 d\Omega^2, \qquad d\Omega^2 = d\bar{\varphi}^2 + \alpha_g^2 dz^2. \tag{3.194}$$

Therefore, we obtain the solution of Einstein's field equation with cosmological constant as,

$$ds^2 = -f d\bar{t}^2 + \frac{dr^2}{f} + r^2 d\bar{\varphi}^2 + \alpha_g^2 r^2 dz^2, \qquad f \equiv \alpha_g^2 r^2 - \frac{b}{\alpha_g r}, \tag{3.195}$$

$$-\infty < \bar{t} < \infty, \quad 0 \le r < \infty, \quad 0 \le \bar{\varphi} < 2\pi, \quad -\infty < z < \infty.$$

where $r$ is the radial coordinate, $b$ is the integral constant which is related to the mass of black string and $\alpha_g^2 \equiv -\frac{1}{3}\Lambda > 0$. One found that the constant $b = 4M$ by using Gauss's law where $M$ is the ADM mass per unit length in $z$ direction. The Eq.(3.195) is called static black string solution. One can see that this metric are diverge at $\alpha_g r = b^{1/3}$ and $\alpha_g r = 0$. Since the Kretschmann scalar is given by,

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 24\alpha_g^4 \big(1 + \frac{b^2}{2\alpha_g^6 r^6}\big), \tag{3.196}$$

the real singularity is located at $\alpha_g r = 0$. It is important to note that the study of the thermodynamics of the black string is mostly performed in the asymptotically AdS spacetime since in dS spacetime the black string has no horizon, then it does not correspond to the thermodynamic system. However, one found that it is possible to exist two horizons in the case of dRGT black string with asymptotically dS spacetime and their thermodynamics also investigated as we will show in the Chapter 5.

# CHAPTER IV

# MASSIVE GRAVITY THEORY

Massive gravity is a theory that modifies GR at the large scale by adding suitable interaction terms interpreted as a graviton mass into the Einstein-Hilbert action. The field theory for massive graviton was firstly proposed by Fierz and Pauli in 1993 [1] by adding the interaction terms at the linearized level of GR. However, it was later found that there was a discontinuity when taking the limit of massless compared with GR, pointed out by van Dam, Veltman, and Zakharov in 1970 [2][3]. This discontinuity invoked further studies on the non-linear generalization of Fierz-Pauli massive gravity. However, in 1972 Boulware and Deser found that additional mass terms usually generate the ghost instability for gravitational theorie [4]. Eventually, the theory of massive gravity without ghost instability was proposed by de Rham, Gabadadze and Tolley (dRGT) in 2010 [5] [6]. We will dedicate this section to explain the construction of this theory.

## 4.1 Massless theory

Massive gravity is the theory that corresponds to a massive spin-2 field, described by a symmetric tensor field $h_{\mu\nu}$. In order to construct the theory with massive graviton, we should start with the massless one. The Lagrangian density of the massless spin-2 field should contain only the kinetic terms. Therefore, the possible contribution of kinetic terms in Lagrangian density should be written as,

$$\mathcal{L}_{kin}^{spin-2} = \partial^\rho h^{\mu\nu}[a_1\partial_\rho h_{\mu\nu} + a_2\partial_{(\mu}h_{\nu)\rho} + a_3\eta_{\mu\nu}\partial_\rho h + a_4\eta_{\rho(\mu}\partial_{\nu)}h], \tag{4.1}$$

where the curve brackets in the subscribe indices denote totally symmetric over the indices and $h$ is the trace of $h_{\mu\nu}$, $h = h^\mu_\nu$. The constant $a_1, a_2$, $a_3$ and $a_4$ are dimensionless coefficients. The constant will be chosen later to avoid the existence of the ghost instability, which is represents a degree of freedom with the wrong sign of kinetic term. To determine these constants, we split the symmetric tensor field $h_{\mu\nu}$ into a transverse tensor field part $h^{(T)}_{\mu\nu}$ and a longitudinal vector field part $h^{(L)}_{\mu}$ as

$$h_{\mu\nu} = h^{(T)}_{\mu\nu} + 2\partial_{(\mu}h^{(L)}_{\nu)}, \quad \text{and} \quad h = h^{(T)} + 2\partial^{\rho}h^{(L)}_{\rho}, \tag{4.2}$$

where transverse part satisfies $\partial^{\mu}h^{(T)}_{\mu\nu} = 0$. After substituting it back to the kinetic term the Eq.(4.1), we thus see the field equation containing the ghost instability appear from higher derivative part. It can be written in the form as,

$$\begin{aligned}\mathcal{L}^{higher-der}_{kin} &= (2a_1 + a_2)h^{(L)\nu}\partial^2\partial^2 h^{(L)}_{\nu} + (2a_1 + 3a_2 + 4a_3 + 4a_4)h^{(L)\nu}\partial^2\partial_{\nu}\partial^{\rho}h^{(L)}_{\rho} \\ &\quad - 2(a_2 + a_4)h^{(T)}_{\mu\nu}\partial^{\mu}\partial^{\nu}\partial^{\rho}h^{(L)}_{\rho} - 2(2a_3 + a_4)h^{(T)}\partial^2\partial^{\rho}h^{(L)}_{\rho},\end{aligned} \tag{4.3}$$

where $\partial^2 = \partial_{\nu}\partial^{\nu} = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}$. In order to eliminate higher derivative terms, all terms in Eq.(4.3) must have vanished. As a result $a_1, a_2$, $a_3$ and $a_4$ must satisfy

$$2a_1 = -a_2 = -2a_3 = a_4. \tag{4.4}$$

Then, the parameter $a_1$ can be set as $a_1 = -1/8$ to follow standard conventions. Eventually, the Lagrangian density for a massless spin-2 field is written as

$$\begin{aligned}\mathcal{L}^{spin-2}_{kin} &= -\frac{1}{8}\partial^{\rho}h^{(T)\mu\nu}\left[\partial_{\rho}h^{(T)}_{\mu\nu} - 2\partial_{(\mu}h^{(T)}_{\nu)\rho} - \eta_{\mu\nu}\partial_{\rho}h^{(T)} + 2\eta_{\rho(\mu}\partial_{\nu)}h^{(T)}\right], \\ &= \frac{1}{8}h^{(T)\mu\nu}\left[\partial^2 h^{(T)}_{\mu\nu} - 2\partial^{\rho}\partial_{(\mu}h^{(T)}_{\nu)\rho} - \eta_{\mu\nu}\partial^2 h^{(T)} + 2\partial_{(\mu}\partial_{\nu)}h^{(T)}\right], \\ &= -\frac{1}{4}h^{(T)\mu\nu}\left(-\frac{1}{2}\left[\partial^2 h^{(T)}_{\mu\nu} - 2\partial^{\rho}\partial_{(\mu}h^{(T)}_{\nu)\rho} + \partial_{\mu}\partial_{\nu}h^{(T)} - \eta_{\mu\nu}\partial^2 h^{(T)}\right]\right) \qquad (4.5) \\ &\quad - \frac{1}{4}h^{(T)\mu\nu}[-\frac{1}{2}\partial_{\nu}\partial_{\mu}h^{(T)}], \\ &= -\frac{1}{4}h^{(T)\mu\nu}\Big( -\frac{1}{2}\left[\partial^2 h^{(T)}_{\mu\nu} - 2\partial^{\rho}\partial_{(\mu}h^{(T)}_{\nu)\rho} + \partial_{\mu}\partial_{\nu}h^{(T)} - \eta_{\mu\nu}\partial^2 h^{(T)}\right] \qquad (4.6) \\ &\quad - \frac{1}{2}\left[\eta_{\mu\nu}\partial_{\rho}\partial_{\sigma}h^{(T)\rho\sigma}\right]\Big), \\ &= -\frac{1}{4}h^{(T)\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h^{(T)}_{\rho\sigma}, \qquad (4.7)\end{aligned}$$

where $\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h^{(T)}_{\rho\sigma} = -\frac{1}{2}\left[\partial^2 h_{\mu\nu} - 2\partial^{\rho}\partial_{(\mu}h_{\nu)\rho} + \partial_{\mu}\partial_{\nu}h + \eta_{\mu\nu}(\partial_{\rho}\partial_{\sigma}h^{\rho\sigma} - \partial^2 h)\right]$ and $\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}$ is the Licherowichz operator. Not only the transverse part $h^{(T)}_{\mu\nu}$ but the whole field of

$h_{\mu\nu}$ also satisfy,

$$\mathcal{L}_{kin}^{spin-2} = -\frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}_{\mu\nu}^{\rho\sigma} h_{\rho\sigma}. \tag{4.8}$$

We also found that the kinetic term is invariant under the linear gauge transformation,

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} \tag{4.9}$$

where $\xi_\mu$ is an arbitrary vector field. One can find the equation of motion by varying the action with respect to $h^{\mu\nu}$,

$$\begin{aligned}
\delta_h S &= \int d^4x \mathcal{L}_{kin}^{spin-2} \qquad\qquad (4.10)\\
&= -\frac{1}{8}\int d^4x \begin{bmatrix} \delta h^{\mu\nu}[\partial^2 h_{\mu\nu} - \partial^\rho\partial_\mu h_{\nu\rho} - \partial^\rho\partial_\nu h_{\mu\rho} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}(\partial_\rho\partial_\sigma h^{\rho\sigma} - \partial^2 h)] \\ +h_{\mu\nu}\partial^2\delta h^{\mu\nu} - h^\mu_\nu\partial_\rho\partial_\mu\delta h^{\nu\rho} - h^\nu_\mu\partial_\rho\partial_\nu\delta h^{\mu\rho} + h^{\mu\nu}\eta_{\rho\sigma}\partial_\mu\partial_\nu\delta h^{\rho\sigma} \\ +h\partial_\rho\partial_\sigma\delta h^{\rho\sigma} - h\eta_{\mu\nu}\partial^2 h\delta h^{\mu\nu} \end{bmatrix}, \\
&= -\frac{1}{8}\int d^4x \begin{bmatrix} \partial^2 h_{\mu\nu} - \partial^\rho\partial_\mu h_{\nu\rho} - \partial^\rho\partial_\nu h_{\mu\rho} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}(\partial_\rho\partial_\sigma h^{\rho\sigma} - \partial^2 h) \\ +\partial^2 h_{\mu\nu} - \partial_\nu\partial_\rho h^\rho_\mu - \partial_\mu\partial_\rho h^\rho_\mu + \eta_{\mu\nu}\partial^\rho\partial_\sigma h^{\rho\sigma} + \partial^\mu\partial_\nu h - \eta_{\mu\nu}\partial^2 h \end{bmatrix} \delta h^{\mu\nu}, \\
&= -\frac{1}{4}\int d^4x \left[\partial^2 h_{\mu\nu} - \partial^\rho\partial_\mu h_{\nu\rho} - \partial^\rho\partial_\nu h_{\mu\rho} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}(\partial_\rho\partial_\sigma h^{\rho\sigma} - \partial^2 h)\right] \delta h^{\mu\nu}.
\end{aligned} \tag{4.11}$$

As a result, the equation of motion can be expressed as

$$\partial^2 h_{\mu\nu} - \partial^\rho\partial_\mu h_{\nu\rho} - \partial^\rho\partial_\nu h_{\mu\rho} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}(\partial_\rho\partial_\sigma h^{\rho\sigma} - \partial^2 h) = 0. \tag{4.12}$$

However, the gauge symmetry in Eq.(4.9) is no longer exist in the massive theory. In addition, the massless spin-2 is compatible with the linearization of GR. One can perturb the metric as,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll \eta_{\mu\nu} \tag{4.13}$$

where is $h_{\mu\nu}$ a symmetric perturbed metric about Minkowski metric, $\eta_{\mu\nu}$. By keeping only the first order in $h_{\mu\nu}$, the Christoffel symbol, Ricci tensor and, Ricci scalar can be written as follows,

$$\Gamma^{(1)\rho}_{\mu\nu} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\rho h_{\mu\nu}) + \frac{1}{2}h^{\mu\nu}(\partial_\mu \eta_{\nu\sigma} + \partial_\nu \eta_{\mu\sigma} - \partial_\rho \eta_{\mu\nu}) \tag{4.14}$$

$$= \frac{1}{2}(\partial_\mu h^\rho_\nu + \partial_\nu h^\rho_\mu - \partial^\rho h_{\mu\nu}) \tag{4.15}$$

$$R^{(1)}_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma^\rho_{\rho\nu}, \tag{4.16}$$

$$= \frac{1}{2}\eta^{\rho\sigma}(\partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\rho\partial_\nu h_{\mu\sigma} - \partial_\rho\partial_\sigma h_{\mu\nu}) - \frac{1}{2}\eta_{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma}, \tag{4.17}$$

$$= \frac{1}{2}(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h), \tag{4.18}$$

$$R^{(1)} = \eta^{\mu\nu}R^{(1)}_{\mu\nu} = \partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h. \tag{4.19}$$

The linearized Einstein tensor can be written as,

$$G^{(1)} = R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)}, \tag{4.20}$$

$$= \frac{1}{2}(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h) - \frac{1}{2}\eta_{\mu\nu}(\partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h). \tag{4.21}$$

Hence, the field equations for the linearized GR in vacuum can be written as

$$\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}(\partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h) = 0. \tag{4.22}$$

This is the same form as one in the Eq.(4.12), thus GR is a theory of a massless spin-2 field. Moreover, the $h_{\mu\nu}$ is able to be decomposed as a scalar mode $h_{00}$, vector mode $h_{0i}$ and tensor mode $h_{ij}$. After substituting this decomposition to the linearized Einstein's field equation in Eq.(4.22), it is found that only the tensor modes are the propagating degrees of freedom. However, by using the linear gauge transformation in Eq.(4.9), one found that the tensor mode can be fixed as $h^i_i = 0$ and $\partial^i h_{ij} = 0$. These eliminate one and three degrees of freedom respectively. Therefore, in the massless theory, there are two propagating degrees of freedom.

### 4.2 Linear massive theory

In this section, we will move our consideration to a massive spin-2 field. The massive theory will consist of the kinetic term and the additional mass term. The simplest contributions for the mass terms are constructed from quadratic order of the field $h_{\mu\nu}$ such as $h_{\mu\nu}h^{\mu\nu}$ and $h^2$. As a result, the general form of Lagrangian of the mass terms can be written as

$$\mathcal{L}_{mass}^{spin-2} = m_g^2(b_1 h_{\mu\nu}h^{\mu\nu} + b_2 h^2), \tag{4.23}$$

where $b_1$ and $b_2$ are dimensionless parameter. $m_g$ is a constant interpreted as the graviton mass. As the same step with the case of massless, we will obtain the Lagrangian density for massive spin-2 field as

$$\mathcal{L}_{mass}^{spin-2} = m_g^2 \left[ \begin{array}{c} b_1 \left( \begin{array}{c} h_{\mu\nu}^{(T)}h^{(T)\mu\nu} + 2h_{\mu\nu}^{(T)}\partial^\mu h^{(L)\nu} + 2h_{\mu\nu}^{(T)}\partial^\nu h^{(L)\mu} \\ +2\partial^\mu h^{(L)\nu}\partial_\mu h_\nu^{(L)} + 2\partial^\mu h^{(L)\nu}\partial_\nu h_\mu^{(L)} \end{array} \right) \\ +b_2((h^{(T)2} + 4h^{(T)}\partial_\rho h^{(L)\rho} + 4\partial_\rho h^{(L)\rho}\partial_\sigma h^{(L)\sigma}) \end{array} \right]. \tag{4.24}$$

From the Eq. (4.24), these mass terms do not contain any higher derivative terms for a tensor field, $h_{\mu\nu}^{(T)}$ and a vector field, $h_\mu^{(L)}$. However, there are more degrees of freedom hiding in the vector field $h_\mu^{(L)}$. To see these modes, we thus choose to decompose longitudinal part as

$$h_\mu^{(L)} = l_\mu^{\perp} + \partial_\mu l^{\parallel}, \tag{4.25}$$

where $l_\mu^{\perp}$ and $l^{\parallel}$ are vector mode and scalar mode respectively. The vector mode satisfies $\partial^\mu l_\mu^{\perp} = 0$. Then, applying this decomposition to the mass term in Eq.(4.24). As a result, we obtain the higher order derivative for the longitudinal part as

$$\begin{aligned} \mathcal{L}_{mass}^{(L)highter-der} &= m_g^2[b_1(2\partial^\mu h^{(L)\nu}\partial_\mu h_\nu^{(L)} + 2\partial^\mu h^{(L)\nu}\partial_\nu h_\mu^{(L)}) + b_2(4\partial_\rho h^{(L)\rho}\partial_\sigma h^{(L)\sigma})], \\ &= m_g^2(b_1 + b_2)l^{\parallel}\partial^2\partial^2 l^{\parallel}. \end{aligned} \tag{4.26}$$

To eliminate higher order derivative term, the condition must be satisfy

$$b_1 = -b_2. \tag{4.27}$$

We also set $b_1 = -1/8$ to obtain the standard convention. Finally, this mass term can be written as

$$\mathcal{L}^{spin-2}_{mass} = -\frac{1}{8}m_g^2(h_{\mu\nu}^2 - h^2). \tag{4.28}$$

The mass terms were proposed by Fierz and Pauli (FP) in 1939 called Fierz-Pauli (FP) mass term. These mass terms break diffeomorphism invariant (gauge symmetry),

$$\begin{aligned}
\mathcal{L}'^{spin-2}_{mass} &= -\frac{1}{8}m_g^2(h'^2_{\mu\nu} - h'^2), \\
&= -\frac{1}{8}m_g^2\left((h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)})(h^{\mu\nu} + 2\partial^{(\mu}\xi_{\nu)}) - (h + 2\partial_\rho\xi^\rho(h + 2\partial_\sigma\xi^\sigma)\right), \\
&= \mathcal{L}^{spin-2}_{mass} - \frac{1}{4}m_g^2\left(2h_{\mu\nu}\partial^\mu\xi^\nu + \partial_\mu\xi_\nu\partial^\mu\xi^\nu + \partial_\mu\xi_\nu\partial^\nu\xi^\mu - 2h\partial_\rho\xi^\rho - 2\partial_\rho\xi^\rho\partial_\sigma\xi^\sigma\right), \\
&\neq \mathcal{L}^{spin-2}_{mass}.
\end{aligned} \tag{4.29}$$

However, there is a procedure to restore the gauge symmetry to the theory which is called the Stückelberg trick as we will discuss in the next section. Therefore, one can write the Lagrangian density for FP massive theory as

$$\mathcal{L}_{FP} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}m_g^2(h_{\mu\nu}^2 - h^2). \tag{4.30}$$

We can count the number of propagating degrees of freedom for the FP massive theory by examining the constraint of the equation of motion of this massive theory which is obtained by varying the FP action with respect to $h^{\mu\nu}$. As a result, we obtain

$$\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h + \eta_{\mu\nu}(\partial^2 h - \partial_\rho\partial_\sigma h^{\rho\sigma}) + m_g^2(h_{\mu\nu} - \eta_{\mu\nu}h) = 0. \tag{4.31}$$

Operating with $\partial^\mu$, one obtains

$$\begin{aligned}
\begin{bmatrix} \partial_\rho\partial^2 h^\rho_\nu + \partial^\mu\partial_\rho\partial_\nu h^\rho_\mu - \partial^\mu\partial^2 h_{\mu\nu} - \partial^2\partial_\nu h \\ +\partial_\nu\partial^2 h - \partial_\nu\partial_\rho\partial_\sigma h^{\rho\sigma} + m_g^2(\partial^\mu h_{\mu\nu} - \partial_\nu h) \end{bmatrix} &= 0, \\
m_g^2(\partial^\mu h_{\mu\nu} - \partial_\nu h) &= 0.
\end{aligned} \tag{4.32}$$

As a result, we have

$$\partial^\mu h_{\mu\nu} = \partial_\nu h. \tag{4.33}$$

Substituting the Eq.(4.33) into the field equation Eq.(4.31), we obtain

$$\partial_\mu \partial_\nu h - \partial^2 h_{\mu\nu} + m_g^2 (h_{\mu\nu} - \eta_{\mu\nu} h) = 0. \tag{4.34}$$

Taking trace of above equation, we then obtain $h = 0$. From the (4.33), we also get $\partial^\mu h_{\mu\nu} = 0$. Substituting constraints $h = 0$ and $\partial^\mu h_{\mu\nu} = 0$ to (4.34), one obtains,

$$(\partial^2 - m_g^2) h_{\mu\nu} = 0. \tag{4.35}$$

One can see that $h_{\mu\nu}$ in four dimensions contains ten independent components with five constraints, four from $\partial^\mu h_{\mu\nu} = 0$ and one from $h = 0$. Therefore, in four dimensions, the FP massive theory contain propagating five degrees of freedom.

### 4.3 Stückelberg trick

One of the differences between massless and massive theories is the existence of linear gauge symmetry. In order to construct the massive theory which is invariant under the gauge transformation, one can use the Stückelberg trick to restore the gauge symmetry to the massive theories [87]. To make the mass term invariant, we start with full Lagrangian density for the FP theory with the matter,

$$\mathcal{L} = -\frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu} h_{\rho\sigma} - \frac{1}{8} m_g^2 (h_{\mu\nu}^2 - h^2) + \frac{1}{2M_{Pl}} h_{\mu\nu} T^{\mu\nu}, \tag{4.36}$$

where a coefficient $1/2M_{Pl}$ is the constant corresponding to Newton's gravitational force and the source term corresponds to the coupling between the spin-2 field $h_{\mu\nu}$ and the matter field $T^{\mu\nu}$. For the Stückelberg trick, one can introduce Stückelberg field $\chi_\mu$ which transforms to preserve under gauge invariant Replacing $h_{\mu\nu}$ by $h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)}$, then substituting these quantities into Eq.(4.36), the Lagrangian density becomes

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}m_g^2\left[(h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)})^2 - (h + 2\partial_\rho\chi^\rho)^2\right] \\
&+ \frac{1}{2}M_{Pl}(h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)})T^{\mu\nu}, \\
= &-\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}m_g^2\Big[(h_{\mu\nu} - h^2) + 2(\partial_\mu\chi_\nu\partial^\mu\chi^\nu - (\partial_\rho\chi^\rho)^2) \\
&+ 4(h_{\mu\nu}\partial^\mu\chi^\nu - h\partial_\rho\chi^\rho)\Big] + \frac{1}{2}M_{Pl}(h_{\mu\nu}T^{\mu\nu} - 2\chi_\mu\partial_\nu T^{\mu\nu}), \\
= &-\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}m_g^2\left[(h_{\mu\nu} - h^2) + \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + 4(h_{\mu\nu}\partial^\mu\chi^\nu - h\partial_\rho\chi^\rho)\right] \\
&+ \frac{1}{2}M_{Pl}(h_{\mu\nu}T^{\mu\nu} - 2\chi_\mu\partial_\nu T^{\mu\nu}),
\end{aligned}
\tag{4.37}
$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu\chi_\nu - \partial_\nu\chi_\mu$ is taken in the same form as Maxwell stress tensor. Note that the matter is general matter, it is not necessary to be conserved. In order to obtain a canonical form of Maxwell's kinetic term, the Stückelberg field would be rescaled as $\chi_\mu \Rightarrow \chi_\mu/m_g$. Therefore, a result is expressed as

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}m_g^2(h_{\mu\nu}^2 - h^2) - \frac{1}{8}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m_g(h_{\mu\nu}\partial^\mu\chi^\nu - h\partial_\rho\chi^\rho) \\
&+ \frac{1}{2M_{Pl}}\Big(h_{\mu\nu}T^{\mu\nu} - \frac{2}{m_g}\chi_\mu\partial_\nu T^{\mu\nu}\Big).
\end{aligned}
\tag{4.38}
$$

It is found that the above Lagrangian density is invariant under the gauge transformations $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$ and $\chi_\mu \to \chi_\mu - \xi_\nu$.

In the massless limit of the FP massive theory the total number of the propagating degrees of freedom is four, two for tensor $h_{\mu\nu}$ and two for vector $\chi_\mu$ meaning that there is a jump in the number of propagating degrees of freedom. To fix it, we have to introduce a Stückelberg scalar field $\pi$ which transforms as $\pi \to \pi - \theta$ where $\theta$ is an arbitrary scalar field. By replacing $\chi_\mu$ with $\chi_\mu + \partial_\mu\pi$, the Lagrangian density becomes,

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}m_g^2(h_{\mu\nu}^2 - h^2) - \frac{1}{8}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m_g(h_{\mu\nu}\partial^\mu\chi^\nu - h\partial_\rho\chi^\rho) \\
&- \frac{1}{2}(h_{\mu\nu}\partial^\mu\partial^\nu\pi - h\partial^2\pi) + \frac{1}{2M_{Pl}}(h_{\mu\nu}T^{\mu\nu} - 2\chi_\mu\partial_\nu T^{\mu\nu} + 2\pi\partial_\mu\partial_\nu T^{\mu\nu}).
\end{aligned}
\tag{4.39}
$$

Note that, we also rescale as $\pi \Rightarrow \pi/m_g$ to obtain the above Lagrangian density. However, we also found that when taking limit $m_g \to 0$ the Stückelberg field will be coupled to the divergence of matter. The problem has arisen if we consider the non conserved matter. It is useful for moving our attention to the conserved matter $\partial_\mu T^{\mu\nu} = 0$. Therefore, the Lagrangian density can be written as,

$$\mathcal{L} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}m_g^2(h^2_{\mu\nu} - h^2) - \frac{1}{8}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m_g(h_{\mu\nu}\partial^\mu\chi^\nu - h\partial_\rho\chi^\rho) \\ - \frac{1}{2}(h_{\mu\nu}\partial^\mu\partial^\nu\pi - h\partial^2\pi) + \frac{1}{2M_{Pl}}h_{\mu\nu}T^{\mu\nu}. \tag{4.40}$$

In order to discuss about the number of propagating degree of freedom, we will consider the Lagrangian density in the other suitable frame with,

$$h_{\mu\nu} = h'_{\mu\nu} + \pi\eta_{\mu\nu}, \qquad \chi_\mu = \chi'_\mu, \qquad \pi = \pi'. \tag{4.41}$$

Therefore, the Lagrangian density for the Eq.(4.40) at the massless limit can be written as

$$\begin{aligned}\mathcal{L}_{m_g\to 0} &= -\frac{1}{4}(h'^{\mu\nu} + \pi'\eta^{\mu\nu})\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}(h'_{\rho\sigma} + \pi'\eta_{\rho\sigma}) - \frac{1}{8}\mathcal{F}'_{\mu\nu}\mathcal{F}'^{\mu\nu} \\ &\quad - \frac{1}{2}[(h'_{\mu\nu} + \pi'\eta_{\mu\nu})\partial^\mu\partial^\nu\pi' - (h' + 4\pi')\partial^2\pi'] + \frac{1}{2M_{Pl}}(h'_{\mu\nu} + \pi'\eta_{\mu\nu})T^{\mu\nu}, \\ &= -\frac{1}{4}h'^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h'_{\rho\sigma} - \frac{1}{8}\mathcal{F}'_{\mu\nu}\mathcal{F}'^{\mu\nu} + \frac{3}{4}\pi'\partial^2\pi' + \frac{1}{2M_{Pl}}h'_{\mu\nu}T^{\mu\nu} + \frac{1}{2M_{Pl}}\pi'T.\end{aligned} \tag{4.42}$$

It is found that this Lagrangian density explicitly propagates two for tensor, two for vector and one for scalar degrees of freedom. The total number of degrees of freedom is five. Nevertheless, one can see that the scalar mode is coupled with the matter in the last term in the Eq.(4.42). The problem emerges because this feature does not exists in massless theory. This indicates that the FP massive theory may not be a good enough theory.

## 4.4 van Dam-Veltman-Zakharov discontinuity

In the previous section, we have already constructed the gauge invariant for FP massive theory and eliminated the problem of the unsmoothness of the degree of freedom at the massless limit. However, there are many aspects about discontinuity at the massless limit. As we know that the massless theory (linearized GR) is nice with observable predictions in the solar system scale, thus massive theory at the massless limit could not contradict this fact. But, the predictions of FP massive theory do not uniformly reduce to linearized GR in the limit $m_g \to 0$, such as the bending of light for massive theory at the massless limit is 3/4 times of the result in massless one. In other words, the gravity in the FP massive theory is weaker than the massless one by 3/4 where taking $m_g \to 0$. The failure of the FP linear massive theory at the massless limit was pointed out by van Dam, Veltman, and Zakharov in 1970 called van Dam-Veltman-Zakharov (vDVZ) discontinuity. However, in 1972, Vainshtein found a mechanism in which the non-linear terms have to be added to solve the vDVZ discontinuity and later known as the Vainshtein mechanism [88, 89]. The non-linear terms can suppress the effect of scalar mode in massless limits at the shot distance.

## 4.5 Non-linear massive theory

As we have know that the kinetic term of the linear massive theory is the linearization of GR, so that it should be promoted to non-linear contribution as

$$\mathcal{L}_{kin}^{spin-2,linear} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} \to \mathcal{L}_{kin,}^{non-linear} = \frac{M_{Pl}^2}{2}R[g]. \tag{4.43}$$

The non-linear version is invariant under the general coordinate transformation, $x^{\mu\nu} \to y^{\mu\nu}$,

$$g_{\mu\nu}(x) \to \frac{\partial y^{\rho}}{\partial x^{\mu}}\frac{\partial y^{\rho}}{\partial x^{\nu}}g_{\rho\sigma}(y(x)). \tag{4.44}$$

In order to construct the mass term for this non-linear massive theory, we have to introduce the other metric additional to the physical metric $g_{\mu\nu}$. Because when dealing with the matrix $g_{\mu\nu}$ any contributions are constants, there are not different from the cosmological constant. Therefore, the massive theory requires the new reference metric in which different form $g_{\mu\nu}$. However, one loses the interpretation of the massive spin-2 field, but still obtains a consistent theory. In our consideration, we choose to introduce the non dynamical reference metric, $f_{\mu\nu}$ (or often known as the fiducial metric),

$$f_{\mu\nu} = g_{\mu\nu} - \frac{H_{\mu\nu}}{M_{Pl}}, \tag{4.45}$$

where $H_{\mu\nu}$ plays the role the perturbed field similar to one in linear case. However, the fiducial metric $f_{\mu\nu}$ does not transform as a tensor under a general coordinate transformation. As a result, the mass terms breaks the gauge symmetry. One can restore the symmetry for $f_{\mu\nu}$ by introduce the Stückelberg scalar field $\psi^{\bar{\mu}}$ which transforms as $\psi^{\bar{\mu}}(x) \to \psi^{\bar{\mu}}(y(x))$, and then promoting the fiducial metric, $f_{\mu\nu}$ to a tensor $\hat{f}_{\mu\nu}$ as,

$$f_{\mu\nu} \Rightarrow \hat{f}_{\mu\nu} = \partial_\mu \psi^{\bar{\rho}} \partial_\nu \psi^{\bar{\sigma}} f_{\bar{\rho}\bar{\sigma}}, \tag{4.46}$$

where the bar index runs over four dimensional spacetime but does not depend on the unbar index. By setting $\psi^{\bar{\mu}} = x^{\bar{\mu}}$, one simply obtains $f_{\mu\nu} = \hat{f}_{\mu\nu}$. It is the unitary gauge. As we can see in the Eq.(4.46) $\hat{f}_{\mu\nu}$ transforms as a tensor under general coordinate transformation and each of $\psi^{\bar{\mu}}$ transform as the scalar field. Then, the extension of non-linear FP mass term is promoted to be

$$\begin{aligned} \mathcal{L}^{non-linear}_{FP,mass} &= -\frac{1}{2} m_g^2 (H_{\mu\nu} H^{\mu\nu} - H^2), \\ &= -\frac{1}{2} m_g^2 M_{Pl}^2 ([(\mathbb{I} - \mathbb{X})^2] - [\mathbb{I} - \mathbb{X}]^2), \end{aligned} \tag{4.47}$$

where $\mathbb{I}$ is the identity matrix and $\mathbb{X}^\mu_\nu$ is a new tensor quantity, defined by $\mathbb{X}^\mu_\nu = g^{\mu\rho} f_{\rho\nu}$. Note that, we can written $H_{\mu\nu}$ in terms of $\mathbb{X}$ as

$$H^\mu_\nu = M_{Pl}(\delta^\mu_\nu - g^{\mu\rho} f_{\rho\nu}) = M_{Pl}(\delta^\mu_\nu - \mathbb{X}^\mu_\nu). \tag{4.48}$$

We have already construct the non-linear mass terms which is invariant under general coordinate transformation. Therefore, the action for the non-linear FP massive gravity theory can be written as

$$S_{FP}^{non-linear} = \int d^4x\sqrt{-g}\frac{M_{Pl}^2}{2}\left[R + m_g^2\left([\mathbb{I}-\mathbb{X}]^2 - [(\mathbb{I}-\mathbb{X})^2]\right)\right] \tag{4.49}$$

However, this non-linear theory still has a problem due to the higher derivative terms which generate the ghost known as the Boulware-Deser (BD) ghost.

### 4.6 Boulware-Deser (BD) ghost

To see the appearance of ghost terms at the non-linear level, we will use the Stückelberg trick by expands the Stückelberg scalar as,

$$\psi^{\bar{\mu}} = x^{\bar{\mu}} - \frac{1}{M_{Pl}}\varphi^{\bar{\mu}}. \tag{4.50}$$

where $x^{\bar{\mu}}$ and $\varphi^{\bar{\mu}}$ are the coordinates and infinitesimal scalar field respectively. Therefore, the reference metric can be written as,

$$f_{\mu\nu} = \partial_\mu(x^{\bar{\rho}} - \frac{1}{M_{Pl}}\varphi^{\bar{\rho}})\partial_\nu(x^{\bar{\sigma}} - \frac{1}{M_{Pl}}\varphi^{\bar{\sigma}})f_{\bar{\rho}\bar{\sigma}}, \tag{4.51}$$

By considering on the flat reference metric $f_{\bar{\rho}\bar{\sigma}} = \eta_{\bar{\rho}\bar{\sigma}}$, we obtain,

$$\begin{aligned} f_{\mu\nu} &= \partial_\mu(x^{\bar{\rho}} - \frac{1}{M_{Pl}}\varphi^{\bar{\rho}})\partial_\nu(x^{\bar{\sigma}} - \frac{1}{M_{Pl}}\varphi^{\bar{\sigma}})\eta_{\bar{\rho}\bar{\sigma}}, \\ &= \eta_{\bar{\rho}\bar{\sigma}}(\partial_\mu x^{\bar{\rho}}\partial_\nu x^{\bar{\sigma}} - \frac{\partial_\mu x^{\bar{\rho}}\partial_\nu\varphi^{\bar{\sigma}}}{M_{Pl}} - \frac{\partial_\nu x^{\bar{\sigma}}\partial_\mu\varphi^{\bar{\rho}}}{M_{Pl}} + \frac{1}{M_{Pl}^2}\partial_\mu\varphi^{\bar{\rho}}\partial_\nu\varphi^{\bar{\sigma}}), \\ &= \eta_{\mu\nu} - \frac{2}{M_{Pl}}\partial_{(\mu}\varphi_{\nu)} + \frac{1}{M_{Pl}^2}\partial_\mu\varphi^{\bar{\rho}}\partial_\nu\varphi^{\bar{\sigma}}\eta_{\bar{\rho}\bar{\sigma}}. \end{aligned} \tag{4.52}$$

Therefore, the fluctuations about flat spacetime, $h_{\mu\nu}$ can be written as

$$h_{\mu\nu} = g_{\mu\nu} - f_{\mu\nu} - \frac{2}{M_{Pl}}\partial_{(\mu}\varphi_{\nu)} + \frac{1}{M_{Pl}^2}\partial_\mu\varphi^{\bar{\rho}}\partial_\nu\varphi^{\bar{\sigma}}\eta_{\bar{\rho}\bar{\sigma}}. \tag{4.53}$$

Hence, in this case, we can write the tensor $H_{\mu\nu}$ in terms of $h_{\mu\nu}$ as

$$\begin{aligned} H_{\mu\nu} &= M_{Pl}(g_{\mu\nu} - f_{\mu\nu}), \\ &= h_{\mu\nu} + 2\partial_{(\mu}\varphi_{\nu)} - \frac{1}{M_{Pl}}\eta_{\rho\sigma}\partial_\mu\varphi^\rho\partial_\nu\varphi_\rho. \end{aligned} \tag{4.54}$$

Then, splitting the field $\varphi^\mu$ into the transverse mode $\chi_\mu$ and longitudinal mode $\pi$, we obtain

$$\varphi^\mu = \frac{1}{m_g}\chi_\mu + \frac{1}{m_g^2}\partial_\mu\pi. \tag{4.55}$$

Substituting the above equation into the Eq.(4.54), we therefore obtain

$$\begin{aligned} H_{\mu\nu} = h_{\mu\nu} &+ \frac{2}{m_g}\partial_{(\mu}\chi_{\nu)} + \frac{2}{m_g^2}\partial_\mu\partial_\nu\pi - \frac{1}{M_{Pl}m_g^2}\partial_\mu\chi^\rho\partial_\nu\chi_\rho - \frac{1}{M_{Pl}m_g^3}\partial_\mu\partial^\rho\pi\partial_\nu\chi_\rho \\ &- \frac{1}{M_{Pl}m_g^3}\partial_\mu\chi^\rho\partial_\nu\partial_\rho\pi - \frac{1}{M_{Pl}m_g^4}\partial_\mu\partial^\rho\pi\partial_\nu\partial_\rho\pi. \end{aligned} \tag{4.56}$$

To see the higher derivative terms, we will ignore tensor and vector mode and only focus on the scalar mode $\pi$ . Therefore, the tensor $\mathbb{X}$ can be expressed as,

$$\mathbb{X}^\mu_{\nu,\pi} = \delta^\mu_\nu - \frac{2}{M_{Pl}m_g^2}\partial^\mu\partial_\nu\pi + \frac{1}{M_{Pl}^2m_g^4}\partial^\mu\partial_\rho\pi\partial^\rho\partial_\nu\pi. \tag{4.57}$$

Substituting the above equation to the mass term in Eq.(4.47) and then the FP mass term will reads,

$$\mathcal{L}^{non-linear}_{FP,mass,\pi} = -\frac{2}{m_g^2}([\Pi^2] - [\Pi]^2) + \frac{2}{M_{Pl}m_g^4}([\Pi^3] - [\Pi][\Pi^2]) + \frac{2}{M_{Pl}^2m_g^6}([\Pi^4] - [\Pi^2]^2), \tag{4.58}$$

where $\Pi^\mu_\nu \equiv \partial^\mu\partial_\nu\pi$. After integration by parts, the first term $[\Pi^2] - [\Pi]^2$ is just a boundary term. The second and third provide higher derivative terms in the equation of motion of $\pi$ which contain a ghost stability called BD ghost. Not only the massive theory with the mass term in Eq.(4.47) contain the BD ghost, but the theory with various mass terms is also proven that there exists this ghost degree of freedom. This non-linear massive theory is unpopular to study. Until 2010, de Rham, Gabadadze and Tolley succeed to construct the appropriate form of the non-linear mass terms which eliminates BD ghost.

### 4.7 dRGT massive gravity theory

The general form of the non-linear massive gravity theory without ghost instability is the de Rham-Gabadadze-Tolley (dRGT) massive gravity theory [5, 6]. The action for dRGT Massive gravity is given by

$$S_{dRGT} = \int d^4x\sqrt{-g}\frac{M_{Pl}^2}{2}\left[R + m_g^2\mathcal{U}(g, f)\right], \tag{4.59}$$

where $\mathcal{U}$ is a potential term characterizing the behavior of the mass term. The suitable form of potential, $\mathcal{U}$ is given by

$$\mathcal{U} = \mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4, \tag{4.60}$$

where $\alpha_3$ and $\alpha_4$ are free parameters of the theory. The potential $\mathcal{U}_2$,$\mathcal{U}_3$ and $\mathcal{U}_4$ can be written as follows,

$$\mathcal{U}_2 = [K]^2 - [K^2], \tag{4.61}$$

$$\mathcal{U}_3 = [K]^3 - 3[K][K^2] + 2[K^3], \tag{4.62}$$

$$\mathcal{U}_4 = [K]^4 - 6[K]^2[K^2] + 8[K][K^3] + 3[K^2]^2 - 6[K^4], \tag{4.63}$$

where

$$K^\mu_\nu = \delta^\mu_\nu - \sqrt{g^{\mu\rho}f_{\rho\nu}} = \delta^\mu_\nu - \mathbb{M}^\mu_\nu. \tag{4.64}$$

The rectangular brackets denote the trace of metric $K^\mu_\mu$,

$$[K^n] = (K^n)^\mu_\mu \qquad \text{and} \qquad (K^n)^\mu_\nu = K^\mu_{\rho_2}K^{\rho_2}_{\rho_3}...K^{\rho(n-1)}_{\rho_n}K^{\rho_n}_\nu \ \text{ for } n \geq 2. \tag{4.65}$$

The above potential is indeed a suitable form in which the ghost instability is eliminated. It is noticed that this ghost-free theory is not constructed only from the physical metric $g_{\mu\nu}$ but also the reference or fiducial metric $f_{\mu\nu}$. The fiducial metric $f_{\mu\nu}$ is non-dynamical field. Therefore, the field equations obtained by the varying the action with respect to $f_{\mu\nu}$ is just the constraints. In other words, $f_{\mu\nu}$ plays a role of the Lagrange multiplier in order to construct the suitable form of the mass terms. To see that there is no ghost appears in dRGT massive gravity, we will consider,

$$\mathbb{X}^{\mu}_{\nu} = g^{\mu\rho} f_{\rho\nu} = (\sqrt{g^{\mu\alpha} f_{\alpha\rho}})(\sqrt{g^{\rho\beta} f_{\beta\nu}}) \tag{4.66}$$

$$= (\delta^{\mu}_{\rho} - K^{\mu}_{\rho})(\delta^{\rho}_{\nu} - K^{\rho}_{\nu}) \tag{4.67}$$

$$= \delta^{\mu}_{\nu} - 2K^{\mu}_{\nu} + K^{\mu}_{\rho} K^{\rho}_{\nu} \tag{4.68}$$

By comparing to the Eq.(4.57), one found that $K^{\mu}_{\nu}$ is proportional to $\Pi^{\mu}_{\nu}$. Therefore, one can see that the combination in the Eq.(4.60) is consistent with no ghost instability. It is useful to construct the mass term in which the scalar mode being the total derivative $\mathcal{L}^{(2)}_{der} = [\Pi^2] - [\Pi]^2$ as in the Galileon theory [90]. The equation of motion of the theory can be obtained by varying the action in Eq.(4.59) with respect to dynamical metric $g_{\mu\nu}$. It can be read as,

$$G_{\mu\nu} + m_g^2 X_{\mu\nu} = 0, \tag{4.69}$$

where $G_{\mu\nu}$ is the Einstein tensor and $X_{\mu\nu}$ is the effective energy-momentum tensor associated with a part from varying the potential term, which can be written explicitly form as

$$\begin{aligned} X_{\mu\nu} = K_{\mu\nu} - [K] g_{\mu\nu} - \alpha \left[ K^2_{\mu\nu} - [K] K_{\mu\nu} + \frac{[K]^2 - [K^2]}{2} g_{\mu\nu} \right] \\ + 3\beta \left[ K^3_{\mu\nu} - [K] K^2_{\mu\nu} + \frac{1}{2} K_{\mu\nu} \left( [K]^2 - [K^2] \right) - \frac{1}{6} g_{\mu\nu} \left( [K]^3 - 3[K][K^2] + 2[K^3] \right) \right]. \end{aligned} \tag{4.70}$$

Here, the free parameters $\alpha_3$ and $\alpha_4$ are redefined in convenient form as

$$\alpha_3 = \frac{\alpha - 1}{3}, \qquad \alpha_4 = \frac{\beta}{4} + \frac{1 - \alpha}{12}. \tag{4.71}$$

According to the Bianchi identity, one also has the constraint for the effective energy-momentum tensor $X_{\mu\nu}$ as

$$\nabla^{\mu} X_{\mu\nu} = 0, \tag{4.72}$$

where $\nabla^{\mu}$ denotes the covariant derivative associated with the metric $g_{\mu\nu}$. Note that these constraints can also be derived from varying the action with respect to the fiducial metric.

### 4.8 Decoupling limit

As we mention before the Vainshtein mechanism is used to screen the effect of the coupling between scalar mode and the matter at a short distance. Hence, there is a specific distance from the source in which the non-linear effect is required which is called the Vainshtein radius, $r_V$. The linear version of the massive gravity theory is valid in the region $r \gg r_V$ while the non-linear effect become important at $r \ll r_V$. Note that, for distances less than the Vainshtein radius, the non-linear massive gravity can be reduced to GR. However, the non-linear massive theory is an effective field theory, valid only in the classical regime. For the quantum regime, we need to use the other quantum theory. This means that the Vainshtein mechanism is no longer valid at the very short distance scale. To simplify, we will consider the static and spherical symmetric source, the Vainshtein radius can be defined by

$$r_v = \frac{1}{\Lambda_\lambda}\left(\frac{M_{source}}{M_{Pl}}\right)^{1/\lambda}. \tag{4.73}$$

where the $\Lambda_\lambda$ is the cutoff scale for non-linear theories. One found that at the scale below $\Lambda_\lambda$, the scalar mode will be strongly coupled to the other fields again which is called the strong coupling scale. This breaks our attention about the classical scale and move to consider the other scale in the quantum theory. To find the strong coupling scale, one can consider the potential term includes generic interactions between the tensor mode $h_{\mu\nu}$, vector mode $\chi_\mu$ and scalar mode $\pi$. The general form of the interaction can be written as

$$\mathcal{L}_{\rm int} = m_g^2 M_{Pl}^2 (h)^{n_h} (\partial\chi)^{n_\chi} (\partial^2\pi)^{n_\pi}, \tag{4.74}$$

where $n_h$, $n_\chi$ and $n_\pi$ are the power of $h$, $\partial\chi$ and $\partial^2\pi$ respectively which are existed because of the decomposition of $H_{\mu\nu}$. Then we normalize these fields as

$$h'_{\mu\nu} = M_{Pl} h_{\mu\nu}, \quad \chi' = m_g M_{Pl} \chi_\mu, \quad \pi' = m_g^2 M_{Pl} \pi. \tag{4.75}$$

The interaction becomes,

$$\begin{aligned}
\mathcal{L}_{\text{int}} &= m_g^2 M_{Pl}^2 \left(\frac{h'}{M_{Pl}}\right)^{n_h} \left(\frac{\partial\chi'}{m_g M_{Pl}}\right)^{n_\chi} \left(\frac{\partial^2\pi'}{m_g^2 M_{Pl}}\right)^{n_\pi}, \\
&= M_{Pl}^{2-n_h-n_\chi-n_\pi} m_g^{2-n_\chi-n_\pi} (h')^{n_h} (\partial\chi')^{n_\chi} (\partial^2\pi')^{n_\pi}, \\
&= \left(M_{Pl}^{\frac{2-n_h-n_\chi-n_\pi}{4-n_h-2n_\chi-3n_\pi}} m_g^{\frac{2-n_\chi-n_\pi}{4-n_h-2n_\chi-3n_\pi}}\right)^{4-n_h-2n_\chi-3n_\pi} (h')^{n_h} (\partial\chi')^{n_\chi} (\partial^2\pi')^{n_\pi}, \\
&= \left[\left(M_{Pl} m_g^{\frac{4-n_h-2n_\chi-3n_\pi}{2-n_h-n_\chi-n_\pi}-1}\right)^{\frac{2-n_h-n_\chi-n_\pi}{4-n_h-2n_\chi-3n_\pi}}\right]^{4-n_h-2n_\chi-3n_\pi} (h')^{n_h} (\partial\chi')^{n_\chi} (\partial^2\pi')^{n_\pi}, \\
&= \left[\left(M_{Pl} m_g^{\lambda-1}\right)^{\frac{1}{\lambda}}\right]^{4-n_h-2n_\chi-3n_\pi} (h')^{n_h} (\partial\chi')^{n_\chi} (\partial^2\pi')^{n_\pi}, \\
&= \Lambda_\lambda^{4-n_h-2n_\chi-3n_\pi} (h')^{n_h} (\partial\chi')^{n_\chi} (\partial^2\pi')^{n_\pi},
\end{aligned} \tag{4.76}$$

where the cutoff scale is

$$\Lambda_\lambda = \left(M_{Pl} m_g^{\lambda-1}\right)^{\frac{1}{\lambda}}, \quad \lambda = \frac{4-n_h-2n_\chi-3n_\pi}{2-n_h-n_\chi-n_\pi}. \tag{4.77}$$

Since, we consider in the interaction terms not normal mass terms, therefore $n_h + n_\chi + n_\pi > 2$. The lowest interaction scale for the arbitrary non-linear massive gravity theory besides the dRGT massive gravity is in the case of $n_h = 0, n_\chi = 0$ and $n_\pi = 3$,

$$\Lambda_\lambda = \Lambda_5 = (M_{Pl} m_g^4)^{1/5}. \tag{4.78}$$

This is the cutoff for such non-linear theories. One can see that in the limit $m_g \to 0$, $M_{Pl} \to \infty$ and $\Lambda_5$ is fixed, the only interaction from the scalar mode exists for this scale while the other interactions will disappear. However, the $\Lambda_5$ does not exist in dRGT massive gravity because the specific form of $\mathcal{U}$ can make the interaction disappear. It is found the next scale is the $\Lambda_4$ which are in the case of $n_h = 0, n_\chi = 0, n_\pi = 4$ and $n_h = 0, n_\chi = 1, n_\pi = 2$. The interaction in the first case is also eliminate by $\mathcal{U}$, while for the second case, the Lagrangian density can be written in the form of the total derivative which is boundary terms. It is found that the cutoff for dRGT massive gravity is,

$$\Lambda_3 = (M_{Pl} m_g^2)^{1/3}, \tag{4.79}$$

see more detail in [89]. One can see that the dRGT massive gravity theory can be use to explain the nature at the scale between the Vainshtein radius and the $\Lambda_3$ scale. For a scale larger than Vainshtein radius, the linear FP massive theory is viable. However, the massive theory is useless at the scale smaller than $\Lambda_3$.

# CHAPTER V

# THERMODYNAMICS OF BLACK STRING FROM RÉNYI ENTROPY IN DE RHAM-GABADAZE-TOLLEY MASSIVE GRAVITY THEORY

The black string is a result from cylindrically symmetric solution of Einstein field equation with cosmological constant analogous to the black hole which is a result from spherically symmetric solution. With respect to the cylindrical symmetry, an event horizon is a cylindrical shell of radius $r$ analogous to the Schwarzschild radius of the black hole. It is found that the thermodynamic properties of the black string can be investigated in the same fashion as in the black hole case. For example, the entropy of the black string is proportional to its area or the temperature can be obtained via the surface gravity at the horizon as $T = \frac{|\kappa|}{2\pi} = \frac{|f'(r)|}{4\pi}$ and then the equivalent laws of thermodynamics are in the same form.

In this chapter, the thermodynamic properties of black string from Rényi entropy in dRGT massive gravity theory are investigated. This is one of the black string solutions, which is significantly different from that one obtained in GR with the cosmological constant. We are also interested in the dS branch of the solution. As a result, the black string with multiple horizons yields the thermodynamic systems with different temperatures, which corresponds to non-equilibrium thermodynamic states. Therefore, in order to study the thermodynamics of the black string, we separate our consideration into two approaches; separated system approach and effective system approach. For the separated case, the thermodynamic systems can be investigated separately by assuming that the systems are separated far enough and the temperatures of the systems do not significantly differ. For the effective case, we can treat the systems as a single system described by the effective thermodynamic quantities.

### 5.1 dRGT black string solution

In this section, we are interested in one of the cylindrically symmetric solutions in dRGT massive gravity theory. The line element corresponding to cylindrical system can be written as,

$$ds^2 = -n(r)dt^2 + 2d(r)dtdr + \frac{dr^2}{f(r)} + L(r)^2 d\Omega^2, \tag{5.1}$$

$$-\infty < t < \infty, \quad 0 \leq r < \infty, \quad 0 \leq \varphi < 2\pi, \quad -\infty < z < \infty, \tag{5.2}$$

where $d\Omega^2 = d\varphi^2 + \alpha_g^2 dz^2$ is a metric on 2-D surface and $\alpha_g$ is a constant in the unit of mass. The solution can be classified into two branches: $d(r) = 0$ or $L(r) = l_0 r$, where $l_0$ is a constant which can be written in terms of the parameters $\alpha$ and $\beta$ [43] as

$$l_0 = \frac{\left(\alpha + 3\beta \pm \sqrt{\alpha^2 - 3\beta}\right)}{2\alpha + 3\beta + 1}. \tag{5.3}$$

For this present work, it is convenient to investigate in the branch $d(r) = 0$. The line element for this branch can be written as

$$ds^2 = -n(r)dt^2 + \frac{dr^2}{f(r)} + L(r)^2 d\Omega^2. \tag{5.4}$$

Let us choose the fiducial metric in the form of

$$f_{\mu\nu} = \mathrm{diag}\Big(0, 0, h(r)^2, h(r)^2 \alpha_g^2\Big), \tag{5.5}$$

where the $h(r)$ plays the role of the radial function similar to the radial coordinate $r$. By using the metric form in the Eq.(5.4), the non vanishing components of the

Christoffel symbol can be written as

$$\begin{aligned}
\Gamma^0_{01} &= \frac{1}{2}g^{00}(\partial_0 g_{01} + \partial_1 g_{00} - \partial_0 g_{01}) = \frac{1}{2}g^{00}(\partial_1 g_{00}),\\
&= \frac{1}{2}(-n)^{-1}\partial_r(-n) = \frac{1}{2}n^{-1}n', && (5.6)\\
\Gamma^1_{00} &= \frac{1}{2}g^{11}(\partial_0 g_{10} + \partial_0 g_{10} - \partial_1 g_{00}) = -\frac{1}{2}g^{11}\partial_1 g_{00},\\
&= -\frac{1}{2}(f)\partial_r(n) = \frac{1}{2}fn' && (5.7)\\
\Gamma^1_{11} &= \frac{1}{2}g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = -\frac{1}{2}g^{11}\partial_1 g_{11},\\
&= -\frac{1}{2}f^{-1}f', && (5.8)\\
\Gamma^1_{22} &= \frac{1}{2}g^{11}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) = -\frac{1}{2}g^{11}\partial_1 g_{22},\\
&= -fLL', && (5.9)\\
\Gamma^1_{33} &= \frac{1}{2}g^{11}(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33})' = -\frac{1}{2}g^{11}\partial_1 g_{33},\\
&= -fLL'\alpha_g^2, && (5.10)\\
\Gamma^2_{12} &= \frac{1}{2}g^{22}(\partial_1 g_{22} + \partial_2 g_{21} - \partial_2 g_{12}) = \frac{1}{2}g^{22}\partial_1 g_{22},\\
&= L^{-1}L', && (5.11)\\
\Gamma^3_{13} &= \frac{1}{2}g^{33}(\partial_1 g_{33} + \partial_3 g_{31}) = \frac{1}{2}g^{33}\partial_1 g_{33},\\
&= L^{-1}L', && (5.12)
\end{aligned}$$

where prime denotes the derivative with respect to $r$. Then, the non vanishing components of the Ricci tensor become,

$$R_{00} = \frac{f'n'}{4} + \frac{fL'n'}{L} - \frac{f(n')^2}{4n} + \frac{fn''}{2}, \tag{5.13}$$

$$R_{11} = -\frac{f'L'}{fL} - \frac{f'n'}{4fn} + \frac{(n')^2}{4n^2} - \frac{2L''}{L} - \frac{n''}{2n}, \tag{5.14}$$

$$R_{22} = -\frac{Lf'L'}{2} - f(L')^2 - \frac{fLL'n'}{2n} - fLL'', \tag{5.15}$$

$$R_{33} = \alpha_g^2 R_{22}, \tag{5.16}$$

and then, one obtains,

$$R^0_0 = g^{00}R_{00} = (-n)^{-1}\left(\frac{f'n'}{4} + \frac{fL'n'}{L} - \frac{f(n')^2}{4n} + \frac{fn''}{2}\right) \tag{5.17}$$

$$R^1_1 = g^{11}R_{11} = f\left(-\frac{f'L'}{fL} - \frac{f'n'}{4fn} + \frac{(n')^2}{4n^2} - \frac{2L''}{L} - \frac{n''}{2n}\right) \tag{5.18}$$

$$R^2_2 = R^3_3 = g^{22}R_{22} = L^{-2}\left(-\frac{Lf'L'}{2} - f(L')^2 - \frac{fLL'n'}{2n} - fLL''\right). \tag{5.19}$$

Therefore, the Ricci scalar can be written as

$$\begin{aligned} R &= R^0_0 + R^1_1 + R^2_2 + R^3_3, \\ &= -\frac{\begin{bmatrix} Lnf'(4nL' + Ln') + f\Big[4n^2(L')^2 + 4LnL'n' \\ +L(-L(n')^2 + 8n^2L'' + 2Lnn'')\Big] \end{bmatrix}}{2L^2n^2}. \end{aligned} \tag{5.20}$$

The components of the Einstein tensor become,

$$G^0_0 = \frac{(LfL' + f((L')^2 + 2LL''))}{L^2}, \tag{5.21}$$

$$G^1_1 = \frac{fL'(nL' + Ln')}{L^2n}, \tag{5.22}$$

$$G^2_2 = G^3_3 = \frac{(nf'(2nL' + Ln') + f(2nL'n' - L(n')^2 + 4n^2L'' + 2Lnn''))}{L4n^2}. \tag{5.23}$$

For computing the effective energy-momentum tensor expressed in the Eq.(4.70), we must be firstly find the quantities $K^\mu_\nu$, $[K]$, $[K^2]$ and $[K^3]$. Let us start with,

$$K^0_0 = K^1_1 = 1, \tag{5.24}$$

$$K^2_2 = K^3_3 = 1 - \frac{h}{L}. \tag{5.25}$$

And then, $(K^2)^\mu_\nu$ and $(K^3)^\mu_\nu$ can be written as,

$$(K^2)^0_0 = (K^2)^1_1 = 1, \tag{5.26}$$

$$(K^2)^2_2 = (K^2)^3_3 = (1 - \frac{h}{L})^2, \tag{5.27}$$

$$(K^3)^0_0 = (K^3)^1_1 = 1 \tag{5.28}$$

$$(K^3)^2_2 = (K^3)^3_3 = (1 - \frac{h}{L})^3. \tag{5.29}$$

Next, we will find the trace for $K$,

$$\begin{aligned}
[K] &= K^0_0 + K^1_1 + K^2_2 + K^3_3, \\
&= 2 + \left(1 - \frac{h}{L}\right) + \left(1 - \frac{\alpha_f h}{\alpha_g L}\right), \qquad (5.30)\\
[K^2] &= (K^2)^0_0 + (K^2)^1_1 + (K^2)^2_2 + (K^2)^3_3, \\
&= 2 + 2\left(1 - \frac{h}{L}\right)^2, \qquad (5.31)\\
[K^3] &= (K^3)^0_0 + (K^3)^1_1 + (K^3)^2_2 + (K^3)^3_3, \\
&= 2 + 2\left(1 - \frac{h}{L}\right)^3. \qquad (5.32)
\end{aligned}$$

Eventually, The component of effective tensor expressed in the Eq.(4.70) can be written as follows,

$$X^0_0 = \frac{-1}{L^2}\begin{bmatrix} h[3\beta(h-L)-L] + L[-(1+3\beta)h + 3(1+\beta)L] \\ +\alpha[h(h-2L) + L(-2h+3L)] \end{bmatrix}, \qquad (5.33)$$

$$X^1_1 = \frac{-1}{L^2}\begin{bmatrix} h[3\beta(h-L)-L] + L[-(1+3\beta)h + 3(1+\beta)L] \\ +\alpha[h(h-2L) + L(-2h+3L)] \end{bmatrix}, \qquad (5.34)$$

$$X^2_2 = X^3_3 = -3(1+\alpha+\beta) + \frac{(1+2\alpha+3\beta)h}{L}. \qquad (5.35)$$

By substituting the component Einstein tensor and effective energy momentum tensor into the Eq.(4.69), and then using the equations for $(0,0)$ and $(1,1)$ components, we have

$$\begin{aligned}
(G^0_0 + m_g^2 X^0_0) - (G^1_1 + m_g^2 X^1_1) &= 0, \qquad (5.36)\\
\frac{(LfL' + f((L')^2 + 2LL''))}{L^2} - \frac{fL'(nL' + Ln')}{L^2 n} &= 0, \qquad (5.37)\\
\frac{1}{Ln}(nf'L' + 2nfL'' - fL'n') &= 0, \qquad (5.38)\\
\frac{n}{l}\frac{d}{dr}\left(\frac{fL'}{n}\right) &= = 0. \qquad (5.39)
\end{aligned}$$

In order to obtain the black string solution with $f(r) = n(r)$, the function $L(r)$ must be proportional to $r$. Hence, we can set $L(r) = r$ for the following investigation. We now have only two independent functions, $f(r)$ and $h(r)$. One can use the two independent equations to find these two functions such that the conservation of the energy momentum tensor in Eq.(4.72) and the $(0,0)$ component of the modified Einstein equation in Eq.(4.69).

By the conservation of the energy momentum tensor, one obtains

$$\nabla_\mu X^\mu_\nu = 0, \tag{5.40}$$

$$\partial_\mu X^\mu_\nu + \Gamma^\mu_{\rho\mu} X^\rho_\mu - \Gamma^\rho_{\mu\nu} X^\mu_\rho = 0, \tag{5.41}$$

$$\frac{h'\left(2r(1+2\alpha+3\beta) - 2h(\alpha+3\beta)\right)}{r^2} = 0. \tag{5.42}$$

Then, two exact solutions of $h(r)$ are given by

$$h(r) = \frac{r(1+2\alpha+3\beta)}{(\alpha+3\beta)}, \tag{5.43}$$

$$h(r) = h_0 = \text{constant}. \tag{5.44}$$

The $(0,0)$ component of the modified Einstein equation can be written in the form as

$$G^0_0 = -m_g^2 X^0_0, \tag{5.45}$$

$$\frac{(rf'+f)}{r^2} = -m_g^2 \begin{pmatrix} \frac{2h-3r}{r} + \alpha\left(\frac{h(2r-h)}{r^2} - \frac{3r-2h}{r}\right) \\ +\beta\left(\frac{3h(r-h)}{r^2} - \frac{3(r-h)}{r}\right) \end{pmatrix}. \tag{5.46}$$

Substituting the solutions of $h(r)$ from Eq.(5.43) and Eq.(5.44) to the above equation, we thus have the two solutions for the horizon function as follows

$$f_1(r) = \left(-\frac{m_g^2(1+\alpha+\alpha^2-3\beta)}{3(\alpha+3\beta)}\right) r^2 - \frac{b}{r}, \tag{5.47}$$

$$f_2(r) = -(m_g^2(1+\alpha+\beta))r^2 - \frac{b}{r} + m_g^2 h_0(1+2\alpha+3\beta)r + m_g^2 h_0^2(\alpha+3\beta), \tag{5.48}$$

where $b$ is an integration constant. The Eq.(5.47) coincides with the Lemos black string in GR with a cosmological constant $\Lambda$. It is already widely investigated. Hence, we will consider only the solution in the Eq.(5.48). For Convenient, one can rewrite the Eq. (5.48) as

$$f_2(r) = \mathcal{F}(r) = -\frac{4M}{r} - m_g^2\left(r^2 - c_1 r - c_0\right), \tag{5.49}$$

where $M = M_{\rm ADM}/\alpha_g$ and $M_{\rm ADM}$ is the Arnowitt-Deser-Misner mass per unit length of the $z$-coordinate. It is important to emphasize that the mass parameter $M$ is actually in the unit of length since $G$ is assigned to be the natural unit for this consideration. The parameters $c_1$ and $c_0$ are expressed in terms of the aforementioned parameters as follows,

$$c_1 \equiv -h\left(\frac{1+2\alpha+3\beta}{1+\alpha+\beta}\right), \quad c_0 \equiv h^2\left(\frac{\alpha+3\beta}{1+\alpha+\beta}\right). \tag{5.50}$$

## 5.2 Horizon structure

The horizons of the black string can be defined in the same way as found in the black hole case by solving $\mathcal{F}(r) = 0$. As a result, the number of possible solutions depends on the sign of $m_g^2$. For $m_g^2 > 0$ the solution becomes the asymptotic dS branch which exists in two horizons, while $m_g^2 < 0$ the solution becomes the asymptotic AdS branch which exists in three horizons. In this work, we will investigate the structure of the horizons by restricting our attention to the asymptotically dS solution. In the limit $c_0 = c_1 = 0$ and $m_g = \alpha_g$, the dRGT black string solution reduces to the Lemos' black string solution,

$$\mathcal{F}(r) = -\frac{4M}{r} - \alpha_g^2 r^2. \tag{5.51}$$

For this case, one found that it is not possible to obtain the horizons because $\mathcal{F}(r)$ is always negative. This means that the effects of $c_0$ and $c_1$ are significantly required for the existence of the horizons. Therefore, the structure of graviton mass is necessarily important. For convenience, we rewrite the function $\mathcal{F}(r)$ in terms of dimensionless quantities by redefining parameters as follows

$$r = x r_V, \quad c_1 = 3\times 2^{2/3} b_1 r_V, \quad c_0 = 3\times 2^{2/3} b_0 r_V^2, \quad r_V = \left(\frac{M}{m_g^2}\right)^{1/3}. \tag{5.52}$$

Here, $r_V$ is a Vainshtien radius at which the graviton mass plays a major part of the gravitational interaction for the radius being much larger than the Vainshtien radius $r \gg r_V$, while it is suppressed at the scale below the Vainshtien radius $r \ll r_V$. As a result, the horizon function can be rewritten as

$$f(x) = \frac{r_V}{M}\mathcal{F}(r) = -\frac{4}{x} - x^2 + \left(3 \times 2^{2/3} b_1\right) x + \left(3 \times 2^{2/3} b_0\right). \tag{5.53}$$

The extremum can be found by solving $f'(x) = 0$, then we obtain

$$3 \times 2^{2/3} b_1 - \frac{4}{x^2} + 2x = 0. \tag{5.54}$$

Therefore, we obtain the extremum as

$$x_0 = \frac{1}{2^{1/3}}\left(b_1 + \frac{b_1^2}{B_1} + B_1\right), \tag{5.55}$$

where

$$B_1 = \left(2 + b_1^3 + 2\sqrt{1 + b_1^3}\right)^{1/3}. \tag{5.56}$$

Substituting the extremum value to the Eq. (5.53), so the value of the function $f(x)$ at extremum is written as

$$f(x_0) = \frac{6}{2^{1/3}} b_0 - \frac{1}{2^{2/3}}\left(b_1 + \frac{b_1^2}{B_1} + B_1\right)^2 + \frac{6b_1}{2^{2/3}}\left(b_1 + \frac{b_1^2}{B_1} + B_1\right) - \frac{4 \times 2^{1/3}}{\left(b_1 + \frac{b_1^2}{B_1} + B_1\right)}. \tag{5.57}$$

One can see that, the parameter $b_0$ (or $c_0$) does not influence the extremum of $f(x)$. In order to obtain the condition for existence of the horizons, one requires that $f(x_0) \geq 0$. As a result, we then find the relation between $b_0$ and $b_1$ to satisfy this condition as follows

$$b_0 \geq \frac{\left(b_1 + \frac{b_1^2}{B_1} + B_1\right)^3 - 6b_1\left(b_1 + \frac{b_1^2}{B_1} + B_1\right)^2 + 8}{6 \times 2^{1/3}\left(b_1 + \frac{b_1^2}{B_1} + B_1\right)}. \tag{5.58}$$

The region for which there exist horizons can be illustrated in Fig. 6. It is obviously to see that it is not possible to have horizon at the origin $(b_1, b_0) = (0, 0)$. Therefore, there is no horizon for usual black string as we have found. It is more convenient for us to consider the case $b_1 \sim 0$, since we can analyze the possibility of finding a deviation from the Lemos' black string. For this case, one has a condition on $b_0$ as $b_0 \gtrsim 1 - 2^{1/3} b_1$.

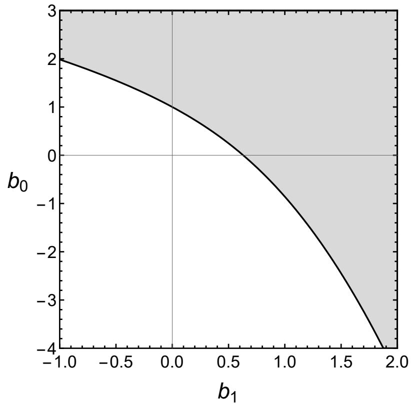


**Figure 6 The region of the existence of the horizons in $(b_1, b_0)$ space, where the shaded area corresponds to a horizon region.**

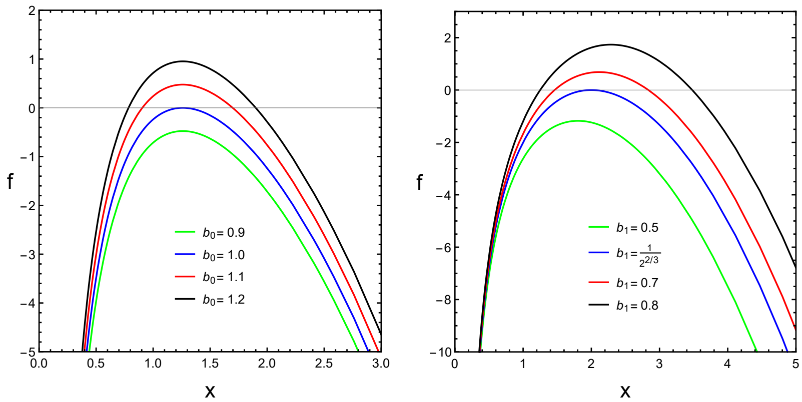


**Figure 7 The left and right panels show the behaviors of the function $f(x)$ for various values of $b_0$ (with fixing $b_1 = 0$) and $b_1$ (with fixing $b_0 = 0$), respectively.**

## 5.3 Separated thermodynamic systems

In this section , we will investigate the thermodynamic properties of the dRGT black string by considering the thermodynamic system of each horizon defined separately. The black string mass $M$ in terms of horizon radius can be found by solving $\mathcal{F}(r) = 0$, and then it can be expressed as

$$M(r) = -\frac{r}{4}m_g^2(r^2 - c_1 r - c_0). \tag{5.59}$$

The Hawking temperature of the dRGT black string can be written in terms of the horizons as

$$T_{b,c} = \pm\frac{m_g^2(c_0 + 2c_1 r_{b,c} - 3r_{b,c}^2)}{4\pi r_{b,c}}. \tag{5.60}$$

In our consideration, we restrict on the dS branch of the solution, there are exist only two horizons: the smaller one denotes by black string horizon, $r_b$ and the larger one denotes by the cosmic horizon, $r_c$. Note that, the quantity with subscript "$b$" (or "$c$") is the quantity defined for the system of black string horizon (or cosmic horizon) . The plus and minus signs in Eq. (5.60) denote the temperatures for black string horizon and cosmic horizon respectively. As a result, the dRGT black string corresponds to two thermodynamic systems with different temperatures. Note also that both temperatures are positive for the whole viable range of parameters.

The entropy of the black string can be defined by using the area law as same as in the black hole case. As a result, the entropy known as the Bekenstein-Hawking entropy can be written as

$$S_{BH} = \frac{\pi r^2}{2}, \tag{5.61}$$

where $S_{BH} = A/(4\alpha_g)$ and $A = 2\pi r \times \alpha_g r$ is the area of the cylinder shell per unit length of the $z$-coordinate. Hence, the above entropy $S_{BH}$ is actually in the unit of the square of length. In order to further study the thermodynamics of the black string with more general form of the first law, let us consider the Smarr formula of the black string by treating the mass parameter $M$ in Eq. (5.59) as a homogeneous function of all parameters in the theory. As a result, we found that $M$ can be written as the homogeneous function with degree 1/2 as $M = M(S_{BH}, m_g^{-2}, c_1^2, c_0)$. By using Euler's theorem, the black string mass can be written as

$$\frac{1}{2}M = S_{BH}\frac{\partial M}{\partial S_{BH}} + m_g^{-2}\frac{\partial M}{\partial m_g^{-2}} + c_1^2\frac{\partial M}{\partial c_1^2} + c_0\frac{\partial M}{\partial c_0}. \tag{5.62}$$

We choose to define the thermodynamic pressure proportional to $m_g^2$ given by

$$P = \frac{3}{8\pi}m_g^2. \tag{5.63}$$

The conjugate variables of $S_{BH}, P, c_1$ and $c_0$ can be respectively computed as follows:

$$T = \pm\frac{\partial M}{\partial S_{BH}} = T_{b,c}, \tag{5.64}$$

$$V = \frac{\partial M}{\partial P} = \frac{2\pi r_h^3}{3}\left(-1 + \frac{c_1}{r_h} + \frac{c_0}{r_h^2}\right), \tag{5.65}$$

$$\Phi_1 = \frac{\partial M}{\partial c_1} = \frac{2}{3}\pi P r_h^2, \tag{5.66}$$

$$\Phi_0 = \frac{\partial M}{\partial c_0} = \frac{2}{3}\pi P r_h. \tag{5.67}$$

Therefore, the Smarr formula can be rewritten as

$$M = \pm 2TS_{BH} - 2PV + \Phi_1 c_1 + 2\Phi_0 c_0. \tag{5.68}$$

It is important to note that the temperature defined in Eq. (5.64) is exactly the same with one defined via the surface gravity in Eq. (5.60). Moreover, the thermodynamic volume in Eq. (5.65) is already absorbed $\alpha_g$ in the same fashion as $M$ and $S$ in the Eqs. (5.59) and (5.61), respectively. Therefore, its unit is the cubic of the length not the volume per unit length in $z$-coordinate. Interestingly, it is possible to obtain the positive thermodynamic volume and pressure for the solution in the dRGT massive gravity while, in the solution in GR with cosmological constant, either volume or pressure should be negative as follows:

$$P_{GR} = \pm\frac{3}{8\pi}m_g^2, \quad V_{GR} = \mp\frac{2\pi r_h^3}{3}. \tag{5.69}$$

Here, all parameters are chosen to have only positive values. Therefore, the structure of the graviton mass can be treated as corrections to the thermodynamic volume. As a result, the first law of thermodynamics corresponding to the dRGT black string can be written as

$$dM = \pm T dS_{BH} + V dP + \Phi_1 dc_1 + \Phi_0 dc_0. \tag{5.70}$$

However, as we have seen, the black string entropy is proportional to the surface area of the black string's horizons, instead of its volume. Therefore, the black string entropy is not an extensive variable. In this context, it is worthwhile to investigate the thermodynamic system by using the non-extensive entropy. In this work, we will use the Rényi entropy which can be written in terms of the Bekenstein-Hawking entropy as,

$$S_R = \frac{1}{\lambda} \ln(1 + \lambda S_{BH}). \tag{5.71}$$

In our work, we focus on the thermodynamic system by fixing $c_1$ and $c_0$. Therefore, the first laws of the thermodynamic system evaluated at the black string horizon and cosmic horizon can be written respectively as

$$dM = T_{R(b)} dS_{R(b)} + V_b dP, \quad dM = -T_{R(c)} dS_{R(c)} + V_c dP. \tag{5.72}$$

It is obvious to see that the thermodynamic volumes $V_{b,c}$ are not correspond to the non-extensivity effect, since both $M$ and $P$ are independent of the non-extensive parameter $\lambda$. As a result, the Rényi temperature for both systems can be written as

$$T_{R(b)} = \left(\frac{\partial M}{\partial S_{R(b)}}\right)_P = \frac{m_g^2(c_0 + 2c_1 r_b - 3r_b^2)}{4\pi r_b}\left(1 + \lambda\frac{\pi r_b^2}{2}\right) = T_b\left(1 + \lambda\frac{\pi r_b^2}{2}\right) \tag{5.73}$$

$$T_{R(c)} = -\left(\frac{\partial M}{\partial S_{R(c)}}\right)_P = \frac{-m_g^2(c_0 + 2c_1 r_c - 3r_c^2)}{4\pi r_c}\left(1 + \lambda\frac{\pi r_c^2}{2}\right) = T_c\left(1 + \lambda\frac{\pi r_c^2}{2}\right). \tag{5.74}$$

Conveniently, let us write temperatures in terms of dimensionless variables as follows

$$\bar{T}_{R(b)} = \frac{r_V^2}{3\times 2^{2/3}M}T_{R(b)} = \frac{\left(b_0+2b_1x-2^{-2/3}x^2\right)(1+x^2\delta)}{4\pi x}, \tag{5.75}$$

$$\bar{T}_{R(c)} = \frac{r_V^2}{3\times 2^{2/3}M}T_{R(c)} = \frac{-\left(b_0+2b_1y-2^{-2/3}y^2\right)(1+y^2\delta)}{4\pi y}, \tag{5.76}$$

where

$$r_b = xr_V, \quad r_c = yr_V, \quad \lambda = \delta\frac{2}{\pi r_V^2}. \tag{5.77}$$

Note that both temperatures in Eq.(5.75) and Eq.(5.76) are positive for the range $0 < r_b < r_c$ or $0 < x < 2^{2/3}\left(b_1+\sqrt{b_1^2+\frac{b_0}{2^{2/3}}}\right)$ and $2^{2/3}\left(b_1+\sqrt{b_1^2+\frac{b_0}{2^{2/3}}}\right) < y < \frac{1}{2^{1/3}}\left(3b_1+\sqrt{6\times 2^{1/3}b_0+9b_1^2}\right)$. For the extremal black string, the parameters $x$ and $y$ are equal to $x=y=2^{2/3}\left(b_1+\sqrt{b_1^2+\frac{b_0}{2^{2/3}}}\right)$. For the case of dS solution in which $m_g^2>0$ and $b_1=b_0=0$, the slope of the temperature is always negative. It implies that system is locally unstable. However, in the case of dRGT with Rényi entropy, it is possible to find the extrema of the temperature. This means that it is possible to have the positive value of slope of the temperature. To find the extremum of $\bar{T}_{R(b)}$, one can solve a condition,

$$\frac{d\bar{T}_{R(b)}}{dx} = 8b_1x\delta + b_0\left(-\frac{2}{x^2}+2\delta\right) - 2^{1/3}(1+3x^2\delta) = 0. \tag{5.78}$$

Note that the explicit expression of the extrema points of $\bar{T}_{R(b)}$ is very lengthy and difficult to analyze. Therefore, we choose to find the extrema points of $\bar{T}_{R(b)}$ by approximating $b_1 \sim 0$ to make it easier to analyze. As a result, the solution can be written as

$$x_{ex\pm} = \frac{2^{5/3}}{3}b_1(1\pm\Delta) \pm \frac{b_0}{b_1\Delta}\left[1+\frac{3}{2}\frac{\delta}{\delta_c}(1\pm\Delta)\right], \tag{5.79}$$

$$\Delta = \sqrt{1-\frac{\delta_c}{\delta}}, \quad \delta_c = \frac{3}{8\times 2^{1/3}b_1^2}. \tag{5.80}$$

To obtain a real positive value of $x_{ex}$, we need $\delta > \delta_c$. Therefore, the dRGT black string will be locally stable if $\delta > \delta_c = 3/(8\times 2^{1/3}b_1^2)$ as shown explicitly in the left panel in Fig. 8. Note that, in the GB limit, there are no extrema for temperature. The temperature profile for varying $b_1$ is illustrated in the right panel in Fig. 8. As

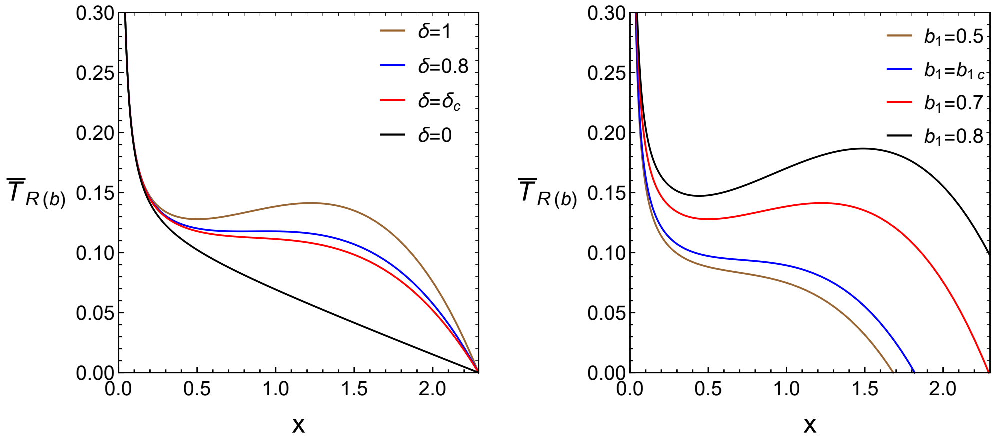


**Figure 8 Left panel shows the temperature profile at black string horizon with various values of $\delta$ by fixing $b_0 = 0.1, b_1 = 0.7$. Right panel shows the temperature profile at black string horizon with various values of $b_1$ by fixing $b_0 = 0.1, \delta = 1$.**

seen in Eq. (5.80), the bound in $\delta$ depends only on the parameter $b_1$. Therefore, it is possible to express the bound in $b_1$ as $b_{1c} = \sqrt{3/(8 \times 2^{1/3}\delta_c)}$. Obviously, the bound in $b_1$ is indeed the lower bound implied from the parameter $b_{1c}$ being inversely proportional to the non-extensive parameter $\delta_c$.

For the thermodynamic system evaluated at the cosmic horizon, the slope of temperature is always positive for the range $2^{2/3}\left(b_1 + \sqrt{b_1^2 + \frac{b_0}{2^{2/3}}}\right) < y < \frac{1}{2^{1/3}}\left(3b_1 + \sqrt{6 \times 2^{1/3}b_0 + 9b_1^2}\right)$. It is possible to find the extrema of $\bar{T}_{R(c)}$ by solving $\frac{d\bar{T}_{R(c)}}{dy} = 0$. However, these points are out of the range $y$ in our consideration. This implies that there are no extrema for temperature evaluated at cosmic horizon. Therefore, the thermodynamic system evaluated at the cosmic horizon is always locally stable since $\bar{T}_{R(c)}$ and $\frac{d\bar{T}_{R(c)}}{dy}$ are always positive. The behavior of temperature at cosmic horizon is shown in the left panel in Fig. 9.

Next, let us discuss on the thermodynamic volumes $V_{b,c}$. Their dimensionless

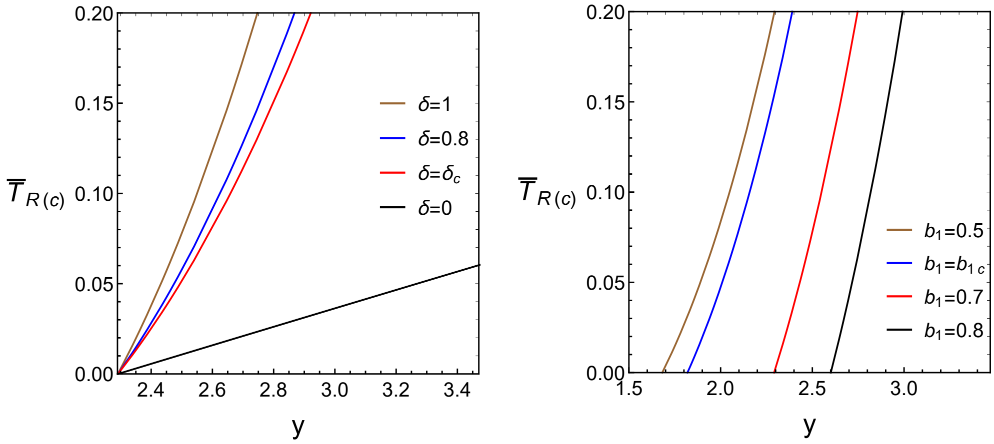


**Figure 9 Left panel shows the temperature profile at cosmic horizon with various values of $\delta$ by fixing $b_0 = 0.1, b_1 = 0.7$ . Right panel shows the temperature profile at cosmic horizon with various values of $b_1$ by fixing $b_0 = 0.1, \delta = 1$ .**

expressions can be written as

$$\bar{V}_b = \frac{1}{\pi r_V^3} V_b = \frac{2}{3} x^3 \left( -1 + 3 \times 2^{2/3} \frac{b_1}{x} + 3 \times 2^{2/3} \frac{b_0}{x^2} \right), \tag{5.81}$$

$$\bar{V}_c = \frac{1}{\pi r_V^3} V_c = \frac{2}{3} y^3 \left( -1 + 3 \times 2^{2/3} \frac{b_1}{y} + 3 \times 2^{2/3} \frac{b_0}{y^2} \right). \tag{5.82}$$

One can see that these volumes can be negative if the horizon radii are large enough. Therefore, it is possible to tune the values of $b_0$ and $b_1$ in which both $V_b$ and $V_c$ are positive within the rang with positive temperature as illustrated in Fig. 10.

The existence of local minimum temperature is provided locally stable-unstable phase transitions. This behavior can be obtained by analyzing the slope of the temperature, which is proportional to the heat capacity. Note that, in our consideration, the slope of the temperature directly infers the sign of the heat capacity. The positiveness of the heat capacity can be determined from the positive slope of the temperature. For the locally thermodynamic stability, the system is required to have positive heat capacity. As $M$ playing the role of enthalpy, the

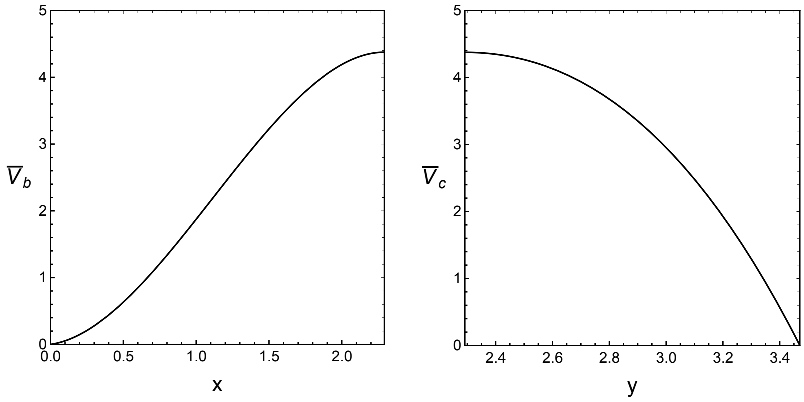


**Figure 10 The volume profile evaluated at black string horizon (left) and cosmic horizon (right) with fixing $b_0 = 0.1, b_1 = 0.7$.**

heat capacity for the system with fixing pressure can be written as

$$C_{R(b,c)} = \pm\left(\frac{\partial M}{\partial T_{R(b,c)}}\right)_P, \tag{5.83}$$

$$\bar{C}_{R(b)} = \frac{x^2(b_0 + 2b_1 x - 2^{-2/3}x^2)}{2^{2/3}b_0\left(\delta x^2 - 1\right) - x^2\left(-4\times 2^{2/3}b_1\delta x + 3\delta x^2 + 1\right)}, \tag{5.84}$$

$$\bar{C}_{R(c)} = \frac{y^2(b_0 + 2b_1 y - 2^{-2/3}y^2)}{2^{2/3}b_0\left(\delta y^2 - 1\right) - y^2\left(-4\times 2^{2/3}b_1\delta y + 3\delta y^2 + 1\right)}. \tag{5.85}$$

The behaviors of heat capacity are shown in Fig. 11. One can see that at the black string horizon, there are three ranges of $x$ in which $\bar{C}_{R(b)}$ is positive for the middle part and the others are negative. Moreover, one found that $\bar{C}_{R(b)}$ changes the sign at extremum points of temperature. As a result, the moderate-sized black string is locally stable while the smaller and larger ones are locally unstable. In other words, there are three possible states of black string. The small black string with higher temperature will radiate thermal energy then it will eventually evaporate away, while the other with the lower temperature will evolve to the moderate-sized black string which lies on the ranges of positive $\bar{C}_{R(b)}$. The large black string with higher temperature will evaporate as losing its mass through thermal radiation and then becomes to moderate-sized eventually. Therefore, the moderated-sized black string state is more stable than the other states. Note that, in the GB limit, it is not possible to have a positive value of the heat capacity as shown in the black line in the right panel in Fig. 11.

For the heat capacity evaluated at the cosmic horizon, there are no divergent points for $\bar{C}_{R(c)}$, since there are no extrema of $\bar{T}_{R(c)}$ for the range of $y$ in our consideration. Therefore, there is no phase transition for the system evaluated at $r_c$. Moreover, one finds that the heat capacity of this system is always positive. This is also compatible with the slope of the temperature that we have analyzed.

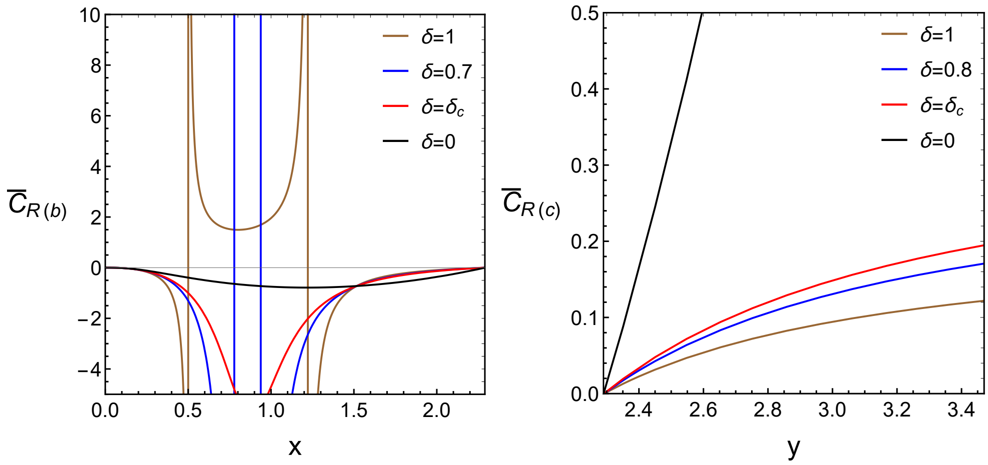


**Figure 11 Left panel shows the heat capacity profile at black string horizon with various values of $\delta$ by fixing $b_0 = 0.1$, $b_1 = 0.7$. Right panel shows the heat capacity profile at black string horizon with various values of $\delta$ by fixing $b_0 = 0.1$, $b_1 = 0.7$.**

We have already analyzed the local stability of the black string by considering the behavior of the heat capacity. Now, we will investigate the global stability by considering the Gibbs free energy. The Gibbs free energy can be expressed as

$$G_{R(b,c)} = M - T_{R(b,c)}S_{R(b,c)}. \tag{5.86}$$

The dimensionless version of the Gibbs free energy for black string and cosmic horizon can be respectively written as

$$\bar{G}_{R(b)} = \frac{G_{R(b)}}{2^{-4/3}M} = x\left(3b_0 + x(3b_1 - \frac{x^2}{2^{2/3}})\right) - \frac{3(b_0 + 2b_1 x - 2^{-2/3}x^2)(1+x^2\delta)\ln(1+x^2\delta)}{2x\delta}, \quad (5.87)$$

$$\bar{G}_{R(c)} = \frac{G_{R(c)}}{2^{-4/3}M} = y\left(3b_0 + x(3b_1 - \frac{y^2}{2^{2/3}})\right) + \frac{3(b_0 + 2b_1 y - 2^{-2/3}y^2)(1+y^2\delta)\ln(1+y^2\delta)}{2y\delta}. \quad (5.88)$$

The system with lower Gibbs free energy at a given temperature prefers to exist compared to those with higher free energy. This state is called being globally stable. For example, if the free energy of the system without black string is zero, thus black string can be formed by the condition $G < 0$. The behavior of Gibbs free energy against the temperature is shown in the left panel in Fig. 12. Notice that the entropy of black string is always positive so that slope of the graph $G_R - T_R$ is always negative, since $\left(\frac{\partial G_{R(b,c)}}{\partial T_{R(b,c)}}\right)_P = -S_{R(b,c)}$. This implies that if entropy keeps increasing, the slope will be more negative. Form the left panel Fig. 12, one can see that there exist two cusps corresponding to two extremum point in the temperature profile denoted by $x_{ex\pm}$. However, the slopes of the Gibbs free energy at these points are still continuous. These points also correspond to the locally stable-unstable phase transitions, since $\left(\frac{\partial^2 G_{R(b)}}{\partial T^2_{R(b)}}\right)_P \propto C_{R(b)}$, where the heat capacity diverges and changes the sign at these points. This is also shown explicitly in the right panel in Fig. 12. The local maximum/minimum of the Gibbs free energy is at the same point with the minimum/maximum of the temperature. We can see that, for the range from $x = 0$ to $x = x_{ex-}$, the Gibbs free energy increases as temperature decreases, while the Gibbs free energy decreases as the temperature increases for the range from $x = x_{ex-}$ to $x = x_{ex+}$, and lastly the Gibbs free energy will increases with temperature decreasing again for the range from $x = x_{ex+}$ to $x = 2^{2/3}\left(b_1 + \sqrt{b_1^2 + \frac{b_0}{2^{2/3}}}\right)$. According to this result, it is possible to obtain the globally stable black string with the dimensionless horizon radius between $x = x_{ex-}$

to $x = x_{ex+}$, since there is a part with negative free energy at a given temperature.

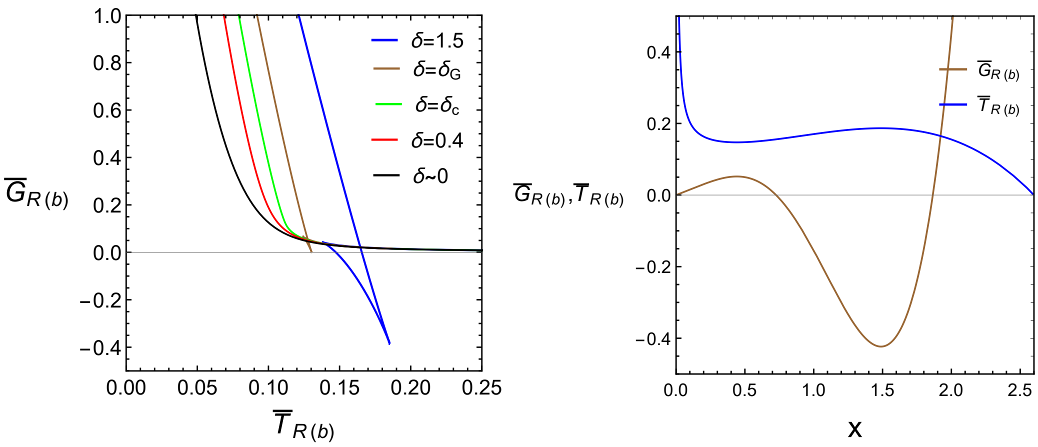


**Figure 12 Left panel shows the Gibbs free energy against the temperature at black string horizon with various values of $\delta$ by fixing $b_0 = 0.1$, $b_1 = 0.7$. Right panel shows the Gibbs free energy and the temperature with respect to $x$ by fixing $b_0 = 0.1$, $b_1 = 0.8$ and $\delta = 1$.**

Moreover, from the left panel in Fig. 12, we can see that there exists a critical value of $\delta$ such that it is not possible to obtain negative Gibbs free energy. Therefore, the lower bound for $\delta$ can be found by requiring the condition $\bar{G}_{R(b)}|_{x_{ex+}} < 0$. This value will be denoted by $\delta_G$. In principle, one can find the expression of $\delta_G$ in terms of parameters $b_0$ and $b_1$, since $x_{ex+}$ depends on $b_0, b_1$ and $\delta$. However, the expression is very lengthy and it is not suitable to show explicitly. In order to obtain the analytical expression of $\delta_G$, we use numerical method evaluating point by point to show that the bound $\delta_G$ slightly depends on the parameter $b_0$ as shown in the left panel in Fig. 13. From this figure, one can see that the approximated value of the bound is still trustable for $b_0 \ll 1$. Therefore, one can use the approximation $b_0 \sim 0$ in order to find the analytic expression of $\delta_G$. By substituting $x_{ex+}$ from Eq. (5.79) to $\bar{G}_{R(b)}$ in Eq. (5.87) and then using approximation $b_0 \ll 1$ and $\Delta \sim 0$ (the approximated free energy denotes as $\bar{G}_{R(\mathrm{app})}$), the bound can be expressed as

$$\delta_G = \frac{9}{8}\frac{\left[e^{\left\{\frac{7}{6}+PL\left(-\frac{7}{6e^{7/6}}\right)\right\}}-1\right]}{2^{1/3}b_1^2} + \frac{b_0}{2b_1^4} = 3\left[e^{\left\{\frac{7}{6}+PL\left(-\frac{7}{6e^{7/6}}\right)\right\}}-1\right]\delta_c + \frac{b_0}{2b_1^4}, \quad (5.89)$$

where $PL(z)$ is the ProductLog function. This function returns the value of $x$ by solving $z = xe^x$. From the right panel in Fig. 13, one can see that the bound from the above expression is very closed and slightly greater than to the numerical result. From this figure, one also see that it is in the same shape with $\delta_c$ but stronger than $\delta_c$.

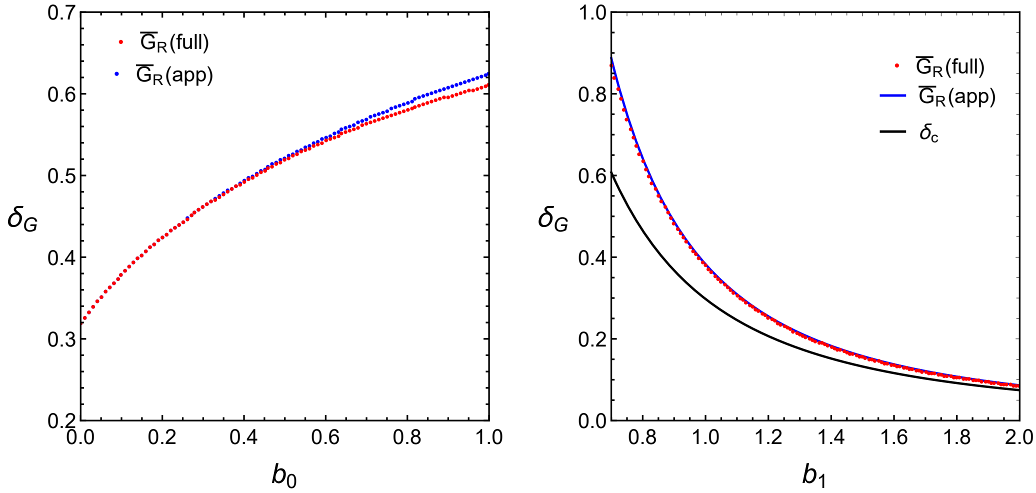


**Figure 13 Left panel shows the comparison of $\delta_G$ obtained from $\bar{G}_R(b)$ in full equation and approximation by fixing $b_1 = 1$. Right panel shows the comparison of $\delta_G$ obtained in full expression and approximation by fixing $b_0 = 0.1$. The black solid curve represents the bound for the local stability $\delta_c$**

In order to have both locally and globally thermodynamic stability, the system at black string horizon must be satisfied the condition $\delta > \delta_G$. It is important to note that there exists a point such that the Gibbs free energy of the non black string state (or hot gas state) and black string state are the same. At this point, the hot gas will transform to the moderate-sized black string. Since the slope of the Gibbs free energy at the transition point is discontinuous, the transition is the first-order phase transition commonly known as the Hawking-Page phase transition. For the system evaluated at cosmic horizon, there are no extrema for $T_{R(c)}$. Therefore, there are no cusps as found in one evaluated at black string horizon. Moreover, $\bar{G}_{R(c)}$ is always negative, and then the black string is stable for the range of $\delta > \delta_G$.

## 5.4 effective thermodynamic systems

For the effective system approach, the whole system is regarded as a single system. The entropy of the effective system is supposed to be an addition of those of two systems as

$$\begin{aligned} S = S_{R(b)} + S_{R(c)} &= \frac{1}{\lambda}\ln\left(1+\lambda\frac{\pi r_b^2}{2}\right) + \frac{1}{\lambda}\ln\left(1+\lambda\frac{\pi r_c^2}{2}\right) \\ &= \frac{1}{\lambda}\ln\left[\left(1+\lambda\frac{\pi r_b^2}{2}\right)\left(1+\lambda\frac{\pi r_c^2}{2}\right)\right]. \end{aligned} \tag{5.90}$$

To obtain the real value of the effective entropy, it requires the condition as

$$\left(1+\lambda\frac{\pi r_b^2}{2}\right)\left(1+\lambda\frac{\pi r_c^2}{2}\right) > 0. \tag{5.91}$$

To satisfy this condition, it is sufficient to restrict our consideration on the positive value of $\lambda$, $\lambda > 0$. In this work, we choose to consider the parameter $M = M(S,P)$ as the enthalpy of the system, since this choice allows us to compare the result with the separated system approach. As a result, the first law for the effective system approach can be written as

$$dM = T_{\text{eff}}dS + V_{\text{eff}}dP, \tag{5.92}$$

where the pressure of this effective system is also defined as the same as one in a separated system approach, i.e., $P = \frac{3}{8\pi}m_g^2$. For the effective system, the sign of heat transfer evaluated at the cosmic horizon is opposite to one at the black string horizon. This is because the observer stays between both horizons. Moreover, the resulting quantities match the first law of thermodynamics. As a result, the effective temperature can be written in terms of $T_{R(b)}$ and $T_{R(c)}$ as

$$\frac{1}{T_{\rm eff}} = \left(\frac{\partial S_{R(b)}}{\partial M}\right)_P - \left(\frac{\partial S_{R(c)}}{\partial M}\right)_P = \frac{1}{T_{R(b)}} + \frac{1}{T_{R(c)}}. \tag{5.93}$$

And then the effective temperature is expressed as

$$T_{\rm eff} = \frac{m_g^2\Big[r_c\left(2c_1 - 3r_c\right) + c_0\Big]\Big[r_b\left(2c_1 - 3r_b\right) + c_0\Big]\left(\pi\lambda r_b^2 + 2\right)\left(\pi\lambda r_c^2 + 2\right)}{8\pi\left(r_b - r_c\right)\left(c_0\left(2 - \pi\lambda r_b r_c\right) + r_b r_c\begin{bmatrix}3\Big\{\pi\lambda\left(r_b r_c + r_b^2 + r_c^2\right) + 2\Big\}\\ -2\pi c_1\lambda\left(r_b + r_c\right)\end{bmatrix}\right)}. \tag{5.94}$$

It is important to note that the effective temperature can be reduced to the black string temperature for the limit $r_c \to \infty$ and to one for cosmic horizon by taking $r_b \to 0$,

$$\lim_{r_c\to\infty} T_{\rm eff} = \frac{m_g^2\left(c_0 + 2c_1 r_b - 3r_b^2\right)}{4\pi r_b}\left(1 + \lambda\frac{\pi r_b^2}{2}\right) = T_{R(b)}, \tag{5.95}$$

$$\lim_{r_b\to 0} T_{\rm eff} = -\frac{m_g^2\left(c_0 + 2c_1 r_c - 3r_c^2\right)}{4\pi r_c}\left(1 + \lambda\frac{\pi r_c^2}{2}\right) = T_{R(c)}. \tag{5.96}$$

Since this effective temperature depends on $r_b$ and $r_c$ which are independent, the slope of the temperature profile does not imply the sign of the heat capacity as we have analyzed in a separated one. This is because the heat capacity is evaluated as the change of temperature with fixing the pressure. Hence, to obtain the temperature profile satisfying the behavior of the heat capacity, we have to find the relation between $r_b$ and $r_c$ such that the pressure is held fixed. By using the horizon equations $\mathcal{F}(r_b) = 0$ and $\mathcal{F}(r_c) = 0$, one obtains the relation as,

$$r_c\mathcal{F}(r_c) - r_b\mathcal{F}(r_b) = 0. \tag{5.97}$$

As a result, $r_c$ can be written in terms of $r_b$ as

$$r_c = \frac{\sqrt{2c_1 r_b - 3r_b^2 + c_1^2 + 4c_0} + r_b - c_1}{2}. \tag{5.98}$$

Substituting $r_c$ from the above equation to $T_{\rm eff}$ in Eq. (5.94), then we obtain $T_{\rm eff} = T_{\rm eff}(r_b)$. Therefore, the dimensionless version for this effective temperature can be written as

$$\bar{T}_{\rm eff} = \frac{r_V^2}{3\times 2^{2/3}M} T_{\rm eff}(r_b)|_{r_b = x r_v} \tag{5.99}$$

$$= \frac{\left[\begin{array}{c} \left(1+x^2\delta\right)\left(x(2\sqrt[3]{4}b_1 - x) + 2^{2/3}b_0\right) \\ \times\left[2^{2/3}b_1\left(\sqrt{3}\beta + 2x\right) + 6\sqrt[3]{2}b_1^2 + 4\sqrt[3]{4}b_0 - x\left(\sqrt{3}\beta + x\right)\right] \\ \times\left[\delta\left(3\sqrt[3]{4}\sqrt{3}\beta b_1 + 18\sqrt[3]{2}b_1^2 + 6\sqrt[3]{4}b_0 - x\left(\sqrt{3}\beta + x\right)\right) + 2\right] \end{array}\right]}{\left[\begin{array}{c} 4\sqrt[3]{4}\pi\left(\sqrt{3}\beta + 3\ 2^{2/3}b_1 - 3x\right) \\ \times\left[\begin{array}{c} x\left(36b_1^3\delta + 2^{2/3}b_1\left(3 - 2\delta x^2\right) + 6\sqrt[3]{2}b_1^2\delta\left(\sqrt{3}\beta + x\right) + \sqrt{3}\beta - x\right) \\ +2b_0\left(12\sqrt[3]{2}b_1\delta x + 2^{2/3}\left(\delta x\left(\sqrt{3}\beta - x\right) + 1\right)\right) \end{array}\right] \end{array}\right]}, \tag{5.100}$$

where $\beta = \sqrt{4\times 2^{2/3}b_0 + 6\sqrt[3]{2}b_1^2 + 2\times 2^{2/3}b_1 x - x^2}$. It is more convenient to use the numerical plot to see the behavior as shown in the Fig. 14. From the left panel in this figure, one can see that there exists the positive slope of the temperature. This implies that there is the suitable size of the black string corresponding to the positive heat capacity. The locus is similar to one for the system evaluated at the black string horizon. It is also seen that the effective temperature is always less than one evaluated at the black string horizon which is compatible with formula (5.93). As a result, at certain size of the stable black string, the effective temperature is always less than one evaluated at the black string horizon. Moreover, from the right panel in Fig. 14, there exists a particular low temperature (e.g. $\bar{T}_3$) at which only the black string in effective system approach will be locally stable while one for the separated system approach is not. For the same argument, there exists a particularly high temperature (e.g. $\bar{T}_1$) at which only the black string in effective system approach will not be locally stable while one for the separated system approach is locally stable. Moreover, for a particular temperature (e.g. $\bar{T}_2$)

at which the systems in both approaches are locally stable, the black string from the effective system approach is always larger than one in the separated system approach. Therefore, this criteria provide us with how to distinguish the thermodynamic description for the dRGT black string if this black string really exists in nature.

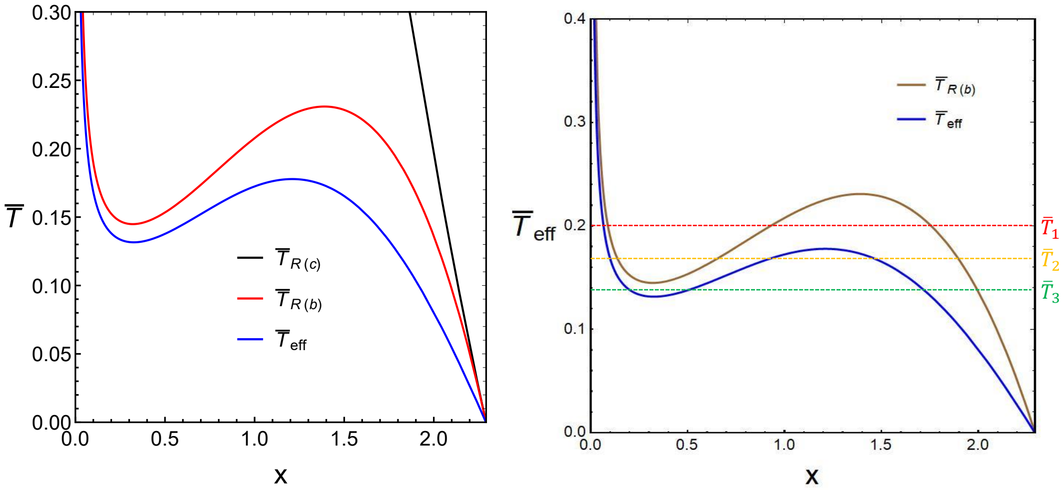


**Figure 14 Left panel shows the comparison of temperatures of the system evaluated at black string horizon, cosmic horizon and effective system by fixing $b_0 = 0.1, b_1 = 0.7$ and $\delta = 2$. Right panel shows the comparison of temperatures of the system evaluated at black string horizon and effective system by fixing $b_0 = 0.1, b_1 = 0.7$ and $\delta = 2$**

Let us consider the effect of non-extensive parameter on the temperature profile for the effective system approach. Since it has the similar locus as one in the separated system approach, it is possible to exist the lower bound of the non-extensive parameter as shown in Fig. 15. In order to find the bound, one can use the same strategy as performed in the previous section by finding the condition to have a positive real solution of the equation $\partial_{r_b} T_{R(b)} = 0$. However, for the effective case, the temperature depends on both $r_b$ and $r_c$. Therefore, one has to find the condition for the existence of the extrema along the direction with fixing pressure.

As a result, the equation for the effective system approach can be written as

$$F(r_b, r_c) = \partial_{r_b} T_{R(b)} + H(r_b, r_c) = 0, \quad H(r_b, r_c) = \partial_{r_c} T_{R(c)} \frac{T^2_{R(b)}}{T^2_{R(c)}} \frac{dr_c}{dr_b}. \tag{5.101}$$

As we have analyzed previously, $\partial_{r_b} T_{R(b)}$ is a convex function and depends only on $r_b$. The additional function $H(r_b, r_c)$ is always negative, since $\partial_{r_c} T_{R(c)} > 0$ and $\frac{dr_c}{dr_b} < 0$. Moreover, since the function $H(r_b, r_c)$ has a part which is divided by $T^2_{R(c)}$, it will be a small function. Note that $T_{R(c)}$ is much greater than $T_{R(b)}$ as found in the left panel in Fig. 14. Therefore, the function $F(r_b, r_c)$ can be written as a convex function subtracted by a small positive function. By using Eq. (5.98), the function, $F(r_b, r_c)$, can be written as a function of only $r_b$. Therefore, one can find the condition on $\delta$ to satisfy the Eq.(5.101), since it is the convex function. By using numerical method, one can find the bound on the non-extensive parameter as

$$\delta_{\text{eff}} \geq 1.254 \left( \frac{3}{8 \times 2^{1/3} b_1^2} \right) = 1.254 \delta_c. \tag{5.102}$$

One can see that the bound from the effective system approach is stronger than one in the separated system approach, $\delta_{\text{eff}} > \delta_c$. This also infers from Eq. (5.101), since the convex function for effective system approach is lower than one in separated system approach. The behavior of the temperature profile with various values of $\delta$ is illustrated in Fig. 15.

Before discussing on the local stability, let us consider the effective volume which is computed from the Eq.(3.178)

$$V_{\text{eff}} = T_{\text{eff}} \left( \frac{V_b}{T_{R(b)}} + \frac{V_c}{T_{R(c)}} \right). \tag{5.103}$$

It can be realized that $V_c = V_c(r_c)$ is indeed the same function as $V_b = V_b(r_b)$, just replacing the different range of the horizon radius, $r_c \to r_b$. Hence, using this fact and Eq. (5.93), the effective is identical to $V_b$ for the variable $r_b$ and to $V_c$ for the variable $r_c$,

$$V_{\text{eff}}(r_b) = V_b(r_b), \quad V_{\text{eff}}(r_c) = V_c(r_c). \tag{5.104}$$

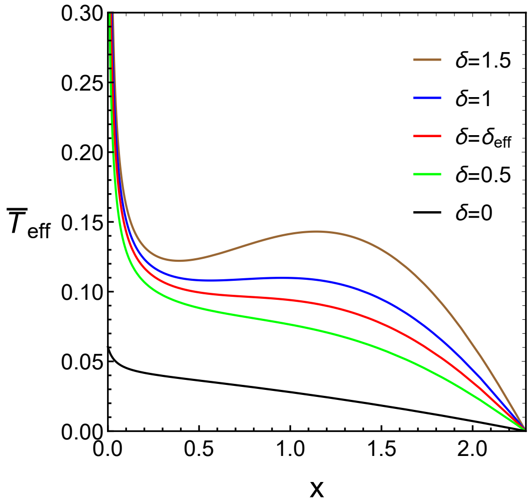


**Figure 15 The effective temperature profile with various values of $\delta$ by fixing $b_0 = 0.1, b_1 = 0.7$.**

It is then possible to choose suitable values of the parameters $b_0$ and $b_1$ corresponding to the effective volume being positive within its viable range. One also notes that this effective volume is independent of the non-extensive parameter $\lambda$ or $\delta$.

Now, let us consider the heat capacity at constant pressure. The thermodynamic system is locally stable if heat capacity is positive. The heat capacity of the effective system can be found by

$$C_{\rm eff} = \left(\frac{\partial M}{\partial T_{\rm eff}}\right)_P = \frac{\partial_{r_b} M dr_b + \partial_{r_c} M dr_c}{\partial_{r_b} T_{\rm eff} dr_b + \partial_{r_c} T_{\rm eff} dr_c} \tag{5.105}$$

$$= \frac{2\pi\mathcal{C}_1^2\,(r_b - r_c)^{\,3}\left(c_0 + 2c_1^2 - 5c_1\,(r_b + r_c) + 3\left(3r_b r_c + r_b^2 + r_c^2\right)\right)}{\left[\begin{array}{c}\mathcal{C}_4 - \mathcal{C}_2\Bigg[\mathcal{C}_1\mathcal{C}_3\left(\mathcal{C}_6 - 2(c_1 - 3r_b)(r_b - r_c)\right)\\ +\mathcal{C}_3\mathcal{C}_5\mathcal{C}_6 r_c(r_c - r_b) - 2\mathcal{C}_1\mathcal{C}_6\pi\lambda r_b(r_b - r_c)(2 + \pi\lambda r_b^2)\Bigg]\end{array}\right]}, \tag{5.106}$$

where

$$\mathcal{C}_1 = c_0\left(2-\pi\lambda r_b r_c\right) + r_b r_c\left(-2\pi c_1\lambda\left(r_b+r_c\right) + 3\pi\lambda\left(r_b r_c + r_b^2 + r_c^2\right) + 6\right), \quad (5.107)$$

$$\mathcal{C}_2 = \left(r_c\left(2c_1 - 3r_c\right) + c_0\right)\left(-r_b - 2r_c + c_1\right), \quad (5.108)$$

$$\mathcal{C}_3 = \left(\pi\lambda r_b^2 + 2\right)\left(\pi\lambda r_c^2 + 2\right), \quad (5.109)$$

$$\begin{aligned}\mathcal{C}_4 = {} & \left(r_b\left(2c_1 - 3r_b\right) + c_0\right)^2\left(-2r_b - r_c + c_1\right)\left(\pi\lambda r_b^2 + 2\right)^2 \\ & \times\left(r_c^2\left(-9\pi\lambda r_c^2 + 4\pi c_1\lambda r_c - 6\right) + c_0\left(\pi\lambda r_c^2 - 2\right)\right),\end{aligned} \quad (5.110)$$

$$\mathcal{C}_5 = 2\pi c_1\lambda\left(2r_b + r_c\right) - 3\pi\lambda\left(2r_b r_c + 3r_b^2 + r_c^2\right) + \pi c_0\lambda - 6, \quad (5.111)$$

$$\mathcal{C}_6 = r_b\left(2c_1 - 3r_b\right) + c_0. \quad (5.112)$$

From the right panel in Fig. 16, one can see that the heat capacity diverges at the extrema of the temperature and the positive part corresponds to the positive slope of the temperature. The dimensionless version of the heat capacity can be written as,

$$\bar{C}_{\text{eff}} = \frac{C_{\text{eff}}}{\pi 2^{2/3} r_V^2} \quad (5.113)$$

$$= \frac{\mathcal{X}_1^2(x-y)^3\left(2^{2/3}b_0 + 12\sqrt[3]{2}\text{b1}^2 - 5\ 2^{2/3}\text{b1}(x+y) + x^2 + 3xy + y^2\right)}{\left[\begin{array}{c}\mathcal{X}_3^2\mathcal{X}_6\left(\delta x^2+1\right)^2\left(3\ 2^{2/3}b_1 - 2x - y\right) - 2\mathcal{X}_1\mathcal{X}_2\mathcal{X}_3\mathcal{X}_4(x-y)\left(\delta y^2+1\right)\delta x \\ +2\mathcal{X}_1\mathcal{X}_2\mathcal{X}_4\left(2^{2/3}b_1 - x\right)\left(\delta x^2+1\right)(x-y)\left(\delta y^2+1\right) \\ +\mathcal{X}_1\mathcal{X}_2\mathcal{X}_3\mathcal{X}_4\left(\delta x^2+1\right)\left(\delta y^2+1\right) + \mathcal{X}_2\mathcal{X}_3\mathcal{X}_4\mathcal{X}_5 y\left(\delta x^2+1\right)(x-y)\left(\delta y^2+1\right)\end{array}\right]}, \quad (5.114)$$

$$\mathcal{X}_1 = 2^{2/3}b_0(\delta xy - 1) - xy\left(-2\ 2^{2/3}b_1\delta(x+y) + \delta x^2 + \delta xy + \delta y^2 + 1\right), \quad (5.115)$$

$$\mathcal{X}_2 = 2^{2/3}b_0 + y\left(2\ 2^{2/3}b_1 - y\right), \quad (5.116)$$

$$\mathcal{X}_3 = x\left(x - 2\ 2^{2/3}b_1\right) - 2^{2/3}b_0, \quad (5.117)$$

$$\mathcal{X}_4 = -3\ 2^{2/3}b_1 + x + 2y, \quad (5.118)$$

$$\mathcal{X}_5 = 2^{2/3} b_0 \delta + 2\ 2^{2/3} b_1 \delta (2x + y) - 3\delta x^2 - 2\delta x y - \delta y^2 - 1, \tag{5.119}$$

$$\mathcal{X}_6 = 2^{2/3} b_0 \left(\delta y^2 - 1\right) + y^2 \left(4\ 2^{2/3} b_1 \delta y - 3\delta y^2 - 1\right), \tag{5.120}$$

$$y = \frac{1}{2}\left(\sqrt{3}\sqrt{4\ 2^{2/3} b_0 + 6\sqrt[3]{2} b_1^2 + 2\ 2^{2/3} b_1 x - x^2} + 3\ 2^{2/3} b_1 - x\right). \tag{5.121}$$

Note that we have used the dimensionless variables for the plot in Fig. 16 and the variable $r_c$ is transformed to $r_b$ by using relation in Eq. (5.98). From the left panel in this figure, one found that the heat capacity is always negative for $\delta < \delta_{\text{eff}}$ and the middle part becomes positive for $\delta > \delta_{\text{eff}}$. This is compatible to the analysis of the temperature profile. Summarily, for the effective system approach, the moderated-sized black string is locally stable for $\delta > \delta_{\text{eff}}$.

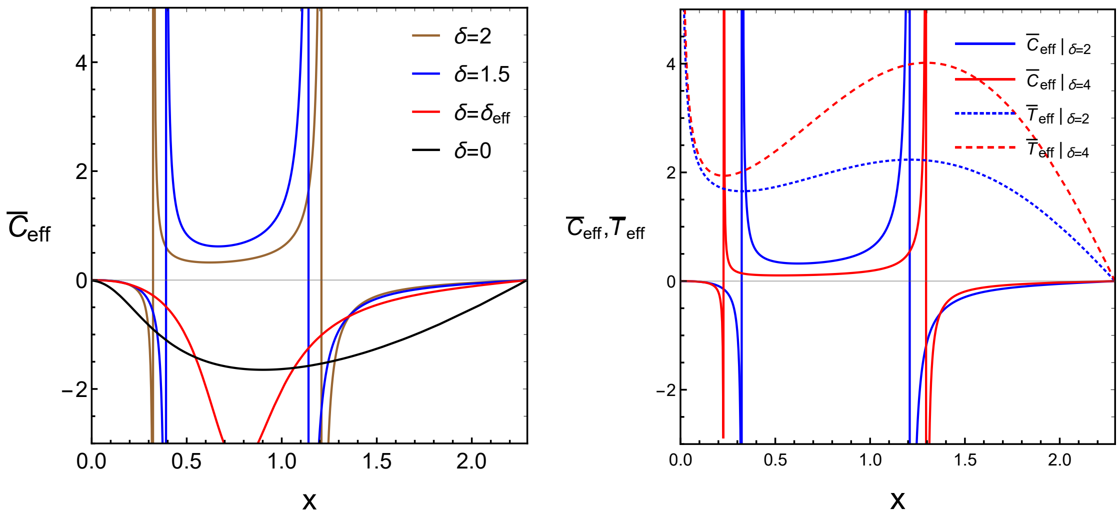


**Figure 16 Left panel shows the effective heat capacity profile with various values of $\delta$ by fixing $b_0 = 0.1$, $b_1 = 0.7$. Right panel shows the effective heat capacity profile compare to the temperature profile by fixing $b_0 = 0.1$, $b_1 = 0.7$.**

Let us move to consider the global stability, which can be determined by considering the value of the Gibbs free energy. According to effective description, the effective Gibbs free energy can be written as

$$G_{\text{eff}} = M - T_{\text{eff}}\Big(S_{R(b)} + S_{R(c)}\Big), \tag{5.122}$$

$$= -\frac{r_b}{4} m_g^2 (r_b^2 - c_1 r_b - c_0) - \frac{m_g^2 \left[\begin{array}{c} (\pi\lambda r_b^2 + 2)\,(\pi\lambda r_c^2 + 2)\,\frac{1}{\lambda}\ln\left[\left(1 + \lambda\frac{\pi r_b^2}{2}\right)\left(1 + \lambda\frac{\pi r_c^2}{2}\right)\right] \\ \Big[r_c\,(2c_1 - 3r_c) + c_0\Big]\Big[r_b\,(2c_1 - 3r_b) + c_0\Big] \end{array}\right]}{8\pi\,(r_b - r_c)\left[\begin{array}{c} c_0\,(2 - \pi\lambda r_b r_c) + \\ r_b r_c \Big[3\Big(\pi\lambda\,(r_b r_c + r_b^2 + r_c^2) + 2\Big) \\ -2\pi c_1 \lambda\,(r_b + r_c)\Big] \end{array}\right]}. \tag{5.123}$$

The behavior of the effective Gibbs free energy is shown in the left panel in Fig. 17. From this figure, it is found that the Gibbs free energy corresponding to the locally stable size of the black string is always negative. Therefore, the locally stable black string is always globally stable. This feature is different from one for the separated system approach in which there is a part of parameter corresponding to the positive value of Gibbs free energy. Hence, there is no other bound of $\delta$ for the Gibbs free energy in effective description. Note that the part between cusps of $\bar{G}_{\text{eff}}$ corresponds to the moderate-sized black string. It is important to note also that the slope of the lines plotted in the left panel in Fig. 17 does not correspond to the entropy, since the negative sign is needed to add in the change in entropy at the cosmic horizon. As a result, the cusps for the effective system approach is not sharp as found in one for the separated system approach. This is also see from the right panel in Fig. 17.

Moreover, it is found that the effective free energy at a certain temperature is more negative compared to one for the system evaluated at the black string horizon as shown in the right panel in Fig. 17. By comparing to the free energy of the hot gas which is zero, it is found that the Gibbs free energy is discontinuous.

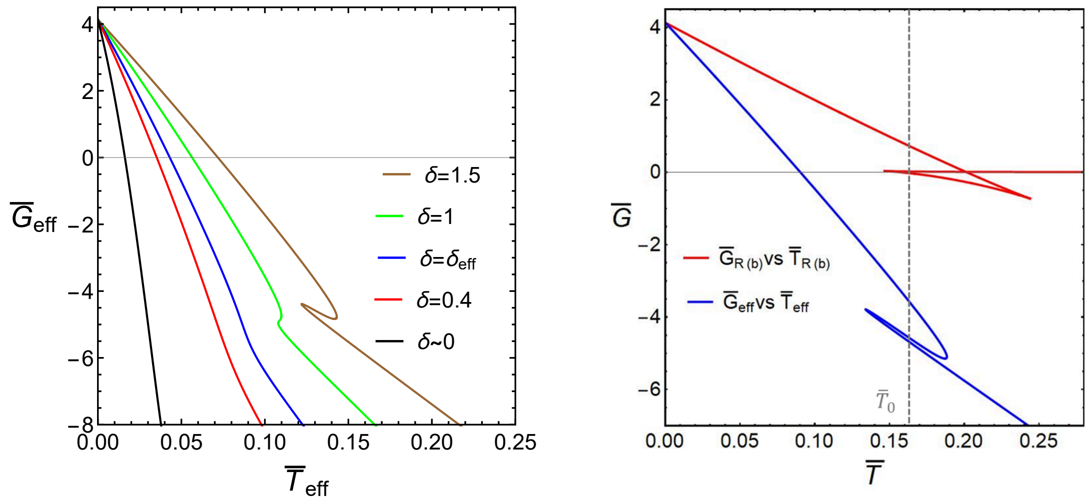


**Figure 17 Left panel shows the profile of the dimensionless Gibbs free energy $\bar{G}_{\text{eff}} = \frac{2^{4/3}}{M} G_{\text{eff}}$ against the effective temperature with various values of $\delta$ by fixing $b_0 = 0.1, b_1 = 0.7$. The right panel shows the comparison of the Gibbs free energy for separated system approach and effective system approach with $b_0 = 0.1, b_1 = 0.7$ and $\delta = 2.15$.**

As a result, for the effective system approach, the hot gas phase has to undergo a zeroth-order phase transition in order to evolve into the moderate-sized stable black string in the effective system. This is one of the key differences to the separated system approach in which the transition is the first-order type as we have discussed.

# CHAPTER VI

# CONCLUSION

In the present work, we aim to investigate the thermodynamic properties of the dRGT black string with asymptotically dS spacetime. When we study the thermodynamic behavior of dRGT black string in the dS branch, there are usually exist two horizons. These two horizons correspond to two thermal systems with generically different temperatures. As a result, the systems are out of thermal equilibrium. Therefore, in this work, the investigation can be performed in two different approaches as follows. Firstly, in the separated system approach, the thermodynamic system can be defined separately and treated to be in a quasi-equilibrium state. Secondly, in the effective system approach, one can treat the systems as a single system described by the effective thermodynamic quantities. For these approaches, the thermodynamic system in consideration can be in the thermal equilibrium. For another obstruction for the dS branch, the system is unstable since the dRGT black string has a negative heat capacity when its thermodynamic quantities are defined based on the Gibbs-Boltzmann statistics. The Rényi entropy can be used to investigate the possibility to obtain the stable dRGT black string. We begin the investigation by showing that the existence of two horizons is a generic property of the dRGT black string with asymptotically dS spacetime. We found that it is possible to find the region for the existence of the horizons in the parameter space $(b_1, b_0)$ as shown in the Fig.6.

For the separated system approach, we examined the non-extensivity by replacing the Rényi entropy with the Gibbs-Boltzmann ones. As a result, we obtained the positive slope of the temperature which corresponds to the positive heat capacity. This suggests that it is possible to obtain the locally stable black string. It is found that the bound on the non-extensive parameter is obey $\delta > \delta_c = 3/(2^{10/3} b_1^2)$. We further investigate the global stability by considering the sign of the Gibbs free energy. By requiring that the Gibbs free energy of the preferred black string must be negative, it is found that the lower bound is stronger than one obtained from local stability, $\delta_G > \delta_c$. Furthermore, we found that in the range $\delta > \delta_G$ it is possible to obtain the first-order Hawking-Page phase transition which is the transition between the thermal radiation or hot gas phase and the stable black string phase. Note that, in the viable range of the black string system $\delta > \delta_G$, the system evaluated at the cosmic horizon is both locally and globally stable.

For the effective system approach, we still restrict on the first law which is in the same form as one in the separated system approach. However, the entropy of the effective system is considered as in the additive form of those of the separated systems. The temperature and volume are treated as effective quantities. As a result, it is possible to find the region of parameter space with positive heat capacity corresponding to a locally stable black string. The lower bound of the non-extensivity parameter can be obtained by analyzing the possibility of the existence of the local extrema of the temperature. We found that the bound in the effective system approach is stronger than one in separated system approach, $\delta_{\text{eff}} = 1.254\delta_c$. This implies that the thermodynamic stability of the black string in the effective system approach requires the non-extensive nature of the system greater than one in the separated system approach. For the global stability, we found that the Gibbs free energy in the range with local stability is always negative. Therefore, there is no further bound on the non-extensive parameter, the locally stable black string is always globally stable. The Hawking-Pages phase transition is found to undergo from hot gas to the black string with the zeroth-order phase transition. This is one of the important results to distinguish between the two approaches since it is the first-order phase transition for the separated system approach.

In conclusion, we found that it is possible to have the thermodynamically stable dRGT black string by considering the Rényi entropy instead of using the

Gibbs-Boltzmann ones. Moreover, we also found the way to distinguish the black string from both approaches. In particular, we found that there exist particular temperatures in which the black string in both approaches will be locally stable. In this case, the black string in the effective system approach is always larger than the one in the separated system approach. Moreover, there exist particular temperatures for which only black string in the effective or separated system approach is stable. As a result, these particular temperatures can be used to distinguish between the two approaches.

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# APPENDIX

# APPENDIX A HOW TO FIND A PROPORTIONAL CONSTANT IN THE EINSTEIN'S FIELD EQUATION

In order to find the constant $k$ in the Einstein's field equation, one can use the fact that GR can be reduced to the Newtonian theory in the Newtonian limit. In Newtonian theory, one has Poisson equation as

$$\vec{\nabla}^2\Phi = 4\pi\rho. \tag{A.1}$$

Consider the trace of Einstein's field equation,

$$g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = R - 2R = kg^{\mu\nu}T_{\mu\nu} \tag{A.2}$$

$$R = -kT \tag{A.3}$$

Then, substituting back to Einstein's field equation, one obtains,

$$R_{\mu\nu} = k(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{A.4}$$

By using the $(0,0)$ component, we can write Einstein's field equation as

$$R_{00} = \frac{1}{2}k\rho, \tag{A.5}$$

where, $T_{00} = \rho$ and $T = g^{00}T_{00} = -\rho$. By comparing the result in the Eq.(3.50) and Newton's second law of motion in Newtonian theory,

$$\vec{F} = -m\vec{\nabla}\Phi = m\vec{a} = m\frac{1}{2}\vec{\nabla}h_{00}, \tag{A.6}$$

we thus obtain $h_{00} = -2\Phi$. Therefore, we can compute $R_{00}$ for $g_{00} = \eta_{00} - h_{00}$ with keep only first order in $h_{00}$ as,

$$R_{00} = -\frac{1}{2}\partial_i\partial^i h_{00} = \vec{\nabla}^2\Phi. \tag{A.7}$$

Comparing the Eq.(A.1), (A.5) and (A.7),

$$R_{00} = \vec{\nabla}^2 \Phi = \frac{1}{2} k\rho = 4\pi\rho, \tag{A.8}$$

we eventually obtain,

$$k = 8\pi. \tag{A.9}$$

# BIOGRAPHY

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