NARESUAN UNIVERSITY
THE INSTITUTE FOR FUNDAMENTAL STUDY
TAH POE SEMINAR SERIES LIV (54) N°5

Characterizing 1D Klein-Fock-Gordon-Majorana particles

Salvatore De Vincenzo [a), b)]

The Institute for Fundamental Study (IF),

Naresuan University, Phitsanulok 65000, Thailand

1:30 – 2:30 p.m. Friday, 20 October 2023

a) ORCID

b) My IF personal webpage

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Abstract

- Theoretically, in (1+1) dimensions, one can have Klein-Fock-Gordon-Majorana (KFGM) particles. More precisely, these are one-dimensional (1D) Klein-Fock-Gordon (KFG) and Majorana particles at the same time. In principle, the wave equations considered to describe such first quantized particles are the standard 1D KFG equation and/or the 1D Feshbach-Villars (FV) equation, each with a real Lorentz scalar potential and some kind of Majorana condition. In this talk, we analyze entirely and systematically the latter assumption; additionally, we introduce specific equations and boundary conditions to characterize these particles. In fact, we write first order in time equations that do not have a Hamiltonian form. We may refer to these equations as the first-order 1D Majorana equations for the 1D KFGM particle. Moreover, each of them leads to –let us say– a (complex) second order in time 1D Majorana equation. In closing, we briefly discuss the nonrelativistic limit of one of the first-order 1D Majorana equations.

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Ettore Majorana in his twenties (*)

–On Saturday March 26, 1938, the director of the Institute of Physics at the University of Naples in Italy, Antonio Carrelli, received a mysterious telegram. It had been sent the previous day from the Sicilian capital Palermo, some 300 km across the Tyrrhenian Sea, and read:

"Don't worry. A letter will follow. Majorana."

–By Sunday the promised letter had reached Carrelli. In it Majorana wrote that he had abandoned his suicidal intentions and would return to Naples, but it revealed no hint of where the illustrious physicist might be. The picture was quickly becoming clear: Majorana had disappeared.

(*) Born: 05/08/1906. Died: Missing since 25/03/1938.

–Worried by these circumstances, Carrelli called his friend Enrico Fermi in Rome, who immediately realized the seriousness of the situation. Fermi was working in his laboratory with the young physicist Giuseppe Cocconi at the time.

–Fermi told Cocconi: *"You see, in the world there are various categories of scientists: there are people of a secondary or tertiary standing, who do their best but do not go very far. There are also those of high standing, who come to discoveries of great importance, fundamental for the development of science* [Fermi considered himself to be in this category.]. *But then there are geniuses like (Galileo) Galilei and Newton. Well, Ettore was one of them. Majorana had what no-one else in the world had"* (**)

(**) S. Esposito, The Physics of Ettore Majorana –Phenomenological, Theoretical, and Mathematical– (Cambridge University Press, Cambridge, 2015).

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Introduction

- In (3+1) dimensions, there is the possibility that a spin-0 particle is its own antiparticle. A typical example of this could be the neutral pion (or neutral pi meson) π^0 (although it is not exactly an elementary particle).

- We may refer to these particles as three-dimensional (3D) Klein-Fock-Gordon-Majorana (KFGM) particles.

- Recall that, in general, a Majorana particle is its own antiparticle, i.e., it is a strictly neutral particle, and the wavefunction that characterizes it is invariant under the respective charge-conjugation operation (within a phase factor).

- Among the known spin-$\frac{1}{2}$ particles, only neutrinos could be of a Majorana nature, i.e., only neutrinos could be Majorana fermions.

- Similarly, because photons (spin-1) and gravitons (spin-2) are also strictly neutral particles, we may say that they are also of a Majorana nature.

- The wave equation intended to describe a strictly neutral 3D KFG spin-0 particle, i.e., a 3D KFGM spin-0 particle, is the 3D Klein-Fock-Gordon (KFG) equation in its standard form (*) with a real-valued Lorentz scalar interaction, but in addition, together with some kind of Majorana condition.

(*)

O. Klein, "Quantentheorie und fünfdimensionale Relativitätstheorie," Zeitschrift für Physik **37**, 895-906 (1926).

V. Fock, "Zur Schrödingerschen Wellenmechanik," Zeitschrift für Physik **38**, 242-50 (1926).

W. Gordon, "Der Comptoneffekt nach der Schrödingerschen Theorie," Zeitschrift für Physik **40**, 117-33 (1926).

- Likewise, the 3D Feshbach-Villars (FV) equation (**) (or the 3D KFG equation in Hamiltonian form) with the scalar potential and the respective Majorana condition may also be used.

(**)

H. Feshbach and F. Villars, "Elementary relativistic wave mechanics of spin 0 and spin 1/2 particles," Rev. Mod. Phys. **30**, 24-45 (1958).

- This way of characterizing a 3D KFGM particle can also be used to describe a one-dimensional (1D) KFGM particle (the latter is a KFGM particle living in 1+1 dimensions). In this case, we may also use the standard 1D KFG equation and/or the 1D FV equation, both with a real-valued scalar potential together with their respective Majorana condition.

- Remark: A neutral 3D KFG particle may not be equal to its antiparticle, for example, a neutral K^0 meson (or neutral kaon) is different from its antiparticle $\overline{\mathrm{K}}^0$; in this case, these two particles carry different internal attributes (different hypercharges) and can be described by –classical– complex fields or complex solutions of the standard 3D KFG equation (The complex field ψ is associated with K^0 and the complex conjugate field ψ^* is associated with $\overline{\mathrm{K}}^0$).

- Remark: If a neutral 3D KFG particle is equal to its antiparticle, then there are no internal attributes that distinguish them; consequently, they must be described by –classical– real fields or real solutions of the standard 3D KFG equation (The same real field $\psi = \psi^*$ is associated with the particle and its antiparticle).

Remark: There is no place for a conserved current four-vector for a strictly neutral 3D KFG particle. In effect, the usual densities

$$\varrho = \varrho(\mathbf{r}, t) = \frac{\mathrm{i}\hbar}{2\mathrm{m}c^2}(\psi^* \,\partial_t\psi - \psi\,\partial_t\psi^*)$$

and

$$\mathbf{j} = \mathbf{j}(\mathbf{r}, t) = -\frac{\mathrm{i}\hbar}{2\mathrm{m}}(\psi^* \,\nabla\psi - \psi\,\nabla\psi^*)$$

cease to exist, i.e., for real solutions of the standard 3D KFG equation we obtain just the trivial solutions

$$\varrho = 0 \quad \text{and} \quad \mathbf{j} = 0.$$

Additionally, the continuity equation ($\partial_t\varrho + \nabla\cdot\mathbf{j} = 0$) is automatically satisfied, but one cannot obtain anything else from this (for example, one cannot obtain the nontrivial result $\int_\Omega \mathrm{d}x\,\varrho = \mathrm{const}$).

Remark: The latter conclusion seems to be a general property of other strictly neutral bosonic particles, for example, it occurs also for photons (spin-1). In fact, the gauge dependent electromagnetic field A^μ is a real vector field and it has zero current density. In this case, the energy flux represented by the Poynting vector is the physical quantity of interest.

A current problem

–The exact composition of dark matter is an open problem that is not exclusive to astrophysics and cosmology but also involves particle physics. A candidate for dark matter particles are axions. Dark matter should be cold and collisionless; thus, axions should have nonrelativistic momenta.

–Hence, a nonrelativistic approach is appropriate to describe axions. Consequently, the problem of taking the nonrelativistic limit of a relativistic (massive) real scalar field arises (such a field coupled to gravity would play the role of dark matter).

–As an example, what is the nonrelativistic limit of the standard 3D KFG equation when its solutions are real? In fact, the solutions of the 3D Schrödinger equation are complex, i.e., the Schrödinger field is always complex. Then, how can one take the nonrelativistic limit of the relativistic KFG real scalar field?

–Actually, because the Schrödinger equation is a complex equation, it would not be surprising that it cannot be derived from the real KFG equation. We will see what our results can say about this.

Intermezzo

- In 1937, Majorana posed the question of whether a 3D spin-$\frac{1}{2}$ particle could be its own antiparticle (*). Majorana noted that if the gamma matrices present in the free Dirac equation

$$\hat{\mathrm{O}}_{\mathrm{D}}\Psi \equiv \left(\mathrm{i}\hat{\gamma}^{\mu}\partial_{\mu} - \frac{\mathrm{m}c}{\hbar}\hat{1}_4\right)\Psi = 0$$

were forced to satisfy the condition $(\mathrm{i}\hat{\gamma}^{\mu})^* = \mathrm{i}\hat{\gamma}^{\mu}$ ($\hat{1}_4$ is the 4×4 identity matrix), then the Dirac equation could have real-valued solutions.

- Remark: The latter result remains valid when a real Lorentz scalar potential is added to the free Dirac equation.

- Majorana found for the first time a set of matrices that satisfy the condition $(\mathrm{i}\hat{\gamma}^{\mu})^* = \mathrm{i}\hat{\gamma}^{\mu}$. Any set of matrices satisfying it defines a Majorana representation.

(*)

E. Majorana, "Teoria simmetrica dell'elettrone e del positrone," Il Nuovo Cimento **14**, 171-84 (1937).

- The particle described by the real-valued solution of the 3D Dirac equation is (usually) called the (3D) Majorana particle, which would be a massive fermion that is its own antiparticle; i.e., a strictly neutral fermion.

- Remark: To date, no elementary particle with a spin of $\frac{1}{2}$ has been identified as a Majorana particle.

- Thus, in the Majorana representation, the equation that describes the Majorana fermion is the Dirac equation together with the reality condition of the wave function, namely,

$$\Psi = \Psi^*.$$

Precisely, in the Majorana representation, Ψ^* is the charge-conjugate wave function of Ψ, i.e.,

$$\Psi^* = \Psi_c\,;$$

therefore, the reality condition expresses the invariance of Ψ under the charge-conjugation operation, i.e.,

$$\Psi = \Psi_c.$$

The latter relation is what defines a Majorana fermion in any representation and is called the Majorana condition.

Characterizing 1D Klein-Fock-Gordon-Majorana particles

Preliminaries

- The 1D KFG wave equation in its standard form is given by

$$\hat{\mathrm{O}}_{\mathrm{KFG}}\,\psi \equiv \left[(\hat{\mathrm{E}} - V)^2 - (c\,\hat{\mathrm{p}})^2 - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S\right]\psi = 0,$$

$\psi = \psi(x,t)$ is a one-component wavefunction, $\hat{\mathrm{E}} = \mathrm{i}\hbar\,\partial/\partial t$ is the energy operator, $\hat{\mathrm{p}} = -\mathrm{i}\hbar\,\partial/\partial x$ is the momentum operator, $V = V(x)$ is the electric potential (energy), and $S = S(x,t) \in \mathbb{R}$ is a real-valued Lorentz scalar potential (energy).

- Thus,

$$\hat{\mathrm{O}}_{\mathrm{KFG}}\,\psi \equiv \left[-\hbar^2\frac{\partial^2}{\partial t^2} - 2V\mathrm{i}\hbar\frac{\partial}{\partial t} + V^2 + \hbar^2 c^2\frac{\partial^2}{\partial x^2} - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S\right]\psi = 0.$$

- The operator $\hat{\mathrm{O}}_{\mathrm{KFG}}$ is real when (a) V is pure imaginary, (b) or zero. In this case, the solutions ψ of $\hat{\mathrm{O}}_{\mathrm{KFG}}\,\psi = 0$ can be chosen to be real, but clearly they do not need to be real. The operator $\hat{\mathrm{O}}_{\mathrm{KFG}}$ is complex when V is real. In this case, the solutions ψ are necessarily complex.

Let us introduce functions φ and χ such that

$$\varphi + \chi = \psi \qquad \text{and} \qquad \varphi - \chi = \frac{1}{\mathrm{m}c^2}(\hat{\mathrm{E}} - V)\psi.$$

Using the latter relations and the standard 1D KFG wave equation one obtains

$$\hat{\mathrm{E}}\,\varphi = \left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi + \chi) + (\mathrm{m}c^2 + V)\varphi,$$

$$\hat{\mathrm{E}}\,\chi = -\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi + \chi) - (\mathrm{m}c^2 - V)\chi.$$

The latter system can be written in matrix form, namely,

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi = \hat{\mathrm{h}}\Psi,$$

where

$$\hat{\mathrm{h}} = \frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + V\,\hat{1}_2 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)$$

is the Hamiltonian operator, $\Psi = \Psi(x,t) = [\,\varphi \;\; \chi\,]^{\mathrm{T}} = [\,\varphi(x,t) \;\; \chi(x,t)\,]^{\mathrm{T}}$ is the two-component wavefunction (the symbol $^{\mathrm{T}}$ represents the transpose of a matrix), $\hat{\tau}_3 = \hat{\sigma}_z$ and $\hat{\tau}_2 = \hat{\sigma}_y$ are Pauli matrices, and $\hat{1}_2$ is the 2×2 identity matrix.

The latter is precisely the 1D FV wave equation with an electric potential and a scalar potential, namely,

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi = \left[\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + V\,\hat{1}_2 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\right]\Psi.$$

The connection between ψ and Ψ is given by

$$\Psi = \begin{bmatrix} \varphi \\ \chi \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \psi + \frac{1}{\mathrm{m}c^2}(\hat{\mathrm{E}} - V)\psi \\ \psi - \frac{1}{\mathrm{m}c^2}(\hat{\mathrm{E}} - V)\psi \end{bmatrix}.$$

Note that even if ψ is a real function, Ψ will be inexorably complex.

Remark: If one has a wave equation of the form $\hat{\mathrm{L}}\Phi \equiv (\hat{\mathrm{E}} - \hat{\mathrm{H}})\Phi = 0$, the operator $\hat{\mathrm{L}}$ is real if $\mathrm{i}\hat{\mathrm{H}} = (\mathrm{i}\hat{\mathrm{H}})^*$, i.e., if $\mathrm{i}\hat{\mathrm{H}}$ is a real operator. Thus, the time-dependent 1D FV wave equation cannot have real solutions.

The charge conjugate of Ψ,

$$\Psi_c \equiv \hat{\tau}_1\Psi^*$$

($\hat{\tau}_1 = \hat{\sigma}_x$ is a Pauli matrix) satisfies the following equation:

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi_c = \hat{\mathrm{h}}_c\Psi_c = \left[\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + V_c\,\hat{1}_2 + S_c\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\right]\Psi_c.$$

Taking the complex conjugate of $\hat{\mathrm{E}}\,\hat{1}_2\Psi = \hat{\mathrm{h}}\Psi$ and using the the results $\hat{\mathrm{E}}^* = -\hat{\mathrm{E}}$ and $(\hat{\mathrm{p}}^2)^* = \hat{\mathrm{p}}^2$, and the facts that $(\hat{\tau}_3+\mathrm{i}\hat{\tau}_2)^* = (\hat{\tau}_3+\mathrm{i}\hat{\tau}_2)$, $(\hat{\tau}_3)^* = \hat{\tau}_3$, and $\hat{\tau}_1\hat{\tau}_3 = -\hat{\tau}_3\hat{\tau}_1$, $\hat{\tau}_1\hat{\tau}_2 = -\hat{\tau}_2\hat{\tau}_1$ ($\Rightarrow (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\hat{\tau}_1 = -\hat{\tau}_1(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)$), and $\hat{\tau}_1^2 = \hat{1}_2$, and finally, using the definition of Ψ_c, we obtain

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi_c = \hat{\mathrm{h}}_c\Psi_c = \left[\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 - V^*\,\hat{1}_2 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\right]\Psi_c,$$

therefore

$$V_c = -V^*\,, \qquad S_c = S.$$

Remark: If we had considered placing a complex scalar potential in the Hamiltonian $\hat{\mathrm{h}}$, then the Hamiltonian $\hat{\mathrm{h}}_c$ would be the one given above but with the replacement $S \to S^*$, and therefore, the corresponding relation would be $S_c = S^*$.

Remark: If one makes $V \in \mathbb{R}$ and $S = 0$, then one has that

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi = \hat{\mathrm{h}}(V)\Psi \qquad \text{and} \qquad \hat{\mathrm{E}}\,\hat{1}_2\Psi_c = \hat{\mathrm{h}}_c(V)\Psi_c = \hat{\mathrm{h}}(-V)\Psi_c,$$

i.e., Ψ describes a 1D KFG particle with one sign of charge, and Ψ_c describes the 1D KFG particle with the opposite sign of charge (i.e., its antiparticle).

- Let us explore the possibility that a 1D KFG particle is its own antiparticle; therefore, it must be an electrically and strictly neutral particle. The condition that defines a particle of this type is customarily given by
$$\Psi = \Psi_c.$$
We refer to this relation as the standard Majorana condition.

- Remark: The latter Majorana condition imposes the following relation between the components of Ψ:
$$\varphi = \varphi_c = \chi^* \qquad (\Leftrightarrow \chi = \chi_c = \varphi^*),$$
i.e., φ and χ are not independent.

- Comparing $\hat{\mathrm{E}}\,\hat{1}_2\Psi = \hat{\mathrm{h}}\Psi$ and $\hat{\mathrm{E}}\,\hat{1}_2\Psi_c = \hat{\mathrm{h}}_c\Psi_c$ (where $\Psi_c = \Psi$ and $V_c = -V^*$), one obtains
$$V = -V^*,$$
i.e., the complex potential V must be pure imaginary, but if V is real-valued, then V must be zero.

- Remark: If we had decided to consider a complex potential S, then, in addition to $V = -V^*$, we would obtain $S = S^*$, i.e., the Majorana condition imposes that S be a real scalar potential.

Thus, the 1D FV wave equation describing a 1D KFG particle that is also a 1D Majorana particle, can be written as follows:

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi = \hat{\mathrm{h}}\Psi = \left[\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + V\,\hat{1}_2 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\right]\Psi,$$

where, if $V \in \mathbb{C}$, then it must be imaginary; and if $V \in \mathbb{R}$, then it must be zero. Likewise, the Lorentz scalar potential S must be real. Additionally, the wavefunction Ψ must have the form $\Psi = [\,\chi^*\ \ \chi\,]^{\mathrm{T}}$ or $\Psi = [\,\varphi\ \ \varphi^*\,]^{\mathrm{T}}$.

Remark: Equivalently, the 1D FV wave equation is invariant under the following substitution:

$$\Psi = [\,\varphi\ \ \chi\,]^{\mathrm{T}} \;\rightarrow\; \Psi_c = [\,\chi^*\ \ \varphi^*\,]^{\mathrm{T}},$$

but the conditions $V = -V^*$ and $S = S^*$ must be satisfied, i.e., if the latter conditions are satisfied, then Ψ and Ψ_c satisfy the same equation (the latter is the equation for the 1D KFGM particle).

The standard Majorana condition, namely,

$$\Psi = \frac{1}{2}\begin{bmatrix} \psi + \frac{1}{\mathrm{m}c^2}(\hat{\mathrm{E}} - V)\psi \\ \psi - \frac{1}{\mathrm{m}c^2}(\hat{\mathrm{E}} - V)\psi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} \psi^* - \frac{1}{\mathrm{m}c^2}(\hat{\mathrm{E}} - V)\psi^* \\ \psi^* + \frac{1}{\mathrm{m}c^2}(\hat{\mathrm{E}} - V)\psi^* \end{bmatrix} = \hat{\tau}_1\Psi^* = \Psi_c,$$

implies the following result:

$$\psi = \psi^*,$$

which is the reality condition for the wavefunction ψ and could be considered the standard Majorana condition for these solutions, i.e., $\psi = \psi^* \equiv \psi_c$, where ψ_c would be the charge conjugate of ψ.

- Remark: The latter relation also arises immediately when using $\varphi + \chi = \psi$ and the Majorana condition in terms of the components of Ψ (i.e., $\varphi = \chi^*$), namely, $\psi = \varphi + \chi = \chi^* + \varphi^* = (\chi + \varphi)^* = \psi^*$.

- The 1D FV wave equation is also invariant under the following substitution:

$$\Psi = \left[\, \varphi \;\; \chi \,\right]^{\mathrm{T}} \;\rightarrow\; -\Psi_c = \left[\, -\chi^* \;\; -\varphi^* \,\right]^{\mathrm{T}},$$

and again the conditions $V = -V^*$ and $S = S^*$ must be satisfied. That is, Ψ and $-\Psi_c$ satisfy the same equation, but this time we obtain the result

$$\psi = -\psi^*.$$

Thus, in this case, the solutions are imaginary, but they can be written real simply by writing $(\psi - \psi^*)/2\mathrm{i} = \psi/\mathrm{i}$.

Summing up, the Majorana condition appears here in two forms, one standard form,

$$\Psi = \Psi_c,$$

and, say, one non-standard form,

$$\Psi = -\Psi_c.$$

In both cases, the one-component solution ψ can (and must) be written real, but additionally, the potentials must satisfy the conditions $V = -V^*$ and $S = S^*$.

Remark: The non-standard Majorana condition imposes the following relation between the components of Ψ:

$$\varphi = -\varphi_c = -\chi^* \qquad (\Leftrightarrow \chi = -\chi_c = -\varphi^*),$$

again, φ and χ are not independent.

Remark: Note that,

$$\text{(a)} \; +\Psi = \Psi_c \quad \Rightarrow \quad \hat{\mathrm{C}}\Psi = +\Psi, \qquad \text{(b)} \; -\Psi = \Psi_c \quad \Rightarrow \quad \hat{\mathrm{C}}\Psi = -\Psi,$$

where $\Psi_c = \hat{\tau}_1\Psi^* \equiv \hat{\mathrm{C}}\Psi$ ($\hat{\mathrm{C}}$ represents the charge conjugation transformation). In principle, the existence of two Majorana conditions defines two specific and different types of 1D KFGM particles (this is the case for 3D KFGM particles). For example, the wavefunction corresponding to the 3D KFGM neutral pion π^0 is an eigenfunction of $\hat{\mathrm{C}}$ with eigenvalue $+1$.

Then, the 1D FV wave equation describing a 1D KFGM particle, can be written as follows:

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi = \hat{\mathrm{h}}\Psi = \left[\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + V\,\hat{1}_2 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\right]\Psi,$$

where $V = -V^*$ and $S = S^*$. Because the components φ and χ of Ψ are not independent, from the latter equation one can write an equation for only one of these components and obtain the other component algebraically.

When $\Psi = \Psi_c$ ($\Rightarrow \chi = \varphi^*$), we obtain the following equation:

$$\hat{\mathrm{E}}\,\varphi = \left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi + \varphi^*) + (\mathrm{m}c^2 + V)\varphi.$$

Alternatively, we can also write an equation for χ, namely,

$$\hat{\mathrm{E}}\,\chi = -\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\chi + \chi^*) - (\mathrm{m}c^2 - V)\chi$$

(and $\varphi = \chi^*$).

Similarly, when $\Psi = -\Psi_c$ ($\Rightarrow \chi = -\varphi^*$), we obtain the following equation:

$$\hat{\mathrm{E}}\,\varphi = \left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi - \varphi^*) + (\mathrm{m}c^2 + V)\varphi.$$

Alternatively, we can also write an equation for χ, namely,

$$\hat{\mathrm{E}}\,\chi = -\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\chi - \chi^*) - (\mathrm{m}c^2 - V)\chi$$

(and $\varphi = -\chi^*$).

Remark: None of the latter four equations for the 1D KFGM particle has the form $\hat{\mathrm{E}}\phi = \hat{\mathrm{H}}\phi$.

A 1D KFGM particle in an interval

Remark: Up to this point, we have not imposed any particular or specific condition on the Hamiltonian $\hat{\mathrm{h}}$, for example, we have not yet imposed on $\hat{\mathrm{h}}$ the condition of formal pseudo hermiticity, i.e.,

$$\hat{\mathrm{h}}_{\mathrm{adj}} \equiv \hat{\tau}_3\,\hat{\mathrm{h}}^\dagger\,\hat{\tau}_3 = \hat{\mathrm{h}}$$

($\hat{\mathrm{h}}_{\mathrm{adj}}$ is the formal generalized adjoint of $\hat{\mathrm{h}}$).

The operator $\hat{\mathrm{h}}_{\mathrm{adj}}$ is given by

$$\hat{\mathrm{h}}_{\mathrm{adj}} = \frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + V^*\,\hat{1}_2 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2).$$

The formula $\hat{\mathrm{h}}_{\mathrm{adj}} \equiv \hat{\tau}_3\,\hat{\mathrm{h}}^\dagger\,\hat{\tau}_3$ can also be formally written as follows:

$$\langle\langle \hat{\mathrm{h}}_{\mathrm{adj}}\Psi, \Phi\rangle\rangle = \langle\langle \Psi, \hat{\mathrm{h}}\Phi\rangle\rangle,$$

where the pseudo inner product is defined by

$$\langle\langle \Psi, \Phi\rangle\rangle \equiv \int_\Omega \mathrm{d}x\,\Psi^\dagger\hat{\tau}_3\Phi$$

($\Omega = [a, b]$ is –an interval–, and $\Psi = [\,\varphi\ \chi\,]^{\mathrm{T}}$ and $\Phi = [\,\zeta\ \xi\,]^{\mathrm{T}}$ are two-component wavefunctions).

Remark: Actually, the relation $\langle\langle \hat{\mathrm{h}}_{\mathrm{adj}}\Psi, \Phi\rangle\rangle = \langle\langle \Psi, \hat{\mathrm{h}}\Phi\rangle\rangle$ defines the generalized adjoint $\hat{\mathrm{h}}_{\mathrm{adj}}$ on an indefinite inner product space. The Hamiltonian operator $\hat{\mathrm{h}}$ is –formally pseudo-Hermitian– because it satisfies the formal relation $\hat{\mathrm{h}}_{\mathrm{adj}} = \hat{\mathrm{h}}$, namely,

$$\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3+\mathrm{i}\hat{\tau}_2)+\mathrm{m}c^2\hat{\tau}_3+V^*\,\hat{1}_2+S\,(\hat{\tau}_3+\mathrm{i}\hat{\tau}_2) = \frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3+\mathrm{i}\hat{\tau}_2)+\mathrm{m}c^2\hat{\tau}_3+V\,\hat{1}_2+S\,(\hat{\tau}_3+\mathrm{i}\hat{\tau}_2).$$

in consequence,

$$V = V^*,$$

and because we are characterizing a 1D KFGM particle, i.e.,

$$V = -V^*,$$

then V must be zero.

- Finally, the 1D FV wave equation that describes a 1D KFGM particle is given by

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi = \hat{\mathrm{h}}\Psi = \left[\,\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\right]\Psi,$$

where the Lorentz scalar potential S is real-valued. When $\Psi = \pm\Psi_c$, then $\Psi = [\,\varphi \quad \pm\,\varphi^*\,]^{\mathrm{T}}$ (or $\Psi = [\,\pm\chi^* \quad \chi\,]^{\mathrm{T}}$).

- The Hamiltonian operator $\hat{\mathrm{h}}$ and its formal generalized adjoint $\hat{\mathrm{h}}_{\mathrm{adj}}$ satisfy the following relation:

$$\langle\langle \hat{\mathrm{h}}_{\mathrm{adj}}\Psi, \Phi\rangle\rangle =$$

$$\langle\langle \Psi, \hat{\mathrm{h}}\Phi\rangle\rangle - \frac{\hbar^2}{2\mathrm{m}}\frac{1}{2}\left[\,((\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\Psi_x)^\dagger\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\Phi - ((\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\Psi)^\dagger\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\Phi_x\,\right]\Big|_a^b,$$

where $[\,g\,]|_a^b \equiv g(b,t) - g(a,t)$, and $\Psi_x \equiv \partial\Psi/\partial x$, etc.

Remark: the latter relation is also true if the Hamiltonians $\hat{\mathrm{h}}$ and $\hat{\mathrm{h}}_{\mathrm{adj}}$ contain, in addition to $S \in \mathbb{R}$, a real electric potential V.

If the boundary conditions imposed on Ψ and Φ at the ends of interval Ω lead to the vanishing of the boundary term, the Hamiltonian $\hat{\mathrm{h}}$, formally satisfying the relation $\hat{\mathrm{h}}_{\mathrm{adj}} = \hat{\mathrm{h}}$, is effectively – pseudo-Hermitian– (or generalized Hermitian). The most general family of boundary conditions leading to the cancellation of this boundary term is given by (we omit the variable t in the boundary conditions hereafter)

$$\left[\begin{array}{c} (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)(\Psi - \mathrm{i}\lambda\Psi_x)(b) \\ (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)(\Psi + \mathrm{i}\lambda\Psi_x)(a) \end{array}\right] = \hat{\mathrm{U}}_{(4\times4)} \left[\begin{array}{c} (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)(\Psi + \mathrm{i}\lambda\Psi_x)(b) \\ (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)(\Psi - \mathrm{i}\lambda\Psi_x)(a) \end{array}\right],$$

where $\lambda \in \mathbb{R}$ is a parameter and $\hat{\mathrm{U}}_{(4\times4)}$ is a 4×4 unitary matrix that can be written as follows:

$$\hat{\mathrm{U}}_{(4\times4)} = \hat{\mathrm{S}}^{\dagger} \left[\begin{array}{cc} \hat{\mathrm{U}}_{(2\times2)} & \hat{0} \\ \hat{0} & \hat{\mathrm{U}}_{(2\times2)} \end{array}\right] \hat{\mathrm{S}},$$

with

$$\hat{\mathrm{S}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

but $\hat{\mathrm{U}}_{(2\times 2)}$ will finally be a 2×2 unitary matrix that depends on three real parameters.

For all the boundary conditions inside the latter general set of boundary conditions, $\hat{\mathrm{h}}$ is a –pseudo-Hermitian– operator, but it is also a –pseudo self-adjoint operator–, that is, $\hat{\mathrm{h}}$ satisfies the relation

$$\langle\langle \hat{\mathrm{h}}\Psi, \Phi\rangle\rangle = \langle\langle \Psi, \hat{\mathrm{h}}\Phi\rangle\rangle.$$

Thus, the functions belonging to the domains of $\hat{\mathrm{h}}$ and $\hat{\mathrm{h}}_{\mathrm{adj}}$ obey the same boundary conditions, and therefore,

$$\hat{\mathrm{h}}_{\mathrm{adj}} = \hat{\mathrm{h}}$$

(in this case, the latter is not just a formal equality).

The following relation can also be written:

$$\langle\langle \hat{\mathrm{h}}_{\mathrm{adj}}\Psi, \Phi\rangle\rangle = \langle\langle \Psi, \hat{\mathrm{h}}\Phi\rangle\rangle - \frac{\hbar^2}{2\mathrm{m}}\left[\,\psi_x^*\,\phi - \psi^*\phi_x\,\right]\Big|_a^b,$$

where $\psi = \varphi + \chi$ and $\phi = \zeta + \xi$ ($\Psi = \left[\,\varphi\ \ \chi\,\right]^{\mathrm{T}}$ and $\Phi = \left[\,\zeta\ \ \xi\,\right]^{\mathrm{T}}$). Additionally, the boundary term in the latter relation can be written as follows:

$$-\frac{\hbar^2}{2\mathrm{m}}\left[\,\psi_x^*\,\phi - \psi^*\phi_x\,\right]\Big|_a^b = \frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle \Psi, \Phi\rangle\rangle,$$

where Ψ and Φ are solutions of the 1D FV wave equation (and ψ and ϕ are the respective solutions of the standard 1D KFG wave equation). Note: $\langle\langle \Psi, \Phi\rangle\rangle$ does not depend on S, but it depends on V. However, its time derivative is independent of the two potentials (provided they are real-valued).

Then, the most general set of pseudo self-adjoint boundary conditions for $\hat{\mathrm{h}} = \hat{\mathrm{h}}_{\mathrm{adj}}$, and consistent with the cancellation of the boundary term, can also be written as

$$\begin{bmatrix} \psi(b) - \mathrm{i}\lambda\psi_x(b) \\ \psi(a) + \mathrm{i}\lambda\psi_x(a) \end{bmatrix} = \hat{\mathrm{U}}_{(2\times 2)}\begin{bmatrix} \psi(b) + \mathrm{i}\lambda\psi_x(b) \\ \psi(a) - \mathrm{i}\lambda\psi_x(a) \end{bmatrix},$$

where $\hat{\mathrm{U}}_{(2\times 2)}$ is precisely the 2×2 unitary matrix that appears in the matrix $\hat{\mathrm{U}}_{(4\times 4)}$ presented above.

- Thus, the pseudo inner product $\langle\langle \Psi, \Phi \rangle\rangle$ is constant, i.e., for all the corresponding solutions ψ and ϕ of the standard 1D KFG wave equation that satisfy any of the boundary conditions included in the general set of pseudo self-adjoint boundary conditions for $\hat{\mathrm{h}}$.

- Then, the standard 1D KFG wave equation that describes a 1D KFGM particle is given by

$$\left[-\hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2 c^2 \frac{\partial^2}{\partial x^2} - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S\right]\psi = 0,$$

where $S \in \mathbb{R}$. When $\Psi = \Psi_c$, then $\psi = \psi^*$, and therefore,

$$\psi \quad \text{and also} \quad \psi^*$$

satisfy the general set of boundary conditions. When $\Psi = -\Psi_c$, then $\psi = -\psi^*$, and therefore,

$$\psi \quad \text{and also} \quad -\psi^*$$

satisfy the general set of boundary conditions. Consequently, the matrix $\hat{\mathrm{U}}_{(2\times2)}$ satisfies the following condition:

$$\hat{\mathrm{U}}^{\mathrm{T}}_{(2\times2)} = \hat{\mathrm{U}}_{(2\times2)},$$

that is, $\hat{\mathrm{U}}_{(2\times2)}$ must additionally be a (complex) symmetric matrix.

If we choose

$$\hat{\mathrm{U}}_{(2\times2)} = \mathrm{e}^{\mathrm{i}\,\mu} \begin{bmatrix} \mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3 & -\mathrm{m}_2 - \mathrm{i}\,\mathrm{m}_1 \\ \mathrm{m}_2 - \mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3 \end{bmatrix},$$

where $\mu \in [0, \pi)$, and the real quantities m_k ($k = 0, 1, 2, 3$) satisfy $(\mathrm{m}_0)^2 + (\mathrm{m}_1)^2 + (\mathrm{m}_2)^2 + (\mathrm{m}_3)^3 = 1$, and impose on $\hat{\mathrm{U}}_{(2\times2)}$ the condition $\hat{\mathrm{U}}^{\mathrm{T}}_{(2\times2)} = \hat{\mathrm{U}}_{(2\times2)}$, we obtain the result $\mathrm{m}_2 = 0$.

Thus, the most general set of pseudo self-adjoint boundary conditions for a 1D KFGM particle, or for the ultimately real solutions of the 1D KFG wave equation, can be written as follows:

$$\begin{bmatrix} \psi(b) - \mathrm{i}\lambda\psi_x(b) \\ \psi(a) + \mathrm{i}\lambda\psi_x(a) \end{bmatrix} = \mathrm{e}^{\mathrm{i}\,\mu} \begin{bmatrix} \mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3 & -\mathrm{i}\,\mathrm{m}_1 \\ -\mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3 \end{bmatrix} \begin{bmatrix} \psi(b) + \mathrm{i}\lambda\psi_x(b) \\ \psi(a) - \mathrm{i}\lambda\psi_x(a) \end{bmatrix},$$

where $\mu \in [0, \pi)$, and $(\mathrm{m}_0)^2 + (\mathrm{m}_1)^2 + (\mathrm{m}_3)^3 = 1$.

Remark: Because $\hat{\mathrm{S}}^{\dagger} = \hat{\mathrm{S}}^{\mathrm{T}}$, but in addition, $\hat{\mathrm{U}}_{(2\times 2)} = \hat{\mathrm{U}}^{\mathrm{T}}_{(2\times 2)}$, we have that

$$\hat{\mathrm{U}}_{(4\times 4)} = \hat{\mathrm{S}}^{\dagger} \begin{bmatrix} \hat{\mathrm{U}}_{(2\times 2)} & \hat{0} \\ \hat{0} & \hat{\mathrm{U}}_{(2\times 2)} \end{bmatrix} \hat{\mathrm{S}} = \hat{\mathrm{U}}^{\mathrm{T}}_{(4\times 4)},$$

then, $\hat{\mathrm{U}}_{(4\times 4)}$ is given by

$$\hat{\mathrm{U}}_{(4\times 4)} = \mathrm{e}^{\mathrm{i}\,\mu} \begin{bmatrix} (\mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3)\hat{1}_2 & -\mathrm{i}\,\mathrm{m}_1\hat{1}_2 \\ -\mathrm{i}\,\mathrm{m}_1\hat{1}_2 & (\mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3)\hat{1}_2 \end{bmatrix}.$$

Some of the boundary conditions included in the general set of pseudo self-adjoint boundary conditions for a 1D particle KFGM are the following:

Boundary condition	Name
$\psi(a) = \psi(b) = 0$	Dirichlet
$\psi_x(a) = \psi_x(b) = 0$	Neumann
$\psi(a) - \lambda\psi_x(a) = 0$ and $\psi(b) + \lambda\psi_x(b) = 0$	Robin
$\psi(a) = \psi(b)$ and $\psi_x(a) = \psi_x(b)$	Periodic
$\psi(a) = -\psi(b)$ and $\psi_x(a) = -\psi_x(b)$	Antiperiodic

Remark: Some boundary conditions that are not suitable for a 1D KFGM particle but are suitable for a 1D KFG particle ($\mathrm{m}_2 \neq 0$) are the following:

Boundary condition	Name
$\psi(a) = \pm \mathrm{i}\psi(b)$ and $\psi_x(a) = \pm \mathrm{i}\psi_x(b)$	Quasi-periodic/antiperiodic
$\psi(a) = \pm \mathrm{i}\lambda\psi_x(b)$ and $\psi(b) = \pm \mathrm{i}\lambda\psi_x(a)$	Quasi-mixed

Then, when $\Psi = \Psi_c$ ($\Rightarrow \chi = \varphi^*$, and therefore, $\psi = \varphi + \chi = \varphi + \varphi^* = 2\,\mathrm{Re}(\varphi)$ and $\psi_x = 2\,(\mathrm{Re}(\varphi))_x$), we obtain the following equation:

$$\hat{\mathrm{E}}\,\varphi = \left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi + \varphi^*) + \mathrm{m}c^2\varphi, \qquad \text{(M1a)}$$

and its solutions must satisfy any of the boundary conditions included in the following general set of boundary conditions:

$$\begin{bmatrix} (\mathrm{Re}(\varphi))(b) - \mathrm{i}\lambda(\mathrm{Re}(\varphi))_x(b) \\ (\mathrm{Re}(\varphi))(a) + \mathrm{i}\lambda(\mathrm{Re}(\varphi))_x(a) \end{bmatrix} =$$

$$= e^{\mathrm{i}\,\mu} \begin{bmatrix} \mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3 & -\mathrm{i}\,\mathrm{m}_1 \\ -\mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3 \end{bmatrix} \begin{bmatrix} (\mathrm{Re}(\varphi))(b) + \mathrm{i}\lambda(\mathrm{Re}(\varphi))_x(b) \\ (\mathrm{Re}(\varphi))(a) - \mathrm{i}\lambda(\mathrm{Re}(\varphi))_x(a) \end{bmatrix}.$$

The following results arise from the latter differential equation:

$$\left[-\hbar^2\frac{\partial^2}{\partial t^2}+\hbar^2c^2\frac{\partial^2}{\partial x^2}-(\mathrm{m}c^2)^2-2\,\mathrm{m}c^2S\right]\mathrm{Re}(\varphi)=0,$$

and

$$\mathrm{Im}(\varphi)=\frac{\hbar}{\mathrm{m}c^2}\frac{\partial}{\partial t}\,\mathrm{Re}(\varphi).$$

Finally, $\varphi=\mathrm{Re}(\varphi)+\mathrm{i}\,\mathrm{Im}(\varphi)$ (and the component χ of Ψ is obtained from the Majorana condition, i.e., $\chi=\varphi^*$).

Alternatively, we can also write an equation for χ, namely,

$$\hat{\mathrm{E}}\,\chi=-\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}+S\right)(\chi+\chi^*)-\mathrm{m}c^2\chi \qquad \text{(M2a)}$$

(and $\varphi=\chi^*$, and therefore, $\psi=\varphi+\chi=\chi^*+\chi=2\,\mathrm{Re}(\chi)$ and $\psi_x=2\,(\mathrm{Re}(\chi))_x$), with the following set of boundary conditions:

$$\begin{bmatrix}(\mathrm{Re}(\chi))(b)-\mathrm{i}\lambda(\mathrm{Re}(\chi))_x(b)\\(\mathrm{Re}(\chi))(a)+\mathrm{i}\lambda(\mathrm{Re}(\chi))_x(a)\end{bmatrix}=$$

$$=\mathrm{e}^{\mathrm{i}\,\mu}\begin{bmatrix}\mathrm{m}_0-\mathrm{i}\,\mathrm{m}_3 & -\mathrm{i}\,\mathrm{m}_1\\-\mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0+\mathrm{i}\,\mathrm{m}_3\end{bmatrix}\begin{bmatrix}(\mathrm{Re}(\chi))(b)+\mathrm{i}\lambda(\mathrm{Re}(\chi))_x(b)\\(\mathrm{Re}(\chi))(a)-\mathrm{i}\lambda(\mathrm{Re}(\chi))_x(a)\end{bmatrix}.$$

The following results arise from the latter equation:

$$\left[-\hbar^2\frac{\partial^2}{\partial t^2}+\hbar^2c^2\frac{\partial^2}{\partial x^2}-(\mathrm{m}c^2)^2-2\,\mathrm{m}c^2 S\right]\mathrm{Re}(\chi)=0,$$

and

$$\mathrm{Im}(\chi)=-\frac{\hbar}{\mathrm{m}c^2}\frac{\partial}{\partial t}\mathrm{Re}(\chi).$$

Finally, $\chi=\mathrm{Re}(\chi)+\mathrm{i}\,\mathrm{Im}(\chi)$ (and the component φ of Ψ is obtained from the Majorana condition, i.e., $\varphi=\chi^*$).

Similarly, when $\Psi=-\Psi_c$ ($\Rightarrow\chi=-\varphi^*$, and therefore, $\psi=\varphi+\chi=\varphi-\varphi^*=2\mathrm{i}\,\mathrm{Im}(\varphi)$ and $\psi_x=2\mathrm{i}\,(\mathrm{Im}(\varphi))_x$), we obtain the following equation:

$$\hat{\mathrm{E}}\,\varphi=\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}+S\right)(\varphi-\varphi^*)+\mathrm{m}c^2\varphi, \qquad \text{(M3a)}$$

and its solutions must satisfy any of the following boundary conditions:

$$\begin{bmatrix}(\mathrm{Im}(\varphi))(b)-\mathrm{i}\lambda(\mathrm{Im}(\varphi))_x(b)\\(\mathrm{Im}(\varphi))(a)+\mathrm{i}\lambda(\mathrm{Im}(\varphi))_x(a)\end{bmatrix}=$$

$$=\mathrm{e}^{\mathrm{i}\,\mu}\begin{bmatrix}\mathrm{m}_0-\mathrm{i}\,\mathrm{m}_3 & -\mathrm{i}\,\mathrm{m}_1\\-\mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0+\mathrm{i}\,\mathrm{m}_3\end{bmatrix}\begin{bmatrix}(\mathrm{Im}(\varphi))(b)+\mathrm{i}\lambda(\mathrm{Im}(\varphi))_x(b)\\(\mathrm{Im}(\varphi))(a)-\mathrm{i}\lambda(\mathrm{Im}(\varphi))_x(a)\end{bmatrix}.$$

The following results arise from the latter differential equation:

$$\left[-\hbar^2\frac{\partial^2}{\partial t^2}+\hbar^2c^2\frac{\partial^2}{\partial x^2}-(\mathrm{m}c^2)^2-2\,\mathrm{m}c^2S\right]\mathrm{Im}(\varphi)=0,$$

and

$$\mathrm{Re}(\varphi)=-\frac{\hbar}{\mathrm{m}c^2}\frac{\partial}{\partial t}\,\mathrm{Im}(\varphi).$$

Finally, $\varphi=\mathrm{Re}(\varphi)+\mathrm{i}\,\mathrm{Im}(\varphi)$ (and the component χ of Ψ is obtained from the Majorana condition, i.e., $\chi=-\varphi^*$).

Alternatively, we can also write an equation for χ, namely,

$$\hat{\mathrm{E}}\,\chi=-\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}+S\right)(\chi-\chi^*)-\mathrm{m}c^2\chi \qquad \text{(M4a)}$$

(and $\varphi=-\chi^*$, and therefore, $\psi=\varphi+\chi=-\chi^*+\chi=2\mathrm{i}\,\mathrm{Im}(\chi)$ and $\psi_x=2\mathrm{i}\,(\mathrm{Im}(\chi))_x$), with the following set of boundary conditions:

$$\begin{bmatrix}(\mathrm{Im}(\chi))(b)-\mathrm{i}\lambda(\mathrm{Im}(\chi))_x(b)\\(\mathrm{Im}(\chi))(a)+\mathrm{i}\lambda(\mathrm{Im}(\chi))_x(a)\end{bmatrix}=$$

$$\mathrm{e}^{\mathrm{i}\,\mu}\begin{bmatrix}\mathrm{m}_0-\mathrm{i}\,\mathrm{m}_3 & -\mathrm{i}\,\mathrm{m}_1\\-\mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0+\mathrm{i}\,\mathrm{m}_3\end{bmatrix}\begin{bmatrix}(\mathrm{Im}(\chi))(b)+\mathrm{i}\lambda(\mathrm{Im}(\chi))_x(b)\\(\mathrm{Im}(\chi))(a)-\mathrm{i}\lambda(\mathrm{Im}(\chi))_x(a)\end{bmatrix}.$$

The following results arise from the latter differential equation:

$$\left[-\hbar^2\frac{\partial^2}{\partial t^2}+\hbar^2 c^2\frac{\partial^2}{\partial x^2}-(\mathrm{m}c^2)^2-2\,\mathrm{m}c^2 S\right]\mathrm{Im}(\chi)=0,$$

and

$$\mathrm{Re}(\chi)=\frac{\hbar}{\mathrm{m}c^2}\frac{\partial}{\partial t}\mathrm{Im}(\chi).$$

Finally, $\chi=\mathrm{Re}(\chi)+\mathrm{i}\,\mathrm{Im}(\chi)$ (and the component φ of Ψ is obtained from the Majorana condition, i.e., $\varphi=-\chi^*$).

Equations (M1a), (M2a), (M3a) and (M4a) could be referred to as the first order in time (non-Hamiltonian) 1D Majorana equations.

Remark: The solutions of equations (M1a), (M2a), (M3a) and (M4a) are complex. Depending on which Majorana condition is used, the real (imaginary) parts of these solutions satisfy the standard 1D KFG equation, but then their imaginary (real) parts are simply the time derivative of the real (imaginary) parts.

Each first-order 1D Majorana equation leads to a second order in time 1D Majorana equation. In fact, applying the operator $\hat{\mathrm{E}}$ to–

both sides of Eq. (M1a), we obtain the following equation:

$$\left[\hat{\mathrm{E}}^2-(c\,\hat{\mathrm{p}})^2-(\mathrm{m}c^2)^2-2\,\mathrm{m}c^2 S\right]\varphi=(\hat{\mathrm{E}}\,S)(\varphi+\varphi^*). \tag{M1b}$$

Similarly, from Eq. (M2a) the following equation is obtained:

$$\left[\hat{\mathrm{E}}^2-(c\,\hat{\mathrm{p}})^2-(\mathrm{m}c^2)^2-2\,\mathrm{m}c^2 S\right]\chi=-(\hat{\mathrm{E}}\,S)(\chi+\chi^*). \tag{M2b}$$

These two equations correspond to the Majorana condition $\Psi=\Psi_c$, that is, $\psi=\psi^*$. If we add Eqs. (M1b) and (M2b), it is confirmed that $\psi=\varphi+\chi$ satisfies the standard 1D KFG equation, as expected.

Similarly, applying the operator $\hat{\mathrm{E}}$ to both sides of Eq. (M3a), gives the following equation:

$$\left[\hat{\mathrm{E}}^2-(c\,\hat{\mathrm{p}})^2-(\mathrm{m}c^2)^2-2\,\mathrm{m}c^2 S\right]\varphi=(\hat{\mathrm{E}}\,S)(\varphi-\varphi^*). \tag{M3b}$$

In the same manner, applying $\hat{\mathrm{E}}$ on Eq. (M4a) gives the following equation:

$$\left[\hat{\mathrm{E}}^2-(c\,\hat{\mathrm{p}})^2-(\mathrm{m}c^2)^2-2\,\mathrm{m}c^2 S\right]\chi=-(\hat{\mathrm{E}}\,S)(\chi-\chi^*). \tag{M4b}$$

The latter two equations correspond to the Majorana condition–

$\Psi = -\Psi_c$, that is, $\psi = -\psi^*$. If we add Eqs. (M3b) and (M4b), it is again found that $\psi = \varphi + \chi$ satisfies the standard 1D KFG equation.

- **Remark**: The second-order 1D Majorana equations reduce to the standard 1D KFG equation when the scalar potential is independent of time.

- **Remark**: Our results can also be extended to the problem of a 1D KFGM particle in a real line with a tiny hole at a point, for example, at $x = 0$ (i.e., $\Omega = \mathbb{R} - \{0\}$). Indeed, all boundary conditions for this problem can be obtained from those corresponding to the particle within the interval $\Omega = [a, b]$ by making the replacements $x = a \rightarrow 0+$ and $x = b \rightarrow 0-$.

On the nonrelativistic limit of one of the 1D Majorana equations

- Let us consider the nonrelativistic limit of one of the first order in time 1D Majorana equations. For example, Eq. (M1a), namely,

$$\hat{\mathrm{E}}\,\varphi = \left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S \right) (\varphi + \varphi^*) + \mathrm{m}c^2 \varphi. \tag{M1a}$$

The latter is completely equivalent to the following equation:

$$\hat{\mathrm{O}}_{\mathrm{KFG}}\,\mathrm{Re}(\varphi) \equiv \left[-\hbar^2\frac{\partial^2}{\partial t^2} + \hbar^2 c^2 \frac{\partial^2}{\partial x^2} - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S\right]\mathrm{Re}(\varphi) = 0,$$

plus the following relation:

$$\mathrm{Im}(\varphi) = \frac{\hbar}{\mathrm{m}c^2}\frac{\partial}{\partial t}\,\mathrm{Re}(\varphi).$$

Finally, $\varphi = \mathrm{Re}(\varphi) + \mathrm{i}\,\mathrm{Im}(\varphi)$ (and the component χ of Ψ is obtained from the Majorana condition, i.e., $\chi = \varphi^*$). Note that the second order in time equation can also be written as follows:

$$\mathrm{Re}\left(\hat{\mathrm{O}}_{\mathrm{KFG}}\,\varphi\right) = \mathrm{Re}\left[\left(-\hbar^2\frac{\partial^2}{\partial t^2} + \hbar^2 c^2 \frac{\partial^2}{\partial x^2} - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S\right)\varphi\right] = 0.$$

And now is when we make the typical ansatz, namely,

$$\varphi = \varphi_{\mathrm{NR}}\,\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t},$$

$$\Rightarrow \quad \partial_t\varphi = \left(\partial_t\varphi_{\mathrm{NR}} - \mathrm{i}\frac{\mathrm{m}c^2}{\hbar}\,\varphi_{\mathrm{NR}}\right)\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}$$

and

$$\Rightarrow \quad \partial_{tt}\varphi = \left[\partial_{tt}\varphi_{\mathrm{NR}} - \mathrm{i}\frac{2\mathrm{m}c^2}{\hbar}\,\partial_t\varphi_{\mathrm{NR}} - \frac{(\mathrm{m}c^2)^2}{\hbar^2}\,\varphi_{\mathrm{NR}}\right]\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}.$$

In the nonrelativistic aproximation we have that

$$|\,\mathrm{i}\hbar\,\partial_t\varphi_{\mathrm{NR}}\,| \ll \mathrm{m}c^2\,|\,\varphi_{\mathrm{NR}}\,| \;\Rightarrow\; |\,\partial_t\varphi_{\mathrm{NR}}\,| \ll \frac{\mathrm{m}c^2}{\hbar}\,|\,\varphi_{\mathrm{NR}}\,|$$

and

$$|\,\mathrm{i}\hbar\,\partial_{tt}\varphi_{\mathrm{NR}}\,| \ll \mathrm{m}c^2\,|\,\partial_t\varphi_{\mathrm{NR}}\,| \;\Rightarrow\; |\,\partial_{tt}\varphi_{\mathrm{NR}}\,| \ll \frac{\mathrm{m}c^2}{\hbar}\,|\,\partial_t\varphi_{\mathrm{NR}}\,|,$$

therefore,

$$\partial_t\varphi \approx -\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}\,\varphi_{\mathrm{NR}}\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}$$

and

$$\partial_{tt}\varphi \approx \left[-\mathrm{i}\frac{2\mathrm{m}c^2}{\hbar}\,\partial_t\varphi_{\mathrm{NR}} - \frac{(\mathrm{m}c^2)^2}{\hbar^2}\,\varphi_{\mathrm{NR}}\right]\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}.$$

Substituting the latter expression into the equation $\mathrm{Re}(\hat{\mathrm{O}}_{\mathrm{KFG}}\,\psi) = 0$, we obtain the following result:

$$\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}\left(-\mathrm{i}\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2\mathrm{m}}\frac{\partial^2}{\partial x^2} + S\right)\varphi_{\mathrm{NR}}\right] = 0. \qquad \text{(MNR1a)}$$

This is not the Schrödinger equation because if we have that $z \in \mathbb{C}$ satisfies $\mathrm{Re}(z) = 0$, then z is not necessarily zero.

Note that the relation giving the imaginary part of φ can also be written as follows:

$$\mathrm{Im}(\varphi) = \frac{\hbar}{\mathrm{m}c^2}\,\mathrm{Re}\left(\frac{\partial}{\partial t}\varphi\right).$$

In the nonrelativistic approximation, we obtain the following result:

$$\mathrm{Im}\left(\varphi_{\mathrm{NR}}\,\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}\right) = \frac{\hbar}{\mathrm{m}c^2}\,\mathrm{Re}\left(-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}\,\varphi_{\mathrm{NR}}\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}\right),$$

but this relation is always true because $\mathrm{Im}(z) = \mathrm{Re}(-\mathrm{i}z)$.

Remark: In field theory, the Schrödinger equations, namely,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi_{\mathrm{NR}} = \left(-\frac{\hbar^2}{2\mathrm{m}}\frac{\partial^2}{\partial x^2} + S\right)\varphi_{\mathrm{NR}} \quad \text{and} \quad -\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi^*_{\mathrm{NR}} = \left(-\frac{\hbar^2}{2\mathrm{m}}\frac{\partial^2}{\partial x^2} + S\right)\varphi^*_{\mathrm{NR}},$$

can be obtained from a Lagrangian density, but φ_{NR} and φ^*_{NR} must be varied independently, i.e., these two functions are treated as independent fields.

Remark: Some authors have considered the nonrelativistic limit–

of the standard real KFG equation, namely,

$$\left[-\hbar^2\frac{\partial^2}{\partial t^2}+\hbar^2c^2\frac{\partial^2}{\partial x^2}-(\mathrm{m}c^2)^2-2\,\mathrm{m}c^2S\right]\psi=0$$

(although without including the scalar potential). They relate the relativistic field ψ to the nonrelativistic field ψ_{NR} as follows:

$$\psi=\psi_{\mathrm{NR}}\,\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}+\psi_{\mathrm{NR}}^*\,\mathrm{e}^{+\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}\in\mathbb{R}.$$

If one inserts this transformation into the real KFG equation, one obtains the Schrödinger equations given before but only if one makes $c\to\infty$ and assumes that ψ_{NR} and ψ_{NR}^* are independent fields.

Remark: If one inserts the transformation given above into the KFG Lagrangian density, one recovers the Schrödinger Lagrangian density but only if one ignores the fast oscillating terms which are proportional to

$$\mathrm{e}^{\pm 2\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}.$$

The latter fact may not be always valid.

Remark: Equation (MNR1a) generates the Schrodinger equations only if φ_{NR} and φ_{NR}^* are assumed to be independent fields.

Some concluding remarks

- The wave equations considered to describe a strictly neutral 1D KFGM particle are the standard 1D KFG equation and/or the 1D FV equation, both with a real Lorentz scalar potential plus their respective Majorana condition.

- Here, the Majorana condition appears in two specific forms, say, one standard and one nonstandard. The imposition of the standard (nonstandard) Majorana condition on the solutions of the 1D FV equation implies that the solutions of the standard 1D KFG equation must be real (imaginary; however, they can also be written real). In any case, the solutions of the 1D FV equation cannot be real.

- Both Majorana conditions determine that the scalar potential must be real. The additional imposition of the formal pseudo hermiticity condition on the FV Hamiltonian together with a Majorana condition determines that the electric potential must be zero.

- If we place a 1D KFGM particle in a finite interval, one has that the FV Hamiltonian is a pseudo self-adjoint operator. Thus, one has a three-parameter general set of boundary conditions for the 1D FV equation and another for the standard (real) 1D KFG equation.

- The latter two general sets of boundary conditions are the same for the two types of Majorana conditions.

- Because of the Majorana condition, the (complex) components of the wavefunction for the 1D FV equation are not independent; hence, we wrote first order in time equations for each of these components and obtained the general sets of pseudo self-adjoint boundary conditions they must obey.

- The latter equations do not have a Hamiltonian form, but any of them alone can model a 1D KFGM particle (in fact, if we know one of the components of the solution of the 1D FV equation, the other component can be obtained algebraically). We may refer to these equations as the first-order 1D Majorana equations for the 1D KFGM particle.

- One can also write (complex) second order in time 1D Majorana equations for each of the components of the 1D FV equation. These equations reduce to the standard 1D KFG equation when the scalar potential does not explicitly depend on time.

- The nonrelativistic limit of one of the first-order 1D Majorana equations can only lead to Schrodinger equations if φ_{NR} and $\varphi_{\mathrm{NR}}^{*}$ are assumed to be independent fields.

Thanks!

Grazie!

¡Gracias!