General Relativity and Gravitation (2024) 56:5
https://doi.org/10.1007/s10714-023-03191-8

RESEARCH

On sufficient conditions for degrees of freedom counting of multi-field generalised Proca theories

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Received: 29 May 2023 / Accepted: 13 December 2023 / Published online: 5 January 2024

Abstract
We derive conditions which are sufficient for theories consisting of multiple vector fields, which could also couple to non-dynamical external fields, to have the required structure of constraints of multi-field generalised Proca theories, so that the number of degrees of freedom is correct. The Faddeev–Jackiw constraint analysis is used and is cross-checked by Lagrangian constraint analysis. To ensure the theory is constraint, we impose a standard special form of Hessian matrix. The derivation benefits from the realisation that the theories are diffeomorphism invariance. The sufficient conditions obtained include a refinement of secondary-constraint enforcing relations derived previously in literature, as well as a condition which ensures that the iteration process of constraint analysis terminates. Some examples of theories are analysed to show whether they satisfy the sufficient conditions. Most notably, due to the obtained refinement on some of the conditions, some theories which are previously interpreted as being undesirable are in fact legitimate, and vice versa. This in turn affects the previous interpretations of cosmological implications which should later be reinvestigated.

Keywords Multi-field generalised Proca theories · Constraint analysis · Diffeomorphism invariance

Contents

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1 Introduction

Physical phenomena are well described fundamentally by field theories, which provide fundamental laws and mechanisms from which phenomena arise. Many important phenomena can be described by theories of vector fields. For example, light can be described as a massless particle arising from the quantisation of a vector gauge field. Force carriers of weak interactions [1–3] are described by vector fields which gain mass due to spontaneous symmetry breaking mechanism [4–7]. Theories describing this type of vector fields have gauge symmetry which is spontaneously broken in the vacuum. More recently, important physical phenomena especially in cosmology can also be described by another type of massive vector field theories, in which gauge symmetry is explicitly broken due to the presence of the explicit mass terms. Furthermore, these theories are considered as effective field theories. Attempts to describe cosmological phenomena, for example, primordial inflation [8] and late-time accelerated expansion [9, 10] using theories of massive vector fields are given for example in [11–13].

In order to obtain a better understanding of vector-field-related phenomena, one of the important steps would be to classify vector theories and give the most general form of theories of each type. In particular, the criteria of the classification would be based on the constraint structure. For theories which describe a single vector field, the most useful types would be theories which generalise Maxwell theory and those which generalise Proca theory [14]. The Dirac–Born–Infeld theory [15, 16] is an example of theory which generalises Maxwell theory. As for theories which generalise Proca theories, the notable constructions, with the aim to describe cosmological phenomena, start from the references [17, 18] (see [19–23] for developments; see also [24] for a review). The idea of [18] is to impose the condition which we will call, for definiteness, "the special Hessian condition", in which time–time and time–space components of Hessian vanishes while determinant of space–space components is non-zero. The reference [25] shows by using constraint analysis that vector theories which are local, diffeomorphism invariance, having Lagrangian containing up to first order derivative in time (which ensures that the theory is free of Ostrogradski instability [26]), as well as passing the special Hessian condition are likely to be generalised Proca theories. Further generalisations to generalised Proca theories are possible, for example, references [27–29] construct beyond generalised Proca theories, references [30–33] construct Proca–Nuevo.[1]

When considering theories which describe systems of multiple vector fields, one would expect that they would simply be describable by several systems of single vector fields arbitrarily interacting with each other. It turns out, however, that the interactions cannot be arbitrary. Further conditions are required. As shown in [34], the special Hessian condition is not sufficient to ensure that theories legitimately contain only the required degrees of freedom. Further conditions called "secondary-constraint enforcing relations" should be imposed. Constructions of theories satisfying the special Hessian conditions and secondary-constraint enforcing relations are given in, for example, [35, 36]. A more ambitious generalisation is provided by [37] in which the systems of any fields, not necessarily vector fields, whose Lagrangian depends up to the first order derivative in the fields are attempted to be classified.

In principle, the formulation presented by [34, 35, 37] still needs small refinements. By nature of constraint analysis, including Lagrangian constraint analysis, time and space are not put on equal footing. Therefore, the analysis is carried out in the way that diffeomorphism invariance is not manifest in most steps (intermediate equations are usually not in the form where spacetime indices are contracted). This should be compensated by making use of the conditions that we will call "diffeomorphism invariance requirements", which are conditions automatically satisfied by any theory which is diffeomorphism invariant. Although trivial for each specific theory, these conditions are helpful for the simplifications of equations in intermediate steps of constraint analysis. Although these requirements are not used in [34, 35, 37], we expect that they are crucial in providing and simplifying sufficient conditions for theories to have the desired number of degrees of freedom. This is in fact demonstrated [25] in the case of single-field generalised Proca theories. We will also demonstrate in our paper in the case of multi-field generalised Proca theories.

In fact, as will be discussed later in this paper, the conditions imposed by [34, 35] to ensure that the theories have secondary constraints are incorrect. There is one term missing from each of these conditions. Generically, this leads to incorrect counting of the number of degrees of freedom. Some theories which are previously interpreted as being undesirable in fact have the desired number of degrees of freedom, and vice versa. In principle, this could consequently lead to incomplete or even incorrect cosmological implications related to multi-field generalised Proca theories.

The goal of this paper is to derive conditions which are sufficient for theories to have the correct number of degrees of freedom as multi-field generalised Proca theories. This is done by using Faddeev–Jackiw constraint analysis [38–41] with the help of diffeomorphism invariance requirements. The steps to obtain the sufficient conditions are as follow. We first impose the special Hessian condition. This ensures that the theories are constraint as well as giving n primary constraints where n is the number

[1] Note that [33] points out that [32] has obtained incorrect secondary constraint. So the result of [32] are not correct. We thank Claudia de Rham for letting us know this recent development and related discussions.

of vector fields in the system. Next, extra conditions should be imposed [34, 35] which ensure that the theories have secondary constraints. The conditions we find actually give the correction to their counterpart obtained in [34, 35]. Further conditions should also be imposed to ensure that the symplectic two-form at the second iteration does not have a zero mode so that the constraint analysis terminates [32, 34, 35, 37]. If a theory passes all these requirements, then it is a multi-field generalised Proca theory.

This paper is organised as follows. In Sect. 2, we consider theories of multiple vector fields which could also couple to non-dynamical external fields.[2] We only consider the theories whose Lagrangians are local, diffeomorphism invariance, depend up to first order derivative of the vector fields and satisfy the special Hessian condition. We then proceed to use Faddeev–Jackiw constraint analysis on these theories and obtain the sufficient conditions for the vector sector to have the expected constraints structure and hence the correct number of degrees of freedom. We then make a cross-check in Sect. 3 by using Lagrangian constraint analysis, which give rise to conditions which after transforming to phase space agree with those obtained in Sect. 2. In Sect. 4, we discuss how to apply the sufficient conditions. In particular, we demonstrate in Sect. 4.1 the usage of these conditions to check example theories previously presented in the literature. Most notably, we provide an example legitimate theory which is previously misinterpreted in the literature as containing extra degrees of freedom. We also provide an example undesirable theory which is previously misinterpreted in the literature as being legitimate. In Sect. 4.2, we argue how the reinterpretations given in Sect. 4.1 would affect the study of cosmological implications previously presented in the literature. In Sect. 5, we provide conclusion and discussion of results and possible future works.

2 Analysis

2.1 Imposing special Hessian condition

For definiteness, we consider theories in 4-dimensional spacetime. However, the analysis of this paper can easily be extended to spacetime with other number of dimensions. We define Lagrangian density $\mathcal{L}$ via

$$S = \int d^4x \, \mathcal{L}. \tag{1}$$

We denote spacetime coordinates by x^μ with $\mu = 0, 1, 2, 3$. We also use other middle lower-case Greek indices $\mu, \nu, \rho \in \{0, 1, 2, 3\}$ to denote spacetime indices. We will denote spatial indices by using middle lower-case Latin indices $i, j, k, l \in \{1, 2, 3\}$. When expressing field we will omit the dependence on time coordinate t. We will also often drop the dependence on space variables $\boldsymbol{x}$ (but keep explicit other space variables e.g. $\boldsymbol{x}'$, $\boldsymbol{y}$, z). So for example φ stands for $\varphi(t, \boldsymbol{x})$, whereas $\varphi(\boldsymbol{y})$ stands for $\varphi(t, \boldsymbol{y})$.

[2] In this paper, the external fields are non-dynamical in the sense to be described in Sect. 2. The consideration of dynamics of the external fields especially gravity is not within the scope of this paper.

We are interested in the class of multi-field generalised Proca theories which is a system of n vector fields A^α_μ with $\alpha = 1, 2, \dots, n$ possibly coupled to external fields, which might also include the metric $g_{\mu\nu}$, and their derivatives. The external fields can be thought of as being predetermined functions on time and space. For example, the system of multiple massive vector fields might be put in a flat or curved backgrounds and might also couple to other external fields. As for the notations, we use beginning lower-case Greek indices $\alpha, \beta, \gamma \in \{1, 2, \dots, n\}$ to denote internal indices for vector fields. The collection of external fields is of the form

$$\{K\} \equiv \left\{ (K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}} \right\}, \tag{2}$$

whereas the collection of external fields and their derivatives is of the form

$$\{K, \partial K, \partial\partial K, \dots\} \equiv \left\{ \partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}} \right\}, \tag{3}$$

where r, r', r'', r''' take values in sets of integers. For example, one of possible cases could be $K^{(1)}$, $\partial_{\nu_1} K^{(1)}$, $\partial_{\nu_1}\partial_{\nu_2} K^{(1)}$, …, $K^{(2)}$, $\partial_{\rho_1} K^{(2)}$, $\partial_{\rho_1}\partial_{\rho_2} K^{(2)}$, …, $(K^{(3)})_{\mu_1\mu_2}$, $\partial_{\rho_1}(K^{(3)})_{\mu_1\mu_2}$, $\partial_{\rho_1}\partial_{\rho_2}(K^{(3)})_{\mu_1\mu_2}$, …, $(K^{(4)})_{\mu_1\mu_2}{}^{\nu_1}$, $\partial_{\rho_1}(K^{(4)})_{\mu_1\mu_2}{}^{\nu_1}$, $\partial_{\rho_1}\partial_{\rho_2}(K^{(4)})_{\mu_1\mu_2}{}^{\nu_1}, \dots, \in \{K, \partial K, \partial\partial K, \dots\}$, in which case, $K^{(1)}$, $K^{(2)}$ are tensors of rank $(0, 0)$, $K^{(3)}$ is a tensor of rank $(0, 2)$, whereas $K^{(4)}$ is a tensor of rank $(1, 2)$. Of course, this is only one of the examples. The analysis in this paper will apply to any types of collection of external fields and their derivatives, provided that the system satisfies the criteria to be specified. Note that for brevity and in order to avoid clutter of notation, we will discard the index structure of the collection of external fields and collection of external fields and their derivatives by simply using the notations defined in Eqs. (2)–(3).

In more details and for definiteness, let us provide further explanations as follows. One may think of the system of interest as being a part of a full system described by the action

$$S_{\text{full}} = \int d^4x \, \mathcal{L}_{\text{full}}, \tag{4}$$

where

$$\begin{aligned} \mathcal{L}_{\text{full}} = \mathcal{L}_{\text{vector sector}} &\left(A^\alpha_\mu, \partial_\mu A^\alpha_\nu, \left\{ \partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}} \right\} \right) \\ &+ \mathcal{L}_{\text{background}} \left(\left\{ \partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}} \right\} \right). \end{aligned} \tag{5}$$

The Lagrangian $\mathcal{L}_{\text{vector sector}}$ contain terms which describe dynamics of A^α_μ as well as the terms describing interaction between A^α_μ and the fields $\{K\} \equiv \{(K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}\}$, whereas the Lagrangian $\mathcal{L}_{\text{background}}$ only contain terms with the fields $\{K\}$ and their derivatives. The equations of motion for the fields $\{K\}$ are schematically given by

$$\frac{\delta \int d^4x \mathcal{L}_{\text{vector sector}}}{\delta (K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}} + \frac{\delta \int d^4x \mathcal{L}_{\text{background}}}{\delta (K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}} = 0. \tag{6}$$

It would be ultimately useful if the dynamics of the full system is fully analysed. However, in some situation, the analysis might be too involved while it is possible to consider simplified situations to gain some insights. In particular, one may consider the situations in which the effect of the first term on LHS of Eq. (6) is negligible. For example, we may consider a system in which the fields A_μ^α is on a fixed curved background in such a way that A_μ^α does not back-react to the background. In this kind of situation, the system of Eq. (6) is approximated as

$$\frac{\delta \int d^4x \mathcal{L}_{\text{background}}}{\delta (K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}} = 0, \tag{7}$$

which can in principle be used to solve for $\{K\}$. Suppose that one of the solutions of (7) is

$$\left(K^{(r)}\right)_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}} = \left(K^{(r)}_{\text{soln}}\right)_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}. \tag{8}$$

The full Lagrangian then becomes

$$\begin{aligned} \mathcal{L}_{\text{full}} = \mathcal{L}_{\text{vector sector}} &\left(A_\mu^\alpha, \partial_\mu A_\nu^\alpha, \left\{\partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (K^{(r)}_{\text{soln}})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}\right\}\right) \\ &+ \mathcal{L}_{\text{background}} \left(\left\{\partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (K^{(r)}_{\text{soln}})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}\right\}\right). \end{aligned} \tag{9}$$

Since the fields $\{K_{\text{soln}}\}$ are already fixed to be some predetermined functions of time and space, at this stage the Lagrangian $\mathcal{L}_{\text{full}}$ only describe the dynamics of A_μ^α. So the only $\mathcal{L}_{\text{vector sector}}$ is relevant to our consideration. That is, we consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{vector sector}} \left(A_\mu^\alpha, \partial_\mu A_\nu^\alpha, \left\{\partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (K^{(r)}_{\text{soln}})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}\right\}\right), \tag{10}$$

in which $\{\partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (K^{(r)}_{\text{soln}})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}\}$ are predetermined functions of time and space. In order to make the notation less cluttered, we may simply drop the subscripts "vector sector" and "soln" and adopt the notation (3) so that Eq. (10) now reads

$$\mathcal{L} = \mathcal{L}\left(A_\mu^\alpha, \partial_\mu A_\nu^\alpha, \{K, \partial K, \partial\partial K, \ldots\}\right). \tag{11}$$

We consider the system (11) simply with the expectation to gain some insights towards the ultimate goal of considering the full system in which the dynamics of all fields are taken into account. Of course the analysis of the full system could be more involved as for example, since the dynamics of gravity is taken into account, one may need to make use of ADM decomposition in constraint analysis.

Having explained the context of the set-up that for theories of interest, we will consider vector sector of the full system, let us now state some initial criteria that we will impose on the theories. We first demand that the full Lagrangian $\mathcal{L}_{\text{full}}$ is local and diffeomorphism invariance. Next, we demand that the vector sector Lagrangian $\mathcal{L}$ which describes the dynamics of the vector fields A_μ^α is free of Ostrogradski instability and depend up to first order derivatives of the vector fields.

In principle, one could also impose some physical conditions on the external fields $\{K\}$. However, since we are only interested in the vector sector Lagrangian $\mathcal{L}$ which does not describe the dynamics of $\{K\}$ and that $\{K, \partial K, \partial\partial K, \ldots\}$ only enters the vector sector Lagrangian as predetermined functions of time and space, conditions to be imposed on the external fields would be independent from the analysis to be given in this paper.

For definiteness, we call the space of the vector fields and their first order time derivatives as the tangent bundle. It would be useful to discuss the different considerations of the vector fields A^α_μ and the external fields. We may describe the tangent bundle by saying that at each given value of $(t, \boldsymbol{x})$, there is a space of $(A^\alpha_\mu, \dot{A}^\alpha_\mu)$. On the other hand, as for the external fields and their derivatives, since they are predetermined functions, each of them describe a real valued indexed object for each given value of $(t, \boldsymbol{x})$.

Next, in order for the vector sector to be constraint, the Hessian condition[3]

$$\det\left(\frac{\partial^2\mathcal{L}}{\partial\dot{A}^\alpha_\mu\partial\dot{A}^\beta_\nu}\right) = 0 \tag{12}$$

should be satisfied. However, in this paper, we will restrict the study to theories satisfying condition

$$\frac{\partial^2\mathcal{L}}{\partial\dot{A}^\alpha_0\partial\dot{A}^\beta_\mu} = 0, \quad \det\left(\frac{\partial^2\mathcal{L}}{\partial\dot{A}^\alpha_i\partial\dot{A}^\beta_j}\right) \neq 0, \tag{13}$$

which would imply the Hessian condition (12). For definiteness, let us call Eq. (13) as "the special Hessian condition". This condition has also been imposed by many references for example [34–36, 42, 43], in order to construct multi-field generalised Proca theories.

By requiring $\partial^2\mathcal{L}/\partial\dot{A}^\alpha_0\partial\dot{A}^\beta_0 = 0$, we see that $\mathcal{L}$ should be at most linear in $\dot{A}^\alpha_0$. Then by using the condition $\partial^2\mathcal{L}/\partial\dot{A}^\alpha_0\partial\dot{A}^\beta_i = 0$, we see that the coefficient of the linear term does not depend on $\dot{A}^\alpha_i$. Then imposing $\det(\partial^2\mathcal{L}/\partial\dot{A}^\alpha_i\partial\dot{A}^\beta_j) \neq 0$ exhausts all the requirements of Eq. (13).

Therefore, theories we consider have Lagrangians of the form

$$\begin{aligned}\mathcal{L} = {} & T\left(A^\alpha_\mu, \partial_i A^\alpha_\mu, \dot{A}^\alpha_i, \{K, \partial K, \partial\partial K, \ldots\}\right) \\ & + U_\beta\left(A^\alpha_\mu, \partial_i A^\alpha_\mu, \{K, \partial K, \partial\partial K, \ldots\}\right)\dot{A}^\beta_0,\end{aligned} \tag{14}$$

subject to

$$\det\left(\frac{\partial^2 T}{\partial\dot{A}^\alpha_i\partial\dot{A}^\beta_j}\right) \neq 0. \tag{15}$$

[3] The determinants in Eqs. (12), (13) and (15) are defined as follows. We combine the two indices of each vector field into one collective index. The matrices appearing within the determinants then have two collective indices. Standard definition for determinant then applies.

Since these theories are diffeomorphism invariance, they satisfy conditions on T, U_β as given in Appendix 1. Further requirements will be imposed in order for the theory to possess the correct number of degrees of freedom. These requirements are known in the literature to allow secondary constraints and to terminate the process of constraint analysis [34, 35, 37, 43]. The conditions which we will present are slightly differed from their counterparts in the literature. These differences, however, are important. Later in this section, we will comment on how and why they differ.

Euler–Lagrange equations for the vector fields are of the form

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{A}_i^\alpha \partial \dot{A}_j^\beta} \ddot{A}_j^\beta + \partial_j \frac{\partial \mathcal{L}}{\partial \partial_j A_i^\alpha} - \frac{\partial \mathcal{L}}{\partial A_i^\alpha} + \cdots = 0, \tag{16}$$

$$\dot{U}_\alpha + \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i A_0^\alpha} - \frac{\partial \mathcal{L}}{\partial A_0^\alpha} = 0, \tag{17}$$

where $\cdots$ are terms which do not contain $\ddot{A}_\mu^\alpha$. Since the Euler–Lagrange equations do not contain time derivative with order higher than two, the theories are free of Ostrogradski instability [26] in the vector sector. Furthermore, it is clear that the systems are free of Ostrogradski instability and are constrained as Euler–Lagrange equations are of second order derivative in time of A_j^β while there is only up to first order derivative in time for A_0^β. In Sect. 3, we will start from these Euler–Lagrange equations and rederive, as a cross-check to the analysis of the present section, secondary-constraint enforcing relations [34, 35]. As to be seen in the analysis, the relations given in [34, 35] miss one term, which would invalidate some of their justifications on behaviour of example theories.

2.2 Faddeev–Jackiw constraint analysis

We require that theories presented in the previous subsection should have the correct number of degrees of freedom. For this, we are going to make use of constraint analysis using the Faddeev–Jackiw method [38–41]. The analysis will give further conditions that the theories should satisfy. We will use the notations and conventions similar to those used in [25, 44].

2.2.1 First iteration

In order to transform from the tangent bundle to phase space, one considers conjugate momenta. Conjugate momenta for the Lagrangian Eq. (14) are

$$\pi_\beta^\mu = \delta_0^\mu U_\beta + \delta_i^\mu \frac{\partial T}{\partial \dot{A}_i^\beta}. \tag{18}$$

These equations allow us to identify primary constraints

$$\Omega_\beta = \pi_\beta^0 - U_\beta. \tag{19}$$

The spatial components of conjugate momenta are given by

$$\pi^i_\beta = \frac{\partial T}{\partial \dot{A}^\beta_i}. \tag{20}$$

Because of the condition (15), these equations can be inverted to give

$$\dot{A}^\beta_i = \Lambda^\beta_i \left(A^\alpha_\mu, \partial_i A^\alpha_\mu, \pi^i_\alpha, \{K, \partial K, \partial\partial K, \ldots\} \right). \tag{21}$$

Since we work in phase space, it would be convenient to define

$$\begin{aligned} &\mathcal{T}\left(A^\alpha_\mu, \partial_i A^\alpha_\mu, \Lambda^\alpha_i, \{K, \partial K, \partial\partial K, \ldots\}\right) \\ &\quad = T\left(A^\alpha_\mu, \partial_i A^\alpha_\mu, \dot{A}^\alpha_i, \{K, \partial K, \partial\partial K, \ldots\}\right)\Big|_{\dot{A}^\alpha_i \to \Lambda^\alpha_i}. \end{aligned} \tag{22}$$

Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= \pi^\mu_\alpha \dot{A}^\alpha_\mu - \mathcal{L} - \dot{\gamma}^\alpha \Omega_\alpha \\ &\approx \pi^i_\alpha \Lambda^\alpha_i - \mathcal{T} - \dot{\gamma}^\alpha \Omega_\alpha, \end{aligned} \tag{23}$$

where γ^α are Lagrange multipliers. Note that the time derivatives of external fields is allowed in the Hamiltonian (through $\mathcal{T}$) since for the system of interest, the external fields are predetermined functions of time and space. So their time derivatives are also predetermined functions. The presence of time-dependent external fields in the Hamiltonian simply means that the Hamiltonian depends explicitly on time. It is also not possible and not relevant to work out the conjugate momenta of the external fields as, apart from the fact that the external fields are predetermined functions, $\mathcal{L}$ does not contain terms describing dynamics of the external fields.

Let us start considering first iteration. First order form of the Lagrangian is given by

$$\mathcal{L}_{FOF} = \pi^\mu_\alpha \dot{A}^\alpha_\mu + \mathcal{L}_v + \dot{\gamma}^\alpha \Omega_\alpha, \tag{24}$$

where

$$\mathcal{L}_v \equiv \mathcal{T} - \pi^i_\alpha \Lambda^\alpha_i. \tag{25}$$

Symplectic variables are

$$\xi^I = \left(A^\alpha_\mu, \pi^\mu_\alpha, \gamma^\alpha\right). \tag{26}$$

Note that since the system of interest only describes the dynamics of A^α_μ, the phase space only contain variables relevant to A^α_μ. On the other hand, the external fields $\{K\}$ are simply predetermined functions of time and space and are not treated as variables. Each of them is a real valued indexed object at each given spacetime position.

Canonical one-form is given by

$$\mathcal{A} = \int d^3\boldsymbol{x} \left(\pi^\mu_\alpha \delta A^\alpha_\mu + \Omega_\alpha \delta\gamma^\alpha\right). \tag{27}$$

So symplectic two-form is

$$\mathcal{F}=\int d^3\boldsymbol{x}\bigg(\delta\pi^\mu_\alpha\wedge\delta A^\alpha_\mu+\delta\pi^0_\alpha\wedge\delta\gamma^\alpha-\frac{\partial U_\alpha}{\partial A^\beta_\mu}\delta A^\beta_\mu\wedge\delta\gamma^\alpha$$
$$-\frac{\partial U_\alpha}{\partial\partial_i A^\beta_\mu}\delta\partial_i A^\beta_\mu\wedge\delta\gamma^\alpha\bigg). \tag{28}$$

Demanding $i_z\mathcal{F}=0$ gives

$$z^{\pi^\mu_\alpha}+\frac{\partial U_\beta}{\partial A^\alpha_\mu}z^{\gamma^\beta}-\partial_i\left(\frac{\partial U_\beta}{\partial\partial_i A^\alpha_\mu}z^{\gamma^\beta}\right)=0, \tag{29}$$
$$z^{A^\alpha_\mu}+\delta^0_\mu z^{\gamma^\alpha}=0, \tag{30}$$
$$z^{\pi^0_\alpha}-\frac{\partial U_\alpha}{\partial A^\beta_\mu}z^{A^\beta_\mu}-\frac{\partial U_\alpha}{\partial\partial_i A^\beta_\mu}\partial_i z^{A^\beta_\mu}=0. \tag{31}$$

In order for these equations to be consistent, the equation

$$\left(\frac{\partial U_\alpha}{\partial A^\beta_0}-\frac{\partial U_\beta}{\partial A^\alpha_0}+\partial_i\frac{\partial U_\beta}{\partial\partial_i A^\alpha_0}\right)z^{\gamma^\beta}+\left(\frac{\partial U_\beta}{\partial\partial_i A^\alpha_0}+\frac{\partial U_\alpha}{\partial\partial_i A^\beta_0}\right)\partial_i z^{\gamma^\beta}=0 \tag{32}$$

has to be satisfied. In fact as analysed in Appendix 1 diffeomorphism invariance requires, among others, Eq. (A10). So we are left with

$$\left(\frac{\partial U_\alpha}{\partial A^\beta_0}-\frac{\partial U_\beta}{\partial A^\alpha_0}+\partial_i\frac{\partial U_\beta}{\partial\partial_i A^\alpha_0}\right)z^{\gamma^\beta}=0. \tag{33}$$

Let us denote

$$q_{\alpha\beta}\equiv\frac{\partial U_\alpha}{\partial A^\beta_0}-\frac{\partial U_\beta}{\partial A^\alpha_0}+\partial_i\frac{\partial U_\beta}{\partial\partial_i A^\alpha_0}. \tag{34}$$

We are particularly interested in the case where $\mathrm{rank}(q_{\alpha\beta})=0$, that is

$$\frac{\partial U_\alpha}{\partial A^\beta_0}-\frac{\partial U_\beta}{\partial A^\alpha_0}+\partial_i\frac{\partial U_\beta}{\partial\partial_i A^\alpha_0}=0. \tag{35}$$

As will be seen later, enforcing these conditions would lead to n secondary constraints. We are only interested in the class of theories with this constraint structure. This class include, for example, a theory of n uncoupled generalised Proca fields (an analysis will be given in Sect. 4.1). On the other hand, if $\mathrm{rank}(q_{\alpha\beta})\neq 0$, and we want the procedure not to terminate after the second iteration, the theory would either have undesired number of degrees of freedom or have first class constraints. Either of these cases are not what we are interested in.

As a cross-check, one may note that after imposing diffeomorphism invariance requirement,

$$[\Omega_\alpha, \Omega_\beta(\boldsymbol{x}')] \approx q_{\alpha\beta}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'). \tag{36}$$

Therefore, the condition Eq. (35) is equivalent to the vanishing of the Poisson's brackets of the primary constraints among themselves. That is

$$[\Omega_\alpha, \Omega_\beta(\boldsymbol{x}')] \approx 0. \tag{37}$$

In Dirac constraint analysis [45, 46], if we demand the primary constraints to be preserved in time we would have, with Hamiltonian density being $\mathcal{H} = \mathcal{H}_0 + u^\beta \Omega_\beta$ where $\mathcal{H}_0 = \pi^i_\alpha \Lambda^\alpha_i - \mathcal{T}$,

$$\int d^3\boldsymbol{x}'[\Omega_\alpha, \mathcal{H}_0(\boldsymbol{x}')] + \int d^3\boldsymbol{x}' u^\beta(\boldsymbol{x}')[\Omega_\alpha, \Omega_\beta(\boldsymbol{x}')] + \frac{\partial \Omega_\alpha}{\partial t} \approx 0. \tag{38}$$

Then since the explicit dependence on time of Ω_α appears in U_α due to the presence of $\{K, \partial K, \partial\partial K, \ldots\}$, we may simply use the chain rule to obtain

$$\begin{aligned} &\int d^3\boldsymbol{x}'[\Omega_\alpha, \mathcal{H}_0(\boldsymbol{x}')] + \int d^3\boldsymbol{x}' u^\beta(\boldsymbol{x}')[\Omega_\alpha, \Omega_\beta(\boldsymbol{x}')] \\ &- \frac{\partial U_\alpha}{\partial \partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}} \partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (\dot{K}^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}} \approx 0, \end{aligned} \tag{39}$$

where it is understood that in the third term on LHS of Eq. (39) there is a sum over the collection of the external fields and their derivatives. If the conditions (37) are not fulfilled, i.e. $\operatorname{rank}(q_{\alpha\beta}) \neq 0$, Eq. (39) would determine some components of u^β. So there will be less than n secondary constraints. In the extreme case where $\operatorname{rank}(q_{\alpha\beta}) = n$, i.e. $\det(q_{\alpha\beta}) \neq 0$, there is no secondary constraint. Furthermore, after classification, it is easy to see that all of these constraints are of second class. So the number of degrees of freedom is less than $3n$, which is not desirable.

Note that in the tangent bundle, Eq. (35) can also be expressed as

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{A}^\alpha_0 \partial A^\beta_0} - \frac{\partial^2 \mathcal{L}}{\partial A^\alpha_0 \partial \dot{A}^\beta_0} + \partial_i \left(\frac{\partial^2 \mathcal{L}}{\partial \partial_i A^\alpha_0 \partial \dot{A}^\beta_0} \right) = 0, \tag{40}$$

which is a correction to the secondary-constraint enforcing relations derived in [34, 35]. Only the last term on the LHS of Eq. (40) is not present in these references. This could be due to the fact that their analysis discards the dependence on spatial derivatives of vector fields. While this is sufficient for the main purpose of counting the number of degrees of freedom, one should be careful with the conditions derived in the process. In order to make use of such conditions, one should appropriately restore the dependence on spatial derivatives of vector fields. It turns out that the restoration in this case is given by the inclusion of the third term on LHS of Eq. (40). As a consequence of the missing term in the secondary-constraint enforcing relations, behaviours of some theories receive incorrect interpretations. For example, a special case of theory presented in [42] is interpreted by [34] to contain extra degrees of freedom. In fact, however, by a careful analysis to be discussed in Sect. 4.1, the theory is a legitimate multi-field generalised Proca theory since it has the desirable number of degrees of freedom.

It would be helpful to first demonstrate that Eq. (40) is indeed satisfied by some simple cases. In particular, it can be shown that Eq. (40) is satisfied by single field generalised Proca theories. In this case, Eq. (40) reduces to

$$\partial_i \left(\frac{\partial^2 \mathcal{L}}{\partial \partial_i A_0 \partial \dot{A}_0} \right) = 0, \tag{41}$$

which is in fact trivially satisfied. The systems of interest as described at the start of Sect. 2 automatically satisfies the diffeomorphism invariant requirement. In particular, consider a diffeomorphism condition, Eq. (A10), which reduces to

$$\frac{\partial^2 \mathcal{L}}{\partial \partial_i A_0 \partial \dot{A}_0} = 0, \tag{42}$$

where we recall that $U \equiv \partial \mathcal{L}/\partial \dot{A}_0$. So by imposing the condition (42), it can be seen that Eq. (41) is trivially satisfied. Having shown that Eq. (40) is satisfied by single field generalised Proca theories, it can immediately be seen that it is also satisfied by separable multi-field generalised Proca theories. See Sect. 4.1.1 for more details.

Let us continue the Faddeev–Jackiw analysis. The zero mode of $\mathcal{F}$ is

$$\begin{aligned} z_1 = {} & z^{\gamma^\alpha} \left(\frac{\delta}{\delta \gamma^\alpha} - \frac{\delta}{\delta A_0^\alpha} \right) + \left(-\frac{\partial U_\alpha}{\partial A_0^\beta} z^{\gamma^\beta} - \frac{\partial U_\alpha}{\partial \partial_i A_0^\beta} \partial_i z^{\gamma^\beta} \right) \frac{\delta}{\delta \pi_\alpha^0} \\ & + \left(-\frac{\partial U_\beta}{\partial A_i^\alpha} z^{\gamma^\beta} + \partial_j \left(\frac{\partial U_\beta}{\partial \partial_j A_i^\alpha} z^{\gamma^\beta} \right) \right) \frac{\delta}{\delta \pi_\alpha^i}, \end{aligned} \tag{43}$$

subject to secondary-constraint enforcing relations (35). Having obtained the zero mode, let us check whether there are further constraints in the system by considering

$$i_{z_1} \int d^3\boldsymbol{x}\, \delta \mathcal{L}_v = \int d^3\boldsymbol{x} \left(-\frac{\partial \mathcal{T}}{\partial A_0^\beta} + \partial_i \frac{\partial \mathcal{T}}{\partial \partial_i A_0^\beta} + \left(\frac{\partial U_\beta}{\partial A_i^\alpha} + \frac{\partial U_\beta}{\partial \partial_j A_i^\alpha} \partial_j \right) \Lambda_i^\alpha \right) z^{\gamma^\beta}, \tag{44}$$

where we have used the identity

$$\pi_\alpha^i = \frac{\partial \mathcal{T}}{\partial \Lambda_i^\alpha}, \tag{45}$$

which is equivalent to Eq. (20). The result from Eq. (44) gives secondary constraints

$$\begin{aligned} \tilde{\Omega}_\beta = {} & \frac{\partial \mathcal{T}}{\partial A_0^\beta} - \partial_i \frac{\partial \mathcal{T}}{\partial \partial_i A_0^\beta} - \left(\frac{\partial U_\beta}{\partial A_i^\alpha} + \frac{\partial U_\beta}{\partial \partial_j A_i^\alpha} \partial_j \right) \Lambda_i^\alpha \\ & - \frac{\partial U_\beta}{\partial \partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}} \partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (\dot{K}^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}} \end{aligned} \tag{46}$$

which, written as functions,

$$\tilde{\Omega}_\beta = \tilde{\Omega}_\beta(A^\alpha_\mu, \partial_i A^\alpha_\mu, \partial_i \partial_j A^\alpha_\mu, \pi^\alpha_i, \partial_i \pi^\alpha_j, \{K, \partial K, \partial\partial K, \dots\}). \tag{47}$$

Note that when reading off the constraint (46), there is also the contribution from external fields as presented in the last term on RHS. This is because the external fields are considered to be functions with explicit dependence on time. So when working out secondary constraints which essentially involves taking derivative of primary constraints with respect to time, the explicit time derivative of the external field should also be taken into account.

2.2.2 Second iteration

Having obtained new constraints from the first iteration, let us start the second iteration by including Lagrange multipliers corresponding to the new constraints. Symplectic variables are

$$\xi^I = (A^\alpha_\mu, \pi^\mu_\alpha, \gamma^\alpha, \tilde{\gamma}^\alpha). \tag{48}$$

Canonical one-form is given by

$$\mathcal{A} = \int d^3\boldsymbol{x}(\pi^\mu_\alpha \delta A^\alpha_\mu + \Omega_\alpha \delta\gamma^\alpha + \tilde{\Omega}_\alpha \delta\tilde{\gamma}^\alpha). \tag{49}$$

So symplectic two-form is

$$\mathcal{F} = \int d^3\boldsymbol{x} \left(\delta\pi^\mu_\alpha \wedge \delta A^\alpha_\mu + \delta\Omega_\alpha \wedge \delta\gamma^\alpha + \delta\tilde{\Omega}_\alpha \wedge \delta\tilde{\gamma}^\alpha\right). \tag{50}$$

We may also denote the constraints and Lagrange multipliers as $\Omega^{(1)}_\alpha \equiv \Omega_\alpha, \Omega^{(2)}_\alpha \equiv \tilde{\Omega}_\alpha, \gamma^\alpha_{(1)} \equiv \gamma^\alpha, \gamma^\alpha_{(2)} \equiv \tilde{\gamma}^\alpha$.

When solving for zero mode of the symplectic two-form $\mathcal{F}$, equations involving Poisson's brackets would arise. In order to easily see this, it will be useful to define the notation for generalised derivatives $\partial_\mathcal{I}$ as follows. Suppose that f and g are functions of $A^\alpha_\mu, \partial_i A^\alpha_\mu, \partial_i \partial_j A^\alpha_\mu, \dots, \pi^\mu_\alpha, \partial_i \pi^\mu_\alpha, \partial_i \partial_j \pi^\mu_\alpha, \dots, \{K, \partial K, \partial\partial K, \dots\}$. So

$$\begin{aligned}\frac{\delta f}{\delta A^\alpha_\mu(\boldsymbol{z})} &= \frac{\partial f}{\partial A^\alpha_\mu}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}) + \frac{\partial f}{\partial\partial_i A^\alpha_\mu}\partial_i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}) + \frac{\partial f}{\partial\partial_i\partial_j A^\alpha_\mu}\partial_i\partial_j\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}) + \cdots \\ &\equiv \frac{\partial f}{\partial\partial_\mathcal{I} A^\alpha_\mu}\partial_\mathcal{I}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}),\end{aligned} \tag{51}$$

where summation over $\mathcal{I}$ is understood. Similarly,

$$\frac{\delta f}{\delta\pi^\mu_\alpha(\boldsymbol{z})} = \frac{\partial f}{\partial\partial_\mathcal{I}\pi^\mu_\alpha}\partial_\mathcal{I}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}). \tag{52}$$

Then in this notation Poisson's bracket can be written as

$$[f, g(\boldsymbol{y})] = (-1)^{|\mathcal{J}|} \frac{\partial f}{\partial \partial_{\mathcal{I}} A^{\alpha}_{\mu}} \partial_{\mathcal{I}} \partial_{\mathcal{J}} \left(\frac{\partial g}{\partial \partial_{\mathcal{J}} \pi^{\mu}_{\alpha}} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \right) - (-1)^{|\mathcal{J}|} \frac{\partial f}{\partial \partial_{\mathcal{I}} \pi^{\mu}_{\alpha}} \partial_{\mathcal{I}} \partial_{\mathcal{J}} \left(\frac{\partial g}{\partial \partial_{\mathcal{J}} A^{\alpha}_{\mu}} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \right), \tag{53}$$

where $|\mathcal{J}|$ is the order of partial derivatives of $\mathcal{J}$, and summation over $\mathcal{I}$ and $\mathcal{J}$ is understood.

Let us then find zero mode of $\mathcal{F}$. Demanding $i_z \mathcal{F} = 0$ gives

$$z^{\pi^{\mu}_{\beta}} - \sum_{\mathfrak{s}=1}^{2} (-1)^{|\mathcal{I}|} \partial_{\mathcal{I}} \left(z^{\gamma^{\alpha}_{(\mathfrak{s})}} \frac{\partial \Omega^{(\mathfrak{s})}_{\alpha}}{\partial \partial_{\mathcal{I}} A^{\beta}_{\mu}} \right) = 0, \tag{54}$$

$$-z^{A^{\beta}_{\mu}} - \sum_{\mathfrak{s}=1}^{2} (-1)^{|\mathcal{I}|} \partial_{\mathcal{I}} \left(z^{\gamma^{\alpha}_{(\mathfrak{s})}} \frac{\partial \Omega^{(\mathfrak{s})}_{\alpha}}{\partial \partial_{\mathcal{I}} \pi^{\mu}_{\beta}} \right) = 0, \tag{55}$$

$$\partial_{\mathcal{I}} z^{A^{\alpha}_{\mu}} \frac{\partial \Omega^{(\mathfrak{s})}_{\beta}}{\partial \partial_{\mathcal{I}} A^{\alpha}_{\mu}} + \partial_{\mathcal{I}} z^{\pi^{\mu}_{\alpha}} \frac{\partial \Omega^{(\mathfrak{s})}_{\beta}}{\partial \partial_{\mathcal{I}} \pi^{\mu}_{\alpha}} = 0, \quad \text{for } \mathfrak{s} = 1, 2. \tag{56}$$

Eliminating $z^{A^{\alpha}_{\mu}}$ and $z^{\pi^{\mu}_{\alpha}}$ and using the identity Eq. (53), we obtain

$$\sum_{\mathfrak{s}=1}^{2} \int d^3\boldsymbol{y} [\Omega^{(1)}_{\alpha}, \Omega^{(\mathfrak{s})}_{\beta}(\boldsymbol{y})] z^{\gamma^{\beta}_{(\mathfrak{s})}}(\boldsymbol{y}) = 0, \tag{57}$$

$$\sum_{\mathfrak{s}=1}^{2} \int d^3\boldsymbol{y} [\Omega^{(2)}_{\alpha}, \Omega^{(\mathfrak{s})}_{\beta}(\boldsymbol{y})] z^{\gamma^{\beta}_{(\mathfrak{s})}}(\boldsymbol{y}) = 0. \tag{58}$$

Note that

$$[\Omega_{\alpha}, \Omega_{\gamma}(\boldsymbol{y})] = \left(-q_{\alpha\gamma} + \left(\frac{\partial \Omega_{\alpha}}{\partial \partial_i A^{\gamma}_0} + \frac{\partial \Omega_{\gamma}}{\partial \partial_i A^{\alpha}_0} \right) \partial_i \right) \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}). \tag{59}$$

Imposing diffeomorphism conditions Eq. (A10) and secondary-constraint enforcing relations Eq. (35), we obtain

$$[\Omega_{\alpha}, \Omega_{\gamma}(\boldsymbol{y})] = 0. \tag{60}$$

Next, after expressing the Poisson's brackets between primary and secondary constraints and substituting this along with Eq. (60) into Eq. (57), one obtains

$$\mathcal{C}_{0\alpha\gamma} z^{\tilde{\gamma}^{\gamma}} + \mathcal{C}^{i}_{1\alpha\gamma} \partial_i z^{\tilde{\gamma}^{\gamma}} + \mathcal{C}^{ij}_{2\alpha\gamma} \partial_i \partial_j z^{\tilde{\gamma}^{\gamma}} = 0, \tag{61}$$

where

$$\mathcal{C}_{0\alpha\gamma} \equiv \frac{\partial \tilde{\Omega}_{\gamma}}{\partial A^{\alpha}_0} - \partial_i \left(\frac{\partial \tilde{\Omega}_{\gamma}}{\partial \partial_i A^{\alpha}_0} \right) + \partial_i \partial_j \left(\frac{\partial \tilde{\Omega}_{\gamma}}{\partial \partial_i \partial_j A^{\alpha}_0} \right)$$

$$-\left(\frac{\partial\Omega_\alpha}{\partial A_k^\beta}+\frac{\partial\Omega_\alpha}{\partial\partial_i A_k^\beta}\partial_i\right)\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\pi_\beta^k}-\partial_j\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_j\pi_\beta^k}\right)\right), \tag{62}$$

$$\begin{aligned}\mathcal{C}^i_{1\alpha\gamma}\equiv&-\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i A_0^\alpha}+2\partial_j\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i\partial_j A_0^\alpha}\right)+\frac{\partial\Omega_\alpha}{\partial A_k^\beta}\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i\pi_\beta^k}\\&-\frac{\partial\Omega_\alpha}{\partial\partial_i A_k^\beta}\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\pi_\beta^k}-\partial_j\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_j\pi_\beta^k}\right)\right)+\frac{\partial\Omega_\alpha}{\partial\partial_j A_k^\beta}\partial_j\left(\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i\pi_\beta^k}\right),\end{aligned} \tag{63}$$

$$\mathcal{C}^{ij}_{2\alpha\gamma}\equiv\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_i\partial_j A_0^\alpha}+\frac{\partial\Omega_\alpha}{\partial\partial_{(i|}A_k^\beta}\frac{\partial\tilde{\Omega}_\gamma}{\partial\partial_{|j)}\pi_\beta^k}. \tag{64}$$

It would be helpful to rewrite Eqs. (62)–(64) in the forms which are easier to use. In particular, one may express $\mathcal{C}_{0\alpha\gamma},\mathcal{C}^i_{1\alpha\gamma},\mathcal{C}^{ij}_{2\alpha\gamma}$ in terms of $\mathcal{T}$ and U_β. However, even with the help of diffeomorphism invariance requirements, the expressions are still not simple to use. It is in fact even better to express these quantities in tangent bundle. We will postpone the presentation of these forms to Sect. 3, where the relevant expressions are given in Eqs. (110)–(111). Nevertheless, we may readily note here that by working in phase space and using diffeomorphism invariance requirements, it can be seen explicitly that

$$\mathcal{C}^i_{1\alpha\gamma}=-\mathcal{C}^i_{1\gamma\alpha},\quad \mathcal{C}^{ij}_{2\alpha\gamma}=0. \tag{65}$$

After using Eq. (65), it can be seen that Eq. (61) becomes

$$\mathcal{C}_{0\alpha\gamma}z^{\tilde{\gamma}^\gamma}+\mathcal{C}^i_{1\alpha\gamma}\partial_i z^{\tilde{\gamma}^\gamma}=0. \tag{66}$$

It is clear that $z^{\tilde{\gamma}^\gamma}=0$ is a solution to Eq. (66). However, the question is whether this solution is unique. If $z^{\tilde{\gamma}^\gamma}=0$ is the unique solution to Eq. (66), then after substituting into Eq. (58), we obtain

$$(\mathcal{C}_{0\gamma\alpha}-\partial_i\mathcal{C}^i_{1\gamma\alpha})z^{\gamma^\gamma}+\mathcal{C}^i_{1\alpha\gamma}\partial_i z^{\gamma^\gamma}=0. \tag{67}$$

As to be discussed in Sect. 3, it can be shown by using diffeomorphism conditions that

$$\mathcal{C}_{0\alpha\gamma}-\mathcal{C}_{0\gamma\alpha}=\partial_i\mathcal{C}^i_{1\alpha\gamma}. \tag{68}$$

So Eq. (67) is equivalent to Eq. (66). If Eq. (66) has the unique solution $z^{\tilde{\gamma}^\gamma}=0$, then $z^{\gamma^\gamma}=0$ should also be the unique solution to Eq. (67). Then by using Eqs. (54)–(55) we obtain $z^{A^\alpha_\mu}=z^{\pi^\mu_\alpha}=0$. So there is no zero mode, and the procedure terminates. Note that the requirement that the constraint analysis should terminate is previously suggested and emphasised in [32, 34, 35, 37]. By using the criteria presented by [47], it can be concluded that the number of degrees of freedom is $3n$ as required.

For definiteness, let us call the condition

$$\mathcal{C}_{0\alpha\gamma}z^{\tilde{\gamma}^\gamma}+\mathcal{C}^i_{1\alpha\gamma}\partial_i z^{\tilde{\gamma}^\gamma}=0\implies \text{unique solution } z^{\tilde{\gamma}^\gamma}=0 \tag{69}$$

as the "completion requirement" since it signals the end of the second iteration. There are two main cases which would satisfy the completion requirement (69):

- Case 1: $\mathcal{C}^i_{1\alpha\gamma} \neq 0$, and the boundary condition that fields should vanish fast enough near spatial infinity (this is the boundary condition which is required in the whole analysis to make integrals of total derivatives vanish) is sufficient to fix the solution to the equation in (69) to be unique.
- Case 2: $\mathcal{C}^i_{1\alpha\gamma} = 0$ and $\det(\mathcal{C}_{0\alpha\gamma}) \neq 0$.

In the case where $\mathcal{C}^i_{1\alpha\gamma} \neq 0$, it is not clear whether the boundary condition would be sufficient to fix the solution to the equation in (69) to be unique. We expect that the analysis should be done separately for each given specific theory. Even then, it would still be quite difficult, if at all possible, to show that the solution is unique. This means that it would not be simple to show whether a given theory with $\mathcal{C}^i_{1\alpha\gamma} \neq 0$ is within the case 1. As for the case where a theory has $\mathcal{C}^i_{1\alpha\gamma} = 0$, it could be very likely that $\det(\mathcal{C}_{0\alpha\gamma}) \neq 0$. This is because the form of $\mathcal{C}_{0\alpha\gamma}$ contains many terms in the expression, which make it difficult for $\mathcal{C}_{0\alpha\gamma}$ to be singular. On the other hand, the requirement $\mathcal{C}^i_{1\alpha\gamma} = 0$ itself would look quite restrictive, which might bring an immediate question as to whether it is possible to find theories within case 2. In fact, as to be explicitly discussed in Sect. 4.1, theories passing this requirement have already appeared in the literature. However, some of them might have been mistakenly ruled out due to the usage of the incorrect version of secondary-constraint enforcing relations [34, 35]. We will only provide one such example.

2.2.3 Matrix form of $\mathcal{F}$

In Faddeev–Jackiw constraint analysis, it is often convenient to consider the matrix form of $\mathcal{F}$. This would allow us to cross-check the analysis at the second iteration and at the same time further justify the completion requirement (69). In order to obtain the components of $\mathcal{F}$, it is convenient to first denote

$$f_{\xi^I} \equiv i_{\frac{\delta}{\delta \xi^I}} \mathcal{F}. \tag{70}$$

From direct calculation, we obtain

$$f_{A^\alpha_\mu} = -\delta\pi^\mu_\alpha + \sum_{\mathfrak{s}=1}^{2} \int d^3\boldsymbol{y}\, \frac{\delta\Omega^{(\mathfrak{s})}_\beta(\boldsymbol{y})}{\delta A^\alpha_\mu} \delta\gamma^\beta_{(\mathfrak{s})}(\boldsymbol{y}), \tag{71}$$

$$f_{\pi^\mu_\alpha} = \delta A^\alpha_\mu + \sum_{\mathfrak{s}=1}^{2} \int d^3\boldsymbol{y}\, \frac{\delta\Omega^{(\mathfrak{s})}_\beta(\boldsymbol{y})}{\delta \pi^\mu_\alpha} \delta\gamma^\beta_{(\mathfrak{s})}(\boldsymbol{y}), \tag{72}$$

$$f_{\gamma^\alpha} = -\int d^3\boldsymbol{y}\, \frac{\delta\Omega_\alpha}{\delta A^\beta_\mu(\boldsymbol{y})} \delta A^\beta_\mu(\boldsymbol{y}) - \int d^3\boldsymbol{y}\, \frac{\delta\Omega_\alpha}{\delta \pi^\mu_\beta(\boldsymbol{y})} \delta\pi^\mu_\beta(\boldsymbol{y}), \tag{73}$$

$$f_{\tilde{\gamma}^\alpha} = -\int d^3\boldsymbol{y}\, \frac{\delta\tilde{\Omega}_\alpha}{\delta A^\beta_\mu(\boldsymbol{y})} \delta A^\beta_\mu(\boldsymbol{y}) - \int d^3\boldsymbol{y}\, \frac{\delta\tilde{\Omega}_\alpha}{\delta \pi^\mu_\beta(\boldsymbol{y})} \delta\pi^\mu_\beta(\boldsymbol{y}). \tag{74}$$

The matrix element of $\mathcal{F}$ can then be obtained by taking interior product of Eqs. (71)–(74) with respect to phase space coordinate basis as follows

$$\mathcal{F}_{IJ}(\boldsymbol{x}, \boldsymbol{x}') = i_{\frac{\delta}{\delta \xi^J(\boldsymbol{x}')}} f_{\xi^I}(\boldsymbol{x}). \tag{75}$$

The matrix form of $\mathcal{F}$ is given by

$$\mathcal{F}(\boldsymbol{x}, \boldsymbol{x}') = \begin{pmatrix} A(\boldsymbol{x}, \boldsymbol{x}') & B(\boldsymbol{x}, \boldsymbol{x}') \\ C(\boldsymbol{x}, \boldsymbol{x}') & D(\boldsymbol{x}, \boldsymbol{x}') \end{pmatrix}, \tag{76}$$

where

$$A(\boldsymbol{x}, \boldsymbol{x}') = \begin{pmatrix} 0 & -\delta_\alpha^\beta \delta_\nu^\mu \\ \delta_\beta^\alpha \delta_\mu^\nu & 0 \end{pmatrix} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'), \tag{77}$$

$$B(\boldsymbol{x}, \boldsymbol{x}') = \begin{pmatrix} \frac{\partial \Omega_\beta}{\partial \partial_{\mathcal{I}} A_\mu^\alpha}(\boldsymbol{x}') & \frac{\partial \tilde{\Omega}_\beta}{\partial \partial_{\mathcal{I}} A_\mu^\alpha}(\boldsymbol{x}') \\ \frac{\partial \Omega_\beta}{\partial \partial_{\mathcal{I}} \pi_\alpha^\mu}(\boldsymbol{x}') & \frac{\partial \tilde{\Omega}_\beta}{\partial \partial_{\mathcal{I}} \pi_\alpha^\mu}(\boldsymbol{x}') \end{pmatrix} \partial_{\mathcal{I}}' \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'), \tag{78}$$

$$C(\boldsymbol{x}, \boldsymbol{x}') = -\begin{pmatrix} \frac{\partial \Omega_\alpha}{\partial \partial_{\mathcal{I}} A_\nu^\beta} & \frac{\partial \Omega_\alpha}{\partial \partial_{\mathcal{I}} \pi_\beta^\nu} \\ \frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_{\mathcal{I}} A_\nu^\beta} & \frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_{\mathcal{I}} \pi_\beta^\nu} \end{pmatrix} \partial_{\mathcal{I}} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'), \tag{79}$$

$$D(\boldsymbol{x}, \boldsymbol{x}') = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{80}$$

where $\partial_{\mathcal{I}}'$ are generalised derivatives with respect to $\boldsymbol{x}'$. One important steps of Faddeev–Jackiw constraint analysis is to find the determinant of $\mathcal{F}$. This determinant would also be useful when working out path integral quantisation as its square root would appear in the path integration measure. By the standard formula of determinant of block matrix, we have

$$\det \mathcal{F} = \det(A) \det(D - CA^{-1}B). \tag{81}$$

By direct calculation, it can be shown that $\det(A) = 1$. So in order to evaluate $\det \mathcal{F}$, one needs to first compute $(D - CA^{-1}B)$. Direct computation gives, after applying Eqs. (65) and (68),

$$\begin{aligned}(D - CA^{-1}B)(\boldsymbol{x}, \boldsymbol{x}') &= \begin{pmatrix} [\Omega_\alpha, \Omega_\beta(\boldsymbol{x}')] & [\Omega_\alpha, \tilde{\Omega}_\beta(\boldsymbol{x}')] \\ [\tilde{\Omega}_\alpha, \Omega_\beta(\boldsymbol{x}')] & [\tilde{\Omega}_\alpha, \tilde{\Omega}_\beta(\boldsymbol{x}')] \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\mathcal{C}_{0\alpha\beta} - \mathcal{C}_{1\alpha\beta}^i \partial_i \\ \mathcal{C}_{0\beta\alpha} - \mathcal{C}_{1\beta\alpha}^i(\boldsymbol{x}') \partial_i & \mathcal{D}_{\alpha\beta}^{\mathcal{I}} \partial_{\mathcal{I}} \end{pmatrix} \\ &\quad \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'), \end{aligned} \tag{82}$$

where $\mathcal{D}_{\alpha\beta}^{\mathcal{I}}$ are functions whose form are not relevant to the analysis of this paper, so we do not provide its explicit form.

In order for $(D - CA^{-1}B)$ to be invertible, the solution w of $(D - CA^{-1}B)w = \psi$ should be unique. Let us denote $w(\boldsymbol{x}') \equiv (u^\beta(\boldsymbol{x}'), v^\beta(\boldsymbol{x}'))^T$, and $\psi(\boldsymbol{x}) \equiv (\chi_\alpha(\boldsymbol{x}), \lambda_\alpha(\boldsymbol{x}))^T$. So

$$-\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right) v^\beta = \chi_\alpha, \tag{83}$$

$$(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i)u^\beta + \mathcal{D}^{\mathcal{I}}_{\alpha\beta}\partial_{\mathcal{I}} v^\beta = \lambda_\alpha. \tag{84}$$

Solution to Eq. (83) is

$$v^\beta = \int d^3\boldsymbol{x}' G^{\beta\gamma}(\boldsymbol{x}, \boldsymbol{x}')\chi_\gamma(\boldsymbol{x}') + v_0^\beta, \tag{85}$$

where $G^{\beta\gamma}(\boldsymbol{x}, \boldsymbol{x}')$ and v_0^β satisfy

$$-\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right) G^{\beta\gamma}(\boldsymbol{x}, \boldsymbol{x}') = \delta^\gamma_\alpha \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'), \tag{86}$$

and

$$-\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right) v_0^\beta = 0. \tag{87}$$

In order for v^β to be the unique solution to Eq. (83), we demand that v_0^β is unique. This is precisely the completion requirement (69).

In the case where $D - CA^{-1}B$ is invertible, the determinant of $\mathcal{F}$ can be determined. In this case, by direct calculation using the standard formula of determinant of block matrix and using the property of determinant of product of square matrices, one obtains

$$\det \mathcal{F} = \{\det[(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i)\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')]\}^2. \tag{88}$$

Demanding that there is no zero mode of $\mathcal{F}$ at the second iteration is equivalent to demanding that $\det \mathcal{F} \neq 0$. So by using Eq. (88), it can be seen that one should demand the differential operator $\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i$ to have no zero mode. This also implies the completion requirement.

The class of theories we consider indeed include the particular theories investigated in [37], in which the conditions called "quantum consistency condition" are derived. The result of our paper suggests that these conditions can indeed be generalised to a larger class of theories. The generalisation is simply the condition we called "completion requirement". The idea is that our differential operator $\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i$ could be thought of as a generalisation to their differential operator $Z_{\alpha\beta}$. We have provided in Eqs. (110)–(111) the formula to directly compute the coefficients $\mathcal{C}_{0\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta}$, which in turn give rise the required differential operator. The quantum consistency condition derived in [37] is $Z_{\alpha\beta} \neq 0$. This seems to demand a differential operator to be non-zero. We suppose that it would be useful to give a slightly clearer interpretation. In particular, one should interpret it as being that the differential operator $Z_{\alpha\beta}$ has no zero mode. This is exactly generalised to our requirement.

Furthermore, by using diffeomorphism invariance requirement, we have shown that $\mathcal{C}_{0\beta\alpha} = \mathcal{C}_{0\alpha\beta} - \partial_i\mathcal{C}^i_{1\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta} = -\mathcal{C}^i_{1\beta\alpha}$. This implies that $\mathcal{C}_{0\beta\alpha} - \mathcal{C}^i_{1\beta\alpha}(\boldsymbol{x}')\partial_i =$

$\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i$, which should be the generalisation to $-Z'_{\beta\alpha} = Z_{\alpha\beta}$ of the theories in [37]. This provides an explanation why the determinant of the symplectic two-form factorises as Eq. (88). For example, in the particular theories of [37], the determinant reduces as $\det\mathcal{F} = (\det(Z\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')))^2 = \det(Z\cdot Z\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')) = \det(-Z'\cdot Z\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'))$, in agreement, modulo a possible minor typographical error, with [37].

An immediate application is that if the theory passes the completion requirement, path integral quantisation can be carried out [44]. In particular, it is possible to read off

$$\sqrt{\det\mathcal{F}} = \det\left[\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\right], \tag{89}$$

which is an expression that appears in the measure of the generating functional in path integral quantisation.

3 Consistency check using Lagrangian constraint analysis

In the previous section, we have presented the criteria for which the theories of n vector fields with Lagrangian of the form Eq. (14) would have $3n$ degrees of freedom, which corresponds to theories of multi-field generalised Proca. In short, the criteria is that the theory should transform in a standard way under diffeomorphism transformation and should satisfy Eqs. (13), (35) and (69).

In this section, we present a consistency check of our result by using Lagrangian constraint analysis developed in [34, 35, 37, 48], and work out the equivalence between the conditions to be obtained in this section with those from the previous section.

In this analysis, it is convenient to define collective coordinates as follows. Let Q^M, Q^α, Q^A be collective for A^α_μ, A^α_0, A^α_i, respectively. The Lagrangian we are interested in is given by

$$\mathcal{L} = \mathcal{L}\left(Q^M, \dot{Q}^M, \partial_i Q^M, \{K, \partial K, \partial\partial K, \ldots\}\right), \tag{90}$$

Euler–Lagrange equations for vector fields are

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{Q}^M}\right) + \partial_i\left(\frac{\partial\mathcal{L}}{\partial\partial_i Q^M}\right) - \frac{\partial\mathcal{L}}{\partial Q^M} \\ &= W_{MN}\ddot{Q}^N + \alpha_M, \end{aligned} \tag{91}$$

where

$$W_{MN} \equiv \frac{\partial^2\mathcal{L}}{\partial\dot{Q}^M\partial\dot{Q}^N}, \tag{92}$$

$$\begin{aligned} \alpha_M = &\frac{\partial^2\mathcal{L}}{\partial\dot{Q}^M\partial Q^N}\dot{Q}^N + \frac{\partial^2\mathcal{L}}{\partial\dot{Q}^M\partial\partial_i Q^N}\partial_i\dot{Q}^N \\ &+ \frac{\partial^2\mathcal{L}}{\partial\dot{Q}^M\partial\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}}\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(\dot{K}^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}} \end{aligned}$$

$$+\partial_i\left(\frac{\partial\mathcal{L}}{\partial\partial_i Q^M}\right)-\frac{\partial\mathcal{L}}{\partial Q^M}. \tag{93}$$

Note that Eqs. (91)–(93) suggest that higher time derivatives on the external fields in the Euler–Lagrange equations for vector fields can be present. This, however, is not problematic since the external fields are non-dynamical since they appear in these equations as predetermined functions. So their time derivatives are also predetermined functions. The special Hessian conditions Eq. (13) give the following conditions on W_{MN}:

$$W_{\alpha N}=0, \quad \det(W_{AB})\neq 0. \tag{94}$$

So Euler Lagrange Eq. (91) can be separated into equations of motion:

$$W_{AB}\ddot{Q}^B+\alpha_A=0, \tag{95}$$

and primary constraints

$$\alpha_\alpha=0. \tag{96}$$

Let M^{AB} be the inverse of W_{AB}. So the equations of motion imply

$$\ddot{Q}^A+M^{AB}\alpha_B=0. \tag{97}$$

Time evolution of constraints is given by, after making use of Eq. (97),

$$\begin{aligned}\dot{\alpha}_\alpha=&\sum_{|\mathcal{I}|=0}^{1}\frac{\partial\alpha_\alpha}{\partial\partial_{\mathcal{I}}\dot{Q}^\beta}\partial_{\mathcal{I}}\ddot{Q}^\beta+\sum_{|\mathcal{I}|=0}^{2}\frac{\partial\alpha_\alpha}{\partial\partial_{\mathcal{I}}Q^M}\partial_{\mathcal{I}}\dot{Q}^M-\sum_{|\mathcal{I}|=0}^{1}\frac{\partial\alpha_\alpha}{\partial\partial_{\mathcal{I}}\dot{Q}^B}\partial_{\mathcal{I}}(M^{BC}\alpha_C)\\&+\frac{\partial\alpha_\alpha}{\partial\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}}\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(\dot{K}^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}.\end{aligned} \tag{98}$$

We demand that the process should not terminate at this stage. So the conditions $\dot{\alpha}_\alpha=0$ should not introduce further dynamics on the vector fields. This means that the expressions with second order derivative in time of Q^β should not appear in Eq. (98). These expressions are $\ddot{Q}^\beta$ and $\partial_i\ddot{Q}^\beta$. From direct calculation, their coefficients are

$$\frac{\partial\alpha_\alpha}{\partial\dot{Q}^\beta}\equiv\frac{\partial^2\mathcal{L}}{\partial\dot{Q}^\alpha\partial Q^\beta}-\frac{\partial^2\mathcal{L}}{\partial Q^\alpha\partial\dot{Q}^\beta}+\partial_i\left(\frac{\partial^2\mathcal{L}}{\partial\partial_i Q^\alpha\partial\dot{Q}^\beta}\right), \tag{99}$$

and

$$\frac{\partial\alpha_\alpha}{\partial\partial_i\dot{Q}^\beta}=\frac{\partial^2\mathcal{L}}{\partial\dot{Q}^\alpha\partial\partial_i Q^\beta}+\frac{\partial^2\mathcal{L}}{\partial\dot{Q}^\beta\partial\partial_i Q^\alpha}. \tag{100}$$

By using a diffeomorphism condition (A10), the coefficient of $\partial_i\ddot{Q}^\beta$ vanishes. So we are left with the terms with $\ddot{Q}^\beta$. In order for the coefficients of these terms to vanish, we should set

$$\frac{\partial\alpha_\alpha}{\partial\dot{Q}^\beta}=0, \tag{101}$$

which turns out to be equivalent to Eq. (35).

Two remarks are in order. The first is that the analysis in [34] does not show explicit dependence on spatial derivatives of fields. While this might be sufficient for the purpose of counting the number of degrees of freedom, the conditions derived in the process are not readily correct until time dependence on spatial derivatives of fields are re-introduced. From their analysis, the last term on RHS of Eq. (99) is missing. This term could be considered as restoring spatial derivatives of fields. The second remark is that the reference [37] does not seem to mention the dependence of $\dot{\alpha}_\alpha$ on $\partial_i \ddot{Q}^\beta$ nor on whether their coefficients disappear. We have learned from the analysis above that diffeomorphism invariance requirement is crucial, at least in the case of multi-field generalised Proca theories that we are analysing, to make the the coefficients disappear. It would be interesting to see whether this behaviour is also the case in the analysis of more general theories given in [37].

Although Lagrangian constraint analysis is more advantageous than Hamiltonian constraint analysis in that it treats time and space on a more equal footing, the nature of constraint analysis still requires that time and space should be treated differently. For example, to see whether there are further constraints, only the time evolution is required. Some information on manifest covariance would then be lost. In order to recover them, one needs to make use of the fact that theories are diffeomorphism invariance (or, in the case of flat spacetime, Lorentz invariance).

Let us continue the analysis. By imposing Eq. (101), we then have n secondary constraints $\phi_\alpha = \dot{\alpha}_\alpha \approx 0$. The next step is to consider the time evolution of ϕ_α. We demand that the condition $\dot{\phi}_\alpha \approx 0$ should not lead to further constraints. For this, $\dot{\phi}_\alpha$ should contain terms with second order derivative in time on Q^β. These terms are

$$\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} \ddot{Q}^\beta + \frac{\partial \phi_\alpha}{\partial \partial_i \dot{Q}^\beta} \partial_i \ddot{Q}^\beta + \frac{\partial \phi_\alpha}{\partial \partial_i \partial_j \dot{Q}^\beta} \partial_i \partial_j \ddot{Q}^\beta \in \dot{\phi}_\alpha. \tag{102}$$

The analysis in [37] does not mention terms with $\partial_i \ddot{Q}^\beta$ and $\partial_i \partial_j \ddot{Q}^\beta$. In principle, these terms are also crucial in determining whether the procedure should be terminated. Analysis of a particular case, for example in [43], also show the dependence of constraints on these terms, especially $\partial_i \ddot{Q}^\beta$.

Let us connect the result in this subsection with the analysis in phase space given in Sect. 2. For this, we first show that by transforming to tangent bundle, $\tilde{\Omega}_\alpha = -\alpha_\alpha$. We start from Eq. (46). Then by using $T = \mathcal{L} - U_\gamma \dot{Q}^\gamma$, and realising that U_α is independent of $\dot{Q}^M$, we obtain[4]

$$\begin{aligned} \tilde{\Omega}_\alpha = -\alpha_\alpha &+ \left(\frac{\partial U_\alpha}{\partial Q^\beta} - \frac{\partial U_\beta}{\partial Q^\alpha} + \partial_i \left(\frac{\partial U_\beta}{\partial \partial_i Q^\alpha} \right) \right) \dot{Q}^\beta \\ &+ \left(\frac{\partial U_\alpha}{\partial \partial_i Q^\beta} + \frac{\partial U_\beta}{\partial \partial_i Q^\alpha} \right) \partial_i \dot{Q}^\beta. \end{aligned} \tag{103}$$

[4] It is understood that LHS of Eq. (103) is actually the pullback of $\tilde{\Omega}_\alpha$ to tangent bundle. Throughout this paper, we do not use different notations to distinguish the functions from their pullbacks as it should be clear from the context.

The second and the third term on RHS vanish due to secondary-constraint enforcing relations (35) and diffeomorphism invariance requirement (A10). This finally gives

$$\tilde{\Omega}_\alpha = -\alpha_\alpha, \tag{104}$$

as required. Then by following the calculations outlined in Appendix 1, we obtain

$$\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} = -\mathcal{C}_{0\alpha\beta}, \quad \frac{\partial \phi_\alpha}{\partial \partial_i \dot{Q}^\beta} = -\mathcal{C}^i_{1\alpha\beta}, \quad \frac{\partial \phi_\alpha}{\partial \partial_i \partial_j \dot{Q}^\beta} = 0. \tag{105}$$

Note in passing that the condition

$$\mathcal{C}_{0\alpha\beta} = \mathcal{C}_{0\beta\alpha} - \partial_i \mathcal{C}^i_{1\beta\alpha}, \tag{106}$$

which is also proven in Appendix 1 is crucial in the derivation of Eq. (105).

Therefore, time evolution of $\dot{\phi}_\alpha$ is of the form

$$\dot{\phi}_\alpha = -\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right)\ddot{Q}^\beta + \cdots, \tag{107}$$

where $\cdots$ are terms with up to first order in time derivative in Q^M. In order for $\dot{\phi} \approx 0$ not to lead to further constraints, we should demand that it is equivalent to

$$\ddot{Q}^\beta + \cdots = 0. \tag{108}$$

This would be possible only when the differential operator

$$\left(\mathcal{C}_{0\alpha\beta} + \mathcal{C}^i_{1\alpha\beta}\partial_i\right) \tag{109}$$

is invertible. Equivalently, this differential operator should have no zero mode. This would lead exactly to the completion requirements (69) given at the end of Sect. 2.2.2.

We have seen that the analysis of Lagrangian constraint analysis agree with the Faddeev–Jackiw constraint analysis. In particular, the functions $\mathcal{C}_{0\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta}$ appear in ones of the important conditions. Having worked with Lagrangian analysis, we are now in a position to express them in a more useful form. They are

$$\mathcal{C}_{0\alpha\beta} = -\frac{\partial \alpha_\alpha}{\partial A_0^\beta} - \partial_j\left(\frac{\partial \alpha^j_\alpha}{\partial \dot{A}_0^\beta}\right) + \frac{\partial \alpha^k_\gamma}{\partial \dot{A}_0^\alpha} M^{\gamma\delta}_{kl} \frac{\partial \alpha^l_\delta}{\partial \dot{A}_0^\beta}, \tag{110}$$

$$\mathcal{C}^i_{1\alpha\beta} = -\frac{\partial \alpha_\alpha}{\partial \partial_i A_0^\beta} - \frac{\partial \alpha^i_\alpha}{\partial \dot{A}_0^\beta} - \frac{\partial \alpha_\alpha}{\partial \dot{A}_i^\beta}, \tag{111}$$

where

$$\frac{\partial \alpha_\alpha}{\partial A_0^\beta} = \partial_\mu\left(\frac{\partial^2 \mathcal{L}}{\partial \partial_\mu A_0^\alpha \partial A_0^\beta}\right) - \frac{\partial^2 \mathcal{L}}{\partial A_0^\alpha \partial A_0^\beta}, \tag{112}$$

$$\frac{\partial \alpha^k_\gamma}{\partial \dot{A}^\beta_0} = \frac{\partial^2 \mathcal{L}}{\partial A^\beta_0 \partial \dot{A}^\gamma_k} + \partial_i \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{A}^\beta_0 \partial \partial_i A^\gamma_k} \right) - \frac{\partial^2 \mathcal{L}}{\partial \dot{A}^\beta_0 \partial A^\gamma_k}, \tag{113}$$

$$\frac{\partial \alpha_\alpha}{\partial \partial_i A^\beta_0} = \partial_\mu \left(\frac{\partial^2 \mathcal{L}}{\partial \partial_\mu A^\alpha_0 \partial \partial_i A^\beta_0} \right) + 2 \frac{\partial^2 \mathcal{L}}{\partial \partial_i A^{[\alpha}_0 \partial A^{\beta]}_0}. \tag{114}$$

4 Application of the sufficient conditions

In the previous sections, we have studied a class of theories of n vector fields, with a possibility to couple to non-dynamical external fields, as described at the beginning of Sect. 2. For the theories in this class, let us call Eqs. (13), (40), (69) as sufficient conditions because if these conditions are satisfied, the theory of interest will have the correct constraint structure as an $n-$field generalised Proca system coupled to non-dynamical external fields. In more details, these conditions are the special Hessian condition (13), secondary-constraint enforcing relation (40), as well as the completion requirement (69) which demands that Eq. (66) contains no zero mode. The completion requirement is the most involved. In order to consider them, one needs to write down the expression of $\mathcal{C}_{0\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta}$. Their explicit forms can be computed by using Eqs. (110)–(114).

In this section, we will demonstrate the use of the criteria presented in Sects. 2–3. We provide a few examples of theories which pass these requirements, as well as an example theory which does not pass, but is previously incorrectly identified in the literature as being legitimate. These examples should be sufficient to serve the purpose. They are, however, far from exhaustive. We expect that many other theories passing these requirements are already presented in the literature, but some of them may have been previously misinterpreted.

4.1 Examples

4.1.1 Separable multi-field generalised Proca theories

One of simple examples is the case where each of the n vector fields in the system does not couple to one another. The system is considered to be separated into n sub-systems of single vector field, possibly coupled to external fields. It could then be expected that one can simply separately apply the constraint analysis on each sub-system. For example, an analysis of [25] confirms that as long as each sub-system describes a generalised Proca field, possibly coupled to external fields, then the vector sector has 3 degrees of freedom.

Direct use of the results presented in Sects. 2–3 can also easily be done. The Lagrangian of the example system takes the form

$$\mathcal{L} = \sum_{\alpha=1}^{n} \mathcal{L}_{(\alpha)}, \tag{115}$$

where for each $\alpha \in \{1, 2, \ldots, n\}$, the sub-Lagrangian $\mathcal{L}_{(\alpha)}$ is a function of only the αth vector field A^α_μ, its first order derivative $\partial_\mu A^\alpha_\nu$, and possibly external fields; but $\mathcal{L}_{(\alpha)}$ does not depend on the βth vector fields nor their derivatives if $\beta \neq \alpha$. After demanding that it satisfies the special Hessian condition (13), we obtain

$$\mathcal{L}_{(\alpha)} = T_{(\alpha)} + U_\alpha \dot{A}^\alpha_0 \quad \text{(no summation over}\alpha). \tag{116}$$

So we have

$$\frac{\partial U_\alpha}{\partial A^\beta_0} = \delta_{\alpha\beta} \frac{\partial U_\alpha}{\partial A^\alpha_0} \quad \text{(no summation over } \alpha), \tag{117}$$

and from Eq. (A10), we have

$$\frac{\partial U_\alpha}{\partial \partial_i A^\beta_0} = 0. \tag{118}$$

Therefore, the secondary-constraint enforcing relations are automatically satisfied. Next, since the derivative of $\mathcal{L}_{(\alpha)}$ with respect to A^β_μ or $\partial_\mu A^\beta_\nu$ vanish if $\alpha \neq \beta$, then $\mathcal{C}_{0\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta}$ are diagonal matrices. In fact, since $\mathcal{C}^i_{1\alpha\beta} = -\mathcal{C}^i_{1\beta\alpha}$, we can conclude that $\mathcal{C}^i_{1\alpha\beta} = 0$. So we have

$$\mathcal{C}_{0\alpha\beta} = \mathcal{C}_{0\alpha\alpha}\delta_{\alpha\beta}, \quad \mathcal{C}^i_{1\alpha\beta} = 0, \quad \text{(no summation over } \alpha). \tag{119}$$

Then in order for Eq. (66) to have no zero mode, we should require

$$\det(\mathcal{C}_{0\alpha\beta}) = \prod_{\alpha=1}^{n} \mathcal{C}_{0\alpha\alpha} \neq 0, \tag{120}$$

which is possible if $\mathcal{C}_{0\alpha\alpha} \neq 0$ for each $\alpha \in \{1, 2, \ldots, n\}$. This means that each subsystem has to be described by a generalised Proca field, possibly coupled to external fields.

4.1.2 A less trivial example

Let us consider an example theory whose Lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_2 \left(A^\alpha_\mu, {A^\alpha}_{\mu\nu}, \{K, \partial K, \partial\partial K, \ldots\} \right), \tag{121}$$

where ${A^\alpha}_{\mu\nu} \equiv \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu$. It is one of the simplest forms of multi-field generalised Proca theories being presented in the literature, see for example [34–36, 42]. We confirm that the theory is indeed legitimate. For this theory,

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A^\alpha_\nu} = -\frac{\partial \mathcal{L}}{\partial \partial_\nu A^\alpha_\mu} = 2 \frac{\partial \mathcal{L}_2}{\partial {A^\alpha}_{\mu\nu}}. \tag{122}$$

This immediately gives $U_\alpha = 0$. So the secondary-constraint enforcing relations (40) is trivially satisfied. Furthermore, $\mathcal{C}_{0\alpha\beta}$ and $\mathcal{C}^i_{1\alpha\beta}$ are simplified to

$$\mathcal{C}_{0\alpha\beta} = \frac{\partial^2 \mathcal{L}_2}{\partial A^\alpha_0 \partial A^\beta_0} - 4 \frac{\partial^2 \mathcal{L}_2}{\partial A^\gamma{}_{0j} \partial A^\alpha_0} M^{\gamma\delta}_{jk} \frac{\partial^2 \mathcal{L}_2}{\partial A^\delta{}_{0k} \partial A^\beta_0}, \quad \mathcal{C}^i_{1\alpha\beta} = 0. \tag{123}$$

It can be seen that, apart from some exceptions, $\det \mathcal{C}_{0\alpha\beta} \neq 0$. So the theory has the required number of degrees of freedom, and hence is an $n-$field generalised Proca theory.

A notable exception is when $\mathcal{L}$ is independent from $A^{\alpha_1}_0$ for $\alpha_1 \in \{1, 2, \ldots, r\}$, where $1 < r \leq n$. While the criteria provided in Sects. 2–3 can only be used to state that this exception is not an $n-$field generalised Proca theory, it should nevertheless intuitively be expected that it describes $(n-r)$ generalised Proca fields while the other r fields might be, provided that it passes some further criteria, generalised Maxwell fields. These criteria, if any, should arise when one considers multi-field generalised Maxwell–Proca theories. While [34, 35, 37] might have already provided the criteria for identifying multi-field generalised Maxwell–Proca theories, we have found in this work that even when restricted to purely (multi-field) Proca theories, their analysis seems to require some non-trivial refinements. So we expect that the refinements to the criteria of multi-field generalised Maxwell–Proca theories are needed. We leave this for future works.

Nevertheless, suppose that we have considered a Lagrangian $\mathcal{L}^{(1)}$ whose $\mathcal{C}^i_{1\alpha\beta}$, denoted $\mathcal{C}^i_{1\alpha\beta}(\mathcal{L}^{(1)})$, is zero while its $\mathcal{C}_{0\alpha\beta}$, denoted $\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(1)})$, is singular. It could still be possible to add to it another Lagrangian $\mathcal{L}^{(2)}$ with $\mathcal{C}^i_{1\alpha\beta}(\mathcal{L}^{(2)}) = 0$ so that the resulting Lagrangian $\mathcal{L}^{(1)} + \mathcal{L}^{(2)}$ might describe an $n-$field generalised Proca theory. This is because, due to Eq. (111), $\mathcal{C}^i_{1\alpha\beta}$ is linear. So $\mathcal{C}^i_{1\alpha\beta}(\mathcal{L}^{(1)} + \mathcal{L}^{(2)}) = \mathcal{C}^i_{1\alpha\beta}(\mathcal{L}^{(1)}) + \mathcal{C}^i_{1\alpha\beta}(\mathcal{L}^{(2)}) = 0$. On the other hand, due to the last term on RHS of Eq. (110), $\mathcal{C}_{0\alpha\beta}$ is non-linear. So $\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(1)} + \mathcal{L}^{(2)}) = \mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(1)}) + \mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(2)}) + \text{non-linear}(\mathcal{L}^{(1)}, \mathcal{L}^{(2)})$. Due to non-linearity of $\mathcal{C}_{0\alpha\beta}$ and of its determinant, it is likely that $\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(1)} + \mathcal{L}^{(2)})$ is not singular even if both $\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(1)})$ and $\mathcal{C}_{0\alpha\beta}(\mathcal{L}^{(2)})$ are singular. Of course, although highly likely to be the case, direct calculations are required in each case to confirm whether this is truly the case.

4.1.3 A legitimate theory previously misinterpreted

Allys et al. [42], actions for multiple vector fields are constructed by using a systematic approach which demands that the special Hessian condition is satisfied. In principle, this is not sufficient to give legitimate theories as further conditions, for example secondary-constraint enforcing relations, are required. The reference [34] points out that one of theories proposed in [42], does not pass secondary-constraint enforcing relations and hence contains extra degrees of freedom. The Lagrangian of this theory is

$$\mathcal{L} = -\frac{1}{4} A^\alpha{}_{\mu\nu} A_\alpha{}^{\mu\nu} - 4\lambda \left(A^{\alpha\sigma} A^\beta_\sigma \partial^\mu A^\alpha_{[\mu} \partial^\nu A^\beta_{\nu]} + A^\alpha_{[\mu} A^\beta_{\nu]} \partial^\mu A^\alpha_\rho \partial^\nu A^{\beta\rho} \right), \tag{124}$$

where λ is a non-zero constant. Actually, since secondary-constraint enforcing relations presented in [34] miss some terms in the expression, in principle, the interpretation being drawn should be revised.

Let us argue that in fact the theory (124) is legitimate. By direct calculation, one obtains

$$\frac{\partial^2\mathcal{L}}{\partial\dot{A}_0^\alpha\partial A_0^\beta}-\frac{\partial^2\mathcal{L}}{\partial\dot{A}_0^\beta\partial A_0^\alpha}=-8\lambda\partial_i\left(A^\alpha_{[0}A^\beta_{i]}\right)=-\partial_i\left(\frac{\partial^2\mathcal{L}}{\partial\dot{A}_0^\beta\partial\partial_i A_0^\alpha}\right),\tag{125}$$

which means that the secondary-constraint enforcing relation (40) is satisfied. Therefore, contrary to the interpretation given in [34], the theory Eq. (124) has secondary constraints. Furthermore, this theory is in fact an $n-$field generalised Proca theory. To see this, one notes that by making direct computation one obtains

$$\mathcal{C}^i_{1\alpha\beta}=0.\tag{126}$$

It can then be checked that if $\lambda\neq 0$, then $\det(\mathcal{C}_{0\alpha\beta})\neq 0$. Therefore, the completion requirement (69) is satisfied.

Of course, the same conclusion can also be reached if one directly starts from the Lagrangian (124) and performs either Hamiltonian or Lagrangian constraint analysis.

We expect that there are also other theories presented in [42] which are legitimate but is previously incorrectly ruled out. A common feature for these theories is that

$$\frac{\partial^2\mathcal{L}}{\partial\dot{A}_0^\alpha\partial A_0^\beta}-\frac{\partial^2\mathcal{L}}{\partial\dot{A}_0^\beta\partial A_0^\alpha}\neq 0,\tag{127}$$

which makes them incorrectly ruled out. So if $\partial^2\mathcal{L}/(\partial\dot{A}_0^\beta\partial\partial_i A_0^\alpha)\neq 0$, then one might try to see if $-\partial_i(\partial^2\mathcal{L}/(\partial\dot{A}_0^\beta\partial\partial_i A_0^\alpha))$ would cancel out with LHS of (127). If this is the case, then one can proceed to check the completion requirement.

4.1.4 An undesired theory previously misinterpreted

After the reference [34] suggests that the special Hessian conditions are not sufficient, and that the secondary-constraint enforcing relations should be satisfied, theories are being proposed in the literature in order to satisfy the required relations. Notable examples are [34–36].

Let us argue that, by using a refined version of secondary-constraint enforcing relations, some of the theories in fact are undesired, i.e. they contain extra degrees of freedom. In particular, we explicitly show one example from [36]. This particular example has the Lagrangian of the form

$$\mathcal{L}=-2A^\alpha{}_{\mu\nu}S^{\beta\mu}{}_\sigma A_{\alpha\rho}A_{\beta\lambda}\epsilon^{\nu\sigma\rho\lambda}+S^\alpha{}_{\mu\nu}S^{\beta\nu}{}_\sigma A_{\alpha\rho}A_{\beta\lambda}\epsilon^{\mu\sigma\rho\lambda},\tag{128}$$

where $S^{\alpha}{}_{\mu\nu} \equiv \partial_\mu A^\alpha_\nu + \partial_\nu A^\alpha_\mu$. By direct calculation, one obtains

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{A}^\alpha_0 \partial A^\beta_0} - \frac{\partial^2 \mathcal{L}}{\partial \dot{A}^\beta_0 \partial A^\alpha_0} = 0 \neq -\partial_i \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{A}^\beta_0 \partial \partial_i A^\alpha_0} \right). \tag{129}$$

Therefore, this theory is in fact undesired.

We expect that there are also other theories presented in the literature which contain extra degrees of freedom but is previously interpreted as being well-behaved. For these theories, $\partial^2\mathcal{L}/(\partial \dot{A}^\alpha_0 \partial A^\beta_0) - \partial^2\mathcal{L}/(\partial \dot{A}^\beta_0 \partial A^\alpha_0) = 0$. So if they are truly undesired, one should find that $-\partial_i(\partial^2\mathcal{L}/(\partial \dot{A}^\beta_0 \partial\partial_i A^\alpha_0)) \neq 0$, which would violate the secondary-constraint enforcing relations (40).

4.2 Cosmological implications

Multi-field generalised Proca theories have been applied for example in [11–13] to explain cosmological phenomena. In some of these studies, the conditions presented by [34, 35] are taken into consideration. However, as we have been discussing, these conditions are incorrect and should be replaced by Eq. (40). In principle, one should then investigate the validation of the cosmological implications presented in [11–13]. In this subsection, we discuss a direction for further investigations on these works.

Rodríguez and Navarro [11], a Lagrangian involving Einstein–Hilbert term, $SU(2)$ Yang–Mills term $\mathcal{L}_{YM}$, and a term called $\alpha\mathcal{L}^1_4$ where α is a constant is considered. Autonomous dynamical system analysis of this model in a homogeneous and isotropic background is studied which allows dark energy and primordial inflation to be discussed. While the dark energy case leads to an interesting result, the primordial inflation case is problematic as the model is strongly sensitive to initial conditions and the value of α. It is then suggest that one should also include a term $\kappa\mathcal{L}^2_4$, where κ is a constant, into the Lagrangian and see if the problem can be evaded.

Let us discuss whether the Lagrangian presented in [11] would pass the sufficient conditions in Sect. 2. Note that for the theory in [11], gravity is dynamical whereas the sufficient conditions we have presented is useful when the gravity is non-dynamical. Nevertheless, a simple check can still be performed in the case of flat spacetime, in which case $\mathcal{L}_{YM}$ is a function of A^α_μ, $A^\alpha{}_{\mu\nu}$, whereas $\mathcal{L}^1_4$ is a function of A^α_μ, $\partial_\mu A^\alpha_\nu$ in such a way that $\partial^2\mathcal{L}^1_4/\partial \dot{A}^\alpha_0 \partial A^\beta_0 = \partial^2\mathcal{L}^1_4/\partial \dot{A}^\beta_0 \partial A^\alpha_0$, $\partial^2\mathcal{L}^1_4/(\partial \dot{A}^\alpha_0 \partial\partial_i A^\beta_0) = 0$. So it can easily be seen from the discussion of Sect. 4.1 that the theory in [11] pass the sufficient conditions.

It would also be interesting to investigate whether the suggestion to include the term $\kappa\mathcal{L}^2_4$ still valid, as far as our sufficient conditions are concerned. So let us also consider the case of flat spacetime. In this case, it can easily be seen that $\mathcal{L}_{YM} + \kappa\mathcal{L}^2_4$ is simply expressible as a summation of the Lagrangians (121) and (124). So indeed the term $\kappa\mathcal{L}^2_4$ can be included to extend the model of [11]. Note on the other hand that if one had used the criteria of [34, 35], the term $\kappa\mathcal{L}^2_4$ would have been incorrectly ruled out.

Gómez and Rodríguez [12], Garnica et al. [13], cosmological implications of multi-field generalised Proca theories are also investigated. It turns out however that some terms of the Lagrangian, for example $\mathcal{L}_4^2$ presented in [11], has been incorrectly ruled out according to the criteria of [34, 35]. But as discussed in the previous paragraph, such a term in fact passes the criteria presented in Sect. 2, so there is no problem with the number of degrees of freedom. It would be interesting to see for example the cosmological implication of the inclusion of $\mathcal{L}_4^2$ to the models of [12, 13].

5 Discussion and conclusion

In this work, we have worked out the sufficient conditions to make a theory describe multi-field generalised Proca theory, possibly coupled to non-dynamical external fields. We focus on a class of theories whose Lagrangians are functions of up to first-order derivative of the vector fields. Furthermore, we demand that the Lagrangian of each theory satisfies the special Hessian condition (13), free of Ostrogradski instability and that it transforms in a standard way under standard diffeomorphism. Theories in this class should also pass the secondary-constraint enforcing relations (35) (or equivalently, Eq. (40)) as well as the completion requirements (69) which can be computed using Eqs. (110)–(114).

As a standard mathematical terminology, one says that a condition C is sufficient for an event E if whenever C is true the event E always occurs. So in this context, we call Eqs. (13), (40), (69) as sufficient conditions for degrees of freedom counting of multi-field generalised Proca theories because if the set of these conditions is satisfied, the theory of interest (being, as described at the beginning of Sect. 2, diffeomorphism invariant and free of Ostrogradski instability) will have the correct constraint structure and hence correct number of degrees of freedom as an $n-$field generalised Proca system possibly coupled to non-dynamical external fields.

We have obtained these conditions by using Faddeev–Jackiw constraint analysis and cross checked using Lagrangian constraint analysis. In the analysis, diffeomorphism invariance requirements, Eqs. (A10), (A16)–(A17) are needed. The diffeomorphism invariance requirements are not extra conditions. They are in fact conditions for which every diffeomorphism invariance theory is satisfied. If one analyses each specific theory one by one, it can be explicitly seen that these requirements are automatically satisfied. However, if one analyses a class of theories at a time, diffeomorphism invariance is less manifest as, by the nature of constraint analysis, time and space are not treated on equal footing. In this case, diffeomorphism invariance requirements help to realise the diffeomorphism invariance that every theory in the class possesses. These requirements are especially useful in simplifying key expressions in intermediate steps. Let us provide two example instances where the usefulness of diffeomorphism invariance requirements when analysing a class of theories are shown.

The first example is that, if the secondary-constraint enforcing relations (35) is imposed, and if one does not know that theories which are diffeomorphism invariant should satisfy Eq. (A10), one would not be able to see, when analysing a class of theories, that Eq. (32) is trivial, and hence would impose Eq. (A10) as another, but is in fact obsolete, secondary-constraint enforcing relations. Another notable example is that diffeomorphism invariance requirements allow us to realise the connection between results from Faddeev–Jackiw constraint analysis and Lagrangian constraint analysis. The diffeomorphism invariance requirements have been helping in simplifying $\mathcal{C}_{0\alpha\beta}$, $\mathcal{C}^{i}_{1\alpha\beta}$, $\mathcal{C}^{ij}_{2\alpha\beta}$ and allowing us to realise that these expressions also appear, after transforming to tangent bundle, in Lagrangian constraint analysis.

Secondary-constraint enforcing relations we have obtained in this paper is a correction to [34, 35]. This means that behaviour of some theories are previously misjudged. We have shown in Sect. 4.1 an example of a legitimate theory previously misinterpreted as containing extra degrees of freedom as well as an example of undesired theory with extra degrees of freedom previously misinterpreted as being legitimate. We leave the work of identifying or constructing all of the theories which pass the secondary-constraint enforcing relations and the completion relations for future. Nevertheless, a consequence can readily be discussed and is provided in Sect. 4.2 which points out that legitimate terms previously misjudged could be reintroduced into models to investigate cosmological implications.

An important future work is to analyse a larger class of theories, not necessarily restricted to those describing only vector fields. In fact, an important step has already been laid out by [37], which gives criteria for counting the number of degrees of freedom for theories with Lagrangians as functions of up to first order derivative in fields. These criteria, however, should be revised because as points out by [33], the analysis of [37] is not correct even in the case of the standard Proca theory. Additionally, as reported in our paper, the analysis of [37] when specialised to multi-field generalised Proca theories misses terms in intermediate steps, for example $\partial_i \ddot{Q}^{\beta}$ and $\partial_i \partial_j \ddot{Q}^{\beta}$ within $\dot{\phi}_{\alpha}$. The corrections are required to address these issues. Once they are taken care of, we expect that the analysis would benefit from the help of diffeomorphism conditions. This is because in constraint analysis, even for Lagrangian constraint analysis, time and space are not treated on an equal footing. So the manifestation of diffeomorphism invariance (or, in case of flat spacetime, Lorentz isometry) is lost in the steps. The manifestation could be recovered with the use of diffeomorphism invariance requirements.

In particular, since the external fields, including gravity, considered in this paper are all non-dynamical, one might attempt to extend the analysis of this paper by considering n vector fields coupled to dynamical gravity and see if it is possible to obtain the criteria for the theory to describe n-field Proca theory, or even Maxwell–Proca theory, coupled to dynamical gravity.

Acknowledgements We are grateful to Sheng-Lan Ko for interests and discussions. We would also like to thank Claudia de Rham for interests, discussions, and comments. Furthermore, we are grateful to anonymous referees for useful comments and suggestions which improve our manuscript. S.J. is supported by the Research Professional Development Project Under the Science Achievement Scholarship of Thailand (SAST).

Data availability No data was analysed or generated in this work.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

Appendix A: Conditions from diffeomorphism invariance

In this appendix, we consider a class of theories described in Sect. 2. Since these theories are diffeomorphism invariant, their Lagrangians would satisfy the conditions to be presented in this appendix.

Under diffeomorphism $x^\mu \mapsto x^\mu - \epsilon^\mu(x)$, the vector fields transform as

$$\delta_\epsilon A^\alpha_\mu = \epsilon^\nu \partial_\nu A^\alpha_\mu + A^\alpha_\nu \partial_\mu \epsilon^\nu, \tag{A1}$$

and the external fields $\{K\}$ transform under standard diffeomorphism. The Lagrangian density transforms as

$$\delta_\epsilon \mathcal{L} = \partial_\mu(\epsilon^\mu \mathcal{L}). \tag{A2}$$

Demanding the expression $\delta_\epsilon \mathcal{L} - \partial_\mu(\epsilon^\mu \mathcal{L})$ to vanish will give rise to useful conditions. In order to evaluate this expression, we begin by recall that $\mathcal{L} = T + U_\alpha \dot{A}^\alpha_0$. Then we consider

$$\begin{aligned}\delta_\epsilon T =& \frac{\partial T}{\partial A^\alpha_\nu}\delta_\epsilon A^\alpha_\nu + \frac{\partial T}{\partial \partial_k A^\alpha_\nu}\partial_k \delta_\epsilon A^\alpha_\nu + \frac{\partial T}{\partial \dot{A}^\alpha_k}\partial_0 \delta_\epsilon A^\alpha_k \\ &+ \frac{\partial T}{\partial \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}} \\ &\times \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\delta_\epsilon(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}.\end{aligned} \tag{A3}$$

Next, let us consider

$$\begin{aligned}-\partial_\mu(\epsilon^\mu T) =& -\partial_\mu \epsilon^\mu T - \epsilon^\mu \left(\frac{\partial T}{\partial A^\alpha_\nu}\partial_\mu A^\alpha_\nu + \frac{\partial T}{\partial \partial_k A^\alpha_\nu}\partial_k \partial_\mu A^\alpha_\nu + \frac{\partial T}{\partial \dot{A}^\alpha_k}\partial_0 \partial_\mu A^\alpha_k\right) \\ &-\epsilon^\mu \frac{\partial T}{\partial \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}} \\ &\times \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\partial_\mu(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}.\end{aligned} \tag{A4}$$

Combining the two expressions, we obtain

$$\begin{aligned}&\delta_\epsilon T - \partial_\mu(\epsilon^\mu T) \\ &= \frac{\partial T}{\partial A^\alpha_\nu} A^\alpha_\mu \partial_\nu \epsilon^\mu + \frac{\partial T}{\partial \partial_k A^\alpha_\nu}\left(\partial_k \epsilon^\mu \partial_\mu A^\alpha_\nu + \partial_k A^\alpha_\mu \partial_\nu \epsilon^\mu + A^\alpha_\mu \partial_k \partial_\nu \epsilon^\mu\right) \\ &+ \frac{\partial T}{\partial \dot{A}^\alpha_k}\left(\dot{\epsilon}^\mu \partial_\mu A^\alpha_k + \dot{A}^\alpha_\mu \partial_k \epsilon^\mu + A^\alpha_\mu \partial_k \dot{\epsilon}^\mu\right) - \partial_\mu \epsilon^\mu T \\ &+ \frac{\partial T}{\partial \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}}\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\delta_\epsilon(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}} \\ &-\epsilon^\mu \frac{\partial T}{\partial \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}} \\ &\times \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\partial_\mu(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}.\end{aligned} \tag{A5}$$

Let us also compute

$$
\begin{aligned}
\delta_\epsilon\left(U_\beta \dot{A}_0^\beta\right) = & \frac{\partial U_\beta}{\partial A_\nu^\alpha}\left(\epsilon^\mu \partial_\mu A_\nu^\alpha + A_\mu^\alpha \partial_\nu \epsilon^\mu\right)\dot{A}_0^\beta + \frac{\partial U_\beta}{\partial \partial_i A_\nu^\alpha}\partial_i\left(\epsilon^\mu \partial_\mu A_\nu^\alpha + A_\mu^\alpha \partial_\nu \epsilon^\mu\right)\dot{A}_0^\beta \\
& + \dot{A}_0^\beta \frac{\partial U_\beta}{\partial \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}}\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\delta_\epsilon(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}} \\
& + U_\beta \partial_0\left(\epsilon^\mu \partial_\mu A_0^\beta + A_\nu^\beta \partial_0 \epsilon^\nu\right),
\end{aligned}
\tag{A6}
$$

and

$$
\begin{aligned}
& -\partial_\mu(\epsilon^\mu U_\beta \dot{A}_0^\beta) \\
& = -\partial_\mu \epsilon^\mu U_\beta \dot{A}_0^\beta - \epsilon^\mu\left(\frac{\partial U_\beta}{\partial A_\nu^\alpha}\partial_\mu A_\nu^\alpha + \frac{\partial U_\beta}{\partial \partial_i A_\nu^\alpha}\partial_i \partial_\mu A_\nu^\alpha\right)\dot{A}_0^\beta \\
& \quad -\epsilon^\mu \dot{A}_0^\beta \frac{\partial U_\beta}{\partial \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}}\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\partial_\mu(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}} \\
& \quad -\epsilon^\mu U_\beta \partial_\mu \dot{A}_0^\beta.
\end{aligned}
\tag{A7}
$$

So we have

$$
\begin{aligned}
& \delta_\epsilon(U_\beta \dot{A}_0^\beta) - \partial_\mu(\epsilon^\mu U_\beta \dot{A}_0^\beta) \\
& = \frac{\partial U_\beta}{\partial A_\nu^\alpha} A_\mu^\alpha \partial_\nu \epsilon^\mu \dot{A}_0^\beta + U_\beta(\dot{\epsilon}^\mu \partial_\mu A_0^\beta + \epsilon^\mu \partial_\mu \dot{A}_0^\beta + \dot{A}_\nu^\beta \partial_0 \epsilon^\nu + A_\nu^\beta \ddot{\epsilon}^\nu) \\
& \quad -\partial_\mu \epsilon^\mu U_\beta \dot{A}_0^\beta - \epsilon^\mu U_\beta \partial_\mu \dot{A}_0^\beta \\
& \quad + \frac{\partial U_\beta}{\partial \partial_i A_\nu^\alpha}(\partial_i \epsilon^\mu \partial_\mu A_\nu^\alpha + \partial_i A_\mu^\alpha \partial_\nu \epsilon^\mu + A_\mu^\alpha \partial_i \partial_\nu \epsilon^\mu)\dot{A}_0^\beta \\
& \quad + \dot{A}_0^\beta \frac{\partial U_\beta}{\partial \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}}\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\delta_\epsilon(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}} \\
& \quad -\epsilon^\mu \dot{A}_0^\beta \frac{\partial U_\beta}{\partial \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}} \\
& \quad \times \partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\partial_\mu(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}.
\end{aligned}
\tag{A8}
$$

Combining Eq. (A5) with Eq. (A8), we obtain the expression for $\delta_\epsilon \mathcal{L} - \partial_\mu(\epsilon^\mu \mathcal{L})$.

We are now ready to obtain useful conditions. Let us note that the expression $\delta_\epsilon \mathcal{L} - \partial_\mu(\epsilon^\mu \mathcal{L})$ is a polynomial in $\dot{A}_0^\alpha$ up to degree two. Consider the term containing $\dot{A}_0^\alpha \dot{A}_0^\beta$ in $\delta_\epsilon \mathcal{L} - \partial_\mu(\epsilon^\mu \mathcal{L})$. It can easily be seen that there is only one term which is

$$
\frac{\partial U_\beta}{\partial \partial_i A_0^\alpha}\partial_i \epsilon^0 \dot{A}_0^\alpha \dot{A}_0^\beta.
\tag{A9}
$$

Demanding this expression to vanish gives

$$\frac{\partial U_\alpha}{\partial\partial_i A_0^\beta}+\frac{\partial U_\beta}{\partial\partial_i A_0^\alpha}=0. \tag{A10}$$

Let us next turn to the coefficients of $\dot{A}_0^\beta$. For this, it would be convenient to consider Eqs. (A5) and (A8) and collect the terms proportional to $\dot{A}_0^\beta$. We have

$$\delta_\epsilon T-\partial_\mu(\epsilon^\mu T)\ni\frac{\partial T}{\partial\partial_k A_0^\alpha}\partial_k\epsilon^0\dot{A}_0^\alpha+\frac{\partial T}{\partial\dot{A}_k^\alpha}\dot{A}_0^\alpha\partial_k\epsilon^0, \tag{A11}$$

and

$$\begin{aligned}
&\delta_\epsilon(U_\beta\dot{A}_0^\beta)-\partial_\mu(\epsilon^\mu U_\beta\dot{A}_0^\beta)\\
&\ni\frac{\partial U_\beta}{\partial A_\nu^\alpha}A_\mu^\alpha\partial_\nu\epsilon^\mu\dot{A}_0^\beta+U_\beta(\dot{\epsilon}^0\dot{A}_0^\beta+\dot{A}_0^\beta\dot{\epsilon}^0)-\partial_\mu\epsilon^\mu U_\beta\dot{A}_0^\beta\\
&\quad+\frac{\partial U_\beta}{\partial\partial_i A_\nu^\alpha}(\partial_i\epsilon^\mu\partial_\mu A_\nu^\alpha+\partial_i A_\mu^\alpha\partial_\nu\epsilon^\mu+A_\mu^\alpha\partial_i\partial_\nu\epsilon^\mu)\Big|_{\dot{A}_0^\alpha=0}\dot{A}_0^\beta\\
&\quad+\dot{A}_0^\beta\frac{\partial U_\beta}{\partial\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}}\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\delta_\epsilon(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}\\
&\quad-\epsilon^\mu\dot{A}_0^\beta\frac{\partial U_\beta}{\partial\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}}\\
&\quad\times\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\partial_\mu(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}.
\end{aligned} \tag{A12}$$

Therefore, we have

$$\begin{aligned}
&\frac{\partial T}{\partial\partial_k A_0^\beta}\partial_k\epsilon^0+\frac{\partial T}{\partial\dot{A}_k^\beta}\partial_k\epsilon^0+\frac{\partial U_\beta}{\partial A_\nu^\alpha}A_\mu^\alpha\partial_\nu\epsilon^\mu+2U_\beta\dot{\epsilon}^0-\partial_\mu\epsilon^\mu U_\beta\\
&\quad+\frac{\partial U_\beta}{\partial\partial_i A_\nu^\alpha}(\partial_i\epsilon^\mu\partial_\mu A_\nu^\alpha+\partial_i A_\mu^\alpha\partial_\nu\epsilon^\mu+A_\mu^\alpha\partial_i\partial_\nu\epsilon^\mu)\Big|_{\dot{A}_0^\alpha=0}\\
&\quad+\frac{\partial U_\beta}{\partial\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}}\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\delta_\epsilon(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}\\
&\quad-\epsilon^\mu\frac{\partial U_\beta}{\partial\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}}\\
&\quad\times\partial_{\rho_1}\cdots\partial_{\rho_{r'''}}\partial_\mu(K^{(r)})_{\mu_1\cdots\mu_{r'}}{}^{\nu_1\cdots\nu_{r''}}=0.
\end{aligned} \tag{A13}$$

Although the above equation looks complicated especially due to the explicit presence of external fields, we will only extract some parts of this equation to obtain the conditions that we will need. These conditions will look much more simple. For example, the dependence on the external fields and their derivatives are only through T and

U_β. We may derive these conditions as follows. Taking derivative of Eq. (A13) with respect to $\dot{A}_j^\alpha$ gives

$$\frac{\partial^2 T}{\partial\partial_k A_0^\beta \partial \dot{A}_j^\alpha} + \frac{\partial^2 T}{\partial \dot{A}_k^\beta \partial \dot{A}_j^\alpha} + \frac{\partial U_\beta}{\partial\partial_k A_j^\alpha} = 0. \tag{A14}$$

Let us take derivative of Eq. (A13) with respect to $\partial_j A_0^\alpha$, then swap the indices α and β, add it to the original equation, and use Eq. (A10), we obtain

$$2\frac{\partial^2 T}{\partial\partial_j A_0^{(\alpha}\partial\partial_k A_0^{\beta)}} + 2\frac{\partial^2 T}{\partial\partial_j A_0^{(\alpha}\partial\dot{A}_k^{\beta)}} + \frac{\partial U_\beta}{\partial\partial_j A_k^\alpha} + \frac{\partial U_\alpha}{\partial\partial_j A_k^\beta} = 0. \tag{A15}$$

Expressing in phase space, the conditions Eqs. (A14)–(A15) become

$$\frac{\partial^2 \mathcal{T}}{\partial\partial_k A_0^\beta \partial \Lambda_j^\alpha} + \frac{\partial^2 \mathcal{T}}{\partial \Lambda_k^\beta \partial \Lambda_j^\alpha} + \frac{\partial U_\beta}{\partial\partial_k A_j^\alpha} = 0, \tag{A16}$$

and

$$2\frac{\partial^2 \mathcal{T}}{\partial\partial_j A_0^{(\alpha}\partial\partial_k A_0^{\beta)}} + 2\frac{\partial^2 \mathcal{T}}{\partial\partial_j A_0^{(\alpha}\partial\Lambda_k^{\beta)}} + \frac{\partial U_\beta}{\partial\partial_j A_k^\alpha} + \frac{\partial U_\alpha}{\partial\partial_j A_k^\beta} = 0. \tag{A17}$$

By substituting Eq. (A16) into Eq. (A17), we obtain

$$\frac{\partial^2 \mathcal{T}}{\partial\partial_{(j|} A_0^\alpha \partial\partial_{|k)} A_0^\beta} - \frac{\partial^2 \mathcal{T}}{\partial \Lambda_{(j}^\alpha \partial \Lambda_{k)}^\beta} = 0. \tag{A18}$$

Appendix B: Expressions of $\partial\phi_\alpha/\partial\partial_{\mathcal{I}}\dot{Q}^\beta$ in phase space

In this appendix, we outline necessary steps to express $\partial\phi_\alpha/\partial\partial_{\mathcal{I}}\dot{Q}^\beta$ in phase space. We use the same set-up and notations as those given in Sects. 2–3. For convenient, let us denote P_A and Λ^A as collective for π_α^i and Λ_i^α, respectively.

The idea is to first express $\partial\phi_\alpha/\partial\partial_{\mathcal{I}}\dot{Q}^\beta$ in terms of α_M. This can be achieved by recalling from Sect. 3 the Eq. (98). Recall also that diffeomorphism invariance requirements and demanding $\dot{\alpha}_\alpha = 0$ to not introduce further dynamics on the vector fields imply that $\partial\alpha_\alpha/\partial\dot{Q}^\beta = 0 = \partial\alpha_\alpha/\partial\partial_i\dot{Q}^\beta$. Furthermore, due to the form of the Lagrangian of interest, we also have $\partial\alpha_\alpha/\partial\partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}\dot{Q}^\beta = 0$ for $l \geq 2$. So

$$\frac{\partial\alpha_\alpha}{\partial\partial_{\mathcal{I}}\dot{Q}^\beta} = 0. \tag{B1}$$

Then since $\phi_\alpha = \dot{\alpha}_\alpha$ we have, from Eqs. (98) and (B1)

$$\phi_\alpha = \sum_{|\mathcal{I}|=0}^{2} \frac{\partial \alpha_\alpha}{\partial \partial_{\mathcal{I}} Q^M} \partial_{\mathcal{I}} \dot{Q}^M - \sum_{|\mathcal{I}|=0}^{1} \frac{\partial \alpha_\alpha}{\partial \partial_{\mathcal{I}} \dot{Q}^B} \partial_{\mathcal{I}} (M^{BC} \alpha_C)$$
$$+ \frac{\partial \alpha_\alpha}{\partial \partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (K^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}} \partial_{\rho_1} \cdots \partial_{\rho_{r'''}} (\dot{K}^{(r)})_{\mu_1 \cdots \mu_{r'}}{}^{\nu_1 \cdots \nu_{r''}}. \quad \text{(B2)}$$

Due to Eq. (B1), it can be seen that ϕ_α depend on $\partial_{\mathcal{I}} \dot{Q}^\beta$ only through the expressions $\partial_{\mathcal{I}} \dot{Q}^M$ and $\partial_{\mathcal{I}} (M^{BC} \alpha_C)$ which appear in the above equation. This gives

$$\frac{\partial \phi_\alpha}{\partial \partial_{\mathcal{I}} \dot{Q}^\beta} = \frac{\partial \alpha_\alpha}{\partial \partial_{\mathcal{I}} Q^\beta} - \frac{\partial \alpha_\alpha}{\partial \partial_{\mathcal{J}} \dot{Q}^A} \frac{\partial \partial_{\mathcal{J}} (M^{AB} \alpha_B)}{\partial \partial_{\mathcal{I}} \dot{Q}^\beta}. \quad \text{(B3)}$$

We then need to compute each expression on RHS of Eq. (B3). For this, let us directly express α_A in terms of Lagrangian then transforming to phase space, but transform α_α to $-\tilde{\Omega}_\alpha$ (cf. Equation (104)). Direct calculations can be given as follows. In order to evaluate $\partial \partial_{\mathcal{J}} (M^{AB} \alpha_B) / \partial \partial_{\mathcal{I}} \dot{Q}^\beta$ we note that $\partial M^{AB} / \partial \partial_{\mathcal{I}} \dot{Q}^\beta = 0$ for $|\mathcal{I}| \geq 0$ whereas α_B depends on $\dot{Q}^\beta$ and $\partial_i \dot{Q}^\beta$ but not on $\partial_{\mathcal{I}} \dot{Q}^\beta$ where $|\mathcal{I}| \geq 2$. By writing $\partial_k \alpha_B$ using chain rule and taking derivative with respect to $\partial_{\mathcal{I}} \dot{Q}^\beta$, we obtain

$$\frac{\partial \partial_k \alpha_B}{\partial \partial_i \partial_j \dot{Q}^\beta} = \frac{\partial \alpha_B}{\partial \partial_l \dot{Q}^\beta} \delta^i_{(k} \delta^j_{l)}, \quad \frac{\partial \partial_j \alpha_B}{\partial \partial_i \dot{Q}^\beta} = \partial_j \frac{\partial \alpha_B}{\partial \partial_i \dot{Q}^\beta} + \delta^i_j \frac{\partial \alpha_B}{\partial \dot{Q}^\beta},$$
$$\frac{\partial \partial_j \alpha_B}{\partial \dot{Q}^\beta} = \partial_j \frac{\partial \alpha_B}{\partial \dot{Q}^\beta}. \quad \text{(B4)}$$

This gives

$$\frac{\partial \partial_k (M^{AB} \alpha_B)}{\partial \partial_i \partial_j \dot{Q}^\beta} = \frac{\partial ((\partial_k M^{AB}) \alpha_B + M^{AB} \partial_k \alpha_B)}{\partial \partial_i \partial_j \dot{Q}^\beta}$$
$$= M^{AB} \frac{\partial \partial_k \alpha_B}{\partial \partial_i \partial_j \dot{Q}^\beta}$$
$$= M^{AB} \frac{\partial \alpha_B}{\partial \partial_l \dot{Q}^\beta} \delta^i_{(k} \delta^j_{l)}, \quad \text{(B5)}$$

$$\frac{\partial \partial_j (M^{AB} \alpha_B)}{\partial \partial_i \dot{Q}^\beta} = \partial_j \left(M^{AB} \frac{\partial \alpha_B}{\partial \partial_i \dot{Q}^\beta} \right) + \delta^i_j M^{AB} \frac{\partial \alpha_B}{\partial \dot{Q}^\beta}, \quad \text{(B6)}$$

$$\frac{\partial \partial_j (M^{AB} \alpha_B)}{\partial \dot{Q}^\beta} = \partial_j \left(M^{AB} \frac{\partial \alpha_B}{\partial \dot{Q}^\beta} \right). \quad \text{(B7)}$$

Next, let us express $\partial \alpha_B / \partial \partial_{\mathcal{I}} \dot{Q}^\beta$ in terms of phase space variables. The calculations will involve $\partial(\partial_j (\partial \mathcal{L} / \partial \partial_j Q^B)) / \partial \partial_{\mathcal{I}} \dot{Q}^\beta$, which can be computed by first using the chain rule for ∂_j and then taking derivative with respect to $\partial_{\mathcal{I}} \dot{Q}^\beta$. The relevant results are

$$\frac{\partial}{\partial \dot{Q}^\beta}\left(\partial_i\left(\frac{\partial \mathcal{L}}{\partial\partial_i Q^B}\right)\right) = \partial_i\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^\beta \partial\partial_i Q^B}\right),$$
$$\frac{\partial}{\partial\partial_i \dot{Q}^\beta}\left(\partial_j\left(\frac{\partial \mathcal{L}}{\partial\partial_j Q^B}\right)\right) = \frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^\beta \partial\partial_i Q^B}. \tag{B8}$$

So

$$\begin{aligned}
\frac{\partial \alpha_B}{\partial \dot{Q}^\beta} &= \frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^B \partial Q^\beta} + \partial_i\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^\beta \partial\partial_i Q^B}\right) - \frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^\beta \partial Q^B}\\
&= \frac{\partial^2 \mathcal{T}}{\partial Q^\beta \partial \Lambda^B} + \partial_i\left(\frac{\partial U_\beta}{\partial\partial_i Q^B}\right) - \frac{\partial U_\beta}{\partial Q^B},
\end{aligned} \tag{B9}$$

$$\begin{aligned}
\frac{\partial \alpha_B}{\partial\partial_i \dot{Q}^\beta} &= \frac{\partial^2 \mathcal{L}}{\partial\partial_i Q^\beta \partial \dot{Q}^B} + \frac{\partial^2 \mathcal{L}}{\partial\partial_i Q^B \partial \dot{Q}^\beta}\\
&= \frac{\partial^2 \mathcal{T}}{\partial\partial_i Q^\beta \partial \Lambda^B} + \frac{\partial U_\beta}{\partial\partial_i Q^B}.
\end{aligned} \tag{B10}$$

Next, let us express $\partial\alpha_\alpha/\partial\partial_{\mathcal{I}} Q^\beta$ in phase space. For this, we first use Eq. (104) to transform α_α to $-\tilde{\Omega}_\alpha$. More precisely, this is

$$\alpha_\alpha = -\tilde{\Omega}_\alpha\left(Q^M, \partial_i Q^M, \partial_i\partial_j Q^M, P_B, \partial_i P_B, \{K, \partial K, \partial\partial K, \ldots\}\right), \tag{B11}$$

such that $P_B = P_B(Q^M, \partial_i Q^M, \dot{Q}^B, \{K, \partial K, \partial\partial K, \ldots\})$, in which both sides of Eq. (B11) are both functions on the tangent bundle. So when taking derivative of α_α with respect to $\partial_{\mathcal{I}} Q^\beta$, we need to also take into account that P_B and $\partial_i P_B$ also depend on $\partial_{\mathcal{I}} Q^\beta$. As part of the intermediate calculations, we need to compute $\partial\partial_k P_B/\partial\partial_{\mathcal{I}} Q^\beta$, which can be done by first writing $\partial_k P_B$ using chain rule, then taking derivative with respect to $\partial_{\mathcal{I}} Q^\beta$. We have

$$\frac{\partial\partial_k P_B}{\partial\partial_i\partial_j Q^\beta} = \frac{\partial P_B}{\partial\partial_l Q^\beta}\delta^i_{(k}\delta^j_{l)}, \quad \frac{\partial\partial_k P_B}{\partial\partial_i Q^\beta} = \partial_k\left(\frac{\partial P_B}{\partial\partial_i Q^\beta}\right) + \delta^i_k \frac{\partial P_B}{\partial Q^\beta},$$
$$\frac{\partial\partial_k P_B}{\partial Q^\beta} = \partial_k\left(\frac{\partial P_B}{\partial Q^\beta}\right). \tag{B12}$$

Then we use Eq. (45), which is equivalent to $P_B = \partial\mathcal{T}/\partial\Lambda^B$. Keeping these in mind, we have

$$\begin{aligned}
\frac{\partial \alpha_\alpha}{\partial\partial_i\partial_j Q^\beta} &= -\frac{\partial \tilde{\Omega}_\alpha}{\partial\partial_i\partial_j Q^\beta} - \frac{\partial \tilde{\Omega}_\alpha}{\partial\partial_{(i|} P_B}\frac{\partial P_B}{\partial\partial_{|j)} Q^\beta}\\
&= -\frac{\partial \tilde{\Omega}_\alpha}{\partial\partial_i\partial_j Q^\beta} - \frac{\partial \tilde{\Omega}_\alpha}{\partial\partial_{(i|} P_B}\frac{\partial^2 \mathcal{T}}{\partial\partial_{|j)} Q^\beta \partial \Lambda^B},
\end{aligned} \tag{B13}$$

$$
\begin{aligned}
\frac{\partial \alpha_\alpha}{\partial \partial_i Q^\beta} &= -\frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_i Q^\beta} - \frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_{\mathcal{I}} P_B} \partial_{\mathcal{I}} \left(\frac{\partial P_B}{\partial \partial_i Q^\beta} \right) - \frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_i P_B} \frac{\partial P_B}{\partial Q^\beta} \\
&= -\frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_i Q^\beta} - \frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_{\mathcal{I}} P_B} \partial_{\mathcal{I}} \left(\frac{\partial^2 \mathcal{T}}{\partial \partial_i Q^\beta \partial \Lambda^B} \right) \\
&\quad - \frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_i P_B} \frac{\partial^2 \mathcal{T}}{\partial Q^\beta \partial \Lambda^B},
\end{aligned}
\tag{B14}
$$

$$
\begin{aligned}
\frac{\partial \alpha_\alpha}{\partial Q^\beta} &= -\frac{\partial \tilde{\Omega}_\alpha}{\partial Q^\beta} - \frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_{\mathcal{I}} P_B} \partial_{\mathcal{I}} \left(\frac{\partial P_B}{\partial Q^\beta} \right) \\
&= -\frac{\partial \tilde{\Omega}_\alpha}{\partial Q^\beta} - \frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_{\mathcal{I}} P_B} \partial_{\mathcal{I}} \left(\frac{\partial^2 \mathcal{T}}{\partial Q^\beta \partial \Lambda^B} \right).
\end{aligned}
\tag{B15}
$$

Finally, let us compute $\partial \alpha_\alpha / \partial \partial_{\mathcal{I}} Q^\beta$. For this, as intermediate steps we compute

$$
\begin{aligned}
&\frac{\partial P_B}{\partial \dot{Q}^A} = \frac{\partial^2 \mathcal{L}}{\partial \dot{Q}^A \partial \dot{Q}^B} = W_{AB}, \quad \frac{\partial \partial_k P_B}{\partial \partial_i \dot{Q}^A} = \delta^i_k \frac{\partial P_B}{\partial \dot{Q}^A} = W_{AB} \delta^i_k, \\
&\frac{\partial \partial_k P_B}{\partial \dot{Q}^A} = \partial_k W_{AB}.
\end{aligned}
\tag{B16}
$$

Then we have

$$
\frac{\partial \alpha_\alpha}{\partial \partial_i \dot{Q}^A} = -\frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_i P_B} W_{AB}, \tag{B17}
$$

$$
\frac{\partial \alpha_\alpha}{\partial \dot{Q}^A} = -\frac{\partial \tilde{\Omega}_\alpha}{\partial \partial_i P_B} \partial_i W_{BA} - \frac{\partial \tilde{\Omega}_\alpha}{\partial P_B} W_{BA}. \tag{B18}
$$

Then by substituting Eqs. (B5)–(B7), (B9)–(B10), (B13)–(B15), (B17)–(B18) into Eq. (B3), we obtain

$$
\begin{aligned}
\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} &= -\mathcal{C}_{0\beta\alpha} + \partial_i \mathcal{C}^i_{1\beta\alpha} - \partial_i \partial_j \mathcal{C}^{ij}_{2\beta\alpha}, \\
\frac{\partial \phi_\alpha}{\partial \partial_i \dot{Q}^\beta} &= \mathcal{C}^i_{1\beta\alpha} - 2\partial_j \mathcal{C}^{ij}_{2\beta\alpha}, \\
\frac{\partial \phi_\alpha}{\partial \partial_i \partial_j \dot{Q}^\beta} &= -\mathcal{C}^{ij}_{2\beta\alpha}.
\end{aligned}
\tag{B19}
$$

By using diffeomorphism invariance requirements, Eq. (65) is realised. This simplifies Eq. (B19). Further simplifications are possible. For this, let us note that using Eqs. (B3)–(B9) and diffeomorphism invariance requirements, one obtains

$$\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} - \frac{\partial \phi_\beta}{\partial \dot{Q}^\alpha} + \partial_i \left(\frac{\partial \phi_\beta}{\partial \partial_i \dot{Q}^\alpha} \right) = \frac{\partial \alpha_\alpha}{\partial Q^\beta} - \frac{\partial \alpha_\beta}{\partial Q^\alpha} + \partial_i \left(\frac{\partial \alpha_\beta}{\partial \partial_i Q^\alpha} + \frac{\partial \alpha_\beta}{\partial \dot{Q}_i^\alpha} + \frac{\partial \alpha_\alpha^i}{\partial \dot{Q}^\beta} \right). \tag{B20}$$

Then by expressing α_M in terms of Lagrangian and using diffeomorphism invariance and secondary-constraint enforcing relations, we obtain

$$\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} - \frac{\partial \phi_\beta}{\partial \dot{Q}^\alpha} + \partial_i \left(\frac{\partial \phi_\beta}{\partial \partial_i \dot{Q}^\alpha} \right) = 0, \tag{B21}$$

which is equivalent to the phase space expression

$$\mathcal{C}_{0\alpha\beta} = \mathcal{C}_{0\beta\alpha} - \partial_i \mathcal{C}^i_{1\beta\alpha}. \tag{B22}$$

Finally, this gives

$$\frac{\partial \phi_\alpha}{\partial \dot{Q}^\beta} = -\mathcal{C}_{0\alpha\beta}, \quad \frac{\partial \phi_\alpha}{\partial \partial_i \dot{Q}^\beta} = -\mathcal{C}^i_{1\alpha\beta}, \quad \frac{\partial \phi_\alpha}{\partial \partial_i \partial_j \dot{Q}^\beta} = 0. \tag{B23}$$

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