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On the cosmology of massive gravity

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Abstract

We present a review of cosmological solutions in nonlinear massive gravity, focusing on the stability of perturbations. Although homogeneous and isotropic solutions have been found, these are now known to suffer from either the Higuchi ghost or a new nonlinear ghost instability. We discuss two approaches to alleviate this issue. By relaxing the symmetry of the background by e.g. breaking isotropy in the hidden sector, it is possible to accommodate a stable cosmological solution. Alternatively, extending the theory to allow for new dynamical degrees of freedom can also remove the conditions which lead to the instability. As examples for this case, we study the stability of self-accelerating solutions in the quasi-dilatonic extension and generic cosmological solutions in the varying mass extension. While the quasi-dilaton case turns out to be unstable, the varying mass case allows stable regimes of parameters. Viable self-accelerating solutions in the varying mass theory yet remain to be found.

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1. Introduction

The search for a finite-range gravity has been a long-standing problem, well motivated by both theoretical and observational considerations. On the theory side, the existence of a theoretically consistent extension of general relativity (GR) by a mass term has been a basic question of classical field theory. After Fierz and Pauli's pioneering attempt in 1939 [1], this issue has attracted a great deal of interest. On the observation side, continuing experimental probes of gravity have revealed new unexpected phenomena at large scales; one of the most profound discovery is the cosmic acceleration, which was found in 1998 [2]. The extremely tiny energy

scale associated with the cosmic acceleration hints that gravity might need to be modified in the infrared (IR). The massive gravity is one of the most interesting attempts in this direction.

However, theoretical and observational consistency of massive gravity theories has been a challenging issue for several decades. Fierz and Pauli's model [1], which extends GR by a linear mass term, suffers from the van Dam–Veltman–Zakharov discontinuity [3, 4]; relativistic and non-relativistic matter couple to gravity with different relative strengths, no matter how small the graviton mass is. Although this problem can be alleviated by nonlinear effects, as suggested by Vainshtein [5], the same nonlinearities lead to ghost instability. Indeed, at the nonlinear level, the theory loses not only the momentum constraint, but also the Hamiltonian constraint and, as a result, the nonlinear theory includes up to six degrees of freedom in the gravity sector. While five of them properly represent the degrees of freedom of a massive spin-2 field in a Poincaré invariant background, the sixth one is the so-called Boulware–Deser (BD) ghost [6].

Adopting the effective field theory approach in the decoupling limit (i.e. $m_g \rightarrow 0$, $M_p \rightarrow \infty$, $\Lambda \rightarrow$ fixed, where Λ is the cutoff of the theory), it was found that the BD ghost is related to the longitudinal mode of the Goldstone bosons associated with the broken general covariance [7]. The construction of a theory free from the BD ghost was only recently achieved by de Rham, Gabadadze and Tolley (dRGT) [8, 9]. It was shown that the Hamiltonian constraint and the associated secondary constraint are restored in this theory, eliminating the BD ghost mode as a result [10–16]⁴.

However, in order for the theory to be theoretically consistent and observationally viable, the absence of the BD ghost is not sufficient. At the very least, a stable cosmological solution is needed.

The purpose of this paper is to review the construction and the stability of cosmological solutions in the context of nonlinear massive gravity. We start with describing the action of dRGT theory in section 2. In section 3, we construct homogeneous and isotropic cosmological solutions that exhibit self-acceleration. We then argue in section 4 that all homogeneous and isotropic solutions in the dRGT theory are unstable and thus cannot describe the universe as we know it. In section 5, we propose three alternative cosmological scenarios to avoid instabilities. One of them is based on the observation that breaking isotropy in the hidden sector (fiducial metric) still allows the isotropic evolution of the visible sector (physical metric) and thus the standard thermal history of the universe. The other two proposals maintain isotropy in both the visible and hidden sectors but are based on extended theories of massive gravity with extra degrees of freedom, such as the quasi-dilaton theory [18] and the varying mass theory [19, 20]⁵.

2. Action

We start by describing the action of dRGT massive gravity theory [9]. In order to have a manifestly diffeomorphism invariant description of the massive gravity, the action is built out of four Stückelberg scalar fields, $\phi^a(x)$, $a = 0, 1, 2, 3$. These four scalars enter the gravity action through a ‘fiducial metric’ defined as

$$f_{\mu\nu} \equiv \bar{f}_{ab}(\phi^c) \partial_\mu \phi^a \partial_\nu \phi^b, \quad (1)$$

⁴ See [17] for the proof of the absence of the BD degree in the bi-metric and multi-metric extensions.

⁵ We note that another possible extension, not considered here, is the bi-metric theory [17], where the fiducial metric is promoted to a second, dynamical metric. The cosmology [21, 22] allows self-acceleration [23]. The cosmological perturbations was studied in [24].

where the ‘reference metric’ $\bar{f}_{ab}(\phi^c)$ is a metric in the field space. The action for the gravity sector is a functional of the physical metric $g_{\mu\nu}$ and the fiducial metric $f_{\mu\nu}$. A necessary condition for the theory to be free from the BD ghost is that the action in the decoupling limit to vanish up to boundary terms when restricted to the longitudinal part of the Stückelberg fields. With this requirement, the most general mass term without the derivatives of $g_{\mu\nu}$ and $f_{\mu\nu}$ is constructed as

$$S_{\text{mass}}[g_{\mu\nu}, f_{\mu\nu}] = M_{\text{Pl}}^2 m_g^2 \int d^4x \sqrt{-g} (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4), \quad (2)$$

with

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{2}([\mathcal{K}]^2 - [\mathcal{K}^2]), & \mathcal{L}_3 &= \frac{1}{6}([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]), \\ \mathcal{L}_4 &= \frac{1}{24}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]), \end{aligned} \quad (3)$$

where a square bracket denotes trace operation and

$$\mathcal{K}_\nu^\mu = \delta_\nu^\mu - (\sqrt{g^{-1}f})^\mu_\nu. \quad (4)$$

It was shown that the theory is free from the BD ghost at the fully nonlinear level even away from the decoupling limit [10–13].

3. FLRW cosmological solution

With the nonlinear massive gravity theory free from the BD ghost in hand, it is important to study its cosmological implications. In this section, we thus construct Friedmann–Lemaître–Robertson–Walker (FLRW) solutions.

3.1. Open FLRW solution with Minkowski reference metric

The original dRGT theory respects the Poincaré symmetry in the field space and thus the reference metric is Minkowski, i.e. $\bar{f}_{ab} = \eta_{ab} = \text{diag}(-1, 1, 1, 1)$. In this subsection, we thus review the FLRW solution with the Minkowski reference metric [25]. This is the first non-trivial FLRW solution in the context of dRGT massive gravity.

In order to find FLRW cosmological solutions, we should adopt an ansatz in which both $g_{\mu\nu}$ and $f_{\mu\nu}$ respect the FLRW symmetry. Since the tensor $f_{\mu\nu}$ is the pullback of the Minkowski metric in the field space to the physical spacetime, such an ansatz would require a flat, closed or open FLRW coordinate system for the Minkowski line element. The Minkowski line element does not admit a closed chart, but it allows an open chart. Thus, while there is no closed FLRW solution, we may hope to find open FLRW solutions. A flat FLRW solution, if it exists, is on the boundary between the closed and open solutions but it was shown in [19] that such a solution does not exist. For these reasons, in the following we shall seek open FLRW solutions.

Motivated by the coordinate transformation from Minkowski coordinates to Milne coordinates, we take the following ansatz for the four Stückelberg scalars:

$$\phi^0 = f(t)\sqrt{1 + |K|\delta_{ij}x^i x^j}, \quad \phi^i = \sqrt{|K|}f(t)x^i, \quad (5)$$

with $K < 0$. This leads to the open FLRW form for the Minkowski fiducial metric,

$$f_{\mu\nu} \equiv \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b = -(f(t))^2\delta_\mu^0\delta_\nu^0 + |K|f(t)^2\Omega_{ij}\delta_\mu^i\delta_\nu^j. \quad (6)$$

As for the physical metric, we adopt the general open ($K < 0$) FLRW ansatz as

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= -N(t)^2 dt^2 + \Omega_{ij} dx^i dx^j, \\ \Omega_{ij} dx^i dx^j &= dx^2 + dy^2 + dz^2 + \frac{K(x dx + y dy + z dz)^2}{1 - K(x^2 + y^2 + z^2)}. \end{aligned} \quad (7)$$

Here, $x^0 = t, x^1 = x, x^2 = y, x^3 = z; \mu, \nu = 0, \dots, 3$ and $i, j = 1, 2, 3$. Without loss of generality, we assume that $\dot{f} > 0, f > 0, a > 0, N > 0$. By substituting the above ansatz into the Einstein–Hilbert action plus the mass term (2), the gravity action up to a boundary term can be written as

$$S_g = \int d^4x \sqrt{\Omega} \left[-3|K|Na - \frac{3\dot{a}^2 a}{N} + m_g^2(L_2 + \alpha_3 L_3 + \alpha_4 L_4) \right], \quad (8)$$

where

$$\begin{aligned} L_2 &= 3a(a - \sqrt{|K|}f)(2Na - \dot{f}a - N\sqrt{|K|}f), \\ L_3 &= (a - \sqrt{|K|}f)^2(4Na - 3\dot{f}a - N\sqrt{|K|}f), \\ L_4 &= (a - \sqrt{|K|}f)^3(N - \dot{f}). \end{aligned} \quad (9)$$

In addition to the gravity action, we also consider a general matter content so that the total action is $S_{\text{tot}} = S_g + S_{\text{matter}}$.

Note that since the above ansatz fully respects the FLRW symmetry, the $(0i)$ components of the equations of motion for $g_{\mu\nu}$ are trivially satisfied; thus, the variation of the action with respect to $N(t)$ and $a(t)$ should correctly give all the non-zero components of the Einstein equation. On the other hand, because of the identity [26]

$$\nabla^\mu \left(\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \right) = \frac{1}{\sqrt{-g}} \frac{\delta S_g}{\delta \phi^a} \partial_\nu \phi^a, \quad (10)$$

the number of independent equations of motion for the Stückelberg scalars is one.

Now let us take the variation of the action with respect to $f(t)$, which contains all non-trivial information about the dynamics of the Stückelberg scalars. It leads to

$$(\dot{a} - \sqrt{|K|}N)[(3 - 2X) + \alpha_3(3 - X)(1 - X) + \alpha_4(1 - X)^2] = 0, \quad (11)$$

where $X \equiv \sqrt{|K|}f/a$. This equation has three solutions. The first one is $\dot{a} = \sqrt{|K|}N$ and corresponds to an empty open universe, i.e. the open FLRW chart of Minkowski spacetime. Thus, this solution is not of our interest. The remaining two solutions are

$$f = \frac{a}{\sqrt{|K|}} X_\pm, \quad X_\pm \equiv \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\alpha_3 + \alpha_4}. \quad (12)$$

Note that these two solutions are singular in the limit $K \rightarrow 0$. This is consistent with the result in [19], i.e. the non-existence of flat FLRW cosmologies. On the other hand, with $K < 0$, by taking the variation of the action with respect to $N(t)$ and using (12), we obtain the following modified Friedmann equation:

$$3H^2 + \frac{3K}{a^2} = \rho_m + \Lambda_\pm, \quad H \equiv \frac{\dot{a}}{Na}, \quad (13)$$

where ρ_m is the energy density of the matter sector and

$$\begin{aligned} \Lambda_\pm &\equiv -\frac{m_g^2}{(\alpha_3 + \alpha_4)^2} \left[1 + \alpha_3 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4} \right] \\ &\quad \times \left[1 + \alpha_3^2 - 2\alpha_4 \pm (1 + \alpha_3) \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4} \right]. \end{aligned} \quad (14)$$

In this way, the graviton mass manifests as the effective cosmological constant Λ_\pm . When $\Lambda_\pm > 0$, the system exhibits self-acceleration. By taking the variation of the action with respect to $a(t)$, we obtain a dynamical equation, which is consistent with the above modified Friedmann equation and the standard conservation equation for matter.

3.2. Flat/closed/open FLRW solutions with general reference metric

In the appendix of [27], the open FLRW solution was generalized to flat/closed/open FLRW solutions by considering a fiducial metric of the general FLRW type. (General reference metrics were first considered in [26] and the absence of the BD ghost in this general setup was proven by [13]. See also [17] for the absence of a ghost in the bi-metric theory.) In this subsection, we describe the general solutions.

The most general fiducial metric consistent with flat ($K = 0$), closed ($K > 0$) or open ($K < 0$) FLRW symmetries is

$$f_{\mu\nu} = -n^2(\varphi^0)\partial_\mu\varphi^0\partial_\nu\varphi^0 + \alpha^2(\varphi^0)\Omega_{ij}(\varphi^k)\partial_\mu\varphi^i\partial_\nu\varphi^j, \quad (15)$$

where n and α are general functions of φ^0 , and $\Omega_{ij}(\varphi^k)$ is defined as in (7) with (x, y, z) replaced by $(\varphi^1, \varphi^2, \varphi^3)$, and the curvature constant K is now either zero, positive or negative. Here, we have used the notation φ^a instead of ϕ^a to make it clear that this form of the fiducial metric may be achieved from the original form (1) by a non-trivial change of variables (as we have explicitly seen in the previous subsection). As for the physical metric, we adopt the ansatz (7) with an arbitrary sign for K .

Similar to the case in the previous subsection, the equation of motion for the Stückelberg fields allows three branches of solutions. In the general setup at hand, the first branch is characterized by $aH = \alpha H_f$, where $H \equiv \dot{a}/(Na)$ and $H_f \equiv \dot{\alpha}/(n\alpha)$ are Hubble expansion rates of the physical and fiducial metrics, respectively. Unfortunately, this branch would not allow non-trivial cosmologies since it does not evade the Higuchi bound [28] and thus linear perturbations around the corresponding solution [26, 29, 30] include a ghost degree in the cosmological history. Therefore, we shall not consider this branch and restrict our attention to the other branches.

The two remaining branches are characterized by $\alpha = X_\pm a$, where X_\pm is the same as in (12). For these two branches, the metric equation of motion is exactly the same as (13) with (14). Surprisingly enough, the modified Friedmann equation (including the value of the effective cosmological constant induced by the graviton mass term) does not depend on the properties of the fiducial metric at all. When $\Lambda_\pm > 0$, the system exhibits self-acceleration.

4. New nonlinear instability of FLRW solutions

In the previous section, we have constructed flat, closed and open FLRW solutions in nonlinear massive gravity with a general FLRW fiducial metric. The construction allows three branches of solutions. However, the first branch characterized by $aH = \alpha H_f$ suffers from the Higuchi ghost at the level of linear perturbations and thus does not allow a non-trivial cosmological history. In this section, we thus consider the other two branches of solutions characterized by $\alpha = X_\pm a$. These solutions evade the Higuchi ghost, but unfortunately we shall see that a new type of ghost instability shows up at nonlinear level [31]. Based on this result, we shall argue that all homogeneous and isotropic FLRW solutions in the dRGT theory are unstable. We shall then propose alternative cosmological scenarios in the following section.

4.1. Linear perturbation

In this subsection, following [27], we shall investigate linear perturbations around a general flat/closed/open FLRW solution (characterized by $\alpha = X_\pm a$) with a general FLRW fiducial metric and an arbitrary matter content. We shall see that time kinetic terms for three among the five graviton degrees of freedom always vanish at the level of the quadratic action, signaling for necessity of nonlinear analysis.

We first define perturbations of four Stückelberg scalars through the exponential mapping, truncating at the second order, as

$$\varphi^a = x^a + \pi^a + \frac{1}{2}\pi^b \partial_b \pi^a + \mathcal{O}(\pi^3). \quad (16)$$

We then perturb the physical metric as

$$g_{00} = -N^2(t)[1 + 2\phi], \quad g_{0i} = N(t)a(t)\beta_i, \quad g_{ij} = a^2(t)[\Omega_{ij} + h_{ij}]. \quad (17)$$

We suppose that $\pi^a, \phi, \beta_i, h_{ij} = \mathcal{O}(\epsilon)$. The following gauge-invariant variables can be constructed out of Stückelberg and metric perturbations:

$$\begin{aligned} \phi^\pi &\equiv \phi - \frac{1}{N}\partial_t(N\pi^0), & \beta_i^\pi &\equiv \beta_i + \frac{N}{a}D_i\pi^0 - \frac{a}{N}\dot{\pi}_i, \\ h_{ij}^\pi &\equiv h_{ij} - D_i\pi_j - D_j\pi_i - 2NH\pi^0\Omega_{ij}, \end{aligned} \quad (18)$$

where D_i is the spatial covariant derivative compatible with Ω_{ij} .

In section 3, we have seen that the mass term acts as an effective cosmological constant at the background level. Hence, we define

$$\tilde{S}_{\text{mass}}[g_{\mu\nu}, f_{\mu\nu}] \equiv S_{\text{mass}}[g_{\mu\nu}, f_{\mu\nu}] + M_{Pl}^2 \int d^4x \sqrt{-g} \Lambda_\pm, \quad (19)$$

where Λ_\pm is specified in (14), and expand \tilde{S}_{mass} instead of S_{mass} . This greatly simplifies the perturbative expansion. As shown in [27], upon using the background equation of motion for the Stückelberg fields but without using the background equation of motion for the physical metric, the quadratic part of \tilde{S}_{mass} is simplified as

$$\tilde{S}_{\text{mass}}^{(2)} = \frac{M_{Pl}^2}{8} \int d^4x N a^3 \sqrt{\Omega} M_{\text{GW}}^2 [(h^\pi)^2 - h_\pi^{ij} h_{ij}^\pi], \quad (20)$$

where

$$M_{\text{GW}}^2 \equiv \pm(r-1)m_g^2 X_\pm^2 \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}, \quad r \equiv \frac{na}{N\alpha} = \frac{1}{X_\pm} \frac{H}{H_f}, \quad (21)$$

X_\pm is given by (12), $h^\pi \equiv \Omega^{ij} h_{ij}^\pi$, $h_\pi^{ij} \equiv \Omega^{ik} \Omega^{jl} h_{kl}^\pi$ and Ω^{ij} is the inverse of Ω_{ij} . This is manifestly gauge-invariant.

What is important here is that the gauge-invariance of $\tilde{S}_{\text{mass}}^{(2)}$ was shown without using the background equation of motion for the physical metric. This means that $\tilde{S}_{\text{mass}}^{(2)}$ is gauge-invariant for any matter content (as far as the matter action does not depend on the Stückelberg fields so that the Stückelberg equation of motion is derived solely from the graviton mass term) and that the remaining part $\tilde{S}_{\text{GR}}^{(2)} \equiv S_{\text{tot}}^{(2)} - \tilde{S}_{\text{mass}}^{(2)}$ of the total (gravity plus matter) quadratic action $S_{\text{tot}}^{(2)}$ is also gauge-invariant by itself. Hence, the remaining part $\tilde{S}_{\text{GR}}^{(2)}$ never depends on the Stückelberg perturbations for any gauge choice. Another important point is that $\tilde{S}_{\text{mass}}^{(2)}$ shown in (20) does not depend on ϕ^π and β_i^π , and hence does not include time derivatives of Stückelberg perturbations. Therefore, for any matter content, the dependence of the total quadratic action on the Stückelberg perturbations is completely given by (20) and the time derivatives of Stückelberg perturbations do not enter the quadratic action at all. This completes the proof of the statement that time kinetic terms for three among five gravity degrees of freedom always vanish at the level of the quadratic action. This proof holds for any matter content [27].

The absence of quadratic kinetic terms for three gravity degrees of freedom shown in this subsection implies that the self-accelerating FLRW solutions evade the Higuchi bound [28] and thus are free from the ghost at the linearized level even when the expansion rate is significantly higher than the graviton mass. At the same time, this signals for the necessity of a nonlinear analysis. In contrast, the first branch solution mentioned in section 3.2, which gives $aH = \alpha H_f$, contains five propagating degrees of freedom. In this case, however, one of these degrees turns out to be the Higuchi ghost [29].

4.2. Nonlinear perturbation

In order to understand the physical content of the FLRW background in dRGT massive gravity, we need to investigate the reason why the kinetic term of one of the scalar modes and two of the vector ones have a vanishing kinetic term. We will show that this feature does not hold in general for the theory, but is rather a consequence of the symmetries of the FLRW background.

Therefore, it is convenient to study another background which has less symmetries than FLRW, but does lead FLRW in some limit. Probably, the simplest implementation of such a background is the axisymmetric Bianchi type-I class of metrics, which can be written as

$$ds^2 = -N^2(t) dt^2 + a^2(t)[e^{4\sigma(t)} dx^2 + e^{-2\sigma(t)} \delta_{ij} dy^i dy^j], \quad (22)$$

where here $i, j \in \{2, 3\}$, and $y^2 = y, y^3 = z$. It is evident that this manifold is not isotropic; however, as σ approaches 0, the spacetime reduces to flat FLRW. Since we want to study the reason why there are missing kinetic terms for the perturbed FLRW fields, we will not consider the above metric as physical. Rather, for the time being, we will merely use it as a tool to study nonlinear perturbations on FLRW. In other words, linear perturbations on an anisotropic manifold will give information equivalent to nonlinear perturbation theory on FLRW. This provides a consistent truncation of the nonlinear perturbations, allowing us to analyze them in a simple way. The goal of this section is to show how the missing kinetic terms will depend on σ , namely we will study their sign and the dispersion relations of the associated perturbation fields.

According to the properties of the perturbation fields under a rotation in the y - z plane, we can decompose such fields into scalar (a.k.a. even) and vector (a.k.a. odd) modes. In particular, we can write the metric for the even modes as

$$ds_{\text{even}}^2 = -N^2(1 + 2\Phi) dt^2 + 2aN dt[e^{2\sigma} \partial_x \chi dx + e^{-\sigma} \partial_i B dy^i] + a^2 e^{4\sigma} (1 + \psi) dx^2 + 2a^2 e^\sigma \partial_x \partial_i \beta dx dy^i + a^2 e^{-2\sigma} [\delta_{ij}(1 + \tau) + \partial_i \partial_j E] dy^i dy^j, \quad (23)$$

whereas the metric for the odd modes reads

$$ds_{\text{odd}}^2 = -N^2 dt^2 + 2a e^{-\sigma} N v_i dt dy^i + 2a^2 e^\sigma \partial_x \lambda_i dx dy^i + a^2 e^{4\sigma} dx^2 + a^2 e^{-2\sigma} (\delta_{ij} + \partial_i h_j) dy^i dy^j, \quad (24)$$

where the transverse condition holds, i.e. $\partial^i v_i = 0 = \partial^i \lambda_i = 0 = \partial^i h_i$. At the same time, we also need to specify the form of the fiducial metric $f_{\mu\nu}$. Since we want to study the FLRW limit, we need the fiducial metric to possess already the FLRW symmetries from the beginning. The anisotropic physical metric corresponds to the simplest deviation from an overall FLRW symmetry for the whole system. Therefore, we assume

$$f_{\mu\nu} = -n(\phi^0)^2 \partial_\mu \phi^0 \partial_\nu \phi^0 + \alpha(\phi^0)^2 (\partial_\mu \phi^1 \partial_\nu \phi^1 + \delta_{ij} \partial_\mu \phi^i \partial_\nu \phi^j), \quad (25)$$

so that we need to give the perturbations of the Stückelberg fields according to even/odd-mode decomposition. In fact, the even-type perturbations of Stückelberg fields read

$$\phi^0 = t + \pi^0, \quad \phi^1 = x + \partial_x \pi^1, \quad \phi^i = y^i + \partial^i \pi. \quad (26)$$

For the odd-mode sector, we consider instead

$$\phi^0 = t, \quad \phi^1 = x, \quad \phi^i = y^i + \pi^i, \quad (27)$$

where $\partial_i \pi^i = 0$.

It is possible to define gauge-invariant fields for even perturbations as follows:

$$\begin{aligned}
\hat{\Phi} &= \Phi - \frac{1}{2N} \partial_t \left(\frac{\tau}{H - \Sigma} \right), & \hat{B} &= B + \frac{e^\sigma}{2a(H - \Sigma)} \tau - \frac{ae^{-\sigma}}{2N} \dot{E}, \\
\hat{\chi} &= \chi + \frac{\tau e^{-2\sigma}}{2a(H - \Sigma)} - \frac{ae^{2\sigma}}{N} \partial_t \left[e^{-3\sigma} \left(\beta - \frac{e^{-3\sigma}}{2} E \right) \right], \\
\hat{\psi} &= \psi - \frac{H + 2\Sigma}{H - \Sigma} \tau - e^{-3\sigma} \partial_x^2 (2\beta - e^{-3\sigma} E), & \hat{E}_\pi &= \pi - \frac{1}{2} E, \\
\hat{\tau}_\pi &= \pi^0 - \frac{\tau}{2N(H - \Sigma)}, & \hat{\beta}_\pi &= \pi^1 - e^{-3\sigma} \left(\beta - \frac{e^{-3\sigma}}{2} E \right),
\end{aligned} \tag{28}$$

where we have defined $\Sigma \equiv \dot{\sigma}/N$.

Then, we can proceed to integrate out all the present auxiliary fields. In general, we can integrate out three modes, that is, $\hat{\Phi}$, \hat{B} and $\hat{\chi}$. However, in the dRGT theory, it is possible to show that also the field $\hat{\tau}_\pi$ can be integrated out. Therefore, there are only three independent fields describing the even-mode perturbations, so we need to study the kinetic matrix of the three remaining fields, $\hat{\psi}$, $\hat{\beta}_\pi$ and \hat{E}_π . As $\sigma \rightarrow 0$, the eigenvalues of the 3×3 kinetic matrix reduce to

$$\kappa_1 \simeq \frac{p_T^4}{8p^4}, \quad \kappa_2 \simeq -\frac{2a^4 M_{\text{GW}}^2 p_L^2}{1-r^2} \sigma, \quad \kappa_3 \simeq -\frac{p_T^2}{2p_L^2} \kappa_2, \tag{29}$$

where we have introduced $r \equiv an/(\alpha N)$, $M_{\text{GW}}^2 \equiv m_g^2(1-r)X^2[(1+2\alpha_3+\alpha_4) - (\alpha_3+\alpha_4)X]$, $X \equiv \alpha/a$, $p_L \equiv k_L/(ae^{2\sigma}) \simeq k_L/a$, $p_T \equiv k_T/(ae^{-\sigma}) \simeq k_T/a$, $k_T^2 \equiv k_y^2 + k_z^2$ and $p^2 \equiv p_L^2 + p_T^2$. The first and most important consideration is that κ_2 and κ_3 have opposite sign. This property implies that a ghost will always be present in the even-mode sector, as the manifold approaches the FLRW limit. Furthermore, both κ_2 and κ_3 vanish in the exact FLRW case. One could wonder whether these modes, if their mass is finite—but non-zero, can be integrated out in this same FLRW limit. If the masses are heavy, then the corresponding modes can be integrated out and the ghost is harmless in general. Otherwise, the ghost will be physical and the theory—at least on FLRW backgrounds—will not be consistent.

In fact, we find

$$\begin{aligned}
\omega_1^2 &\simeq p^2 + M_{\text{GW}}^2, \\
\omega_2^2 &\simeq \left(\frac{r^2 - 1}{24\sigma} \right) \left[\sqrt{(10p^2 + p_T^2)^2 - 8p_L^2 p_T^2} - (2p^2 + 3p_T^2) \right], \\
\omega_3^2 &\simeq -\omega_2^2 + \frac{1-r^2}{12\sigma} (2p_L^2 + 5p_T^2),
\end{aligned} \tag{30}$$

with $\omega_2^2 \omega_3^2 < 0$ in general. Since there is not any mass gap (i.e. there always exists some value of the momenta for which the frequency vanishes) for $\omega_{2,3}^2$, we conclude that the ghost is physical and cannot be integrated out from the Lagrangian. Therefore, the FLRW background is not viable in the dRGT massive gravity theory. Note that the first mode corresponds to the massive gravitational wave. Even though the theory succeeds in removing the BD ghost and giving the tensor mode a mass, it does not accept a stable FLRW solution. This result agrees with the nonlinear analysis of [32], where the cubic kinetic terms are shown to be non-vanishing.

We conclude this section by studying the odd modes. For these modes, a procedure similar to the one followed for the even modes leads to two independent fields. Therefore, we confirm the expected presence of five dynamical degrees of freedom for this theory (3 even modes + 2 odd ones). The kinetic terms and the frequencies for these two independent odd modes read

$$\kappa_1 = \frac{a^4 p_L^2 p_T^4}{2p^2}, \quad \kappa_2 = \frac{a^4 p_T^2 M_{\text{GW}}^2}{4(1-r^2)} \sigma, \quad \omega_1^2 = p^2 + M_{\text{GW}}^2, \quad \omega_2^2 = c_{\text{odd}}^2 p^2, \tag{31}$$

where $c_{\text{odd}}^2 = (1 - r^2)/(2\sigma)$. Therefore, we find a massive tensor mode, and a healthy massless propagating mode (at speed c_{odd}), provided that $(1 - r)\sigma > 0$.

5. Toward healthy massive cosmologies

The appearance of the nonlinear ghost shown in subsection 4.2 originates from the fact that quadratic kinetic terms exactly vanish: the kinetic terms show up at the cubic order and can become negative. The disappearance of kinetic terms at the quadratic order was shown upon using the background equation of motion for the Stückelberg fields but without using other background equations. One can actually show that the off-shell quadratic kinetic terms have coefficients proportional to $J_\phi \equiv (3 - 2X) + \alpha_3(3 - X)(1 - X) + \alpha_4(1 - X)^2$, where $X \equiv \alpha/a$, and that the self-accelerating FLRW solution is characterized by $J_\phi = 0$ (or $X = X_\pm$ with X_\pm shown in (12)).

For this reason, in order to find a stable cosmological background, one needs to detune the proportionality between the quadratic kinetic terms and the Stückelberg equation of motion characterizing the self-accelerating background. One way to achieve this would be to relax the FLRW symmetry by a deformation of the background. This possibility will be considered in subsection 5.1. We shall find that relatively large deformation by anisotropy in the hidden sector (fiducial metric) may render the background solution stable. Another possibility would be to maintain the FLRW symmetry but to change the Stückelberg equation of motion by adding extra dynamical degrees of freedom to the theory. We shall thus consider the quasi-dilaton extension in subsection 5.2 and the varying mass extension in subsection 5.3. While self-accelerating FLRW solutions in the quasi-dilaton theory turn out to be unstable, the varying mass case allows some stable regimes of parameters.

Before presenting our results, we note that other extensions, such as the bi-metric theory [17], where both metrics are dynamical, may also give rise to FLRW-type cosmologies [21–23], although perturbation analysis in [24] indicates that such cosmologies, in the presence of perfect fluids, may develop instabilities.

5.1. Anisotropic FLRW solution

As argued above, the appearance of the nonlinear ghost shown in subsection 4.2 is a consequence of the FLRW symmetry and the structure of the theory; in order to obtain a stable solution within the same theory, the FLRW symmetry needs to be relaxed.

An inhomogeneous background solution was obtained in [19], where the observable universe is approximately FLRW for a horizon size smaller than the Compton length of graviton. Similar solutions with inhomogeneities in the Stückelberg sector, meaning that the physical metric and the fiducial metric do not have common isometries acting transitively, were found in [33]. Note that those inhomogeneous solutions cannot be isotropic everywhere since isotropy at every point implies homogeneity. Note also that cosmological perturbations can in principle probe inhomogeneities in the Stückelberg sector. For example, generic spherically symmetric solutions are isotropic only when they are observed from the origin.

The goal of this subsection is, following [34], to introduce an alternative option, where the assumption of isotropy is dropped but homogeneity is kept. In a region with a relatively large anisotropy, we find an attractor solution. On the attractor, the physical metric is still isotropic, and the background geometry is of FLRW type. Hence, the thermal history of the standard cosmology can be accommodated in this class of solutions. However, the Stückelberg field configuration is anisotropic, which may lead to effects at the level of the perturbations, suppressed by smallness of the graviton mass.

5.1.1. Fixed point solutions. In this subsection, we review anisotropic FLRW solutions in the dRGT theory with the de Sitter reference metric [34]⁶.

The fiducial metric is obtained by taking (15) and setting $K = 0$ and $H_f \equiv \dot{\alpha}/(n\alpha) = \text{constant}$. For the physical metric, we consider axisymmetric anisotropic extension of a flat FLRW, i.e. the axisymmetric Bianchi type-I metric, given by

$$g_{\mu\nu}^{(0)} dx^\mu dx^\nu = -dt^2 + a^2(t)[e^{4\sigma(t)} dx^2 + e^{-2\sigma(t)} \delta_{ij} dy^i dy^j], \quad (32)$$

where indices $i, j = 2, 3$ correspond to the coordinates on the y - z plane.

The Stückelberg equation gives

$$J_\phi^{(x)}(H + 2\Sigma - H_f e^{-2\sigma} X) + 2J_\phi^{(y)}(H - \Sigma - H_f e^\sigma X) = 0, \quad (33)$$

where

$$\begin{aligned} J_\phi^{(x)} &\equiv (3 + 3\alpha_3 + \alpha_4) - 2(1 + 2\alpha_3 + \alpha_4)e^\sigma X + (\alpha_3 + \alpha_4)e^{2\sigma} X^2, \\ J_\phi^{(y)} &\equiv (3 + 3\alpha_3 + \alpha_4) - (1 + 2\alpha_3 + \alpha_4)(e^{-2\sigma} + e^\sigma)X + (\alpha_3 + \alpha_4)e^{-\sigma} X^2, \end{aligned} \quad (34)$$

$H \equiv \dot{a}/a$, $\Sigma \equiv \dot{\sigma}$ and $X \equiv \alpha/a$. The metric equations of motion are given by

$$\begin{aligned} 3(H^2 - \Sigma^2) - \Lambda &= m_g^2[-(6 + 4\alpha_3 + \alpha_4) + (3 + 3\alpha_3 + \alpha_4)(2e^\sigma + e^{-2\sigma})X \\ &\quad - (1 + 2\alpha_3 + \alpha_4)(e^{2\sigma} + 2e^{-\sigma})X^2 + (\alpha_3 + \alpha_4)X^3], \\ \dot{\Sigma} + 3H\Sigma &= \frac{m_g^2}{3}(e^{-2\sigma} - e^\sigma)X[(3 + 3\alpha_3 + \alpha_4) - (1 + 2\alpha_3 + \alpha_4)(e^\sigma + r)X \\ &\quad + (\alpha_3 + \alpha_4)r e^\sigma X^2], \end{aligned} \quad (35)$$

where

$$r \equiv \frac{na}{\alpha} \equiv \frac{1}{XH_f} \left(\frac{\dot{X}}{X} + H \right). \quad (36)$$

We now look for solutions that are anisotropic ($\Sigma = 0$, $\sigma = \sigma_0 \neq 0$) and undergo a de Sitter ($\dot{H} = 0$, $H = H_0$) expansion, which implies that the remaining parameters are also constant, i.e. $X = X_0$, $r = r_0$. Excluding the fixed points which are isotropic $\sigma_0 = 0$ and those which exist only for special values of parameters (i.e. in a measure-zero subspace of the parameter space), we find a characteristic relation for the anisotropic fixed point, $r_0 = e^{-2\sigma_0}$, or,

$$X_0 = \frac{H_0}{H_f} e^{2\sigma_0}. \quad (37)$$

The remaining two equations allow us to determine X_0 and σ_0 . One can show that this solution is stable against homogeneous linear perturbations if [34]

$$M_\sigma^2 \equiv -\frac{3M^2 \tilde{M}^2 (9H_0^2 + \tilde{M}^2)}{\tilde{M}^4 + 9H_0^2 (3M^2 - \tilde{M}^2)} > 0, \quad (38)$$

where [35]

$$\begin{aligned} M^2 &\equiv \frac{H_0 m_g^2}{3H_f^3} [H_0^2 e^{3\sigma_0} (1 + 2e^{3\sigma_0})(\alpha_3 + \alpha_4) - 2H_0 H_f (1 + e^{3\sigma_0} + e^{6\sigma_0})(1 + 2\alpha_3 + \alpha_4) \\ &\quad + H_f^2 (2 + e^{3\sigma_0})(3 + 3\alpha_3 + \alpha_4)], \\ \tilde{M}^2 &\equiv -\frac{3H_0 m_g^2}{2H_f^3} [H_0^2 e^{6\sigma_0} (\alpha_3 + \alpha_4) - 2H_0 H_f e^{3\sigma_0} (1 + 2\alpha_3 + \alpha_4) + H_f^2 (3 + 3\alpha_3 + \alpha_4)]. \end{aligned} \quad (39)$$

⁶ The adoption of the de Sitter reference metric here is due to the flat spatial curvature associated with the Bianchi type-I metric. We remind the reader that the Minkowski reference metric cannot be put into a flat FLRW form (see section 3.1).

5.1.2. Linear perturbations. The linear perturbation theory around axisymmetric Bianchi type-I backgrounds in dRGT theory was formulated in [35]. The formulation can be used to calculate the coefficients of the kinetic terms for the five gravity degrees of freedom. Similarly to the off-shell kinetic terms around isotropic FLRW solutions mentioned in the second paragraph of section 5, it turns out that the kinetic term of one of the five degrees is proportional to $J_\phi^{(x)}$ and other two are proportional to $J_\phi^{(y)}$. The remaining two correspond to the standard two polarizations of the tensor graviton and thus they always have finite and positive kinetic terms.

By studying (33), we see that on the fixed point (37), $J_\phi^{(x)} \neq 0$ while $J_\phi^{(y)} = 0$. Hence, the kinetic terms for two of the expected five gravity degrees of freedom vanish, signaling for necessity of nonlinear analysis.

5.1.3. Nonlinear perturbations. Due to the broken $SO(3)$ symmetry, we can no longer use the standard scalar/vector/tensor decomposition for the perturbations. However, the axisymmetry of the background allows us to use $SO(2)$ symmetry in the classification. The following analysis is based on [35].

Even modes. The even-mode perturbations are introduced according to the decomposition (23) and once the non-dynamical degrees are integrated out, there are generically three dynamical degrees of freedom. However, once the background is fixed to be the anisotropic attractor solution, one of these modes has a vanishing kinetic term. On the other hand, we can still analyze the properties of the higher order kinetic terms by considering homogeneous deformations around the attractor solution, characterized by

$$\sigma = \sigma_0 + \epsilon \sigma_1 + \mathcal{O}(\epsilon^2), \quad \Sigma = \epsilon \Sigma_1 + \mathcal{O}(\epsilon^2) = \epsilon \dot{\sigma}_1 + \mathcal{O}(\epsilon^2), \quad (40)$$

where the background-physical-metric coefficients are given by $g_{tt} = -N^2$, $g_{xx} = a^2 e^{4\sigma}$, $g_{yy} = g_{zz} = a^2 e^{-2\sigma}$. Furthermore, we have defined Σ as $\Sigma \equiv \dot{\sigma}$, and expanded both σ and Σ on the attractor solution. After diagonalization, the kinetic terms become

$$\begin{aligned} \kappa_1 &\simeq \left[\frac{8 p^4}{p_T^4} - \frac{8 \tilde{M}^4}{\tilde{M}^4 + 9 H_0^2 (3 M^2 - \tilde{M}^2)} \right]^{-1}, \\ \kappa_2 &\simeq \frac{2 a_0^4 e^{8\sigma_0} \tilde{M}^2 p_L^2 [9 H_0^2 p^4 (\tilde{M}^2 - 3 M^2) + \tilde{M}^4 p_L^2 (-2 p^2 + p_L^2)]}{\tilde{M}^2 p_L^2 (\tilde{M}^2 - 3 p^2)^2 - 9 H_0^2 (\tilde{M}^2 - 3 M^2) [6 p^4 + \tilde{M}^2 (-4 p^2 + p_L^2)]}, \\ \kappa_3 &\simeq - \frac{3 \tilde{M}^2 e^{2\sigma_0} a_0^4 p_T^2 [3 M^2 (9 H_0^2 + \tilde{M}^2) \sigma_1 + 2 H_0 (9 M^2 - 2 \tilde{M}^2) \Sigma_1]}{(1 - e^{6\sigma_0}) [\tilde{M}^4 - 27 H_0^2 (3 M^2 - \tilde{M}^2)]}, \end{aligned} \quad (41)$$

where p_L and p_T are the components of the physical momentum vector along the \hat{x} direction and on the y - z plane, respectively, while $p^2 \equiv p_L^2 + p_T^2$. Furthermore, we have introduced the mass scale \tilde{M}^2 as in equation (39). Generically, the absence of ghosts imposes momentum-dependent conditions. However, one can ensure stability at all scales by adopting the sufficient condition

$$\tilde{M}^2 < 0, \quad M^2 < \frac{\tilde{M}^2 (9 H_0^2 - \tilde{M}^2)}{27 H_0^2} < 0, \quad (42)$$

under which, both κ_1 and κ_2 can be made positive. For a parameter set which satisfies (42), the no-ghost condition for the third mode (with order ϵ kinetic term) becomes

$$\sigma_0 [-3 |M^2| (9 H_0^2 - |\tilde{M}^2|) \sigma_1 + 2 H_0 (2 |\tilde{M}^2| - 9 |M^2|) \Sigma_1] < 0, \quad (43)$$

which depends linearly on the homogeneous deformations σ_1 and Σ_1 around the fixed point. Thus, regardless of the value of M^2 and \tilde{M}^2 , there could always be a region where σ_1 and

Σ_1 conspire to render the third mode a ghost. On the other hand, if the initial conditions are such that the system is close to the attractor, it is possible to connect the evolution of Σ_1 algebraically to that of σ_1 and obtain a regime where one can avoid the instability. By considering the equation of motion for σ_1 ,

$$\dot{\Sigma}_1 + 3H_0 \Sigma_1 + M_\sigma^2 \sigma_1 = 0, \quad (44)$$

we first note that condition (38) for the stability of the fixed point against homogeneous perturbations, combined with the conditions (42), yields

$$9H_0^2 - |\tilde{M}^2| > 0. \quad (45)$$

To satisfy condition (43), we suppose that the system is in the attractor regime, so that $\Sigma_1 \propto \sigma_1$ and that σ_1 does not change sign during the course of evolution. This scenario can be attained if the friction term in (44) dominates over the mass term, i.e.

$$9H_0^2 > 4M_\sigma^2. \quad (46)$$

Then, solving equation (44) and evaluating the solution at late times, we find the relation

$$\Sigma_1 \simeq \left(-\frac{3}{2}H_0 + \sqrt{\frac{9}{4}H_0^2 - M_\sigma^2} \right) \sigma_1. \quad (47)$$

Thus, in this regime, condition (43) can in principle be satisfied by choosing the appropriate sign for σ_1 .

Odd modes. The odd-mode perturbations are introduced according to the decomposition (24) and once the non-dynamical degree is integrated out (another mode can be gauged away), there are generically two dynamical degrees of freedom. On the anisotropic attractor solution, the kinetic term of one of these modes vanishes, and as we did for the even modes, we consider homogeneous deviations from the fixed point to determine the conditions for nonlinear stability.

After diagonalization, the kinetic terms become

$$\begin{aligned} \kappa_1 &\simeq \frac{a_0^4 e^{-4\sigma_0} p_L^2 p_T^4}{2 p^2}, \\ \kappa_2 &\simeq -\frac{3\tilde{M}^2 e^{2\sigma_0} a_0^4 p_T^2 [3M^2 (9H_0^2 + \tilde{M}^2)\sigma_1 + 2H_0 (9M^2 - 2\tilde{M}^2)\Sigma_1]}{4(1 - e^{6\sigma_0}) [\tilde{M}^4 - 27H_0^2 (3M^2 - \tilde{M}^2)]}. \end{aligned} \quad (48)$$

The first kinetic term is always positive, whereas the second mode acquires a kinetic term proportional to the deviation from the fixed point. In fact, up to a numerical factor, κ_2 above is the same as the kinetic term of the third mode in equation (42), so if the conditions discussed in the even sector are satisfied, the odd sector will also be stable.

5.2. Extended theory I: quasi-dilaton

The quasi-dilaton theory is obtained by introducing a scalar field σ associated with a dilation-like global symmetry to the dRGT action (σ has different meaning than the previous subsection),

$$\sigma \rightarrow \sigma - \alpha M_{\text{pl}}, \quad \phi^a \rightarrow e^\alpha \phi^a. \quad (49)$$

The action compatible with this symmetry is given in Einstein frame as [18]⁷

$$\begin{aligned} S = \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g} \left[R - 2\Lambda + 2m_g^2 (\mathcal{L}_2(\tilde{\mathcal{K}}) + \alpha_3 \mathcal{L}_3(\tilde{\mathcal{K}}) + \alpha_4 \mathcal{L}_4(\tilde{\mathcal{K}})) \right. \\ \left. - \frac{\omega}{M_{\text{pl}}^2} \partial_\mu \sigma \partial^\nu \sigma + \mathcal{L}_{\text{matter}} \right], \end{aligned} \quad (50)$$

⁷ There is an additional term allowed by the symmetry, $\int d^4x \sqrt{-f} e^{4\sigma/M_p}$, which does not change the conclusions of the present discussion (see [36, 37] for details).

where \mathcal{L}_2 , \mathcal{L}_3 and \mathcal{L}_4 are given in equation (3), but the building block tensor (4) is replaced with

$$\mathcal{K}_\nu^\mu \rightarrow \bar{\mathcal{K}}_\nu^\mu \equiv \delta_\nu^\mu - e^{\sigma/M_{\text{Pl}}} (\sqrt{g^{-1}f})_\nu^\mu. \quad (51)$$

5.2.1. Self-accelerating solutions. We adopt the Minkowski reference metric and the flat FLRW ansatz for the physical metric as

$$f_{\mu\nu} = -n^2(t) \delta_\mu^0 \delta_\nu^0 + \delta_{ij} \delta_\mu^i \delta_\nu^j, \quad g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j. \quad (52)$$

The equations of motion for the Stückelberg fields yield

$$(1 - X) X [3 + 3\alpha_3 + \alpha_4 - (3\alpha_3 + 2\alpha_4) X + \alpha_4 X^2] = \frac{\text{constant}}{a^4}, \quad (53)$$

where $X \equiv e^{\sigma/M_{\text{Pl}}}/a$, leading to four attractors: $X = 0$, $X = 1$ and $X = X_\pm$ with

$$X_\pm = \frac{3\alpha_3 + 2\alpha_4 \pm \sqrt{9\alpha_3^2 - 12\alpha_4}}{2\alpha_4}. \quad (54)$$

Among them, $X = 0$ and $X = 1$ leads to either strong coupling or instability [18]. We thus consider $X = X_\pm$ as backgrounds. Along these branches of solutions, the (modified) Friedmann equation becomes

$$\left(3 - \frac{\omega}{2}\right) H^2 = \Lambda + \Lambda_\pm, \quad (55)$$

where the graviton mass manifests as the effective cosmological constant

$$\Lambda_\pm = -\frac{m_g^2}{2\alpha_4^3} [9(3\alpha_3^4 - 6\alpha_3^2\alpha_4 + 2\alpha_4^2) \pm \alpha_3(9\alpha_3^2 - 12\alpha_4)^{3/2}]. \quad (56)$$

From (55), we immediately see that a sensible cosmology requires $\omega < 6$. Finally, the equation of motion for the quasi-dilaton field gives

$$r \equiv n a = 1 + \frac{\omega H^2}{m_g^2 X^2 [\alpha_3 (X - 1) - 2]}. \quad (57)$$

5.2.2. Perturbations. We now introduce perturbations as [36]

$$\begin{aligned} \sigma &= M_{\text{Pl}} [\log(aX) + \delta\sigma], \quad \delta g_{00} = -2\Phi, \quad \delta g_{0i} = a(B_i^T + \partial_i B), \\ \delta g_{ij} &= a^2 \left[2\Psi \delta_{ij} + \left(\partial_i \partial_j - \frac{\delta_{ij}}{3} \partial_l \partial^l \right) E + \frac{1}{2} (\partial_i E_j^T + \partial_j E_i^T) + h_{ij}^{TT} \right], \end{aligned} \quad (58)$$

while we fix the unitary gauge $\delta\phi^a = 0$, where B_i^T and E_i^T are transverse and h_{ij}^{TT} is transverse and traceless. With respect to the dRGT theory, we have an additional scalar field, so in total, we expect two tensor, two vector and two scalar degrees, once the non-dynamical modes are integrated out.

Tensor perturbations. The quadratic action for the tensor modes reduces to

$$S_T = \frac{M_{\text{Pl}}^2}{8} \int d^3k a^3 dt \left[|\dot{h}_{ij}^{TT}|^2 - \left(\frac{k^2}{a^2} + M_{\text{GW}}^2 \right) |h_{ij}^{TT}|^2 \right], \quad (59)$$

where

$$M_{\text{GW}}^2 \equiv \frac{m_g^2 (r-1) X^3}{X-1} + H^2 \omega \left(\frac{r}{r-1} + \frac{2}{X-1} \right). \quad (60)$$

Generically, $M_{\text{GW}} \sim \mathcal{O}(H)$ so even if the tensor modes are tachyonic, the timescale of their instability is of the order of the age of the universe [36].

Vector perturbations. For the vector modes, the quadratic action is

$$S_V = \frac{M_{\text{Pl}}^2}{16} \int d^3k a^3 dt k^2 \left[\frac{|\dot{E}_i^T|^2}{\left(1 + \frac{k^2(r^2-1)}{2a^2 H^2 \omega}\right)} - M_{\text{GW}}^2 |E_i^T|^2 \right]. \quad (61)$$

We see from (61) that if $(r^2 - 1)/\omega < 0$, there is a critical momentum above which the vector modes have ghost instability. Therefore, the UV cut-off scale of the effective theory Λ_{UV} should be lower than this critical (physical) momentum to ensure the stability of the system: $\Lambda_{\text{UV}}^2 (1 - r^2)/(H^2 \omega) < 2$. In addition, the frequency in the canonical normalization yields a further condition on avoiding tachyonic instability, which arises if $M_{\text{GW}}^2 > 0$ and $(r^2 - 1)/\omega > 0$. The growth rate of this instability can be made lower than or at most of the cosmological scale for all physical momenta below the UV cut-off Λ_{UV} , provided that [36]

$$\Lambda_{\text{UV}}^2 \lesssim \frac{2H^2 \omega}{r^2 - 1}. \quad (62)$$

Scalar perturbations. After integrating out $\delta g_{0\mu}$ as well as the would-be BD degree, the scalar sector contains two coupled modes. The kinetic part of the quadratic action is formally written as

$$S_S \ni \int \frac{d^3k}{2} a^3 dt [K_{11} |\dot{Y}_1|^2 + K_{22} |\dot{Y}_2|^2 + K_{12} (\dot{Y}_1 \dot{Y}_2^* + \dot{Y}_2 \dot{Y}_1^*)], \quad (63)$$

where Y_1 and Y_2 are particular linear combinations of Ψ and $\delta\sigma$. For our purposes, it is enough to study the determinant of the kinetic matrix, given by

$$\det K \equiv K_{11}K_{22} - K_{12}^2 = \frac{3k^6 \omega^2 a^4 H^4}{\left[\omega a^2 H^2 - \frac{4k^2}{(6-\omega)}\right] (r-1)^2}, \quad (64)$$

The absence of ghost degrees in the scalar sector requires $\det K > 0$ as a necessary condition. We first note that the determinant is always negative if $\omega < 0$. Along with the condition obtained from (55), we thus obtain $0 < \omega < 6$ as a necessary condition.

Furthermore, demanding that $\det K > 0$ for all physical momenta below the UV cutoff of the theory, we obtain

$$\frac{\Lambda_{\text{UV}}}{H} < \frac{\sqrt{\omega(6-\omega)}}{2} < \frac{3}{2}, \quad (65)$$

where we have used the condition $0 < \omega < 6$ to obtain the last inequality. Unfortunately, (65) is not acceptable since it would imply that the UV cut-off scale would be lower than the cosmological scale and that the theory would not be applicable to cosmology. Therefore, we conclude that for physical wavelengths shorter than cosmological scales, $\det K < 0$ and one of the two degrees of freedom is a ghost [36].

It can also be checked (see [36] for details) that energies of the ghost mode are not parametrically higher than $H \sim m_g$. This signals the presence of ghost instabilities in the regime of validity of the effective field theory.

5.3. Extended theory II: varying mass

A further way of extending the dRGT theory is to allow the parameters of the theory to vary with a scalar field σ . The action in this case is [20],

$$S = \int d^4x \sqrt{-g} \left\{ M_{\text{Pl}}^2 \left[\frac{R}{2} - \Lambda + m_g^2(\sigma) [\mathcal{L}_2 + \alpha_3(\sigma) \mathcal{L}_3 + \alpha_4(\sigma) \mathcal{L}_4] \right] - \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - V(\sigma) + \mathcal{L}_m \right\}, \quad (66)$$

where \mathcal{L}_2 , \mathcal{L}_3 and \mathcal{L}_4 are given by equation (3).

5.3.1. Background. As in the previous subsection, we adopt the Minkowski reference metric and the flat FLRW ansatz for the physical metric (52). The equations of motion for the Stückelberg fields yield

$$\frac{m_g^2(X-1)}{X^3} [3 - 3(X-1)\alpha_3 + (X-1)^2\alpha_4] = \text{constant}, \quad (67)$$

where $X \equiv 1/a$ and $r \equiv an$. Due to the assumptions of flat space and Minkowski reference metric, if m_g and $\alpha_{3,4}$ are time independent, the solution $X = \text{constant}$ does not allow any non-trivial cosmologies (see the second paragraph of subsection 3.1).

By defining

$$\begin{aligned} \rho_m &\equiv M_{\text{Pl}}^2 m_g^2 (X-1) [6 + 4\alpha_3 + \alpha_4 - X(3 + 5\alpha_3 + 2\alpha_4) + X^2(\alpha_3 + \alpha_4)], \\ p_m &\equiv M_{\text{Pl}}^2 m_g^2 [6 + 4\alpha_3 + \alpha_4 - (2+r)X(3 + 3\alpha_3 + \alpha_4) \\ &\quad + (1+2r)X^2(1 + 2\alpha_3 + \alpha_4) - rX^3(\alpha_3 + \alpha_4)], \\ Q &\equiv M_{\text{Pl}}^2 m_g^2 \dot{\sigma} (X-1)^2 \left\{ \alpha_3' (4 - X - 3rX) + \alpha_4' (X-1)(rX-1) \right. \\ &\quad \left. + \frac{2m_g'}{m_g} \left[3 - (X-1)\alpha_3 + \frac{rX-1}{X-1} [3 - 3(X-1)\alpha_3 + (X-1)^2\alpha_4] \right] \right\}, \\ \rho_\sigma &\equiv \frac{\dot{\sigma}^2}{2} + V, \quad p_\sigma \equiv \frac{\dot{\sigma}^2}{2} - V, \end{aligned} \quad (68)$$

we can write the set of background equations of motion in the following form:

$$\begin{aligned} 3H^2 &= \Lambda + \frac{1}{M_{\text{Pl}}^2} (\rho_\sigma + \rho_m), \quad \dot{H} = -\frac{1}{2M_{\text{Pl}}^2} [(\rho_\sigma + p_\sigma) + (\rho_m + p_m)], \\ \dot{\rho}_m + 3H(\rho_m + p_m) &= -Q, \quad \dot{\rho}_\sigma + 3H(\rho_\sigma + p_\sigma) = Q. \end{aligned} \quad (69)$$

where prime denotes differentiation with respect to σ .

Although dynamical analysis for these equations have been studied in the literature [38], there is not yet a simple self-accelerating solution in the varying parameter massive gravity. In the following, we do not assume any specific evolution and keep the functions $m_g(\sigma)$, $\alpha_3(\sigma)$ and $\alpha_4(\sigma)$ generic.

5.3.2. Perturbations. We now introduce perturbations, following [36]. The metric is decomposed as in equation (58) and we adopt the unitary gauge as $\delta\phi^a = 0$, while the scalar field is perturbed as

$$\sigma = \langle\sigma\rangle + M_{\text{Pl}} \delta\sigma. \quad (70)$$

Tensor perturbations. The tensor action reduces to

$$S_T = \frac{M_{\text{Pl}}^2}{8} \int d^3k a^3 dt \left[|\dot{h}_{ij}^{TT}|^2 - \left(\frac{k^2}{a^2} + M_{\text{GW}}^2 \right) |h_{ij}^{TT}|^2 \right], \quad (71)$$

where

$$M_{\text{GW}}^2 = \frac{(r-1)X^2}{(X-1)^2} \left[m_g^2 (X-1) - \frac{\rho_m}{M_{\text{Pl}}^2} \right] - \left(\frac{1}{r-1} + \frac{2X}{X-1} \right) \frac{\rho_m + p_m}{M_{\text{Pl}}^2}. \quad (72)$$

The stability of long wavelength tensor modes is ensured by $M_{\text{GW}}^2 > 0$.

Vector perturbations. For the vector modes, the action is

$$S_V = \frac{M_{\text{Pl}}^2}{16} \int d^3k a^3 dt k^2 \left[\frac{|\dot{E}_i^T|^2}{\left(1 - \frac{k^2 (r^2-1) M_{\text{Pl}}^2}{2a^2 (\rho_m + p_m)} \right)} - M_{\text{GW}}^2 |E_i^T|^2 \right]. \quad (73)$$

By requiring that the kinetic term is positive for all physical momenta below the cut-off scale of the theory Λ_{UV} , we obtain the stability condition for the vector modes as

$$\frac{\Lambda_{UV}^2 (1 - r^2)}{H^2 R} < 2, \quad R \equiv -\frac{\rho_m + p_m}{M_{Pl}^2 H^2}. \quad (74)$$

Under the above condition, we can further analyze the stability of the vector sector, by introducing a time reparametrization which renders the modes E_i^T canonical, and then requiring that their frequency is an increasing function. This procedure yields a sufficient (but not necessary) condition for stability

$$\left[1 + \frac{1}{8NH} \frac{d}{dt} \ln \left(\frac{RM_{GW}^2}{r^2 - 1} \right) \right] \frac{\Lambda_{UV}^2 (1 - r^2)}{H^2 R} < \frac{3}{2} + \frac{1}{4NH} \frac{d \ln (M_{GW}^2)}{dt}. \quad (75)$$

Scalar perturbations. As in the quasi-dilaton theory, we integrate out the nondynamical degrees and are left with two coupled modes in the scalar sector. The kinetic part of the action is formally

$$S_S \ni \int \frac{d^3 k}{2} a^3 dt [K_{11} |\dot{Y}_1|^2 + K_{22} |\dot{Y}_2|^2 + K_{12} (\dot{Y}_1 \dot{Y}_2^* + \dot{Y}_2 \dot{Y}_1^*)], \quad (76)$$

where Y_1 and Y_2 are linear combinations of Ψ and $\delta\sigma$. For our purposes, it is enough to study $\det K = K_{11}K_{22} - K_{12}^2$, whose explicit form is

$$\det K = \frac{3 M_{Pl}^2 a^2 k^6 (\rho_m + p_m)^2 (\rho_\sigma + p_\sigma - 6 M_{Pl}^2 H^2)}{(r - 1)^2 \left[4 M_{Pl}^4 H^2 \frac{k^2}{a^2} - (\rho_m + p_m)(\rho_\sigma + p_\sigma - 6 M_{Pl}^2 H^2) \right]}. \quad (77)$$

By requiring that the determinant is positive, we see that in order to avoid a ghost degree of freedom, the momenta in the range $0 \leq k/a \leq \Lambda_{UV}$ should all satisfy

$$\left(\frac{\rho_\sigma + p_\sigma}{4 M_{Pl}^2 H^2} - \frac{3}{2} \right)^{-1} \frac{k^2}{a^2} > \frac{\rho_m + p_m}{M_{Pl}^2}. \quad (78)$$

Explicit diagonalization of the system shows that this condition is actually a sufficient condition to avoid ghost instabilities in the scalar sector [36].

For a background solution which can effectively describe the late time acceleration, we can assume a de Sitter like expansion, i.e. $|\dot{H}| \ll H^2$. With these considerations, the stability requirement for the scalar sector becomes even simpler,

$$R + \frac{4}{R - 6} \frac{k^2}{H^2 a^2} > 0, \quad (79)$$

where R is defined in (74). If indeed all the physical momenta below the cut-off scale Λ_{UV} satisfy (79) and if we suppose $\Lambda_{UV}/H > 3/2$ so that the theory is applicable to cosmological scales, then the no-ghost condition for scalar perturbations in the regime $|\dot{H}| \ll H^2$ becomes simply

$$R > 6. \quad (80)$$

6. Summary and discussion

The extension of GR by a mass term has been studied for several decades. Nonetheless, a self-consistent nonlinear massive gravity theory with five propagating degrees of freedom, dubbed the dRGT theory, has been proposed only recently.

In this paper, we reviewed several cosmological solutions in the context of the dRGT theory [8, 9]. We have firstly described open FLRW solutions with a Minkowski reference metric.

By considering a general FLRW-form fiducial metric, the branch of open FLRW solutions was generalized to FLRW solutions with a general spatial curvature. However, for all of these FLRW-type cosmological solutions, the kinetic terms of three among five gravity degrees of freedom vanish at the level of the quadratic action. This phenomenon is a consequence of the symmetry of the FLRW background. On analyzing the behavior of the nonlinear perturbations by considering a consistent truncation, it was then shown that there is always at least one ghost (among the five degrees of freedom) in the gravity sector.

We have then discussed two approaches toward healthy cosmologies in massive gravity. One proposal is to introduce relatively large anisotropy in the configuration of Stückelberg fields, which form the hidden sector of the theory. We considered the fixed point solution named as ‘anisotropic FLRW’, a solution with the FLRW symmetry in the visible sector (physical metric) but with anisotropy in the hidden sector. Performing a nonlinear analysis around the anisotropic fixed point yields that anisotropic FLRW solutions can be ghost-free for a range of parameters and initial conditions. The second proposal discussed here consists of introducing an extra degree of freedom coupled to the hidden sector. As examples for this possibility, we have considered the quasi-dilaton theory and the varying mass model. For the quasi-dilaton theory, the self-accelerating background turns out to be unstable. On the other hand, in the varying mass case, there is a regime of parameters in which a stable cosmological evolution is possible, although viable self-accelerating solutions yet remain to be found.

Besides the stability investigation, the study of observational signals from graviton mass, although not included in this review, is also important. For example, in [39], it was found that graviton mass may leave a prominent feature with a sharp peak in the stochastic gravitational wave spectrum. The position and height of the peak may tell us information about the graviton mass today and the duration of the inflationary period.

Last but not least, as a developing field, massive gravity still leaves many intriguing unsolved questions. One of the most interesting questions is the construction of a possible UV completion of massive gravity. One of the potential directions to this end would be to seek a mechanism that realizes the specific structure of the graviton mass term as a consequence of a spontaneous symmetry breaking. Another important question is the fate of super-luminal mode [40] in the gravity sector. It is generically expected that in the massless limit, observable effects of the super-luminal mode should disappear and GR should be recovered, provided that the mode is excited by a fixed amount of matter source. Thus, it should be possible to obtain an observational upper bound on the graviton mass although it is probably not stronger than $m_g < O(H_0)$.

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