Qualifying Examination

Academic Year 2023 (2566) : PART 2 : ROUND 2

26 JAN 2024

Name:...

Student ID:..

Part 2: Subjects Relating to student's Research Field : General Relativity and Cosmology

Instructions and cautions

1. Choose 3 out of the following 6 questions relating student's field.
2. Write down detailed and clear solution to problems of your choice.
3. Complete answer are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	

Name:...

1 Manifold, Curvature, Gravitation, Spherically symmetric solution, Black holes

Question 1.

1.1). Prove the following relation and find S

$$\nabla_X(\nabla_Y Z^\alpha) - \nabla_Y(\nabla_X Z^\alpha) = S Z^\alpha + R^\alpha{}_{\rho\mu\nu} X^\mu Y^\nu Z^\rho,$$

where $\nabla_B = B^\mu \nabla_\mu$ and $R^\alpha{}_{\rho\mu\nu}$ is the curvature tensor. [6 points]

1.2). Show the degrees of freedom for Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ and Ricci tensor $R_{\mu\nu}$ in the five-dimensional spacetimes. One needs to declare the reductions with the symmetry properties. [4 points]

For marker: Name.. Signature...

2 Field theory of gravity

Question 2.

For the gravitational theory with Palatini formalism, the connection is promoted to be the dynamical field.

2.1) What are the violated assumptions in General relativity of Einstein [2 points]

2.2) For the Einstein-Hilbert action, find the equations of motion (You can use Palatini equation directly: $\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\rho_{\mu\rho}$). [4 points]

2.3) Explain the advantages and disadvantages of this formalism compare to General Relativity of Einstein. [4 points]

3 Linearized gravity, Gravitational wave

Question 3.

In the linearized Einstein field equations, they can be written interm of wave equations andgauge conditions as follows

$$\begin{aligned} \partial_\rho \partial^\rho h^{\mu\nu} &= 0, \\ \partial_\mu h^{\mu\nu} &= 0, \quad \partial_\rho \partial^\rho \xi^\mu = 0. \end{aligned}$$

3.1) What is the number degrees of freedom and explain how to count it.

[1 point]

3.2) Show that the solution is a wave function which is transverse and propagates with speed of light. [2 point]

3.3) From the formula of the quadrupole moment of the source $I^{ij} = \int T^{00} y^i y^j d^3y$, find the exact form of the equal-mass (M) circular binary star with radius (a) and constant angular frequency (Ω). [3 points]

3.4) Explain logical step in order to obtain the expression for the energy loss due to the gravitational wave from this binary star. [4 points]

Name:..

4 Physical Foundations of Cosmology: cosmological principle, perfect fluid, ΛCDM model

Question 4.

4.1) List the important matter and energy components in the ΛCDM model, and explain briefly what are roles of each components in the universe. [4 points]

4.2) Compute the energy density ρ_d of dark energy in terms of redshift z for the case where its equation of state parameter $w_d(z)$ depends on z. [6 points]

5 Inflationary cosmology: models of cosmic inflation, necessary conditions for cosmic inflation

Question 5.

5.1) Compute the slow-roll parameter $\epsilon \equiv -\frac{\dot{H}}{H^2}$ in terms of the potential V of inflaton in the slow-roll limit, where H is the Hubble parameter. [4 points]

5.2) What is the graceful exit in inflationary scenario? What are conditions on evolution of ϵ and H required to have graceful exit? [3 points]

5.3) Show in detail whether an inflationary model has graceful exit if an inflaton is a phantom field. [3 points]

6 Late-time cosmology: models of accelearting universe, problem of those models, basic of modified theories of gravity

Question 6.

6.1) What is the cosmic coincidence problem? How can this problem be alleviated?[3 points]

6.2) What is the chameleon mechanism? How can this mechanism be realized? [3 points]

6.3) Compute the redshift at which the universe starts to accelerate for the spatially flat Friedmann universe containing dark matter and dark energy with constant equation of state parameter w_d. Note that you can express your result in terms of w_d and the present values of the densities parameter of dark energy and dark matter. [4 points]

Qualifying Examination

Round 2/2566 (2/2023)

30 August 2021

Name:..

Student ID:...

Part 2: Subjects Relating to student's Research Field : General Relativity and Cosmology

Instructions and cautions

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1 Manifold, Curvature, Gravitation, Spherically symmetric solution, Black holes

Question 1.

1.1). Prove the following relation and find S

$$\nabla_X(\nabla_Y Z^\alpha) - \nabla_Y(\nabla_X Z^\alpha) = S Z^\alpha + R^\alpha{}_{\rho\mu\nu} X^\mu Y^\nu Z^\rho,$$

where $\nabla_B = B^\mu \nabla_\mu$ and $R^\alpha{}_{\rho\mu\nu}$ is the curvature tensor. [6 points]

Sol.

$$\begin{aligned}
\nabla_X\left(\nabla_Y Z^\alpha\right) &= X^\mu \nabla_\mu Y^\nu \nabla_\nu Z^\alpha, \\
&= X^\mu \partial_\mu \left(Y^\nu \partial_\nu Z^\alpha + Y^\nu \Gamma^\alpha_{\rho\nu} Z^\rho\right) + X^\mu \Gamma^\alpha_{\sigma\mu}\left(Y^\nu \partial_\nu Z^\sigma + Y^\nu \Gamma^\sigma_{\rho\nu} Z^\rho\right), \\
&= X^\mu \left(\partial_\mu Y^\nu\right) \partial_\nu Z^\alpha + X^\mu Y^\nu \partial_\mu \partial_\nu Z^\alpha \\
&\quad + X^\mu \left(\partial_\mu Y^\nu\right) \Gamma^\alpha_{\rho\nu} Z^\rho + X^\mu Y^\nu \left(\partial_\mu \Gamma^\alpha_{\rho\nu}\right) Z^\rho + X^\mu Y^\nu \Gamma^\alpha_{\rho\nu} \partial_\mu Z^\rho \\
&\quad + X^\mu Y^\nu \Gamma^\alpha_{\sigma\mu} \partial_\nu Z^\sigma + \Gamma^\alpha_{\sigma\mu} \Gamma^\sigma_{\rho\nu} X^\mu Y^\nu Z^\rho.
\end{aligned}$$

$$\begin{aligned}
\nabla_Y\left(\nabla_X Z^\alpha\right) &= Y^\mu \left(\partial_\mu X^\nu\right) \partial_\nu Z^\alpha + Y^\mu X^\nu \partial_\mu \partial_\nu Z^\alpha \\
&\quad + Y^\mu \left(\partial_\mu X^\nu\right) \Gamma^\alpha_{\rho\nu} Z^\rho + Y^\mu X^\nu \left(\partial_\mu \Gamma^\alpha_{\rho\nu}\right) Z^\rho + Y^\mu X^\nu \Gamma^\alpha_{\rho\nu} \partial_\mu Z^\rho \\
&\quad + Y^\mu X^\nu \Gamma^\alpha_{\sigma\mu} \partial_\nu Z^\sigma + \Gamma^\alpha_{\sigma\mu} \Gamma^\sigma_{\rho\nu} Y^\mu X^\nu Z^\rho.
\end{aligned}$$

Therefore

$$\nabla_X(\nabla_Y Z^\alpha) - \nabla_Y(\nabla_X Z^\alpha) = \left(X^\mu \left(\partial_\mu Y^\nu\right) - Y^\mu \left(\partial_\mu X^\nu\right)\right) \nabla_\nu Z^\alpha + R^\alpha{}_{\rho\mu\nu} X^\mu Y^\nu Z^\rho,$$

where

$$S = [X^\mu \left(\partial_\mu Y^\nu\right) - Y^\mu \left(\partial_\mu X^\nu\right)] \nabla_\nu.$$

Note that it is equivalent to $S = [X^\mu \left(\nabla_\mu Y^\nu\right) - Y^\mu \left(\nabla_\mu X^\nu\right)] \nabla_\nu$ if torsion free.

1.2). Show the degrees of freedom for Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ and Ricci tensor $R_{\mu\nu}$ in the five-dimensional spacetimes. One needs to declare the reductions with the symmetry properties. [4 points]

Sol.

In general, the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ in five-dimensional spacetimes includes $5^4 = 625$ degrees of freedom.

For the anti-symmetry of $\mu\nu$ and $\alpha\beta$ ($R_{\mu\nu\alpha\beta} = R_{\nu\mu\beta\alpha} = -R_{\mu\nu\beta\alpha} = -R_{\nu\mu\alpha\beta}$), firstly, $\mu = \nu$ and $\alpha = \beta$ terms vanish, there for $5^4 - 2 \times 5^3 + 5^2 = 400$ degrees of freedom left.

Still for the anti-symmetry of $\mu\nu$ and $\alpha\beta$, for $\mu \neq \nu$ and $\alpha \neq \beta$, we have $\frac{400}{4} = 100$ degrees of freedom left.

For the symmetry of the first two and the last two indexes ($R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$), we have $\frac{100-10}{2} + 10 = 55$ degrees of freedom left.

Lastly, for the cyclic relation

$$R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha} = 0,$$

which provide five conditional equations that minus five more degrees of freedom in five-dimensional spacetimes. Therefore, the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$ includes $55-5 = 50$ degrees of freedom in general five-dimensional spacetimes.

For the Ricci tensor, we have $R_{\mu\nu} = R_{(\mu\nu)}$, therefore we have $\frac{5^2-5}{2} + 5 = 15$ degrees of freedom in five-dimensional spacetimes.

2 Field theory of gravity

Question 2.

For the gravitational theory with Palatini formalism, the connection is promoted to be the dynamical field.

2.1) What are the violated assumptions in General relativity of Einstein [2 points]

Solution:

1. Metric compatibility
2. Only the metic tensor is the mediator of the gravitational interacton.

2.2) For the Einstein-Hilbert action, find the equations of motion (You can use Palatini equation directly: $\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\rho_{\mu\rho}$). [4 points]

Solution:

Varying the action with respect to $g_{\mu\nu}$ yields the Einstein equation,

$$\frac{\delta \bar{S}_g}{\delta g^{\mu\nu}} = \int d^4x\sqrt{-g}\left(\bar{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\bar{R}\right).$$

Since

$$\delta R_{\mu\nu} = \bar{\nabla}_\rho \delta\bar{\Gamma}^\rho{}_{\mu\nu} - \bar{\nabla}_\nu \delta\bar{\Gamma}^\rho{}_{\mu\rho},$$

then, varying the action with respect to the connection $\bar{\Gamma}^\rho{}_{\mu\nu}$ gives

$$\begin{aligned}
\delta_\Gamma \bar{S}_g &= \int d^4x\sqrt{-g}\; g^{\mu\nu}\left(\bar{\nabla}_\rho \delta\bar{\Gamma}^\rho{}_{\mu\nu} - \bar{\nabla}_\nu \delta\bar{\Gamma}^\rho{}_{\mu\rho}\right), \\
&= \int d^4x\sqrt{-g}\;\left(\bar{\nabla}_\rho\left(g^{\mu\nu}\delta\bar{\Gamma}^\rho{}_{\mu\nu}\right) + \bar{\nabla}_\nu\left(g^{\mu\nu}\delta\bar{\Gamma}^\rho{}_{\mu\rho}\right)\right) \\
&\qquad - \left(\delta\bar{\Gamma}^\rho{}_{\mu\rho}\left(\bar{\nabla}_\nu g^{\mu\nu}\right) - \delta\bar{\Gamma}^\rho{}_{\mu\nu}\left(\bar{\nabla}_\rho g^{\mu\nu}\right)\right), \\
&= \int d^4x\sqrt{-g}\;\left(\delta^\nu_\rho \delta^\lambda_\sigma \bar{\nabla}_\lambda g^{\mu\sigma} - \bar{\nabla}_\rho g^{\mu\nu}\right)\delta\bar{\Gamma}^\rho{}_{\mu\nu}.
\end{aligned}$$

Therefore, the equation of motion for varying the connection is $\bar{\nabla}_\rho g^{\mu\nu} = 0$.

2.3) Explain the advantages and disadvantages of this formalism compare to General Relativity of Einstein. [4 points]

Solution:

Advatage:

1. No assumtion for metric compatibility
2. Easer for computing the equation of montion.
3. Do not need the first derivative of the field to be vanished at the boundary.

Disavantage:

1. No describtion of curvature
2. It is difficult to define how matter couples to the gravity

3 Linearized gravity, Gravitational wave

Question 3.

In the linearized Einstein field equations, they can be written in terms of wave equations and gauge conditions as follows

$$
\begin{aligned}
\partial_\rho \partial^\rho h^{\mu\nu} &= 0, \\
\partial_\mu h^{\mu\nu} &= 0, \quad \partial_\rho \partial^\rho \xi^\mu = 0.
\end{aligned}
$$

3.1) What is the number degrees of freedom and explain how to count it.

[1 point]

Solution:
Number degrees of freedom is 2. There are 10 degrees fo freedom from $\partial_\rho \partial^\rho h^{\mu\nu} = 0$ and there are 4 constraints from $\partial_\mu h^{\mu\nu} = 0$ as well as 4 from $\partial_\rho \partial^\rho \xi^\mu = 0$. So that totally there are $10 - 4 - 4 = 2$.

3.2) Show that the solution is a wave function which is transverse and propagates with speed of light. [2 point]
Solution:
General solution to the wave equation is

$$
h^{\mu\nu} = A^{\mu\nu} e^{ik_\rho x^\rho}.
$$

Substituting this general solution into the wave equation , one has

$$
\Box \bar{h}^{\mu\nu} = 0, \quad \Rightarrow \quad -k_\rho k^\rho A^{\mu\nu} = 0, \quad \Rightarrow \quad k_\rho k^\rho = 0.
$$

From this equation, it follows that this gauge field is massless, since $k_\rho k^\rho = 0$ corresponds to $p_\rho p^\rho = 0$, where p^ρ is the four-momentum of the particle. This equation also provide us the speed of the propagation as follows

$$
k_\rho k^\rho = -\left(k^0\right)^2 + |\vec{k}|^2 = 0 \;\; \Rightarrow \;\; \frac{\omega}{c} = \frac{2\pi}{\lambda} \;\; \Rightarrow \;\; v = c. \tag{1}
$$

Applying the solution to the constraint $\partial_\mu h^{\mu\nu} = 0$, one obtains

$$k_\mu A^\mu = 0. \tag{2}$$

This equation suggests us that the vector field correspond to the transverse wave.

3.3) From the formula of the quadrupole moment of the source $I^{ij} = \int T^{00} y^i y^j d^3y$, find the exact form of the equal-mass (M) circular binary star with radius (a) and constant angular frequency (Ω). [3 points]

Solution:

The quadupole moment can be written in terms of mass density as

$$\begin{aligned} I^{ij}(ct) &= \int T^{00} y^i y^j d^3y, \\ &= \int c^2 \rho y^i y^j d^3y, \\ &= \int c^2 \rho x^i x^j d^3x, \end{aligned}$$

where the mass density can be written as

$$\begin{aligned} \rho_A &= M\delta(x^3)\left[\delta(x^1 - a\cos\Omega t)\delta(x^2 - a\sin\Omega t)\right], \\ \rho_B &= M\delta(x^3)\left[\delta(x^1 + a\cos\Omega t)\delta(x^2 + a\sin\Omega t)\right], \\ \rho &= M\delta(x^3)\left[\delta(x^1 - a\cos\Omega t)\delta(x^2 - a\sin\Omega t) + \delta(x^1 + a\cos\Omega t)\delta(x^2 + a\sin\Omega t)\right]. \end{aligned}$$

Then each components can be obtained as

$$
\begin{aligned}
I^{11}(ct) &= c^2 \int \rho \left(x^1\right)^2 d^3x, \\
&= Mc^2 \left[(a\cos\Omega t)^2 + (-a\cos\Omega t)^2\right], \\
&= Mc^2 \left(2a^2\cos^2\Omega t\right), \\
&= Mc^2a^2 \left(1+\cos 2\Omega t\right), \\
I^{22}(ct) &= c^2 \int \rho \left(x^2\right)^2 d^3x, \\
&= Mc^2 \left(2a^2\sin^2\Omega t\right), \\
&= Mc^2a^2 \left(1-\cos 2\Omega t\right), \\
I^{12}(ct) &= c^2 \int \rho x^1 x^2 d^3x, \\
&= Mc^2 \left(\left(-a\cos\Omega t\right)\left(-a\sin\Omega t'\right) + \left(a\cos\Omega t\right)\left(a\sin\Omega t\right)\right), \\
&= Mc^2 \left(2a^2\cos\Omega t\sin\Omega t\right), \\
&= Mc^2a^2\sin 2\Omega t, \\
&= I^{21}(ct), \\
I^{3i}(ct) &= I^{i3}(ct) = 0, \\
I^{ij}(ct) &= Mc^2a^2 \begin{pmatrix} 1+\cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & 1-\cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

3.4) Explain logical step in order to obtain the expression for the energy loss due to the gravitational wave from this binary star. [4 points]

Solution:

1. Find relation between the source and the wave

$$
h^{ij}(t,\vec{x}) = \frac{2G}{c^6 r}\ddot{I}^{ij},
$$

2. Find relation between the wave and the energy flux

$$
F(\hat{x}) = \frac{c^3\omega^2}{32\pi G}\left\langle h^{ij}_{TT}(\hat{x})h^{TT}_{ij}(\hat{x})\right\rangle.
$$

3. Find rate flow of the energy

$$\begin{aligned} \frac{dE}{dt} &= -L_{GW} = -\int F(\hat{r})r^2 d\Omega = -\int F(\hat{r})r^2 \sin^2\theta \, d\theta \, d\phi, \\ &= -\frac{c^4 r^2}{32\pi G}\int \left\langle \partial_t h^{ij}_{TT} \partial_r h^{TT}_{ij} \right\rangle \sin^2\theta \, d\theta \, d\phi. \end{aligned}$$

4. Compute the rate flow of the energy and remember that we have to compute it by using tranverse and traceless gauge. In this case, we have to project the source to compatible to this gauge.

4 Physical Foundations of Cosmology: cosmological principle, perfect fluid, ΛCDM model

4.1) List the important matter and energy components in the ΛCDM model, and explain briefly what are roles of each components in the universe. [4 points]

Solution:

The important matter and energy components in ΛCDM are massless neutrino, photon, baryon, dark matter, and cosmological constant. The main rule of neutrino is to control the interaction rate of weak interaction in the early universe. The photon has contribution to the recombination processes. The baryons are protons and neutrons which are the nuclei of atoms. The dark matter is introduced to explain the rotation curves of galaxy. The cosmological constant is a possible form of energy that can drive an accelerated expansion of the late-time universe.

4.2) Compute the energy density ρ_d of dark energy in terms of redshift z for the case where its equation of state parameter $w_d(z)$ depends on z. [6 points]

Solution:

From energy conservation,

$$\dot{\rho}_d = -3H(1 + w_d(z))\rho_d = -3\frac{\dot{a}}{a}(1 + w_d(z))\rho_d \,, \tag{3}$$

where a dot denotes derivative with respect to time t. Using $z = 1/a - 1$,w we can write the above equation as

$$\frac{\dot{\rho}_d}{\rho_d} = 3\frac{\dot{z}}{1+z}(1 + w_d(z)) \,, \tag{4}$$

which can be integrated as

$$\ln \rho_d = 3\int dz' \frac{1 + w_d(z')}{1 + z'} + C \,, \quad \rightarrow \quad \rho_d = C \exp\left(3\int dz' \frac{1 + w_d(z')}{1 + z'}\right) \,. \tag{5}$$

where C is an integration constant.

5 Inflationary universe: models of cosmic inflation, necessary conditions for cosmic inflation

5.1) Compute the slow-roll parameter $\epsilon \equiv -\frac{\dot{H}}{H^2}$ in terms of the potential V of inflaton in the slow-roll limit, where H is the Hubble parameter. [4 points]

Solution:
In the slow-roll limit, the Friedmann equation and the evolution equation for the inflaton are

$$3M_{\rm Pl}^2 H^2 \simeq V \,, \quad 3H\dot{\phi} \simeq -\frac{dV}{[}d\phi] \equiv -V_{,\phi} \,, \tag{6}$$

where $M_{\rm Pl} \equiv 1/\sqrt{8\pi G}$ is the reduced Planck mass. Differentiating the Friedmann equation with respect to time, and using the inflaton equation to express $V_{,\phi}$ in terms of H, we get

$$\dot{H} \simeq \frac{V_{,\phi}\dot{\phi}}{6M_{\rm Pl}^2 H} \simeq -\frac{V_{,\phi}^2}{18M_{\rm Pl}^2 H^2} \,. \tag{7}$$

After using the Friedmann equation again, we obtain

$$\epsilon = -\frac{\dot{H}}{H^2} \simeq \frac{V_{,\phi}^2}{18M_{\rm Pl}^2 H^4} \simeq \frac{V_{,\phi}^2}{2M_{\rm Pl}^2 V^2} \,. \tag{8}$$

5.2) What is the graceful exit in inflationary scenario? What are conditions on evolution of ϵ and H required to have graceful exit? [3 points]

Solution:
The graceful exit is the natural end of inflationary epoch. The inflationary models can have the graceful exit if ϵ increases from a value that much smaller than unity during the initial stage of inflation to unity at the end of inflation. This implies that H slowly decreases during inflation. The graceful exit can be achieved if the decreasing rate of H increases.

2.3) Show in detail whether an inflationary model has graceful exit if an inflaton is a phantom field. [3 points]

Solution:

The energy density ρ_f and pressure P_f of the phantom field are given by

$$\rho_f = -\frac{1}{2}\dot{\phi}^2 + V\,, \quad \text{and} \quad P_f = -\frac{1}{2}\dot{\phi}^2 - V\,, \tag{9}$$

Hence, the energy conservation yields

$$\dot{\rho}_f = -3H(\rho_f + P_f) = 3H\dot{\phi}^2\,. \tag{10}$$

Differentiating the Friedmann equation with respect to time, we get

$$\dot{H} = \frac{1}{6M_{\rm Pl}^2}\dot{\rho}_f > 0\,. \tag{11}$$

This indicates that H increases with time, so that there is no graceful exit.

6 Late-time cosmology: models of accelerating universe, problems of those models, basic of modified theories of gravity

6.1) What is the cosmic coincidence problem? How can this problem be alleviated? [3 points]

Solution:
The cosmic coincidence problem is the puzzle why the energy densities of dark energy and dark matter are in the same order of magnitude at present, even though they evolve differently and by construction they are independent. This problem could be alleviated if the cosmic evolution has suitable fixed points, or dark energy can couple with dark matter through unknown form of the interaction.

6.2) What is the chameleon mechanism? How can this mechanism be realized? [3 points]

Solution:
The chameleon mechanism is introduced for suppressing the fifth force that is delivered by scalar field dark energy or scalar degree of freedom from modified theories of gravity. The effective mass of the scalar field dark energy has to be small, and consequently it can deliver the long distance force called fifth force which drives an accelerated expansion of the late-time universe. Nevertheless, recently, such fifth force is not detected on the earth and solar system. Hence, there is an assumption that the fifth force is suppressed in the region which has high density of matter, and the fifth force can propagate as long distance in the region which has low density of matter. Based on this assumption, the chameleon mechanism can be realized if there is a direct interaction between dark energy and dark matter.

6.3) Compute the redshift at which the universe starts to accelerate for the spatially flat Friedmann universe containing dark matter and dark energy with constant equation of state parameter w_d. Note that you can express your result in terms of w_d and the present values of the densities parameter of dark energy and dark matter. [4 points]

Solution:
From the acceleration equation, we have

$$\frac{\ddot{a}}{a} = -\frac{M_{\rm Pl}^2}{6}\left(\rho_m + (1+3w_d)\rho_d\right) \tag{12}$$

The universe starts to accelerate when $\ddot{a} = 0$, so that the above equation gives

$$\rho_m = -(1 + 3w_d)\rho_d \,, \tag{13}$$

when an acceleration starts. Using the conservation of energy, the above equation becomes

$$\begin{aligned} \rho_m^0 (1+z)^3 &= -(1+3w_d)\rho_d^0 (1+z)^{3(1+w_d)} \,, & (14) \\ (1+z)^{3w_d} &= -\frac{1}{1+3w_d}\frac{\rho_m^0}{\rho_d^0} \,, & (15) \\ z &= -1 + \left[-\frac{1}{1+3w_d}\frac{\Omega_m^0}{\Omega_d^0} \right]^{1/(3w_d)} \,, & (16) \end{aligned}$$