Theoretical and Mathematical Physics, **218**(3): 357–369 (2024)

ONE-PARAMETER DISCRETE-TIME CALOGERO–MOSER SYSTEM

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We present a new type of integrable one-dimensional many-body systems called a one-parameter Calogero–Moser system. At the discrete level, the Lax pairs with a parameter are introduced and the discrete-time equations of motion are obtained as together with the corresponding discrete-time Lagrangian. The integrability property of this new system can be expressed in terms of the discrete Lagrangian closure relation by using a connection with the temporal Lax matrices of the discrete-time Ruijsenaars–Schneider system, an exact solution, and the existence of a classical r-matrix. As the parameter tends to zero, the standard Calogero–Moser system is recovered in both discrete-time and continuous-time forms.

Keywords: one-parameter, discrete-time Calogero–Moser system, discrete-time Ruijsenaars–Schneider system, closure relation

DOI: 10.1134/S0040577924030012

1. Introduction

The Calogero–Moser (CM) system is a mathematical model that describes the motion in a one-dimensional system of particles interacting via long-range forces [1], [2]. The CM system is an integrable system, which exhibits rich symmetries and has a sufficient number of conserved quantities, according to Liouville's integrability notion, to construct the exact solutions. We give the equations motion of the CM system for the simplest type of interaction, known as the rational case,

$$\ddot{x}_i = \sum_{j=1}^{N} \frac{1}{(x_i - x_j)^3}, \qquad i = 1, \dots, N, \tag{1.1}$$

where x_i is the position of the ith particle.

The Ruijsenaars–Schneider (RS) system is another integrable one-dimensional system of particles with a long-range interaction [3], [4]. In the case of the simplest interaction, namely, the rational case, the equations of motion are given by

$$\ddot{x}_i = \sum_{j=1}^{N} \dot{x}_i \dot{x}_j \left(\frac{1}{x_i - x_j + \lambda} + \frac{1}{x_i - x_j - \lambda} - \frac{2}{x_i - x_j} \right), \qquad i = 1, \dots, N, \tag{1.2}$$

where λ is a parameter. In the limit $\lambda \to 0$, the CM system is recovered. Then the RS system can be treated as a "one-parameter generalization" of the CM system.

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Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 218, No. 3, pp. 415–429, March, 2024. Received June 10, 2023. Revised August 2, 2023. Accepted August 10, 2023.

In 1994, a time-discretized version of the CM system was introduced by Nijhoff and Pang [5]. In the rational case, the discrete-time equations of motion are given by

$$\sum_{k=1}^{N}\left(\frac{1}{x_i-\tilde{x}_k}+\frac{1}{x_i-\underset{\sim}{x}_k}\right)-\sum_{\substack{k=1,\\ k\neq i}}^{N}\frac{2}{x_i-x_k}=0, \tag{1.3}$$

where $\tilde{x}_i = x_i(n+1)$ is a forward shift and $\underset{\sim}{x}_i = x_i(n-1)$ is a backward shift. The integrability of the system can be captured in the same sense as for the continuous system in terms of a classical r-matrix, the existence of exact solutions, and the existence of sufficiently many invariants. Soon after that, the time-discretized version of the RS system was introduced [6]. In the rational case, the discrete-time equations of motion are given by

$$\prod_{\substack{j=1,\\ j\neq i}}^{N}\frac{x_i-x_j+\lambda}{x_i-x_j-\lambda}=\prod_{j=1}^{N}\frac{(x_j-\tilde{x}_j)(x_i-\underset{\sim}{x}_j+\lambda)}{(x_j-\underset{\sim}{x}_j)(x_i-\tilde{x}_j+\lambda)}. \tag{1.4}$$

Again, in the limit $\lambda \to 0$, the discrete-time CM system is recovered. Of course, the discrete-time RS system (1.4) can also be treated as the "one-parameter generalization" of the discrete-time CM system (1.3).

Recently, a new hallmark for integrability was proposed known as the multi-dimensional consistency. On the level of the discrete-time equations of motion, the multi-dimensional consistency can be inferred as the consistency around the cube [7], [8]. On the level of the Hamiltonians, it can be expressed in terms of the Hamiltonian commuting flows as a direct consequence of the involution in Liouville's integrability [9]. Alternatively, on the level of Lagrangians, the multi-dimensional consistency can be expressed in terms of the Lagrangian closure relation as a direct result of the variation of the action with respect to independent variables. Because the closure relation for Lagrangian 1-form plays a major role in this paper as an integrability criterion, we give its derivation here.

We let $\mathbf{n}$ be a vector in the lattice and $\mathbf{e}_i$ be a unit vector in the ith direction. An elementary shift in the ith direction on the lattice is defined as $\mathbf{n} \to \mathbf{n}+\mathbf{e}_i$. Therefore, the discrete-time Lagrangians can be expressed in the form

$$\mathcal{L}_i(\mathbf{n}) = \mathcal{L}_i(\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n}+\mathbf{e}_i)), \tag{1.5}$$

where $\mathbf{x} = \{x_1, \ldots, x_N\}$. The discrete-time action is defined as

$$S = S[\mathbf{x}(\mathbf{n}) : \Gamma] = \sum_{\mathbf{n}\in\Gamma}\mathcal{L}_i(\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n}+\mathbf{e}_i)), \tag{1.6}$$

where Γ is an arbitrary discrete curve (see Fig. 1). We next consider another discrete curve Γ' sharing the same endpoints with Γ, with the action given by

$$S' = S[\mathbf{x}(\mathbf{n}) : \Gamma'] = \sum_{\mathbf{n}\in\Gamma'}\mathcal{L}_i(\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n}+\mathbf{e}_i)). \tag{1.7}$$

Of course, this can be viewed as the variation of the independent variables $\mathbf{n} \to \mathbf{n}+\Delta\mathbf{n}$ of the action

$$\begin{aligned} S' = S &- \mathcal{L}_i(\mathbf{x}(\mathbf{n}+\mathbf{e}_j), \mathbf{x}(\mathbf{n}+\mathbf{e}_i+\mathbf{e}_j)) + \mathcal{L}_i(\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n}+\mathbf{e}_i)) + \\ &+ \mathcal{L}_j(\mathbf{x}(\mathbf{n}+\mathbf{e}_i), \mathbf{x}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_i)) - \mathcal{L}_j(\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n}+\mathbf{e}_j)). \end{aligned} \tag{1.8}$$

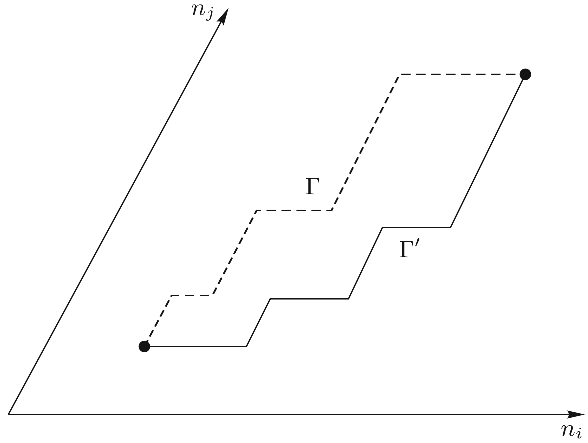

Fig. 1. Arbitrary curves on the space of independent discrete variables.

The least action principle requires that $\delta S = S' - S = 0$, whence

$$
\begin{aligned}
0 = {} & \mathcal{L}_i(\mathbf{x}(\mathbf{n}+\mathbf{e}_j), \mathbf{x}(\mathbf{n}+\mathbf{e}_i+\mathbf{e}_j)) - \mathcal{L}_i(\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n}+\mathbf{e}_i)) - \\
& - \mathcal{L}_j(\mathbf{x}(\mathbf{n}+\mathbf{e}_i), \mathbf{x}(\mathbf{n}+\mathbf{e}_j+\mathbf{e}_i)) + \mathcal{L}_j(\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n}+\mathbf{e}_j)),
\end{aligned} \tag{1.9}
$$

which is the closure relation for the discrete-time Lagrangian 1-form.

Equivalently, for a two-dimensional lattice (see Fig. 2), Eq. (1.9) can be reexpressed in the form

$$
\widehat{\mathcal{L}(\mathbf{x},\widetilde{\mathbf{x}})} - \mathcal{L}(\mathbf{x},\widetilde{\mathbf{x}}) - \widetilde{\mathcal{L}(\mathbf{x},\widehat{\mathbf{x}})} + \mathcal{L}(\mathbf{x},\widehat{\mathbf{x}}) = 0. \tag{1.10}
$$

In this paper, we propose a new type of one-parameter CM systems, besides the RS system, and study its integrability in terms of the existence of an exact solution, a classical r-matrix, and the closure relation. The structure of this paper is as follows. In Sec. 2, the two compatible one-parameter discrete-time CM systems are obtained from the Lax equations. In Sec. 3, the discrete-time Lagrangians are also established and the closure relation is directly obtained via the connection between the RS temporal Lax matrices and the Lagrangian. In Sec. 4, the classical r-matrix for the one-parameter discrete-time CM system is considered. In Sec. 5, an exact solution is derived. In Sec. 6, the continuum limit is performed in the one-parameter discrete-time CM system, resulting in the one-parameter continuous-time CM system. In Sec. 7, we summarize and discuss possible further investigations.

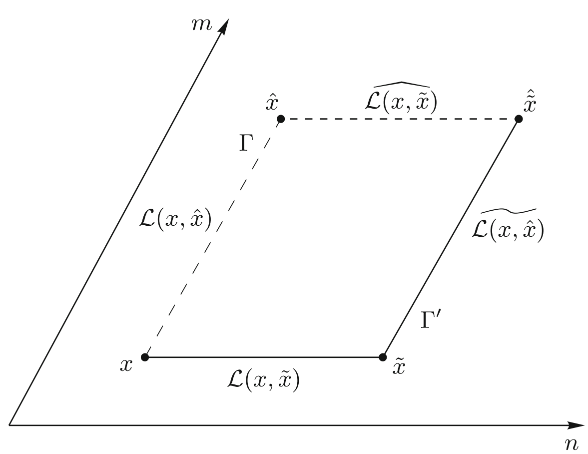

Fig. 2. Local variation of a discrete curve on the space of two independent variables.

2. One-parameter discrete-time CM system

In this section, we construct the discrete-time CM system with a parameter λ. First, we introduce the spatial Lax matrix $\mathbf{L}_\lambda$ with two temporal matrices $\mathbf{M}$ and $\mathbf{N}$ as

$$\mathbf{L}_\lambda = \sum_{i,j=1}^{N} \frac{1}{x_i - x_j + \lambda} E_{ij}, \tag{2.1a}$$

$$\mathbf{M} = \sum_{i,j=1}^{N} \frac{1}{\tilde{x}_i - x_j} E_{ij}, \tag{2.1b}$$

$$\mathbf{N} = \sum_{i,j=1}^{N} \frac{1}{\hat{x}_i - x_j} E_{ij}, \tag{2.1c}$$

where $x_i = x_i(n,m)$ is the position of the ith particle, N is the number of particles in the system, and E_{ij} is the matrix with the entries $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Here, $\hat{x}_i = x_i(m+1)$ is a forward shift and $\underset{\hat{}}{x}_i = x_i(m-1)$ is a backward shift.

Discrete flow-n direction. The compatibility between (2.1a) and (2.1b) gives

$$\widetilde{\mathbf{L}_\lambda}\mathbf{M} = \mathbf{M}\mathbf{L}_\lambda.$$

This gives a set of equations

$$\sum_{i,j=1}^{N}\sum_{k,\ell=1}^{N} \frac{1}{(\tilde{x}_i - \tilde{x}_j + \lambda)} \frac{1}{(\tilde{x}_k - x_\ell)} E_{ij}E_{k\ell} = \sum_{i,j=1}^{N}\sum_{k,\ell=1}^{N} \frac{1}{(\tilde{x}_i - x_j)} \frac{1}{(x_k - x_\ell + \lambda)} E_{ij}E_{k\ell},$$
$$\sum_{i,\ell=1}^{N}\sum_{k=1}^{N} \frac{1}{(\tilde{x}_i - \tilde{x}_k + \lambda)(\tilde{x}_k - x_\ell)} E_{i\ell} = \sum_{i,\ell=1}^{N}\sum_{k=1}^{N} \frac{1}{(\tilde{x}_i - x_k)(x_k - x_\ell + \lambda)} E_{k\ell}.$$

Canceling common factors, we obtain

$$\sum_{k=1}^{N}\left(\frac{1}{\tilde{x}_i - \tilde{x}_k + \lambda} - \frac{1}{\tilde{x}_i - x_k}\right) = \sum_{k=1}^{N}\left(\frac{1}{x_k - x_\ell + \lambda} - \frac{1}{\tilde{x}_k - x_\ell}\right). \tag{2.2}$$

We see that both sides of Eq. (2.2) are independent, and therefore is holds if

$$\sum_{k=1}^{N}\left(\frac{1}{\tilde{x}_i - \tilde{x}_k + \lambda} - \frac{1}{\tilde{x}_i - x_k}\right) \equiv \widetilde{p}, \tag{2.3}$$

where $p = p(n)$ is independent of the particle indices and is a function of the discrete time n. Taking a backward shift in (2.3), we obtain

$$\sum_{k=1}^{N}\left(\frac{1}{x_i - x_k + \lambda} - \frac{1}{x_i - \underset{\sim}{x}_k}\right) = p. \tag{2.4}$$

Automatically, in the right-hand side of (2.2), we have

$$p = \sum_{k=1}^{N}\left(\frac{1}{x_\ell - \tilde{x}_k} - \frac{1}{x_\ell - x_k - \lambda}\right). \tag{2.5}$$

From (2.4) and (2.5), we then readily see that

$$\sum_{k=1}^{N}\left(\frac{1}{x_i-\tilde{x}_k}+\frac{1}{x_i-\underset{\sim}{x}_k}\right)-\sum_{k=1}^{N}\left(\frac{1}{x_i-x_k+\lambda}+\frac{1}{x_i-x_k-\lambda}\right)=0, \tag{2.6}$$

which we regard as a one-parameter discrete-time CM system in the n-direction. In the limit $\lambda\to 0$, we obtain

$$\sum_{k=1}^{N}\left(\frac{1}{x_i-\tilde{x}_k}+\frac{1}{x_i-\underset{\sim}{x}_k}\right)-\sum_{\substack{k=1,\\ k\neq i}}^{N}\frac{2}{x_i-x_k}=0, \tag{2.7}$$

which is nothing but a standard discrete-time CM system in the n direction.

Discrete flow-*m* direction. The compatibility between (2.1a) and (2.1c) gives

$$\widehat{\mathbf{L}_\lambda}\mathbf{M}=\mathbf{M}\mathbf{L}_\lambda.$$

This gives a set of equations

$$\sum_{i,j=1}^{N}\sum_{k,\ell=1}^{N}\frac{1}{(\hat{x}_i-\hat{x}_j+\lambda)}\frac{1}{(\hat{x}_k-x_\ell)}E_{ij}E_{k\ell}=\sum_{i,j=1}^{N}\sum_{k,\ell=1}^{N}\frac{1}{(\hat{x}_i-x_j)}\frac{1}{(x_k-x_\ell+\lambda)}E_{ij}E_{k\ell},$$

$$\sum_{i,\ell=1}^{N}\sum_{k=1}^{N}\frac{1}{(\hat{x}_i-\hat{x}_k+\lambda)(\hat{x}_k-x_\ell)}E_{i\ell}=\sum_{i,\ell=1}^{N}\sum_{k=1}^{N}\frac{1}{(\hat{x}_i-x_k)(x_k-x_\ell+\lambda)}E_{k\ell}.$$

Again, canceling common factors, we obtain

$$\sum_{k=1}^{N}\left(\frac{1}{\hat{x}_i-\hat{x}_k+\lambda}-\frac{1}{\hat{x}_i-x_k}\right)=\sum_{k=1}^{N}\left(\frac{1}{x_k-x_\ell+\lambda}-\frac{1}{\hat{x}_k-x_\ell}\right). \tag{2.8}$$

The situation is similar to the preceding one. Both sides of Eq. (2.8) are independent, and it holds if

$$\sum_{k=1}^{N}\left(\frac{1}{\hat{x}_i-\hat{x}_k+\lambda}-\frac{1}{\hat{x}_i-x_k}\right)\equiv\hat{q}, \tag{2.9}$$

where $q=q(m)$ is independent of particle indices and is a function of the discrete time m. Taking a backward shift in (2.9), we obtain

$$\sum_{k=1}^{N}\left(\frac{1}{x_i-x_k+\lambda}-\frac{1}{x_i-\underset{\wedge}{x}_k}\right)=q. \tag{2.10}$$

From the right-hand side of (2.8), we then have

$$q=\sum_{k=1}^{N}\left(\frac{1}{x_\ell-\hat{x}_k}-\frac{1}{x_\ell-x_k-\lambda}\right). \tag{2.11}$$

Therefore, Eqs. (2.10) and (2.11) give

$$\sum_{k=1}^{N}\left(\frac{1}{x_i-\hat{x}_k}+\frac{1}{x_i-\underset{\wedge}{x}_k}\right)-\sum_{k=1}^{N}\left(\frac{1}{x_i-x_k+\lambda}+\frac{1}{x_i-x_k-\lambda}\right)=0, \tag{2.12}$$

which we regard as a one-parameter discrete-time CM system in the m-direction. In the limit $\lambda\to 0$, we obtain the equation

$$\sum_{k=1}^{N}\left(\frac{1}{x_i-\hat{x}_k}+\frac{1}{x_i-\underset{\wedge}{x}_k}\right)-\sum_{\substack{k=1,\\ k\neq i}}^{N}\frac{2}{x_i-x_k}=0, \tag{2.13}$$

which is a discrete-time CM system in the m direction.

Commutativity between discrete flows. Two discrete-time dynamics are consistent if the compatibility between (2.1b) and (2.1c) holds in the form

$$\widehat{\mathbf{M}}\mathbf{N} = \widetilde{\mathbf{N}}\mathbf{M}.$$

This gives a set of equations

$$p - q = \sum_{k=1}^{N}\left(\frac{1}{x_i - \tilde{x}_k} - \frac{1}{x_i - \hat{x}_k}\right), \tag{2.14}$$

$$p - q = \sum_{k=1}^{N}\left(\frac{1}{x_i - \underset{\sim}{x}_k} - \frac{1}{x_i - \underset{\wedge}{x}_k}\right), \tag{2.15}$$

which we call corner equations. Comparing (2.14) and (2.15), we obtain

$$\sum_{k=1}^{N}\left(\frac{1}{x_i - \tilde{x}_k} + \frac{1}{x_i - \underset{\sim}{x}_k}\right) = \sum_{k=1}^{N}\left(\frac{1}{x_i - \hat{x}_k} - \frac{1}{x_i - \underset{\wedge}{x}_k}\right), \tag{2.16}$$

which is a constraint equation relating two discrete flows.

3. Integrability: the closure relation

In this section, we show that the one-parameter discrete-time CM systems in the preceding section are integrable in the sense that their discrete-time Lagrangians satisfy the closure relation as a consequence of the least action principle with respect to the independent variables [10]–[17].

It is not difficult to see that Eqs. (2.6) and (2.12) can be obtained from the discrete Euler–Lagrange equations [12]

$$\frac{\partial \widetilde{\mathcal{L}_n(x,\tilde{x})}}{\partial x_i} + \frac{\partial \mathcal{L}_n(x,\tilde{x})}{\partial \tilde{x}_i} = 0, \qquad \frac{\partial \widehat{\mathcal{L}_m(x,\hat{x})}}{\partial x_i} + \frac{\partial \mathcal{L}_m(x,\hat{x})}{\partial \hat{x}_i} = 0, \tag{3.1}$$

where

$$\begin{aligned}\mathcal{L}_n(x,\tilde{x}) &= -\sum_{i,j=1}^{N} \ln|x_i - \tilde{x}_j| + \sum_{i,j=1}^{N} \ln|x_i - x_j + \lambda| + p(\Xi - \widetilde{\Xi}),\\ \mathcal{L}_m(x,\hat{x}) &= -\sum_{i,j=1}^{N} \ln|x_i - \hat{x}_j| + \sum_{i,j=1}^{N} \ln|x_i - x_j + \lambda| + q(\Xi - \widehat{\Xi}).\end{aligned} \tag{3.2}$$

Here, $\Xi = \sum_{i=1}^{N} x_i$ is the center-of-mass variable.

To show that the Lagrangian closure relation for the one-parameter discrete-time CM model holds, we use a connection between the temporal Lax matrix and the Lagrangian, as we did have in the case of the standard discrete-time CM model [12]. An interesting point is that in the system under discussion, we can obtain the discrete-time Lagrangian from the relation $\mathcal{L}(x,\tilde{x}) = \ln|\det \mathbf{M}_{\mathrm{RS}}|$ (see the appendix for an explicit computation), where $\mathbf{M}_{\mathrm{RS}}$ is a temporal matrix for the RS model given by

$$\mathbf{M}_{\mathrm{RS}} = \sum_{i,j=1}^{N} \frac{\tilde{h}_i h_j}{\tilde{x}_i - x_j + \lambda} E_{ij}, \tag{3.3}$$

where $h_i = h_i(n,m)$ are auxiliary variables, which can be determined [6]. We suppose that there is another temporal matrix given by

$$\mathbf{N}_{\mathrm{RS}} = \sum_{i,j=1}^{N} \frac{\hat{h}_i h_j}{\hat{x}_i - x_j + \lambda} E_{ij}, \tag{3.4}$$

and the matrices $\mathbf{M}_{\mathrm{RS}}$ and $\mathbf{N}_{\mathrm{RS}}$ satisfy the relation

$$\widehat{\mathbf{M}}_{\mathrm{RS}}\mathbf{N}_{\mathrm{RS}} = \widetilde{\mathbf{N}}_{\mathrm{RS}}\mathbf{M}_{\mathrm{RS}}. \tag{3.5}$$

Calculating the determinant and taking the logarithm, we obtain

$$\ln|\det\widehat{\mathbf{M}}_{\mathrm{RS}}| + \ln|\det\mathbf{N}_{\mathrm{RS}}| = \ln|\det\widetilde{\mathbf{N}}_{\mathrm{RS}}| + \ln|\det\mathbf{M}_{\mathrm{RS}}|, \tag{3.6}$$

which yields closure relation (1.10).

4. Integrability: the classical r-matrix

In this section, we construct the classical r-matrix for the one-parameter discrete-time CM system. We first rewrite the spatial Lax matrix as

$$\mathbf{L}_\lambda = \sum_{i=1}^{N}\frac{1}{\lambda}E_{ii} - \sum_{\substack{i,j=1,\\ j\neq i}}^{N}\frac{1}{x_i - x_j + \lambda}E_{ij}. \tag{4.1}$$

Next, we recall the spatial Lax matrix of the standard CM system [18] given by

$$\mathbf{L} = \sum_{i=1}^{N}P_iE_{ii} - \sum_{\substack{i,j=1,\\ j\neq i}}^{N}\frac{1}{x_i - x_j}E_{ij}, \tag{4.2}$$

where P_i is the momentum variable for ith particle. With this structure, we find that the classical r-matrix can be computed from the relation

$$\{\mathbf{L} \overset{\otimes}{,} \mathbf{L}\} = [r_{12}, \mathbf{L}\otimes \mathbb{1}] - [r_{12}, \mathbb{1}\otimes\mathbf{L}], \tag{4.3}$$

where r_{12} is the classical r-matrix for the CM system. Comparing (4.1) with (4.2), we immediately find the classical r-matrix r_{12}^{λ} for the one-parameter discrete-time CM system by replacing $P_i \to \frac{1}{\lambda}$ and $\frac{1}{x_i - x_j} \to \frac{1}{x_i - x_j + \lambda}$:

$$\{\mathbf{L}_\lambda \overset{\otimes}{,} \mathbf{L}_\lambda\} = [r_{12}^{\lambda}, \mathbf{L}_\lambda\otimes \mathbb{1}] - [r_{12}^{\lambda}, \mathbb{1}\otimes\mathbf{L}_\lambda]. \tag{4.4}$$

We note that in the limit $\lambda \to 0$, the classical r-matrix r_{12}^{λ} does not yield the standard classical r-matrix. This problem arises from the fact that the spatial Lax matrix (4.1) is a fake one because it does not provide the integrals of motion via the relation $I_n = \frac{1}{n!}\,\mathrm{Tr}(\mathbf{L}_\lambda)^n$.

5. Integrability: an exact solution

In this section, we construct the exact solution $\{x_i(n)\}$ with initial values $\{x_i(0)\}$ and $\{x_i(1) = \tilde{x}_i(0)\}$. We first rewrite the Lax matrices as

$$\mathbf{XL} - \mathbf{LX} + \lambda\mathbf{L} = \mathbf{E}, \tag{5.1}$$

$$\widetilde{\mathbf{X}}\mathbf{M} - \mathbf{MX} = \mathbf{E}, \tag{5.2}$$

where $\mathbf{X} = \sum_{i=1}^{N} x_iE_{ii}$ and $\mathbf{E} = \sum_{i=1}^{N} E_{ij}$. In addition, we have

$$(\widetilde{\mathbf{L}} - \mathbf{M})\mathbf{E} = 0, \tag{5.3}$$

$$\mathbf{E}(\mathbf{L} - \mathbf{M}) = 0, \tag{5.4}$$

which also gives the equations of motion. We set $\mathbf{M} = \widetilde{\mathbf{U}}\mathbf{U}^{-1}$ and $\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}$, where $\mathbf{U}$ is an invertible matrix. Equation(5.2) then leads to

$$
\begin{aligned}
&\widetilde{\mathbf{X}}\widetilde{\mathbf{U}}\mathbf{U}^{-1} - \widetilde{\mathbf{U}}\mathbf{U}^{-1}\mathbf{X} = \mathbf{E},\\
&\mathbf{U}^{-1}\widetilde{\mathbf{X}}\widetilde{\mathbf{U}}\mathbf{U}^{-1} - \widetilde{\mathbf{U}}^{-1}\widetilde{\mathbf{U}}\mathbf{U}^{-1}\mathbf{X} = \widetilde{\mathbf{U}}^{-1}\mathbf{E},\\
&\widetilde{\mathbf{U}}^{-1}\widetilde{\mathbf{X}}\widetilde{\mathbf{U}}\mathbf{U}^{-1}\mathbf{U} - \mathbf{U}^{-1}\mathbf{X}\mathbf{U} = \widetilde{\mathbf{U}}^{-1}\mathbf{E}\mathbf{U},\\
&\widetilde{\mathbf{U}}^{-1}\widetilde{\mathbf{X}}\widetilde{\mathbf{U}} - \mathbf{U}^{-1}\mathbf{X}\mathbf{U} = \widetilde{\mathbf{U}}^{-1}\mathbf{E}\mathbf{U},\\
&\widetilde{\mathbf{Y}} - \mathbf{Y} = \widetilde{\mathbf{U}}^{-1}\mathbf{E}\mathbf{U},
\end{aligned}
\tag{5.5}
$$

where $\mathbf{Y} = \mathbf{U}^{-1}\mathbf{X}\mathbf{U}$. We also find that (5.1) gives

$$
\begin{aligned}
&\mathbf{X}\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1} - \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}\mathbf{X} + \lambda\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1} = \mathbf{E},\\
&\mathbf{X}\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}\mathbf{U} - \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}\mathbf{X}\mathbf{U} + \lambda\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}\mathbf{U} = \mathbf{E}\mathbf{U},\\
&\mathbf{U}^{-1}\mathbf{X}\mathbf{U}\boldsymbol{\Lambda} - \mathbf{U}^{-1}\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}\mathbf{X}\mathbf{U} + \mathbf{U}^{-1}\lambda\mathbf{U}\boldsymbol{\Lambda} = \mathbf{U}^{-1}\mathbf{E}\mathbf{U},\\
&\mathbf{U}^{-1}\mathbf{X}\mathbf{U}\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\mathbf{U}^{-1}\mathbf{X}\mathbf{U} + \lambda\mathbf{U}^{-1}\mathbf{U}\boldsymbol{\Lambda} = \mathbf{U}^{-1}\mathbf{E}\mathbf{U},\\
&\mathbf{U}^{-1}\mathbf{X}\mathbf{U}\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\mathbf{U}^{-1}\mathbf{X}\mathbf{U} + \lambda\boldsymbol{\Lambda} = \mathbf{U}^{-1}\mathbf{E}\mathbf{U},\\
&\mathbf{Y}\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\mathbf{Y} + \lambda\boldsymbol{\Lambda} = \mathbf{U}^{-1}\mathbf{E}\mathbf{U},
\end{aligned}
\tag{5.6}
$$

and (5.3) gives

$$
\begin{aligned}
&(\widetilde{\mathbf{U}}\boldsymbol{\Lambda}\widetilde{\mathbf{U}}^{-1} - \widetilde{\mathbf{U}}\mathbf{U}^{-1})\mathbf{E} = 0,\\
&\widetilde{\mathbf{U}}\boldsymbol{\Lambda}\widetilde{\mathbf{U}}^{-1}\mathbf{E} - \widetilde{\mathbf{U}}\mathbf{U}^{-1}\mathbf{E} = 0,\\
&\widetilde{\mathbf{U}}^{-1}\widetilde{\mathbf{U}}\boldsymbol{\Lambda}\widetilde{\mathbf{U}}^{-1}\mathbf{E} - \widetilde{\mathbf{U}}^{-1}\widetilde{\mathbf{U}}\mathbf{U}^{-1}\mathbf{E} = 0,\\
&\boldsymbol{\Lambda}\widetilde{\mathbf{U}}^{-1}\mathbf{E} - \mathbf{U}^{-1}\mathbf{E} = 0,\\
&\mathbf{U}^{-1}\mathbf{E}\mathbf{U} = \boldsymbol{\Lambda}\widetilde{\mathbf{U}}^{-1}\mathbf{E}\mathbf{U}.
\end{aligned}
\tag{5.7}
$$

Substituting (5.7) in (5.6), we obtain

$$
\mathbf{Y}\boldsymbol{\Lambda} - \boldsymbol{\Lambda}\mathbf{Y} + \lambda\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\widetilde{\mathbf{U}}^{-1}\mathbf{E}\mathbf{U}. \tag{5.8}
$$

To eliminate the invertible matrix $\mathbf{U}$ and $\mathbf{E}$ in the right-hand side of (5.8), we use Eq. (5.4), which can be expressed in the form

$$
\begin{aligned}
&\mathbf{E}(\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1} - \widetilde{\mathbf{U}}\mathbf{U}^{-1}) = 0,\\
&\mathbf{E}\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1} - \mathbf{E}\widetilde{\mathbf{U}}\mathbf{U}^{-1} = 0,\\
&\mathbf{E}\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}\mathbf{U} - \mathbf{E}\widetilde{\mathbf{U}}\mathbf{U}^{-1}\mathbf{U} = 0,\\
&\mathbf{E}\mathbf{U}\boldsymbol{\Lambda} - \mathbf{E}\widetilde{\mathbf{U}} = 0,\\
&\mathbf{U}^{-1}\mathbf{E}\widetilde{\mathbf{U}} = \mathbf{U}^{-1}\mathbf{E}\mathbf{U}\boldsymbol{\Lambda}.
\end{aligned}
\tag{5.9}
$$

Because $\mathbf{U}^{-1}\mathbf{E}\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}^{-1}\mathbf{E}\mathbf{U}$, we have

$$
\widetilde{\mathbf{U}}^{-1}\mathbf{E}\mathbf{U} = \mathbf{U}^{-1}\mathbf{E}\mathbf{U}\boldsymbol{\Lambda}. \tag{5.10}
$$

Substituting (5.9) in (5.5), we find

$$
\widetilde{\mathbf{Y}} - \mathbf{Y} = \mathbf{U}^{-1}\mathbf{E}\mathbf{U}\boldsymbol{\Lambda}. \tag{5.11}
$$

Rearranging (5.8), we obtain

$$
\begin{aligned}
&\boldsymbol{\Lambda}^{-1}\mathbf{Y}\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}\mathbf{Y} + \boldsymbol{\Lambda}^{-1}\lambda\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}\widetilde{\mathbf{U}}^{-1}\mathbf{E}\mathbf{U},\\
&\boldsymbol{\Lambda}^{-1}\mathbf{Y}\boldsymbol{\Lambda} - \mathbf{Y} + \lambda = \widetilde{\mathbf{U}}^{-1}\mathbf{E}\mathbf{U}.
\end{aligned}
\tag{5.12}
$$

Substituting (5.5) in (5.12), we obtain

$$\mathbf{\Lambda}^{-1}\mathbf{Y}\mathbf{\Lambda} - \mathbf{Y} + \lambda = \widetilde{\mathbf{Y}}\mathbf{Y}, \qquad \widetilde{\mathbf{Y}} = \mathbf{\Lambda}^{-1}\mathbf{Y}\mathbf{\Lambda} + \lambda. \tag{5.13}$$

Hence, if we proceed in steps over n, we find that

$$\widetilde{\widetilde{\mathbf{Y}}} = \mathbf{\Lambda}^{-1}\widetilde{\mathbf{Y}}\mathbf{\Lambda} + \lambda = \mathbf{\Lambda}^{-1}[\mathbf{\Lambda}^{-1}\mathbf{Y}\mathbf{\Lambda} + \lambda]\mathbf{\Lambda} + \lambda = (\mathbf{\Lambda}^{-1})^2\mathbf{Y}\mathbf{\Lambda}^2 + 2\lambda$$

and so on, with $\mathbf{Y}(n) = (\mathbf{\Lambda})^{-n}\mathbf{Y}\mathbf{\Lambda}^n + n\lambda$. Of course, for the m-steps,

$$\mathbf{Y}(m) = (\mathbf{\Lambda})^{-m}\mathbf{Y}\mathbf{\Lambda}^m + m\lambda. \tag{5.14}$$

Then, for arbitrary (n,m)-steps, we have

$$\mathbf{Y}(n,m) = (p+\mathbf{\Lambda})^{-n}(q+\mathbf{\Lambda})^{-m}\mathbf{Y}(0,0)(q+\mathbf{\Lambda})^m + (p+\mathbf{\Lambda})^n + (n+m)\lambda. \tag{5.15}$$

It is not difficult to show that in the limit $\lambda \to 0$, we obtain the solution

$$\mathbf{Y}(n,m) = (p+\mathbf{\Lambda})^{-n}(q+\mathbf{\Lambda})^{-m}\mathbf{Y}(0,0)(q+\mathbf{\Lambda})^m(p+\mathbf{\Lambda})^n, \tag{5.16}$$

which is nothing but a standard solution of the discrete-time CM system [5].

6. The continuum limit

In this section, we consider the continuum limit of the one-parameter discrete-time CM system discussed in the previous sections. Because there are two discrete-time variables (n,m), we can perform a naive continuum limit [5] with respect to each of these variables, resulting in a one-parameter continuous-time CM system. To proceed with such a continuum limit, we define $x_i = Z_i + n\Delta$, where Δ is a small parameter. Consequently, we also have $\tilde{x}_i = \widetilde{Z}_i + (n+1)\Delta$ and $\underset{\sim}{x}_i = \underset{\sim}{Z}_i + (n-1)\Delta$. Then Eq. (2.6) becomes

$$\sum_{k=1}^{N}\left(\frac{1}{Z_i - \widetilde{Z}_k - \Delta} + \frac{1}{Z_i - \underset{\sim}{Z}_k + \Delta}\right) - \sum_{\substack{i,k=1,\\ k\neq i}}^{N}\left(\frac{1}{Z_i - Z_k + \lambda} + \frac{1}{Z_i - Z_k - \lambda}\right) = 0 \tag{6.1}$$

or

$$\begin{aligned}&\left(\frac{1}{Z_i - \widetilde{Z}_i - \Delta} + \frac{1}{Z_i - \underset{\sim}{Z}_i + \Delta}\right) - \\ &- \sum_{\substack{i,k=1,\\ k\neq i}}^{N}\left(\frac{1}{Z_i - \widetilde{Z}_k - \Delta} + \frac{1}{Z_i - \underset{\sim}{Z}_k + \Delta} - \frac{1}{Z_i - Z_k + \lambda} - \frac{1}{Z_i - Z_k - \lambda}\right) = 0.\end{aligned} \tag{6.2}$$

Expanding, we obtain

$$\begin{aligned}\widetilde{Z}_i &= Z_i + \varepsilon\frac{dZ_i}{dt} + \frac{\varepsilon^2}{2}\frac{d^2Z_i}{dt^2} + \cdots, \\ \underset{\sim}{Z}_i &= Z_i - \varepsilon\frac{dZ_i}{dt} + \frac{\varepsilon^2}{2}\frac{d^2Z_i}{dt^2} + \cdots,\end{aligned} \tag{6.3}$$

where ε is the time-step parameter. Then the first two terms in (6.2) can be expressed in the form

$$\frac{1}{Z_i - \widetilde{Z}_i - \Delta} + \frac{1}{Z_i - \underset{\sim}{Z}_i + \Delta} = \frac{\varepsilon^2}{\Delta^2}\frac{d^2Z_i}{dt^2} + \cdots. \tag{6.4}$$

We also find that

$$\sum_{\substack{k=1,\\k\neq i}}^{N}\left(\frac{1}{Z_i-\widetilde{Z}_k-\Delta}+\frac{1}{Z_i-\underset{\sim}{Z}_k+\Delta}\right)=$$
$$=\sum_{\substack{k=1,\\k\neq i}}^{N}\left(\frac{2}{Z_i-Z_k}+\frac{1}{(Z_i-Z_k)^3}\left(\varepsilon^2\frac{dZ_k}{dt}+2\varepsilon\Delta\frac{dZ_k}{dt}+\Delta^2\right)+\cdots\right). \tag{6.5}$$

If $\varepsilon\approx\Delta^2$, then

$$\sum_{\substack{k=1,\\k\neq i}}^{N}\left(\frac{1}{Z_i-\widetilde{Z}_k-\Delta}+\frac{1}{Z_i-\underset{\sim}{Z}_k+\Delta}\right)\approx\sum_{\substack{k=1,\\k\neq i}}^{N}\left(\frac{2}{Z_i-Z_k}+\frac{2\Delta^2}{(Z_i-Z_k)^3}\right). \tag{6.6}$$

Finally, the continuous version of the one-parameter CM system is given by

$$\frac{d^2Z_i}{dt^2}+\sum_{\substack{i,k=1,\\k\neq i}}^{N}\left(g'\left[\frac{2}{Z_i-Z_k}-\frac{1}{Z_i-Z_k+\lambda}-\frac{1}{Z_i-Z_k-\lambda}\right]+\frac{2g}{(Z_i-Z_k)^3}\right)=0, \tag{6.7}$$

where $g\equiv\Delta^4/\varepsilon^2$ and $g'\equiv\Delta^2/\varepsilon^2$. Therefore, in the limit $\lambda\to 0$, we have

$$\frac{d^2Z_i}{dt^2}+2g\sum_{\substack{i,k=1,\\k\neq i}}^{N}\frac{1}{(Z_i-Z_k)^3}=0, \tag{6.8}$$

which is actually the standard continuous CM system.

With (6.7), the Lagrangian is given by

$$\mathscr{L}_\lambda=\sum_{i=1}^{N}\frac{\partial Z_i}{\partial t}-\frac{1}{2}\sum_{\substack{i,k=1,\\k\neq i}}^{N}\frac{g}{(Z_i-Z_k)^2}-g'\sum_{\substack{i,k=1,\\k\neq i}}^{N}\big(\ln|Z_i-Z_k+\lambda|+\ln|(Z_i-Z_k)|\big) \tag{6.9}$$

with the Euler–Lagrange equation

$$\frac{\partial\mathscr{L}_\lambda}{\partial Z_i}-\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}_\lambda}{\partial\left(\frac{\partial Z_i}{\partial t}\right)}\right)=0. \tag{6.10}$$

Of course, in the limit $\lambda\to 0$,

$$\lim_{\lambda\to 0}\mathscr{L}_\lambda=\mathscr{L}=\sum_{i=1}^{N}\frac{\partial Z_i}{\partial t}+\sum_{\substack{i,k=1,\\k\neq i}}^{N}\frac{g}{(Z_i-Z_k)^2}, \tag{6.11}$$

the standard Lagrangian for the CM system is recovered.[1] In addition, the Hamiltonian of the one-parameter continuous-time CM system can be written in the form

$$\mathscr{H}_\lambda=\sum_{i=1}^{N}P_i^2+\frac{1}{2}\sum_{\substack{i,k=1,\\k\neq i}}^{N}\frac{g}{(Z_i-Z_k)^2}+g'\sum_{\substack{i,k=1,\\k\neq i}}^{N}\big(\ln|Z_i-Z_k+\lambda|+\ln|Z_i-Z_k|\big), \tag{6.12}$$

where $P_i=\partial Z_i/\partial t$ is the momentum variable for the ith particle.

[1] We note that the CM system in this equation comes with the opposite sign compared with the standard one.

7. Summary

In this paper, we proposed a new type of integrable one-dimensional many-body system called a one-parameter or deformed discrete-time CM system. In the limit $\lambda \to 0$, the standard CM system is recovered in both discrete and continuous cases. In Fig. 3, we provide a diagram of the connections among all CM-type systems. We would rank our model on the same level as the RS system because both systems contain a parameter.

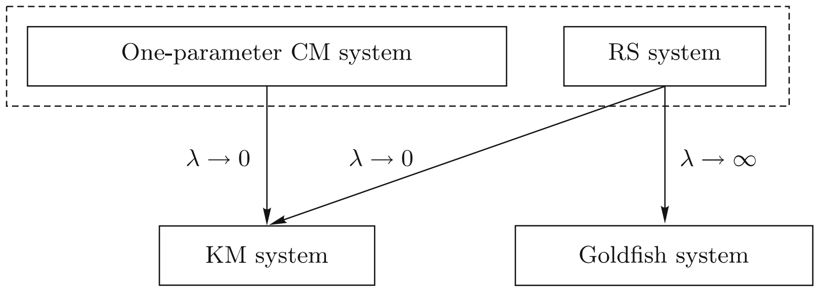

Fig. 3. Connection between the one-parameter CM, RS, KM, and Goldfish systems.

We also note that the continuous system obtained in Sec. 6 is just the first one in CM hierarchy [12]. A question then arises: how do the other systems deformed in the hierarchy deform? Moreover, one also can try to study the integrability condition and the quantum properties of the system. Further investigation is needed, and we shall address these points elsewhere.

Appendix: The connection between the Lagrangian and the $\mathbf{M}_{\mathrm{RS}}$ matrix of the RS model

In this appendix, we derive the connection between the one-parameter discrete-time Lagrangian and the RS model matrix

$$\mathbf{M}_{\mathrm{RS}} = \sum_{i,j=1}^{N} \frac{\tilde{h}_i h_j}{\tilde{x}_i - x_j + \lambda} E_{ij}. \tag{A.1}$$

For simplicity, we start with the case of a 2×2 matrix given by

$$\mathbf{M}_{\mathrm{RS}} = \begin{bmatrix} \dfrac{\tilde{h}_1 h_1}{\tilde{x}_1 - x_1 + \lambda} & \dfrac{\tilde{h}_1 h_2}{\tilde{x}_1 - x_2 + \lambda} \\ \dfrac{\tilde{h}_2 h_1}{\tilde{x}_2 - x_1 + \lambda} & \dfrac{\tilde{h}_2 h_2}{\tilde{x}_2 - x_2 + \lambda} \end{bmatrix}.$$

We then compute the determinant

$$\begin{aligned}\det \mathbf{M}_{\mathrm{RS}} &= \frac{\tilde{h}_1 h_1 \tilde{h}_2 h_2}{(\tilde{x}_1 - x_1 + \lambda)(\tilde{x}_2 - x_2 + \lambda)} - \frac{\tilde{h}_2 h_1 \tilde{h}_1 h_2}{(\tilde{x}_2 - x_1 + \lambda)(\tilde{x}_1 - x_2 + \lambda)} = \\ &= h_1 \tilde{h}_1 h_2 \tilde{h}_2 \left[\frac{1}{(\tilde{x}_1 - x_1 + \lambda)(\tilde{x}_2 - x_2 + \lambda)} - \frac{1}{(\tilde{x}_2 - x_1 + \lambda)(\tilde{x}_1 - x_2 + \lambda)} \right] = \\ &= h_1 \tilde{h}_1 h_2 \tilde{h}_2 \left[\frac{(\tilde{x}_2 - x_1 + \lambda)(\tilde{x}_1 - x_2 + \lambda) - (\tilde{x}_1 - x_1 + \lambda)(\tilde{x}_2 - x_2 + \lambda)}{\prod_{i,j=1,2} (\tilde{x}_i - x_j + \lambda)} \right].\end{aligned}$$

This equation can be further simplified as follows:

$$\det \mathbf{M}_{\mathrm{RS}} = h_1 \tilde{h}_1 h_2 \tilde{h}_2 \left[\frac{(\tilde{x}_2 - \tilde{x}_1)(x_1 - x_2)}{\prod_{i,j=1,2} (\tilde{x}_i - x_j + \lambda)} \right]. \tag{A.2}$$

Recalling relations [13], we have

$$h_i^2 = -\frac{\prod_{j=1}^{N}(x_i - x_j + \lambda)(x_i - \tilde{x}_j - \lambda)}{\prod_{i,j=1,j\neq i}^{N}(x_i - x_j)\prod_{j=1}^{N}(x_i - \tilde{x}_j)},$$
$$\tilde{h}_i^2 = -\frac{\prod_{j=1}^{N}(\tilde{x}_i - x_j + \lambda)(\tilde{x}_i - \tilde{x}_j - \lambda)}{\prod_{i,j=1,j\neq i}^{N}(\tilde{x}_i - \tilde{x}_j)\prod_{j=1}^{N}(\tilde{x}_i - x_j)},$$

and therefore, for $i, j = 1, 2$,

$$h_1^2 = -\frac{(x_1 - x_1 + \lambda)(x_1 - x_2 + \lambda)(x_1 - \tilde{x}_1 - \lambda)(x_1 - \tilde{x}_2 - \lambda)}{(x_1 - x_2)(x_1 - \tilde{x}_1)(x_1 - \tilde{x}_2)},$$
$$\tilde{h}_1^2 = \frac{(\tilde{x}_1 - x_1 + \lambda)(\hat{x}_1 - x_2 + \lambda)(\tilde{x}_1 - \tilde{x}_1 - \lambda)(\tilde{x}_1 - \tilde{x}_2 - \lambda)}{(\tilde{x}_1 - \tilde{x}_2)(\tilde{x}_1 - x_1)(\tilde{x}_1 - x_2)},$$
$$h_2^2 = -\frac{(x_2 - x_1 + \lambda)(x_2 - x_2 + \lambda)(x_2 - \tilde{x}_1 - \lambda)(x_2 - \tilde{x}_2 - \lambda)}{(x_2 - x_1)(x_2 - \tilde{x}_1)(x_2 - \tilde{x}_2)},$$
$$\tilde{h}_2^2 = \frac{(\tilde{x}_2 - x_1 + \lambda)(\hat{x}_2 - x_2 + \lambda)(\tilde{x}_2 - \tilde{x}_1 - \lambda)(\tilde{x}_2 - \tilde{x}_2 - \lambda)}{(\tilde{x}_2 - \tilde{x}_1)(\tilde{x}_2 - x_1)(\tilde{x}_2 - x_2)}.$$

Taking the logarithm gives

$$\ln|h_1| = \frac{1}{2}[\ln|\lambda| + \ln|x_1 - x_2 + \lambda| + \ln|x_1 - \tilde{x}_1 - \lambda| +$$
$$+ \ln|x_1 - \tilde{x}_2 - \lambda| - \ln|x_1 - x_2| - \ln|x_1 - \tilde{x}_1| - \ln|x_1 - \tilde{x}_2,$$
$$\ln|\tilde{h}_1| = \frac{1}{2}\big[\ln|\lambda| + \ln|\tilde{x}_1 - x_1 + \lambda| + \ln|\tilde{x}_1 - x_2 + \lambda| -$$
$$- \ln|\tilde{x}_1 - \tilde{x}_2 - \lambda| - \ln|\tilde{x}_1 - \tilde{x}_2| - \ln|\tilde{x}_1 - x_1| - \ln|\tilde{x}_1 - x_2|],$$
$$\ln|h_2| = \frac{1}{2}[\ln|\lambda| + \ln|x_2 - x_1 + \lambda| + \ln|x_2 - \tilde{x}_1 - \lambda| +$$
$$+ \ln|x_2 - \tilde{x}_2 - \lambda| - \ln|x_2 - x_1| - \ln|x_2 - \tilde{x}_1| - \ln|x_2 - \tilde{x}_2|],$$
$$\ln|\tilde{h}_2| = \frac{1}{2}[\ln|\lambda| + \ln|\tilde{x}_2 - x_1 + \lambda| + \ln|\tilde{x}_2 - x_2 + \lambda| +$$
$$+ \ln|\tilde{x}_2 - \tilde{x}_1 - \lambda| - \ln|\tilde{x}_2 - \tilde{x}_1| - \ln|\tilde{x}_2 - x_1| - \ln|\tilde{x}_2 - x_2|].$$

Hence,

$$\det \mathbf{M}_{\mathrm{RS}} = \ln|h_1| + \ln|\tilde{h}_1| + \ln|h_2| + \ln|\tilde{h}_2| + \ln|\tilde{x}_2 - \tilde{x}_1| + \ln|x_1 - x_2| - \sum_{i,j=1,2} \ln|\tilde{x}_i - x_j + \lambda| =$$
$$= 2\ln|\lambda| - \sum_{i,j=1,2} \ln|\tilde{x}_i - x_j + \lambda| = \sum_{i,j=1,2} \ln|x_i - x_j + \lambda| - \sum_{i,j=1,2} \ln|x_i - \tilde{x}_j|.$$

Obviously, for N particles or an $N \times N$ matrix, we have

$$\det \mathbf{M}_{\mathrm{RS}} = \sum_{i,j=1}^{N} \ln|x_i - x_j + \lambda| - \sum_{i,j=1}^{N} \ln|x_i - \tilde{x}_j|, \tag{A.3}$$

which is indeed the discrete-time Lagrangian for the one-parameter CM system.

Funding. Umpon Jairuk thanks the Rajamangala University of Technology Thanyaburi (RMUTT) for financial support under the Personnel Development Fund in 2023.

Conflict of interest. The authors of this work declare that they have no conflicts of interest.

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