NARESUAN UNIVERSITY
The Institute for Fundamental Study (IF)

LINEAR ALGEBRA, LINEAR SPACES, EXAMPLES, SPECTRAL DESCOMPOSITION (M1)

FOURIER SERIES, FOURIER TRANSFORM, DIRAC DELTA, DISTRIBUTIONS (M2)

SUMMER SCHOOL
Einstein's Term 2024

Homework assignment

(Final update: May 20, 2024)

1st) Consider the following discrete orthonormal basis in $\mathcal{L}^2(\Omega)$, where Ω is the interval $[0, \ell]$:

$$\psi_a(x) = \sqrt{\frac{1}{\ell}} \exp\left(\mathrm{i}\frac{2\pi a}{\ell} x\right), \quad a = \cdots, -2, -1, 0, +1, +2, \cdots . \tag{1.1}$$

Every function $\psi(x) \in \mathcal{L}^2(\Omega)$ can be expanded in one and only one way in terms of the $\{\psi_a(x)\}$:

$$\psi(x) = \sum_{a=-\infty}^{\infty} C_a \, \psi_a(x). \tag{1.2}$$

As you know, the coefficients of the latter series are given by the formula $C_a = \langle \psi_a, \psi \rangle$. Also note that $\psi(x)$ satisfies the periodic boundary condition, i.e., $\psi(0) = \psi(\ell)$. (a) Try to obtain the typical Fourier series from the series given above, namely,

$$\psi(x) = A_0 + \sum_{a=1}^{\infty} A_a \cos\left(\frac{2\pi a}{\ell} x\right) + \sum_{a=1}^{\infty} B_a \sin\left(\frac{2\pi a}{\ell} x\right), \tag{1.3}$$

with the coefficients A_0, A_a and B_a written in terms of the coefficients C_a, namely,

$$A_0 = \sqrt{\frac{1}{\ell}}\, C_0 = \frac{1}{\ell} \int_0^{\ell} \mathrm{d}x \, \psi(x) \,, \tag{1.4}$$

$$A_a = \sqrt{\frac{1}{\ell}}\, (C_a + C_{-a}) = \frac{2}{\ell} \int_0^{\ell} \mathrm{d}x \, \cos\left(\frac{2\pi a}{\ell} x\right) \psi(x) \,, \tag{1.5}$$

$$B_a = \sqrt{\frac{1}{\ell}}\, \mathrm{i}(C_a - C_{-a}) = \frac{2}{\ell} \int_0^{\ell} \mathrm{d}x \, \sin\left(\frac{2\pi a}{\ell} x\right) \psi(x) \,. \tag{1.6}$$

(b) Demonstrate that $\psi(x)$ is real if and only if $C_{-a} = C_a^*$ (the asterisk *denotes the complex conjugate, as usual), and therefore, the coefficients A_0, A_a and B_a are real.

2nd) Try to demonstrate the following integral representation for the Dirac delta:

$$\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} = \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} = \lim_{a\to 0} \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}kx} \mathrm{e}^{-a|x|} = 2\pi\, \delta(k). \tag{2.1}$$

Note that, I am proposing you to add a regularizing factor to derive the expression. You will also need to use the following representation of the Dirac delta:

$$\lim_{a\to 0} \frac{1}{\pi} \frac{a}{a^2 + k^2} = \delta(k). \tag{2.2}$$

Note: In both limits, the parameter a approaches zero from the positive side.

3rd) As presented in class, if $\psi(x)$ is a (real or complex) function of the variable x, its Fourier transform $\phi(k)$, if it exist, is defined by

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi(x)\, \mathrm{e}^{-\mathrm{i}kx} \equiv \mathrm{FT}[\psi(x)], \tag{3.1}$$

and the inverse formula is:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k\, \phi(k)\, \mathrm{e}^{+\mathrm{i}kx} \equiv (\mathrm{FT})^{-1}[\phi(k)]. \tag{3.2}$$

(a) Demonstrate the following property:

$$\mathrm{FT}[\psi(cx)] = \frac{1}{|c|}\, \phi\left(k/c\right), \tag{3.3}$$

where c is a (real) constant. (b) In particular, $\mathrm{FT}[\psi(-x)] = \phi(-k)$, and together with $\mathrm{FT}[\psi(x)] = \phi(k)$ (see Eq. (3.1)), it follows that if the function $\psi(x)$ has a definite parity, its Fourier transform $\phi(k)$ has the same parity. Prove it!

4th) (a) Let us consider the following function:

$$\psi(x) = \frac{1}{\sqrt{2a}} \left[\, \Theta(x+a) - \Theta(x-a)\, \right], \tag{4.1}$$

where $a > 0$ and $\Theta(x)$ is the Heaviside step function, namely, $\Theta(x < 0) = 0$ and $\Theta(x > 0) = 1$. Find the Fourier transform of $\psi(x)$, namely,

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \psi(x)\, \mathrm{e}^{-\mathrm{i}kx}. \tag{4.2}$$

(b) Is the function $\psi(x)$ normalized to unity? In other words, is $\psi(x)$ a square-integrable function with norm equal to one? (c) But furthermore, is also $\phi(k)$ a square-integrable function with norm equal to one? Hint: You may find the following integral useful:

$$\int_0^{+\infty} \mathrm{d}u \, \frac{\sin^2(u)}{u^2} = \frac{\pi}{2}. \tag{4.3}$$

5th) Prove the following result (in fact, it is a theorem): If $\phi(k)$ and $\varphi(k)$ are the respective Fourier transforms of the square-integrable functions $\psi(x)$ and $\chi(x)$, one has that

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \chi^*(x) \, \psi(x) = \int_{-\infty}^{+\infty} \mathrm{d}k \, \varphi^*(k) \, \phi(k). \tag{5.1}$$

A particular case of this result is the conservation of the norm

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, |\psi(x)|^2 = \int_{-\infty}^{+\infty} \mathrm{d}k \, |\phi(k)|^2 , \tag{5.2}$$

that is, a function and its Fourier transform have the same norm. This result is called the Parseval-Plancherel formula.

6th) (a) Demonstrate that the Fourier transform of the Heaviside step function $\Theta(x)$ is given by

$$\phi(k) = \mathrm{FT}[\Theta(x)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x \, \Theta(x) \, \mathrm{e}^{-\mathrm{i}kx} = \frac{(-\mathrm{i})}{\sqrt{2\pi}} \lim_{a\to 0} \frac{1}{k - \mathrm{i}a}, \tag{6.1}$$

(remember that $\Theta(x < 0) = 0$ and $\Theta(x > 0) = 1$). The limit can be evaluated using the following identity:

$$\lim_{a\to 0} \frac{1}{k \pm \mathrm{i}a} = \mathrm{P.V.} \left(\frac{1}{k}\right) \mp \mathrm{i}\pi\delta(k)$$

$$\left[\text{i.e., } \lim_{a\to 0} \int_{\mathbb{R}} \mathrm{d}k \, \frac{f(k)}{k \pm \mathrm{i}a} = \mathrm{P.V.} \int_{\mathbb{R}} \mathrm{d}k \, \frac{f(k)}{k} \mp \mathrm{i}\pi \int_{\mathbb{R}} \mathrm{d}k \, f(k)\delta(k) \right.$$
$$\left. = \mathrm{P.V.} \int_{\mathbb{R}} \mathrm{d}k \, \frac{f(k)}{k} \mp \mathrm{i}\pi f(0) \right]. \tag{6.2}$$

Certainly, $f(x)$ is a regular function at $k = 0$ (in these limits, the parameter approaches zero from the positive side), and P.V. represents the Cauchy principal value of the integral to the right of the symbol P.V., namely,

$$\mathrm{P.V.} \int_{\mathbb{R}} \mathrm{d}k \, \frac{f(k)}{k} \equiv \lim_{\epsilon\to 0} \int_{-\infty}^{-\epsilon} \mathrm{d}k \, \frac{f(k)}{k} + \lim_{\epsilon\to 0} \int_{+\epsilon}^{+\infty} \mathrm{d}k \, \frac{f(k)}{k} \quad (\epsilon > 0). \tag{6.3}$$

(b) Find the inverse Fourier transform of $\phi(k)$

$$(\mathrm{FT})^{-1}[\phi(k)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}k \, \phi(k) \, \mathrm{e}^{+\mathrm{i}kx}. \tag{6.4}$$

Did you find what you expected? Hint:

$$\frac{1}{\pi}\int_{\mathbb{R}} \mathrm{d}k\, \frac{\sin(kx)}{k} = \operatorname{sgn}(x) = 2\,\Theta(x) - 1. \tag{6.5}$$

7th) Let $\psi = \psi(x)$ be a function on which the operator $\hat{\mathrm{p}}^{-1}$ acts, where $\hat{\mathrm{p}} = -\mathrm{i}\hbar\, \mathrm{d}/\mathrm{d}x$ is the momentum operator. Let $\phi = \phi(x)$ be the result of this operation:

$$\phi(x) = \hat{\mathrm{p}}^{-1}\, \psi(x). \tag{7.1}$$

(a) How does $\hat{\mathrm{p}}^{-1}$ act on the function $\psi(x)$? I will tell you how to answer this question. If $f(p)$ and $g(p)$ are, respectively, the Fourier transforms of $\psi(x)$ and $\phi(x)$, we can write the following relations:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \mathrm{d}p\, f(p)\, \exp\left(+\mathrm{i}\frac{p}{\hbar}x\right) \Rightarrow f(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \mathrm{d}x\, \psi(x)\, \exp\left(-\mathrm{i}\frac{p}{\hbar}x\right) \tag{7.2}$$

and

$$\phi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \mathrm{d}p\, g(p)\, \exp\left(+\mathrm{i}\frac{p}{\hbar}x\right) \Rightarrow g(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \mathrm{d}x\, \phi(x)\, \exp\left(-\mathrm{i}\frac{p}{\hbar}x\right). \tag{7.3}$$

Show that

$$g(p) = p^{-1} f(p). \tag{7.4}$$

Multiplying the latter relation by $\exp(+\mathrm{i}px/\hbar)/\sqrt{2\pi\hbar}$, and then integrating over p, the following result is obtained:

$$\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \mathrm{d}p\, g(p)\, \exp\left(+\mathrm{i}\frac{p}{\hbar}x\right) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \mathrm{d}p\, \frac{1}{p}\, f(p)\, \exp\left(+\mathrm{i}\frac{p}{\hbar}x\right),$$

or even better

$$\phi(x) = \frac{\mathrm{i}}{2\pi\hbar}\int_{-\infty}^{+\infty} \mathrm{d}x' \left\{\int_{-\infty}^{+\infty} \mathrm{d}p\, \frac{1}{p}\, \sin\left[\frac{p}{\hbar}(x - x')\right]\right\} \psi(x') \tag{7.5}$$

(where expressions included in Eqs. (7.2) and (7.3) were used). Show that the formula (7.5) can be written as follows:

$$\phi(x) = \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{x} \mathrm{d}x'\, \psi(x') - \frac{\mathrm{i}}{2\hbar}\int_{-\infty}^{+\infty} \mathrm{d}x'\, \psi(x'). \tag{7.6}$$

Note: Formula (7.4) shows that the function $g(p)$ has, in general, a pole at $p = 0$. To avoid this result, the condition $f(p = 0) = 0$ could be imposed, which implies that $\int_{-\infty}^{+\infty} \mathrm{d}x'\, \psi(x') = 0$. Thus, the result in Eq. (7.6) can be written as follows:

$$\phi(x) = \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{x} \mathrm{d}x'\, \psi(x'). \tag{7.7}$$

Clearly, the operator $\hat{\mathrm{p}}^{-1}$ is an integral operator of the form

$$\hat{\mathrm{p}}^{-1} = \frac{\mathrm{i}}{\hbar} \int_{-\infty}^{x} \mathrm{d}x' \left(\, \right), \tag{7.8}$$

and acts on a function of x (which should be placed inside the parenthesis in (7.8) as a function of x'). (b) Final note: Verify, using formula (7.7), that $\hat{\mathrm{p}}\,\phi(x) = \psi(x)$.

8th) In a given representation, or basis, a bounded operator can be represented by a matrix. If we change the representation, the operator will be represented by a different matrix. On the other hand, if we write the matrix in the basis of its own eigenvectors, then the matrix is diagonal. Write the Pauli matrix $\hat{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in the basis of its eigenvectors. Which matrix did you find?

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