Qualifying Examination

Academic Year 2023 (2566) : PART 1 : ROUND 2

24 JAN 2024

Name:...

Student ID:...

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 8 questions .
2. Write down detailed and clear solution to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

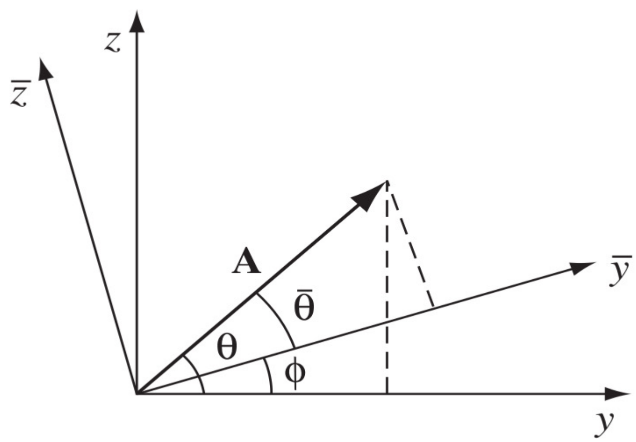
Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	
7	
8	

Name:..

1 Vector Calculus

Question 1.

From the figure



$$A_y = A\cos\theta, \text{ and } A_z = A\sin\theta$$

1.1) Write down the components of $\bar{A}$ in the rotation axes and in the matrix form.

1.2) Show that the two-dimensional rotation matrix preserves dot product. (Hint: show that $\bar{A}_y\bar{B}_y + \bar{A}_z\bar{B}_z = A_yB_y + A_zB_z$)

[10 marks]

Name:...

2 Linear Algebra

Question 2.

1. Solve the following set of linear equations using a matrix approach:

$$
\begin{aligned}
2x - 5y + 3z &= 7 \\
3x - 2y + 4z &= 5 \\
x + 3y + 2z &= -2
\end{aligned}
$$

Solving them using the Gaussian elimination, invert matrix or Cramer's rule are all acceptable. [5 points]

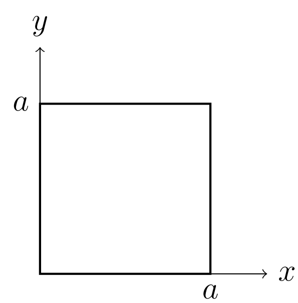


Figure 1: A thin square plate

2. In rigid body problems, it is useful to calculate a tensor of inertia I. For a real symmetric tensor I, there always exists an orthogonal matrix R which diagonalises the tensor I giving

$$I_D = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = RIR^T.$$

Now let's consider a thin square plate whose tensor of inertia is

$$I = \frac{ma^2}{12} \begin{pmatrix} 4 & -3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Diagonalise the provided matrix to find out the eigenvalues (I_1, I_2, I_3) as well as the transformation matrix R. [5 points]

3 Group Theory/Lie Algebra

Question 3.

The Lorentz group is defined as the set of transformations $x^\mu \mapsto \lambda^\mu{}_\nu x^\nu$, which leaves invariant the quantity

$$\eta_{\mu\nu} x^\mu x^\nu = t^2 - x^2 - y^2 - z^2.$$

3.1) Show that, equivalently, Λ such that $\eta = \Lambda^T \eta \Lambda$ is a Lorentz transformation. [1 point]

3.2) As is well known, the infinitesimal generators of the Lorentz group divide into two types, those that generate rotations and those that generate boosts:

$$J^j = i\epsilon_{jkl} x_k \partial_l, \qquad K^j = -i(x_0 \partial_j + x_j \partial_0), \tag{1}$$

where 0 is the time component and $j, k = 1, 2, 3$.

How are the K^j's generators of Lorentz boost? Let

$$B_x(\eta) = \exp\left(-i\eta K^1\right) = \exp\left[-\eta(t\partial_x + x\partial_t)\right].$$

Show that its action for an arbitrary function $\psi(t, x)$ of the spacetime coordinates (reducing to two dimension for simplicity) is

$$B(\eta)\psi(t, x) = \psi(t\cosh\eta - x\sinh\eta, x\cosh\eta - t\sinh\eta).$$

Hint: Start from the expression $K^1 = -i\frac{\partial}{\partial\xi}$, where $t = r\sinh\xi$ and $x = r\cosh\xi$. (You don't have to derive this expression.)

Some useful identities:

$$\sinh(\xi - \eta) = \sinh\xi\cosh\eta - \cosh\xi\sinh\eta,$$
$$\cosh(\xi - \eta) = \cosh\xi\cosh\eta - \sinh\xi\sinh\eta.$$

[3 points]

3.2) Compute the commutators between K^j and all the six generators (1). Do the set of all boosts from a group? [6 points]

4 Group Theory/Lie Algebra

Question 4.

Let $\mathfrak{g}$ be a matrix Lie algebra, and $[X, Y] \equiv XY - YX$ be the matrix commutator between elements X, Y in the Lie algebra.

4.1) Show that the map ad defined by $\text{ad}_X(Y) \equiv [X, Y]$ is a representation of the Lie algebra. This is called the *adjoint representation.* [2 points]

4.2) Now specialize to the Lie algebra $\mathfrak{su}(2)$ generated by the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which obey the commutation relation (assuming Einstein summation convention)

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l.$$

Let $|j\rangle = \sigma_j$, $j = 1, 2, 3$ be the orthonormal basis of the adjoint representation. Write down the matrix representation of $\text{ad}_{\sigma_1}, \text{ad}_{\sigma_2}$ and ad_{σ_3} in this basis. Are they Hermitian? Are they unitary? [3 points]

4.3) Compute $\exp\left(i\lambda\, \text{ad}_{\sigma_j}/2\right)$ for $j = 1, 2, 3$ by directly exponentiating the matrix. Verify that $\exp\left(i\lambda\, \text{ad}_{\sigma_3}/2\right)|1\rangle$ gives the same result as $e^X\sigma_1 e^{-X}$, where $X = i\lambda\sigma_3/2$. (This is the special case of the identity $e^X Y e^{-X} = e^{\text{ad}_X}(Y)$.)

[5 points]

5 Function of Complex Variables

Question 5.

1. Find all the roots of $z^4 = 1$. [2 points]

2. Let $f : \mathbb{C} \to \mathbb{C}, u : \mathbb{C} \to \mathbb{R}, v : \mathbb{C} \to \mathbb{R}$. The function f is analytic if it can be expressed as $f(x, y) = u(x, y) + iv(x, y)$, where u and v satisfies Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

   Consider the case where $u(x, y) = x^3 - 3xy^2$. Find $v(x, y)$ which makes $f(x, y)$ analytic. [3 points]

3. With the help of contour integral and residue theorem, evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}.$$

   [5 points]

6 Integral Transforms

Question 6.

6.1). Let us consider the following function:

$$\psi(x) = \frac{1}{\sqrt{2a}} \left[\, \Theta(x+a) - \Theta(x-a) \,\right],$$

where $a > 0$ and $\Theta(x)$ is the Heaviside step function, namely, $\Theta(x<0) = 0$ and $\Theta(x>0) = 1$. Find the Fourier transform of $\psi(x)$, namely,

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x \, \psi(x) \, \mathrm{e}^{-\mathrm{i}kx}.$$

[7 marks.]

6.2). Is the function $\psi(x)$ normalized to unity? In other words, is $\psi(x)$ a square-integrable function with norm equal to one?

[1 mark.]

6.3). But furthermore, is also $\phi(k)$ a square-integrable function with norm equal to one? Please justify your answer. Hint: You may find some of the following integrals useful:

$$\int_0^{+\infty} \mathrm{d}u \, \frac{\sin(u)}{u} = \frac{\pi}{2}\,, \quad \int_0^{+\infty} \mathrm{d}u \, \frac{\sin(u)}{u^2} = \infty\,, \quad \int_0^{+\infty} \mathrm{d}u \, \frac{\sin^2(u)}{u} = \infty\,, \quad \int_0^{+\infty} \mathrm{d}u \, \frac{\sin^2(u)}{u^2} = \frac{\pi}{2}.$$

[2 marks.]

7 Integral Transforms

Question 7.

In synchronous gauge the line element is

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i\mathrm{d}x^j\right],$$

where τ is the conformal time and $a(\tau)$ is the scale factor. The metric perturbation h_{ij} can be decomposed into a trace part $h \equiv h_{ii}$ and a traceless part $h^{||}_{ij}$ where $h_{ij} = h\delta_{ij}/3 + h^{||}_{ij}$. $h^{||}_{ij}$ can be written in terms of a scalar field μ,

$$h^{||}_{ij} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\mu.$$

We will be working in Fourier space k and introduce two scalar fields $h(\tau, \mathbf{k})$ and $\eta(\tau, \mathbf{k})$. We can write the scalar mode of h_{ij} as a Fourier integral

$$h_{ij}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\left[\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j h(\tau, \mathbf{k}) + \left(\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j - \frac{1}{3}\delta_{ij}\right)6\eta(\tau, \mathbf{k})\right],$$

$$\mathbf{k} = k\hat{\mathbf{k}}.$$

Note that h denotes the trace of h_{ij} in both the real space and Fourier space.

7.1) Write $\mu(\tau, \mathbf{k})$ in terms of $h(\tau, \mathbf{k})$ and $\eta(\tau, \mathbf{k})$. [5 points]

7.2) The component-00 of the perturbed Einstein equation is

$$\frac{1}{2}\partial^i\partial^j h^{||}_{ij} - \frac{1}{3}\nabla^2 h + \mathcal{H}h' = 8\pi G a^2\delta\rho,$$

where $\mathcal{H} \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau}$ and $'$ denotes derivative with respect to τ. Show that in Fourier space it is equivalent to

$$k^2\eta - \frac{1}{2}\mathcal{H}h' = -4\pi G a^2\delta\rho.$$

[5 points]

8 Ordinary Differential Equations

Question 8.

8.1). Find a real general solution to the following ordinary differential equation:

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}y(x) - \lambda^4 y(x) = 0,$$

where λ is a positive constant.

[5 marks.]

8.2). Impose the following boundary conditions on the general solution you wrote in the previous question:

$$y(x) = 0\,, \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) = 0\,, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L > 0.$$

What solutions did you find?

[5 marks.]

Solution

2.) Linear Algebra

2.1) We can turn a set of linear equations into the matrix equation

$$\begin{pmatrix} 2 & -5 & 3 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix} \quad (1)$$

There are many ways to solve the equation

<u>Method 1</u> Use Guassian elimination

We turn (1) to augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -5 & 3 & 7 \\ 3 & -2 & 4 & 5 \\ 1 & 3 & 2 & -2 \end{array}\right)$$

Then we try to change the augmented matrix into echelon or reduced row echelon form.

$$\left(\begin{array}{ccc|c} 2 & -5 & 3 & 7 \\ 3 & -2 & 4 & 5 \\ 1 & 3 & 2 & -2 \end{array}\right) \xrightarrow{R_1/2} \left(\begin{array}{ccc|c} 1 & -2.5 & 1.5 & 3.5 \\ 3 & -2 & 4 & 5 \\ 1 & 3 & 2 & -2 \end{array}\right)$$

$$\downarrow R_2 - 3R_1, \; R_3 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & -2.5 & 1.5 & 3.5 \\ 0 & 5.5 & -0.5 & -5.5 \\ 0 & 0 & 1 & 0 \end{array}\right) \xleftarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -2.5 & 1.5 & 3.5 \\ 0 & 5.5 & -0.5 & -5.5 \\ 0 & 5.5 & 0.5 & -5.5 \end{array}\right)$$

$$\xrightarrow[R_2 + 0.5R_3]{R_1 - 1.5R_3} \left(\begin{array}{ccc|c} 1 & -2.5 & 0 & 3.5 \\ 0 & 5.5 & 0 & -5.5 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2 / 5.5} \left(\begin{array}{ccc|c} 1 & -2.5 & 0 & 3.5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\downarrow R_1 + 2.5R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Therefore, $x = 1$, $y = -1$ and $z = 0$

<u>Method 2</u> Finding the inverse matrix

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$$

where $\operatorname{adj}(A) = C^T$ with C being cofactor matrix.

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,

$$C = \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

Now go back to our problem

$$A = \begin{pmatrix} 2 & -5 & 3 \\ 3 & -2 & 4 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\therefore C = \begin{pmatrix} -16 & -2 & 11 \\ 19 & 1 & -11 \\ -14 & 1 & 11 \end{pmatrix}$$

$$\text{adj}(A) = C^T = \begin{pmatrix} -16 & 19 & -14 \\ -2 & 1 & 1 \\ 11 & -11 & 11 \end{pmatrix}$$

$$\det(A) = 11$$

$$\therefore A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \frac{1}{11} \begin{pmatrix} -16 & 19 & -14 \\ -2 & 1 & 1 \\ 11 & -11 & 11 \end{pmatrix}$$

$\therefore \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Therefore $x = 1, \; y = -1, \; z = 0$

<u>Method 3</u> Cramer's rule

For a matrix equation

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

By Cramer's rule,

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Again, finally we find

$x = 1, \; y = -1, \; z = 0$

2.2) Consider $I = \frac{ma^2}{12} \begin{pmatrix} 4 & -3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

Diagonalize the matrix I by solving

$$\det (I - \mathbb{1}\tilde{\lambda}) = 0$$

$$\det \left(I - \mathbb{1} \left(\frac{ma}{12}\right)^2 \lambda \right) = 0$$

$$\therefore \det \begin{pmatrix} 4-\lambda & -3 & 0 \\ -3 & 4-\lambda & 0 \\ 0 & 0 & 8-\lambda \end{pmatrix} = 0$$

$$\therefore (4-\lambda)^2 (8-\lambda) - 9(8-\lambda) = 0$$

$$(1-\lambda)(7-\lambda)(8-\lambda) = 0$$

$$\lambda = 1, 8$$

We then find the eigenvectors.

For $\lambda = 1$

$$\begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

$\therefore \quad u_1 = u_2 = \frac{1}{\sqrt{2}}, \quad u_3 = 0$

For $\lambda = 7$,
$$\begin{pmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = 0$$

$\therefore \quad u_1' = -u_2' = 1/\sqrt{2}, \quad u_3' = 0$

For $\lambda = 8$,
$$\begin{pmatrix} -4 & -3 & 0 \\ -3 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1'' \\ u_2'' \\ u_3'' \end{pmatrix} = 0$$

$\therefore \quad u_1'' = u_2'' = 0, \quad u_3'' = 1$

Therefore, the transformation matrix is

$$R = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3.1) By definition

[1 pt]

$$\eta_{\mu\nu}(\Lambda^\mu{}_\alpha x^\alpha)(\Lambda^\nu{}_\beta x^\beta) = \eta_{\mu\nu} x^\mu x^\nu$$

$$\parallel$$

$$(\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta)\, x^\alpha x^\beta$$

$$(\Lambda^T)_\alpha{}^\mu \eta_{\mu\nu} \Lambda^\nu{}_\beta$$

Since the equality holds for any x, we have that $\Lambda^T \eta \Lambda = \eta$.

(3.1)

[3 pts]

$$K = -i(t\partial_x + x\partial_t)$$

$$= -i\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi}\frac{\partial}{\partial t}\right)$$

$$= -i\frac{\partial}{\partial \xi}$$

$t = r\sinh\xi$, $\frac{\partial t}{\partial \xi} = x$

$x = r\cosh\xi$, $\frac{\partial x}{\partial \xi} = t$

The student doesn't have to derive this.

Define $\varphi(\xi) := \psi(t(\xi, r), x(\xi, r))$.

$B(\eta)\varphi(\xi) = \varphi(\xi - \eta)$ We know this either from QM or doing the Taylor expansion of $\exp\left(-\eta\frac{\partial}{\partial \xi}\right)$

$$= \psi[r\sinh(\xi-\eta), r\cosh(\xi-\eta)]$$

$$= \psi[r\sinh\xi\cosh\eta - r\cosh\xi\sinh\eta,\ r\cosh\xi\cosh\eta - r\sinh\xi\sinh\eta]$$

$$= \psi(t\cosh\eta - x\sinh\eta,\ x\cosh\eta - t\sinh\eta)$$

(3.3)

[6 pts] $j,k = 1,2,3$

$$[x_j, \partial_k] f = x_j \partial_k f - \partial_k (x_j f)$$
$$= x_j \partial_k f - \delta_{jk} f - x_j \partial_k f$$
$$\Rightarrow [x_j, \partial_k] = -\delta_{jk}$$

$$[x_0, \partial_j] f = x_0 \partial_j f - x_0 \partial_j f = 0$$
$$[x_j, \partial_0] f = x_j \partial_0 f - x_j \partial_0 f = 0$$

$$[x_\mu \partial_\nu, x_\rho \partial_\sigma] = x_\mu [\partial_\nu, x_\rho] \partial_\sigma + x_\mu x_\rho [\partial_\nu, \partial_\sigma]$$ ($\to 0$)

μ, ν, ρ, σ: 0,1,2,3

$$+ [x_\mu, x_\rho] \partial_\sigma \partial_\nu + x_\rho [x_\mu, \partial_\sigma] \partial_\nu$$ ($[x_\mu, x_\rho] \to 0$)
$$= \delta_{\nu\rho} x_\mu \partial_\sigma - \delta_{\mu\sigma} x_\rho \partial_\nu$$

$$\therefore [K^j, J^k] = \epsilon_{klm} \left(\underbrace{[x_0 \partial_j, x_l \partial_m]}_{\delta_{jl} x_0 \partial_m} + \underbrace{[x_j \partial_0, x_l \partial_m]}_{-\delta_{jm} x_l \partial_0} \right)$$

$$= \epsilon_{kjm} x_0 \partial_m - \epsilon_{klj} x_l \partial_0$$

Change dummy index to l

$$= -\epsilon_{jkl} x_0 \partial_l - \epsilon_{jkl} x_l \partial_0 = \boxed{-i \epsilon_{jkl} K^l}$$

and $[K^j, K^k] = -[x_0\partial_j + x_j\partial_0, x_0\partial_k + x_k\partial_0]$

$$= -[x_0\partial_j, x_k\partial_0] - [x_j\partial_0, x_0\partial_k]$$

$$= -\delta_{jk}x_0\partial_0 + x_k\partial_j - x_j\partial_k + \delta_{jk}x_0\partial_0$$

$$= \boxed{-i\epsilon_{jkl}J_l}$$

The boosts don't from a group because the commutators of their generators are not closed (i.e. a sequence of boosts become a boost + a rotation).

(4.1) For ad to be a Lie algebra rep, we need to show that [2 pts] it preserves the commutator

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$$

Let both sides act on $Z \in \mathfrak{g}$:

$$[[X,Y],Z] \stackrel{?}{=} [X,[Y,Z]] - [Y,[X,Z]]$$
$$= [X,[Y,Z]] + [Y,[Z,X]]$$
$$0 \stackrel{?}{=} [X,[Y,Z]] + [Y,[Z,X]] - [[X,Y],Z]$$
$$\stackrel{?}{=} [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]].$$

But the last line is the Jacobi's identity which holds in any Lie algebra by definition, so the statement is proved. □

(4.2) The whole point of this problem is to translate [3 pts] $[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l$ to the form of a linear map acting on σ_k, and the answer is

$$\mathrm{ad}_{\sigma_x} = 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \mathrm{ad}_{\sigma_y} = 2\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad \mathrm{ad}_{\sigma_z} = 2\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Evidently, they are Hermitian but not unitary (as they [multiplied by i] have to exponentiate to unitaries).

(4.3) In the relavant 2x2 subspace, $\frac{i\lambda}{2}\mathrm{ad}_{\sigma_j}$ is $\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}$, $\forall j$

$\begin{bmatrix} 0 & -\lambda \\ -\lambda & 0 \end{bmatrix}^2 = -\lambda^2 \mathbb{1}$ and so on. So the series for $\exp(\frac{i\lambda}{2}\mathrm{ad}_{\sigma_j})$ alternates.

$$\exp\left(\frac{i\lambda}{2}\mathrm{ad}_{\sigma_j}\right) = \mathbb{1} + \lambda \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \frac{\lambda^2 \mathbb{1}}{2!} - \frac{\lambda^3}{3!}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \cdots$$

$$= \sum_{k \text{ even}} \frac{(-\lambda)^k}{k!}\mathbb{1} + \sum_{k \text{ odd}} \frac{(-\lambda)^k}{k!}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ ← typo, has to be λ

$$\therefore \exp\left(\frac{i\lambda}{2}\mathrm{ad}_{\sigma_x}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$\exp\left(\frac{i\lambda}{2}\mathrm{ad}_{\sigma_y}\right) = \begin{bmatrix} \cos\theta & 0 & -i\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\exp\left(\frac{i\lambda}{2}\mathrm{ad}\,\sigma_z\right) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ $e^{i\lambda \mathrm{ad}_{\sigma_z}/2}|1\rangle$

$= \cos\theta|1\rangle - \sin\theta|2\rangle$

$\Leftrightarrow \cos\theta\,\sigma_x - \sin\theta\,\sigma_y$

$$e^{i\pi\sigma_z/2}\sigma_x e^{-i\pi\sigma_z/2} = (\overset{\cos\frac{\pi}{2}}{\cos} + i\overset{\sin\frac{\pi}{2}}{\sin}\sigma_z)\sigma_x(\cos - i\sin\sigma_z)$$

$$= \cos^2\sigma_x + i\sin\cos\underbrace{\sigma_z\sigma_x}_{i\sigma_y} - i\sin\cos\underbrace{\sigma_x\sigma_z}_{-i\sigma_y} + \sin^2\underbrace{\sigma_z\sigma_x\sigma_z}$$

$$= \underbrace{(\cos^2 - \sin^2)}_{\cos\pi}\sigma_x - \underbrace{2\sin\cos}_{\sin\pi}\sigma_y$$ ✓ [5 pts]

Qualifying Examination, January 2024

The Institute for Fundamental Study

Solution for problem 5

5) Function of Complex Variables

5.1) Find all the roots of $z^4 = 1$. [2 points]

5.2) Let $f : \mathbb{C} \to \mathbb{C}, u : \mathbb{C} \to \mathbb{R}, v : \mathbb{C} \to \mathbb{R}$. The function f is analytic if it can be expressed as $f(x, y) = u(x, y) + iv(x, y)$, where u and v satisfies Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Consider the case where $u(x, y) = x^3 - 3xy^2$. Find $v(x, y)$ which makes $f(x, y)$ analytic. [3 points]

5.3) With the help of contour integral and residue theorem, evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}.$$

[5 points]

Solution

Soln5.1) From $z^4 = 1$, we have $z^2 = \pm i$. Since $i = e^{\pi/2} = e^{5\pi/2}$, and $-i = e^{3\pi/2} = e^{7\pi/2}$. Then the roots are $z = e^{\pi/4}, e^{3\pi/4}, e^{5\pi/4}, e^{7\pi/4}$, or $z = (1 + i)/\sqrt{2}, (-1 + i)/\sqrt{2}, (-1 - i)/\sqrt{2}, (1 - i)/\sqrt{2}$.

[1 point for method.
Other correct methods are also qualified for this point,
for example, by drawing points on complex plane.]

[1 point for correct result.
Either form is allowed.]

Soln5.2) By direct calculation, we obtain

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy.$$

[1 point]

By integrating the first equation above, we obtain

$$v(x, y) = 3x^2y - y^3 + g(x),$$

where $g(x)$ is to be determined. [1 point]

Then by taking derivative of the above equation with respect to x, we obtain

$$\frac{\partial v}{\partial x} = 6xy + g'(x).$$

Comparing with the Cauchy-Riemann equations, we see that

$$g'(x) = 0.$$

So $v(x, y) = 3x^2y - y^3 + k$, where $k \in \mathbb{C}$. [1 point.
NB: full point is also awarded
even if k is some specific complex number
e.g. $k = 0$.]

Solution for problem 5

Soln5.3) Let C be a closed curved consists of 2 parts: (1) : a straight line from $z = -R$ to $z = R$; (2) : a semicircle in the upper half-plane with centre at $z = 0$ and radius R. [1 point]

Consider

$$\oint_C \frac{dz}{z^2 + 1}.$$

The poles are at $z = \pm i$. [1 point]

Note that

$$\oint_C = \int_{(1)} + \int_{(2)}.$$

We have, from residue theorem

$$\oint_C = 2\pi i \frac{1}{i + i} = \pi.$$

[1 point]

By direct calculation, we have

$$\int_{(2)} = \int_0^\pi \frac{d\theta \; iRe^{i\theta}}{R^2 e^{2i\theta} + 1},$$

which vanishes as $R \to \infty$. [1 point]

Then since

$$\lim_{R\to\infty} \int_{(1)} = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1},$$

and after combining everything, we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \pi.$$

[1 point]

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QUALIFYING EXAMINATION (QE)
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Answer Nº 6
(Final update: January 15, 2024)

Answer Nº 6.
6.1). Note, first, that $\Theta(x+a) = \Theta(x-(-a)) = 1$, for $x > -a$ and $\Theta(x+a) = \Theta(x-(-a)) = 0$, for $x < -a$. Likewise, $\Theta(x-a) = 1$, for $x > a$ and $\Theta(x-a) = 0$, for $x < a$. Consequently, $\Theta(x+a) - \Theta(x-a) = 1$, for $|x| < a$ and $\Theta(x+a) - \Theta(x-a) = 0$, for $|x| > a$. Also, $\psi(x) = 1/\sqrt{2a}$, for $|x| < a$ and $\psi(x) = 0$, for $|x| > a$. Then, the Fourier transform of $\psi(x)$ is given by

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \psi(x)\, \mathrm{e}^{-ikx} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} \mathrm{d}x\, \frac{1}{\sqrt{2a}}\, \mathrm{e}^{-ikx} = \sqrt{\frac{a}{\pi}}\, \frac{\sin(ka)}{ka}. \tag{1}$$

[7 marks.]
6.2). The function $\psi(x)$ is normalized to unity, i.e., $\psi(x)$ belongs to the space of square integrable functions $\mathcal{L}^2(\mathbb{R}, \mathrm{d}x)$. In effect

$$\int_{\mathbb{R}} \mathrm{d}x\, |\psi(x)|^2 = \int_{-a}^{+a} \mathrm{d}x\, \left(\frac{1}{\sqrt{2a}}\right)^2 = \frac{1}{2a}\, 2a = 1. \tag{2}$$

[1 mark.]
6.3). The Fourier transform $\phi(k)$ is also normalized to unity, i.e., $\phi(k)$ belongs to the space of square integrable functions $\mathcal{L}^2(\mathbb{R}, \mathrm{d}k)$. This is because there is an isomorphism between $\mathcal{L}^2(\mathbb{R}, \mathrm{d}x)$ and $\mathcal{L}^2(\mathbb{R}, \mathrm{d}k)$. Precisely, the Parseval-Plancherel theorem realizes this isomorphism, namely,

$$\int_{\mathbb{R}} \mathrm{d}x\, |\psi(x)|^2 = \int_{\mathbb{R}} \mathrm{d}k\, |\phi(k)|^2 \,. \tag{3}$$

For completeness, here is the proof of this theorem:

$$\int_{\mathbb{R}} \mathrm{d}x\, |\psi(x)|^2 = \int_{\mathbb{R}} \mathrm{d}x\, \psi^*(x)\, \psi(x)$$

$$= \int_{\mathbb{R}} \mathrm{d}x\, \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}k\, \phi(k)\, \mathrm{e}^{+ikx}\right)^* \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\lambda\, \phi(\lambda)\, \mathrm{e}^{+i\lambda x}$$

$$= \int_{\mathbb{R}} \mathrm{d}k \int_{\mathbb{R}} \mathrm{d}\lambda\, \phi^*(k)\, \phi(\lambda) \left(\frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{+i(\lambda-k)x}\right)$$

$$= \int_{\mathbb{R}} \mathrm{d}k \int_{\mathbb{R}} \mathrm{d}\lambda\, \phi^*(k)\, \phi(\lambda)\, \delta(\lambda - k)$$

$$= \int_{\mathbb{R}} \mathrm{d}k\, \phi^*(k)\phi(k) = \int_{\mathbb{R}} \mathrm{d}k\ |\phi(k)|^2 . \quad (4)$$

Thus, if $\int_{\mathbb{R}} \mathrm{d}x\ |\psi(x)|^2 = 1$, then it is verified that $\int_{\mathbb{R}} \mathrm{d}k\ |\phi(k)|^2 = 1$. Certainly, this last definite integral can also be calculated explicitly using the following result:

$$\int_0^{+\infty} \mathrm{d}u\, \frac{\sin^2(u)}{u^2} = \frac{\pi}{2} \quad \Rightarrow \quad \int_{-\infty}^{+\infty} \mathrm{d}u\, \frac{\sin^2(u)}{u^2} = 2 \int_0^{+\infty} \mathrm{d}u\, \frac{\sin^2(u)}{u^2} = 2\, \frac{\pi}{2} = \pi. \quad (5)$$

In effect,

$$\int_{\mathbb{R}} \mathrm{d}k\ |\phi(k)|^2 = \frac{a}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}k\, \frac{\sin^2(ka)}{(ka)^2} = \frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{d}u}{a}\, \frac{\sin^2(u)}{u^2} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}u\, \frac{\sin^2(u)}{u^2} = \frac{1}{\pi}\, \pi = 1. \quad (6)$$

[2 marks.]

SDeV/SDeV

Question 7.

7.1) In synchronous gauge the line element is

$$\mathrm{d}s^2 = a^2(\tau)\left[-\mathrm{d}\tau^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i \mathrm{d}x^j\right],$$

where τ is the conformal time and $a(\tau)$ is the scale factor. The metric perturbation h_{ij} can be decomposed into a trace part $h \equiv h_{ii}$ and a traceless part $h^{||}_{ij}$ where $h_{ij} = h\delta_{ij}/3 + h^{||}_{ij}$. $h^{||}$ can be written in terms of a scalar field μ,

$$h^{||}_{ij} = \left(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\mu.$$

We will be working in Fourier space k and introduce two fields $h(\tau, \mathbf{k})$ and $\eta(\tau, \mathbf{k})$. We can write the scalar mode of h_{ij} as a Fourier integral

$$h_{ij}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j h(\tau, \mathbf{k}) + \left(\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j - \frac{1}{3}\delta_{ij}\right) 6\eta(\tau, \mathbf{k})\right],$$

$$\mathbf{k} = k\hat{\mathbf{k}}.$$

Note that h denotes the trace of h_{ij} in both the real space and Fourier space.

Write $\mu(\tau, \mathbf{k})$ in terms of $h(\tau, \mathbf{k})$ and $\eta(\tau, \mathbf{k})$.

[5 points]

Solution The Fourier transform of $h(\tau,\vec{x})$ and $\mu(\tau,\vec{x})$

$$h(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} h(\tau,\vec{k})$$

and

$$\mu(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \mu(\tau,\vec{k})$$

$$\rightarrow h^{||}_{ij}(\tau,\vec{x}) = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\mu(\tau,\vec{x})$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left(-k_i k_j + \frac{1}{3}\delta_{ij} k^2\right)\mu(\tau,\vec{k})$$

$$h^{||}_{ij}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)(-k^2)\mu(\tau,\vec{k})$$

$\longrightarrow$

$$h_{ij}(\tau,\vec{x}) = \frac{1}{3}\delta_{ij} h(\tau,\vec{x}) + h^{||}_{ij}(\tau,\vec{x})$$

$$h_{ij}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left[\frac{1}{3}\delta_{ij} h(\tau,\vec{k}) + \left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)(-k^2)\mu(\tau,\vec{k})\right]$$

By comparison with

$$h_{ij}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[\hat{k}_i \hat{k}_j h(\tau,\vec{k}) + (\hat{k}_i \hat{k}_j - \frac{1}{3}\delta_{ij}) 6\eta(\tau,\vec{k}) \right]$$

$\longrightarrow$

$$\frac{1}{3}\delta_{ij} h(\tau,\vec{k}) + (\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})(-k^2)\mu(\tau,\vec{k})$$
$$= \hat{k}_i\hat{k}_j h(\tau,\vec{k}) + (\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}) 6\eta(\tau,\vec{k})$$

$$\longrightarrow \mu(\tau,\vec{k}) = -\frac{1}{k^2}\left(h(\tau,\vec{k}) + 6\eta(\tau,\vec{k}) \right)$$

7.2) The component-00 of the perturbed Einstein equation is

$$\frac{1}{2}\partial^i\partial^j h^{||}_{ij} - \frac{1}{3}\nabla^2 h + \mathcal{H}h' = 8\pi G a^2 \delta\rho,$$

where $\mathcal{H} \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\tau}$ and $'$ denotes derivative with respect to τ. Show that in Fourier space it is equivalent to

$$k^2\eta - \frac{1}{2}\mathcal{H}h' = -4\pi G a^2 \delta\rho.$$

[5 points]

Solution From

$$h^{||}_{ij}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} (\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})(-k^2)\mu(\tau,\vec{k})$$

Using the result form the previous part

$$h^{||}_{ij}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} (\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})\left(h(\tau,\vec{k}) + 6\eta(\tau,\vec{k}) \right)$$

$$\frac{1}{2}\partial^i\partial^j h^{||}_{ij}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left(-\frac{1}{2}k^i k^j\right)(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})\left(h(\tau,\vec{k}) + 6\eta(\tau,\vec{k}) \right)$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left(-\frac{1}{2}\right)\left(k^2 - \frac{1}{3}k^2\right)\left(h(\tau,\vec{k}) + 6\eta(\tau,\vec{k}) \right)$$

$$\frac{1}{2}\partial^i\partial^j h^{\parallel}_{ij}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left(-\frac{k^2}{3} h(\tau,\vec{k}) - 2k^2\eta(\tau,\vec{k}) \right)$$

$$-\frac{1}{3}\nabla^2 h(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{k^2}{3} h(\tau,\vec{k})$$

$$\mathcal{H} h'(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \mathcal{H} h'(\tau,\vec{k})$$

$\longrightarrow$

$$\frac{1}{2}\partial^i\partial^j h^{\parallel}_{ij}(\tau,\vec{x}) - \frac{1}{3}\nabla^2 h(\tau,\vec{x}) + \mathcal{H} h'(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left(\mathcal{H} h'(\tau,\vec{k}) - 2k^2\eta(\tau,\vec{k}) \right)$$

$$\longrightarrow \quad \mathcal{H} h' - 2k^2\eta = 8\pi G a^2 \delta\rho$$

$$\longrightarrow \quad k^2\eta - \frac{1}{2}\mathcal{H} h' = -4\pi G a^2 \delta\rho \,.$$

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Answer Nº 8
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Answer Nº 8.
8.1). Assume that $y = \mathrm{e}^{mx}$ is a solution, where m is a constant. Substituting this solution into the ordinary differential equation (ODE) yields the following relation:

$$(m^4 - \lambda^4)\,\mathrm{e}^{mx} = 0, \tag{1}$$

which implies that

$$(m^4 - \lambda^4) = 0 \;\Rightarrow\; (m^2)^2 - (\lambda^2)^2 = 0 \;\Rightarrow\; (m^2 - \lambda^2)(m^2 + \lambda^2) = 0. \tag{2}$$

Therefore, the solutions of the latter algebraic equation are given by $m = \pm\lambda$ and $m = \pm\mathrm{i}\lambda$. Thus, the following functions are solutions of the ODE:

$$y_1(x) = \mathrm{e}^{\lambda x} \quad y_2(x) = \mathrm{e}^{-\lambda x} \quad y_3(x) = \mathrm{e}^{\mathrm{i}\lambda x} \quad y_4(x) = \mathrm{e}^{-\mathrm{i}\lambda x}, \tag{3}$$

and they are clearly linearly independent. The solutions $y_3(x)$ and $y_4(x)$ are complex solutions, so one can combine them to obtain two real solutions. In effect, the following two solutions are real:

$$y_3(x) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{e}^{-\mathrm{i}\lambda x}\right) = \cos(\lambda x) \qquad y_4(x) = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\lambda x} - \mathrm{e}^{-\mathrm{i}\lambda x}\right) = \sin(\lambda x). \tag{4}$$

Although the solutions $y_1(x)$ and $y_2(x)$ given in Eq. (3) are real (and could also be used), it is worthwhile to rewrite them, namely,

$$y_1(x) = \frac{1}{2}\left(\mathrm{e}^{\lambda x} + \mathrm{e}^{-\lambda x}\right) = \cosh(\lambda x) \qquad y_2(x) = \frac{1}{2}\left(\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}\right) = \sinh(\lambda x). \tag{5}$$

These two solutions are particularly useful when one has homogeneous boundary conditions. Then, a real general solution of the ODE is as follows:

$$y(x) = C_1\cosh(\lambda x) + C_2\sinh(\lambda x) + C_3\cos(\lambda x) + C_4\sin(\lambda x), \tag{6}$$

where C_1, C_2, C_3, and C_4 are constants.
[5 marks.]

8.2). By imposing the boundary conditions $y(x = 0) = 0$ and $(\mathrm{d}^2y/\mathrm{d}x^2)\,(x = 0) = 0$ on the general solution given in Eq. (6), one obtains respectively the following relations:

$$C_1 + C_3 = 0\,, \quad C_1 - C_3 = 0, \tag{7}$$

consequently, $C_1 = C_3 = 0$. Substituting the latter result into the general solution and imposing on it the boundary conditions $y(x = L) = 0$ and $(\mathrm{d}^2y/\mathrm{d}x^2)\,(x = L) = 0$, one obtains $C_2 = 0$. Finally, the solutions satisfying the boundary conditions are given by

$$y(x) = C_4 \sin(\lambda x), \tag{8}$$

where $\lambda = N\pi/L$, $N = 1, 2, 3, \ldots$.
[5 marks.]

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