NARESUAN UNIVERSITY
The Institute for Fundamental Study (IF)

FOURIER TRANSFORM AND THE DIRAC DELTA FUNCTION (M2)
SUMMER SCHOOL
Einstein's Term 2023

Homework assignment
(Final update: May 29, 2023)

1st) Consider the following discrete orthonormal basis in $\mathcal{L}^2(\Omega)$, where Ω is the interval $[0, \ell]$:

$$\psi_a(x) = \sqrt{\frac{1}{\ell}} \exp\left(\mathrm{i}\frac{2\pi a}{\ell}x\right), \quad a = \cdots, -2, -1, 0, +1, +2, \cdots . \tag{1.1}$$

Every function $\psi(x) \in \mathcal{L}^2(\Omega)$ can be expanded in one and only one way in terms of the $\{\psi_a(x)\}$:

$$\psi(x) = \sum_{a=-\infty}^{\infty} C_a \, \psi_a(x). \tag{1.2}$$

As you know, the coefficients of the latter series are given by the formula $C_a = \langle \psi_a, \psi \rangle$. Also note that $\psi(x)$ satisfies the periodic boundary condition, i.e., $\psi(0) = \psi(\ell)$. (a) Try to obtain the typical Fourier series from the series given above, namely,

$$\psi(x) = A_0 + \sum_{a=1}^{\infty} A_a \cos\left(\frac{2\pi a}{\ell}x\right) + \sum_{a=1}^{\infty} B_a \sin\left(\frac{2\pi a}{\ell}x\right), \tag{1.3}$$

with the coefficients A_0, A_a and B_a written in terms of the coefficients C_a, namely,

$$A_0 = \sqrt{\frac{1}{\ell}}\, C_0 = \frac{1}{\ell}\int_0^{\ell} \mathrm{d}x \, \psi(x)\,, \tag{1.4}$$

$$A_a = \sqrt{\frac{1}{\ell}}\,(C_a + C_{-a}) = \frac{2}{\ell}\int_0^{\ell} \mathrm{d}x \, \cos\left(\frac{2\pi a}{\ell}x\right)\psi(x)\,, \tag{1.5}$$

$$B_a = \sqrt{\frac{1}{\ell}}\,\mathrm{i}(C_a - C_{-a}) = \frac{2}{\ell}\int_0^{\ell} \mathrm{d}x \, \sin\left(\frac{2\pi a}{\ell}x\right)\psi(x)\,. \tag{1.6}$$

(b) Demonstrate that $\psi(x)$ is real if and only if $C_{-a} = C_a^*$ (the asterisk *denotes the complex conjugate, as usual), and therefore, the coefficients A_0, A_a and B_a are real.
[3+1=4 Pts.]

2nd) Try to demonstrate the following integral representation for the Dirac delta:

$$\int_{\mathbb{R}} \mathrm{d}x \, \mathrm{e}^{\mathrm{i}kx} = \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{\mathrm{i}kx} = \lim_{a\to 0}\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-a|x|} = 2\pi\,\delta(k). \tag{2.1}$$

Note that, I am proposing you to add a regularizing factor to derive the expression. You will also need to use the following representation of the Dirac delta:

$$\lim_{a\to 0} \frac{1}{\pi} \frac{a}{a^2 + k^2} = \delta(k). \tag{2.2}$$

Note: In both limits, the parameter a approaches zero from the positive side.
[4 Pts.]

3rd) As presented in class, if $\psi(x)$ is a (real or complex) function of the variable x, its Fourier transform $\phi(k)$, if it exist, is defined by

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi(x)\, \mathrm{e}^{-\mathrm{i}kx} \equiv \mathrm{FT}[\psi(x)], \tag{3.1}$$

and the inverse formula is:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k\, \phi(k)\, \mathrm{e}^{+\mathrm{i}kx} \equiv (\mathrm{FT})^{-1}[\phi(k)]. \tag{3.2}$$

(a) Demonstrate the following property:

$$\mathrm{FT}[\psi(cx)] = \frac{1}{|c|}\, \phi\left(k/c\right), \tag{3.3}$$

where c is a (real) constant. (b) In particular, $\mathrm{FT}[\psi(-x)] = \phi(-k)$, and together with $\mathrm{FT}[\psi(x)] = \phi(k)$ (see Eq. (3.1)), it follows that if the function $\psi(x)$ has a definite parity, its Fourier transform $\phi(k)$ has the same parity. Prove it!
[3+1=4 Pts.]

4th) Prove the following result (in fact, it is a theorem): If $\phi(k)$ and $\varphi(k)$ are the respective Fourier transforms of the square-integrable functions $\psi(x)$ and $\chi(x)$, one has that

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \chi^*(x)\, \psi(x) = \int_{-\infty}^{+\infty} \mathrm{d}k\, \varphi^*(k)\, \phi(k). \tag{4.1}$$

A particular case ($\psi = \chi$) of this result is the conservation of the norm

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \left|\psi(x)\right|^2 = \int_{-\infty}^{+\infty} \mathrm{d}k\, \left|\phi(k)\right|^2, \tag{4.2}$$

that is, a function and its Fourier transform have the same norm. This result is called the Parseval-Plancherel formula.
[4 Pts.]

5th) (a) Demonstrate that the Fourier transform of the Heaviside step function $\Theta(x)$ is given by

$$\phi(k) = \mathrm{FT}[\Theta(x)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \Theta(x)\, \mathrm{e}^{-\mathrm{i}kx} = \frac{(-\mathrm{i})}{\sqrt{2\pi}} \lim_{a\to 0} \frac{1}{k - \mathrm{i}a}, \tag{5.1}$$

(remember that $\Theta(x < 0) = 0$ and $\Theta(x > 0) = 1$). The limit can be evaluated using the following identity:

$$\lim_{a\to 0} \frac{1}{k \pm \mathrm{i}a} = \mathrm{P.V.} \left(\frac{1}{k}\right) \mp \mathrm{i}\pi\delta(k)$$

$$\Bigg[\text{i.e.,} \lim_{a\to 0} \int_{\mathbb{R}} \mathrm{d}k\, \frac{f(k)}{k \pm \mathrm{i}a} = \mathrm{P.V.} \int_{\mathbb{R}} \mathrm{d}k\, \frac{f(k)}{k} \mp \mathrm{i}\pi \int_{\mathbb{R}} \mathrm{d}k\, f(k)\delta(k)$$

$$= \mathrm{P.V.} \int_{\mathbb{R}} \mathrm{d}k\, \frac{f(k)}{k} \mp \mathrm{i}\pi f(0) \Bigg]. \tag{5.2}$$

Certainly, $f(x)$ is a regular function at $k = 0$ (in these limits, the parameter approaches zero from the positive side), and P.V. represents the Cauchy principal value of the integral to the right of the symbol P.V., namely,

$$\mathrm{P.V.} \int_{\mathbb{R}} \mathrm{d}k\, \frac{f(k)}{k} \equiv \lim_{\epsilon\to 0} \int_{-\infty}^{-\epsilon} \mathrm{d}k\, \frac{f(k)}{k} + \lim_{\epsilon\to 0} \int_{+\epsilon}^{+\infty} \mathrm{d}k\, \frac{f(k)}{k} \quad (\epsilon > 0). \tag{5.3}$$

(b) Find the inverse Fourier transform of $\phi(k)$

$$(\mathrm{FT})^{-1}[\phi(k)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}k\, \phi(k)\, \mathrm{e}^{+\mathrm{i}kx}. \tag{5.4}$$

Did you find what you expected? Hint:

$$\frac{1}{\pi} \int_{\mathbb{R}} \mathrm{d}k\, \frac{\sin(kx)}{k} = \mathrm{sgn}(x) = 2\,\Theta(x) - 1. \tag{5.5}$$

[2+2=4 Pts.]

SDeV/SDeV (*)

(*) `https://bit.ly/3DM8jJE`