QUANTUM SCATTERING BY THE DYNAMICAL PRINCIPLE

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Anajak Phonchantuek

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ABSTRACT

Yukawa scattering, includes the effecting mass in the interaction to incident particle, is pedagogically interpreted by the Schwinger's quantum dynamical principle involving the generating functions. These functions are finally replaced by a functional differential operation. As for the results, we got the asymptotically free Green function that explains the behavior of the Yukawa potential when the mass parameter is increasing. In particular, it can also lead to scattering amplitude and differential cross section, respectively.

CHAPTER I

INTRODUCTION

1.1 Literature review

Presently, many discovery of elementary particles were found from the scattering by accelerating those particle to hit each other, such as electron, neutron, proton or nucleus. These occurred particles were observed and detected in the experiment. Physicists had discovered many particle from the detector, shows and tracks the particle's trajectories on screen, along late 19th century until now. Historically, they found the electron, nucleus (consists of proton and neutron), the photon (by Compton scattering), mesons, antiparticles, neutrinos (carry out the momentum energy), strange particles and quarks, respectively. Finally, the standard model was established to interpret all of particles [2]. This theory includes many rules of number ratio which explain the reaction probability, like the chemical reaction. Moreover, the toy model (Feynman diagram), presented by R. P. Feynman, simplifies all this scattering phenomena.[13, 14]

In the field of quantum theory, many methods have been applied to explain a behavior of the particles traveling; propagator. One of all, the quantum dynamical principle (QDP) provides a formalism for quantum physics which is developed and proposed by J. S. Schwinger [17, 18, 19, 20, 21, 22, 23]. It has been proved to be the powerful and elegant tool of the elementary particles dynamics and, furthermore, high-energy physics. It gives an expression for the variation of a transformation function, $\delta\langle \mathbf{a}t|\mathbf{b}t'\rangle$, from a B at time t' to an A at time t, whereas occurring from any changing of the parameters of Hamiltonian such as *masses, coupling constants, charges, external sources, etc.* This method is derived from the functional differentiation that respects to the general Hamiltonian, depending on time. It bases on the source theory, imposed by Schwinger [23]. These sources generate the coordinates or degrees of freedom for any dynamical system that depends on time evolution. Our interested system is satisfied the Hamiltonian with sources, like $H(\mathbf{q}, \mathbf{p}, t) = H_0 - \mathbf{q} \cdot \mathbf{F}(t) + \mathbf{p} \cdot \mathbf{S}(t)$, where H_0 is the free Hamiltonian which only includes the momenta, $\mathbf{p}^2/2m$. This generating function (source) is finally replaced by the functional differential. For the specific transformations are $\langle \mathbf{q}t|\mathbf{q}'t'\rangle$, $\langle \mathbf{q}t|\mathbf{p}t'\rangle$ or $\langle \mathbf{p}t|\mathbf{p}'t'\rangle$ in $\mathbf{q}, \mathbf{p}$ language. These transformations functions lead to the propagator, as the final result, that we have to evaluate it. This propagator actually describes the behavior of particle's motion, evolved on time. The quantum dynamical principle is a wideness length tool that can be applied to many problems of quantum physics, including the particle propagator through the potential problem, the forced harmonic oscillator problems, Bose/Fermi excitations. Furthermore, surprisingly, it can also apply to the path integrals [5, 15]. Comparing with the path integrals method, proposed by R. P. Feynman, it is an infinite continual integrating which uses many approximations for scattering case, such as estimating a range [12]. In a case of the dynamical principle, however, we have to do many step of mathematics but it is elegant and powerful when we actually want to use this transformation function in the closed-form for propagator calculating.

For theory of scattering, the classical one is considered an impact parameter b, and a scattering angle θ, when given a small impact parameter, will get the greater scattering angle. In the measurement, the particle incident with a cross-sectional area, σ will scatter corresponding solid angle, Ω. The ratio between a cross -sectional and a solid angle is the differential (scattering) cross section, denoted as $D(\theta) = d\sigma/d\Omega$. Also, for the quantum scattering which is measured within the solid angle, we deal with a wave traveling scatter the potential and get the outgoing spherical wave. So, we have to work with scattering amplitude, $f(\theta)$ that is the probability of scattering [1]. Finally, we get the differential cross section from that scattering amplitude by taking the absolute square. In this case, the Born approximation which applies and describes the scattering amplitude of incident particles, is very useful. Obviously, it leads to the Born series, explains a situation that the incident particles are effected by the force of potential with many times. If we need correcting the Born approximation, it has to deal in higher order of this scattering. The quantum dynamical principle also leads to this approximation by deriving the transformation function, gives a propagator. It gives us the modified Born approximation which is multiplied with a particle's trajectory [5, 7], eventually. Thus, this method inspires us applying to other potential such as Yukawa potential.

For the condense-matter field, so-called the material sciences, they work with the charge particles (electrons) scattered the potential of the semiconductors, valence-band holes, and analyze the radial wave functions from the Lippman-Schwinger integral equation by dealing with the Yukawa potential [11]. Many application in this field is always based on the quantum theory that consider the potential or a band energy, depends on the material kind.

1.2 Objectives

For this thesis, our objective is how to use the method of Quantum Dynamical principle in a case of Yukawa scattering. And, this method is applied to Yukawa potential to obtain the asymptotically free Green function which leads to the scattering amplitude and differential cross section eventually.

1.3 Frameworks

In this project, we apply the quantum dynamical principle to deal with the Yukawa potential, called the screened Coulomb potential that has the mass of particles involving. A detail about the basic of scattering is provided in the Chapter II. In the Chapter III, we simplify and explain a process of the quantum dynamical principle, a visual concept of this method and, in particular, the simple source theory concept. The free particle and the asymptotically free Green functional propagators, by using the QDP, are in the Chapter IV. It is about the Coulomb scattering, an interaction between the charge particles, and then we obtained the transition amplitude. For the Chapter V, it is about the calculation of this method (QDP) which using Yukawa potential to carry out the asymptotically free Green function. The last chapter is a conclusion that about the obtained propagator and differential cross section, given in Chapter VI.

CHAPTER II

THE THEORY OF SCATTERING

Now a day, many particles, also known as “standard model” of elementary particles, are discovered by the experimental scattering system in the frame of laboratory. In this chapter, we present about the theory of scattering, begins with classical and becomes to quantum. Generally, the system of a scattering consists of two particles, are the own potential particle (a target) and the in-coming particle (an incident). In the classical scattering, it deals with the differential cross section. But in the case of the quantum scattering, it performs with the scattering amplitude. The detail is given in the following section.

2.1 Classical Scattering

Let’s imagine about the scattering system, in normal sense, when you kick a smaller football to the bigger one and then its trajectory will bend from the straight line. In experimental, a particle collides on the target (Scattering center) with the momentum energy E and **impact parameter** b. After scattering, it emits at some **scattering angle** θ -(Figure 1). When we give the small impact parameter, we will get the large scattering angle.

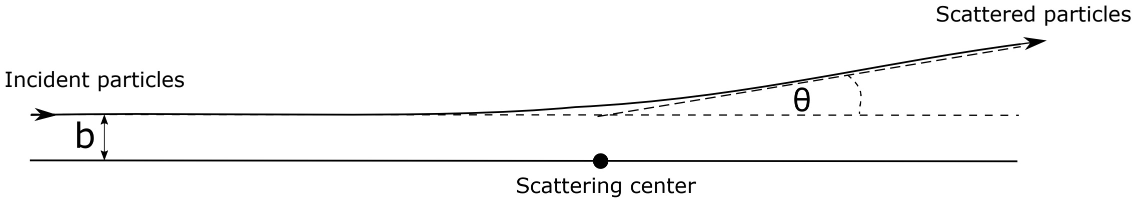

Figure 1 The classical scattering with the impact parameter b and the scattering angle θ.

The particles incident with an infinitesimal of cross-sectional area $d\sigma$ is scattered from the scattering center with an infinitesimal solid angle $d\Omega$. $d\sigma$ and $d\Omega$ are proportional value which equal to the **differential cross-section**, (Figure 2),

$$D(\theta) \equiv \frac{d\sigma}{d\Omega} \ . \tag{2.1}$$

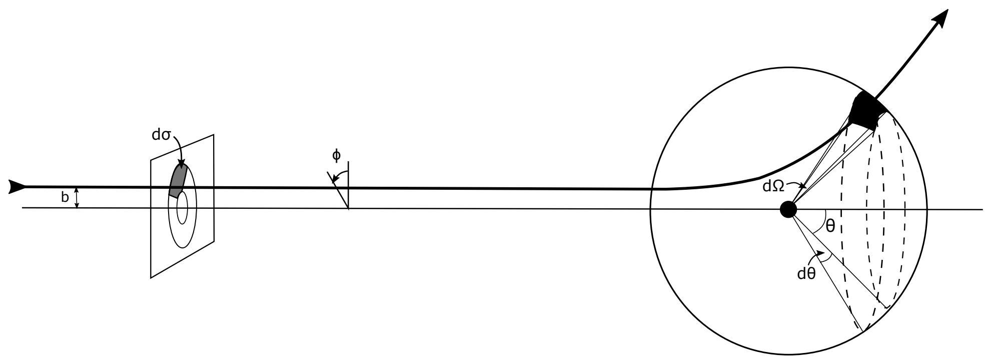

Figure 2 Particles incident in the cross-section area $d\sigma$ scatter to the solid angle $d\Omega$.

For the impact parameter and the azimuthal angle ϕ, $d\sigma = bdbd\phi$ and $d\Omega = \sin\theta d\theta d\phi$, thus

$$D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \ . \tag{2.2}$$

The absolute of $db/d\theta$ in Eq.(2.2) because θ is a function of b which decreasing.

After integrating over solid angles for $D(\theta)$, we get the **total cross-section** is

$$\sigma \equiv \int D(\theta) d\Omega. \tag{2.3}$$

In the experiment, a target is "soft" such as the Coulomb potential of nucleus, however, for normal sense with football kicking is "hard-sphere" which about incoming particles with the impact parameter b might miss the target. The soft-sphere is not about hitting or missing.

Lastly, we assume that have an incident particles beam, it is called intensity (**luminosity**)

$$\mathcal{L} \equiv \frac{\textbf{number of incident particles}}{\textbf{unit area}} / \textbf{unit time} \,. \tag{2.4}$$

From Figure 2, the number of particles passing an $d\sigma$ area per unit time and scattering through the solid angle $d\Omega$, is

$$dN = \mathcal{L} d\sigma = \mathcal{L} D(\theta) d\Omega \,. \tag{2.5}$$

Therefore

$$D(\theta) = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega} \tag{2.6}$$

which is the definition of the differential cross-section in the laboratory that is easy to be measured [1].

2.2 Quantum Scattering

In a case of the quantum scattering theory, we are thinking about an incident plane wave, $\psi(z) = Ae^{ikz}$ travels in the z direction, which enters a potential and scatters that producing an outgoing spherical wave [1] - (Figure 3). Therefore, we have to look for the solution of the Schrödinger equation. It is given in the form as

$$\psi(r, \theta) \approx A \left[e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right], \tag{2.7}$$

where radius r is large, an azimuthal ϕ of target is symmetric, and $f(\theta)$ is the **scattering amplitude**.

The **wave number** k is associated to the energy of the incident particles that is

$$k \equiv \frac{\sqrt{2mE}}{\hbar} \,. \tag{2.8}$$

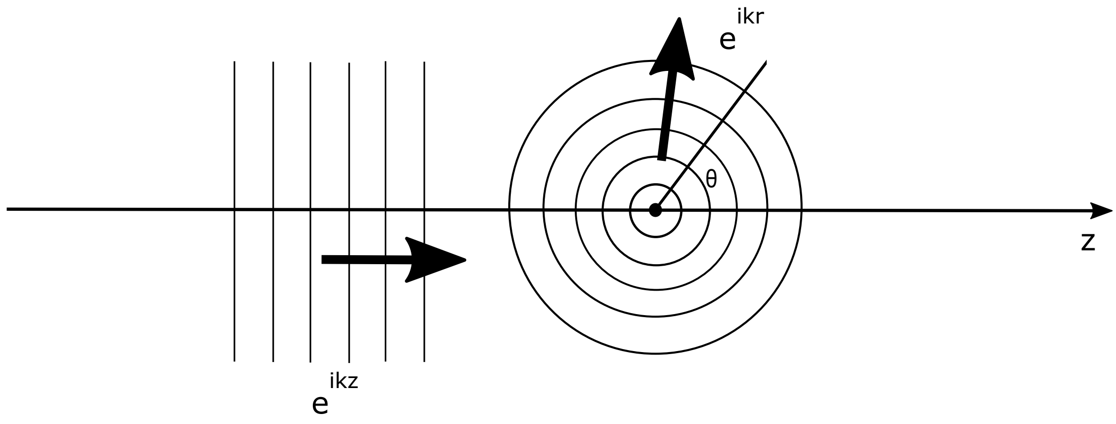

Figure 3 Wave scattering which an incoming plane wave produces an outgoing spherical wave.

This scattering amplitude $f(\theta)$ gives us the **probability of scattered at an angle** θ that is related to the differential cross-section. For the probability of the incident particles through an infinitesimal cross-section area $d\sigma$ at speed v in time dt is given as

$$dP = |\psi_{incident}|^2 dV = |A|^2 (vdt) d\sigma. \tag{2.9}$$

For the probability of the scattered particles passing a solid angle $d\Omega$ is

$$dP = |\psi_{scattered}|^2 dV = \frac{|A|^2 |f|^2}{r^2} (vdt) r^2 d\Omega. \tag{2.10}$$

The following result, by comparing Eq.(2.9) and (2.10), $d\sigma = |f|^2 d\Omega$, is

$$D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2. \tag{2.11}$$

The differential cross-section equals to the absolute square of the scattering amplitude which solving from the Schrödinger equation.

This problem can use many techniques to evaluate the scattering amplitude, such as the **partial wave analysis** and the **Born approximation**. However, in this thesis, we will give you the detail of the Born approximation method so you can read more content about the partial wave analysis in the J.Griffiths' book [1].

2.3 The Born approximation

We gently start with the time-independent Schrödinger equation to find the integral form of this equation,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi. \tag{2.12}$$

It is simply written as

$$(\nabla^2 + k^2)\psi = \mathcal{Q}, \tag{2.13}$$

where

$$k \equiv \frac{\sqrt{2mE}}{\hbar} \quad \text{and} \quad \mathcal{Q} \equiv \frac{2m}{\hbar^2}V\psi \tag{2.14}$$

This is the inhomogeneous differential equation, called the **Helmholtz equation**, with $\mathcal{Q}$ is depends on ψ. Thus, Eq.(2.13) obviously becomes

$$(\nabla^2 + k^2)G(\mathbf{r}) = \delta^3(\mathbf{r}) \tag{2.15}$$

We have to find $G(r)$, called **Green's function**, which solves the above equation. So, the ψ is expressed, integral form, as

$$\psi(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r_0})\mathcal{Q}(\mathbf{r_0})\mathrm{d}^3\mathbf{r_0}, \tag{2.16}$$

and then we substitute Eq.(2.16) in Eq.(2.13). We get

$$\begin{aligned}(\nabla^2 + k^2)\psi(\mathbf{r}) &= \int \left[(\nabla^2 + k^2)G(\mathbf{r} - \mathbf{r_0})\right]\mathcal{Q}(\mathbf{r_0})\mathrm{d}^3\mathbf{r_0} \\ &= \int \delta^3(\mathbf{r} - \mathbf{r_0})\mathcal{Q}(\mathbf{r_0})d^3\mathbf{r_0} = \mathcal{Q}(\mathbf{r})\end{aligned} \tag{2.17}$$

The first, the Fourier transform for $G(\mathbf{r})$ is

$$G(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int e^{i\mathbf{s}\cdot\mathbf{r}}g(\mathbf{s})\mathrm{d}^3\mathbf{s}. \tag{2.18}$$

Taking $(\nabla^2 + k^2)$ operator to the above equation, so

$$(\nabla^2 + k^2)G(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int \left[(\nabla^2 + k^2)e^{i\mathbf{s}\cdot\mathbf{r}}\right]g(\mathbf{s})\mathrm{d}^3\mathbf{s}. \tag{2.19}$$

From the operator ∇^2, we get

$$\nabla^2 e^{i\mathbf{s}\cdot\mathbf{r}} = -s^2 e^{i\mathbf{s}\cdot\mathbf{r}}, \tag{2.20}$$

and the Fourier transforms of the delta function in Eq.(2.15) is

$$\delta^3(\mathbf{r}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{s}\cdot\mathbf{r}}\mathrm{d}^3\mathbf{s}. \tag{2.21}$$

So, Eq.(2.15) becomes

$$\frac{1}{(2\pi)^{3/2}}\int(-s^2+k^2)e^{i\mathbf{s}\cdot\mathbf{r}}g(\mathbf{s})\mathrm{d}^3\mathbf{s} = \frac{1}{(2\pi)^3}\int e^{i\mathbf{s}\cdot\mathbf{r}}\mathrm{d}^3\mathbf{s}. \tag{2.22}$$

The result is

$$g(\mathbf{s}) = \frac{1}{(2\pi)^{3/2}(k^2-s^2)}. \tag{2.23}$$

Then, we substitute $g(\mathbf{s})$ back into Eq.(2.18), so

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{s}\cdot\mathbf{r}}\frac{1}{(k^2-s^2)}\mathrm{d}^3\mathbf{s}. \tag{2.24}$$

By using the **Complex analysis**, a detail about calculating is given in Appendix.A, we get

$$G(\mathbf{r}) = -\frac{e^{ikr}}{4\pi r}. \tag{2.25}$$

Eq.(2.15) possible add any function $G_0(\mathbf{r})$ which satisfies the homogeneous Helmholtz equation,

$$(\nabla^2+k^2)G_0(\mathbf{r}) = 0. \tag{2.26}$$

Of course, the result of $(G+G_0)$ satisfies Eq.(2.15).

For Eq.(2.16), the solution of the Schrödinger equation becomes

$$\boxed{\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int\frac{e^{ik|\mathbf{r}-\mathbf{r_0}|}}{|\mathbf{r}-\mathbf{r_0}|}V(\mathbf{r_0})\psi(\mathbf{r_0})\mathrm{d}^3\mathbf{r_0}} \tag{2.27}$$

where ψ_0 satisfies the free-particle Schrödinger equation, as

$$(\nabla^2+k^2)\psi_0 = 0. \tag{2.28}$$

Eq.(2.27) is the **intergral form of the Schrödinger equation** for any potential.

2.3.1 The First Born Approximation

Assume $V(\mathbf{r_0})$ is localized that $\mathbf{r_0}\approx 0$ and we have to calculate $\psi(\mathbf{r})$ at far away from the scattering center. So, $|\mathbf{r}| \gg |\mathbf{r_0}|$ for all points in the integral of Eq.(2.27), therefore, it is

$$|\mathbf{r}-\mathbf{r_0}|^2 = r^2 + r_0^2 - 2\mathbf{r}\cdot\mathbf{r_0} \approx r^2\left(1-2\frac{\mathbf{r}\cdot\mathbf{r_0}}{r^2}\right), \tag{2.29}$$

then

$$|\mathbf{r}-\mathbf{r_0}| \approx r - \hat{r}\cdot\mathbf{r_0}. \tag{2.30}$$

Let

$$\mathbf{k} \equiv k\hat{r}, \tag{2.31}$$

so

$$e^{ik|\mathbf{r}-\mathbf{r_0}|} \cong e^{ikr}e^{-i\mathbf{k}\cdot\mathbf{r_0}}, \tag{2.32}$$

and we got

$$\frac{e^{ik|\mathbf{r}-\mathbf{r_0}}}{|\mathbf{r}-\mathbf{r_0}|} \cong \frac{e^{ikr}}{r}e^{-i\mathbf{k}\cdot\mathbf{r_0}}. \tag{2.33}$$

For an incident plane wave that scattering, we give

$$\psi_0(\mathbf{r}) = Ae^{ikz}. \tag{2.34}$$

A large r, by input Eq.(2.33), thus, it is given as

$$\psi(\mathbf{r}) \cong Ae^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\mathbf{k}\cdot\mathbf{r_0}}V(\mathbf{r_0})\psi(\mathbf{r_0})\mathrm{d}^3\mathbf{r_0}. \tag{2.35}$$

Comparing between Eq.(2.35) and Eq.(2.7), the above equation is a general form. So, we get the scattering amplitude as

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\mathbf{k}\cdot\mathbf{r_0}}V(\mathbf{r_0})\psi(\mathbf{r_0})\mathrm{d}^3\mathbf{r_0}. \tag{2.36}$$

In a case of that depending on the **Born approximation**, we suppose the incident plane wave is not under influenced by the potential, so

$$\psi(\mathbf{r_0}) \approx \psi_0(\mathbf{r_0}) = Ae^{ikz_0} = Ae^{i\mathbf{k'}\cdot\mathbf{r_0}}, \tag{2.37}$$

where $\mathbf{k'} \equiv k\hat{z}$. Then, we finally get the Born approximation that is

$$\boxed{f(\theta,\phi) \cong -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k'}-\mathbf{k})\cdot\mathbf{r_0}}V(\mathbf{r_0})\mathrm{d}^3\mathbf{r_0}.} \tag{2.38}$$

This method is very useful for many scattering cases which easily gives us the scattering amplitude.

2.3.2 The Born series

The Born approximation likes the **impulse approximation**, the meaning of impulsing is a ball hits a wall and returned by the impulse of force, in the case of classical scattering. With the force from an impulse to the particle which entering the potential, we get

$$I = \int F_{\perp} dt. \tag{2.39}$$

The incident particles, with the momentum p, pass through the scattering center by the scattering angle θ, is very small, that is

$$\theta \cong \tan^{-1}(I/p). \tag{2.40}$$

From the integral form of the solution of the Schrödinger equation is

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int g(\mathbf{r} - \mathbf{r_0}) V(\mathbf{r_0}) \psi(\mathbf{r_0}) \mathrm{d}^3\mathbf{r_0}, \tag{2.41}$$

where

$$g(\mathbf{r}) \equiv -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}, \tag{2.42}$$

which is the Green's function. The ψ_0 is the incoming plane wave without the potential influencing, is the 0^{th} order Born approximation. So, Eq.(2.42) is simpler rewritten as

$$\psi = \psi_0 + \int gV\psi, \tag{2.43}$$

where V is the scattering potential.
When, we substitute Eq.(2.43) into itself, thus it becomes

$$\psi = \psi_0 + \int gV\psi_0 + \int\int gVgV\psi. \tag{2.44}$$

Repeating the process again, we obtain a series for ψ, is

$$\psi = \psi_0 + \int gV\psi_0 + \int\int gVgV\psi_0 + \int\int\int gVgVgV\psi_0 + \cdots . \tag{2.45}$$

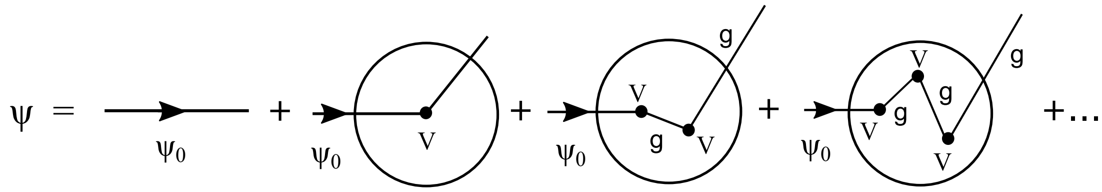

Figure 4 The representational diagram of the Born series.

This Eq.(2.45) generates for us the higher-order correction. The Born series, represented in Figure 4, was the inspiration for Feynman's formulation of relativistic quantum mechanics, Feynman diagrams.[1]

CHAPTER III

THE QUANTUM DYNAMICAL PRINCIPLE

The quantum dynamical principle (QDP) is, also called the quantum action principle (QAP), proposed by Julian S. Schwinger. It easily apply to many case of quantum physics. In the working frame, it takes an expression of the variation of transformation function, $\delta \langle \mathbf{a}t|\mathbf{b}t'\rangle$, from a B state at an initial time t' to an A state at a final time t. This $\langle \mathbf{a}t|\mathbf{b}t'\rangle$ is given by changing on the parameters of a Hamiltonian that depends on *masses, coupling constants, prescribed frequencies, external sources*, etc.,[5]. The specific transformation functions is considered in the term of $\langle \mathbf{q}t|\mathbf{q}'t'\rangle$ and $\langle \mathbf{q}t|\mathbf{p}t'\rangle$ which $\mathbf{q}$ and $\mathbf{p}$ are the general coordinate.

The QDP is the powerful method for the development of quantum physics that can apply to many problem, one possible as approaching over the "path integral" and, moreover, the Quantum field theory.

Before we start the detail and mathematical background of this method. We would like to present the concept of the external source which is early given.

3.1 The Simple Source Theory

At this state, we will give the detail about the "Source theory" in the scope of Quantum theory that is the reduced concept from Quantum field theory, presented by Julian S. Schwinger [16, 18, 23].

3.1.1 The Visual concept of the External Source

First of all, we would like to tell reader about the sources theory which are defined from the mathematical source or numerical function, $\mathbf{F}(x)$. The physical source meaning is a creation of the physical properties that is own by the created particles.

Comparing with the normal situation in real life, you can see an object with the light that reflects and go to your eyes. In this case, the light (particle-like) emits from a candle or a light bulbs (the sources) and travels to incident our eyes (the detector) given in Figure 5. Here, these sources mean that it generates, emits or stimulates something from nothing.

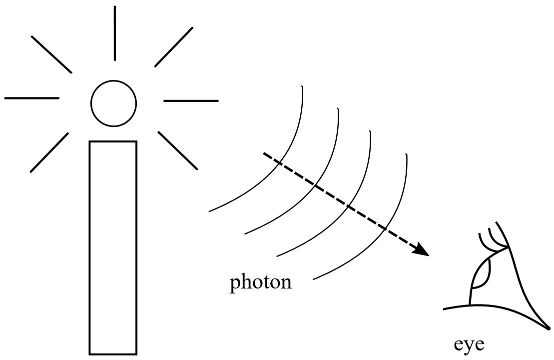

Figure 5 The normal situation of the source and dectector.

Moreover, this external source also means an external force, depends on time, where it affects only in a temporary time. An example, the harmonic oscillator is added the external source. The details about this system is given in next subsection.

In the quantum interpretation, we give the Fourier transform of source as

$$\mathbf{F}(\mathbf{p}) = \int \mathrm{d}x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\mathbf{F}(\mathbf{x}). \tag{3.1}$$

This thesis, the sources are $\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$, $t' \leq \tau < t$, which is the generator of general coordinates, is called the degrees of freedom, and also cause the transition of $|\mathbf{p}t'\rangle$ to $|\mathbf{q}t\rangle$ state. For transformation function $\langle\mathbf{q}t|\mathbf{p}t'\rangle$, we impose the Hamiltonian equation that is

$$H(\tau) = -\mathbf{q}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau). \tag{3.2}$$

The image of sources which generates the physical states, is given in the Figure 6 below

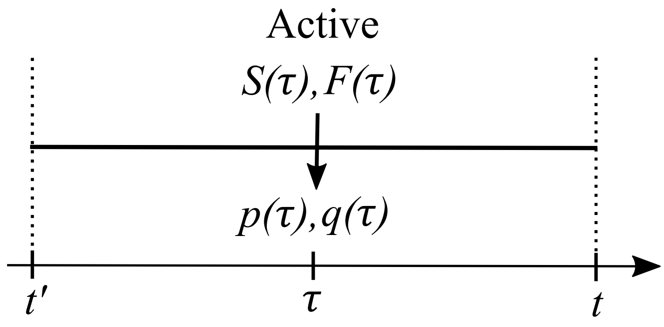

Figure 6 The Generated $\mathbf{q}(\tau), \mathbf{p}(\tau)$ state by the sources.

In particular, these two sources are activated over the interval of time between t' and t. Time is possible expanded to $-\infty$ and ∞. This situation begins from $-\infty$ then approaches t'. For the $\mathbf{F}(\tau)$ source generates the $\mathbf{q}(\tau)$ state and the $\mathbf{p}(\tau)$ state is generated by the $\mathbf{S}(\tau)$ source. This time-dependent source is similar to the situation that the external force stimulates the system. An example, the box is gentle pushed by our force which affects its position, $\mathbf{x}$, infinitesimal moving to $\mathbf{x} + \delta\mathbf{x}$. According to previous, the source generates, emits or stimulates for the system at an initial state to reach a final state, founded by detector, so, this event makes sense to an experiment.

3.1.2 The External Source

At the beginning, we introduce the Hamiltonian, for Harmonic oscillator, that include the external force (or source). This force is *linear coupling* with the disturbance which is the time-dependent source. It is written as

$$H(t) = \frac{\mathbf{P}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{x}^2 - \sqrt{\frac{2m\omega}{\hbar}}\mathbf{x} \cdot F(t) \tag{3.3}$$

where $F(t)$ is the external force. Then, by using the annihilation and creation operators, $(a, a^\dagger)$, that are defined as

$$a = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \mathbf{q}} + \mathbf{q}\right) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial \mathbf{q}} + \mathbf{q}\right).$$

Defining $\mathbf{x} = \sqrt{\hbar/m\omega}\mathbf{q}$, for removing a unit, an above equation can be rewritten in the form of

$$H(t) = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) - F(t)(a + a^\dagger) \tag{3.4}$$

which consists $(a + a^\dagger)$, coupling with $F(t)$. This equation is the prototype of field theory in zero-dimension space at time $t = 0$ [5].

The external source leads to the disturbance of system that provides the transition between the different states, means that ground-states to excited states. But, somehow, the state still stay in their old state, initial state.

We set this force is "on" after T_1 time and "off" at T_2 time where $T_2 > T_1$, so we draw it down in Figure 7. Denoting, the external force vanishes in $t \leqslant T_1$ and $t \geqslant T_2$.

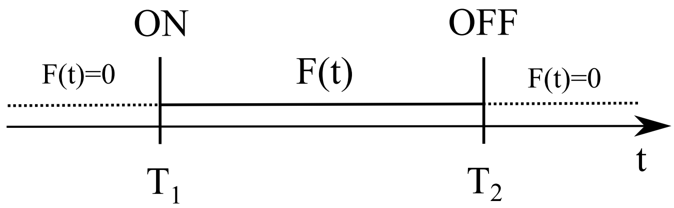

Figure 7 The external force is swicthed ON and OFF at T_1 and T_2 times, respectively.

We consider the chosen ground-state which is

$$|\psi(T_1)\rangle \equiv |0_-\rangle\,, \tag{3.5}$$

where this $|0_-\rangle$ notation is borrowed from quantum field theory.

To find the possible transitions, we have to solve the time-dependent Schrödinger equation for the final states, $|\psi(T_2)\rangle$, which is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)\,|\psi(t)\rangle\,. \tag{3.6}$$

Consequently, we can make the ansatz as

$$|\psi(t)\rangle = \exp\left[-\frac{i}{\hbar}\left(\xi(t)a^\dagger + \eta(t)\right)\right]|0_-\rangle\,, \tag{3.7}$$

where $\xi(t)$ and $\eta(t)$ are any numbering functions that satisfy

$$\xi(T_1) = 0, \qquad \eta(T_1) = 0 \tag{3.8}$$

for initial conditions. Then, Eq.(3.7) is substituted in Eq.(3.6) where we use the identity which is

$$ae^{\xi(t)a^\dagger} = e^{\xi(t)a^\dagger}(a+\xi)$$

and the truth that $a\,|0_-\rangle = 0$. Finally, we obtain

$$\left[\dot{\xi}(t)a^\dagger + \dot{\eta}(t)\right]|0_-\rangle = \left[\hbar\omega\left(\frac{1}{2} - \frac{i}{\hbar}\xi(t)a^\dagger\right) - F(t)\left(-\frac{i}{\hbar}\xi(t) + a^\dagger\right)\right]|0_-\rangle\,. \tag{3.9}$$

This result leads to the solution of $\xi(t)$ and $\eta(t)$ by using the integration factor of differential equation and the initial condition in Eq.(3.8). After integrating, $\xi(t)$ and $\eta(t)$ are written as

$$\xi(t) = -e^{-i\omega t}\int_{-\infty}^{t} \mathrm{d}t'\, e^{i\omega t'}F(t'), \tag{3.10}$$

$$\eta(t) = \frac{\hbar\omega}{2}(t - T_1) - \frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t'\int_{-\infty}^{\infty}\mathrm{d}t''\, e^{-i\omega(t''-t')}F(t'')\Theta(t-t'')\Theta(t''-t')F(t'). \tag{3.11}$$

The integrating bound of these two equations are expanded to infinity time that covers $T_1 \leqslant t \leqslant T_2$ time interval. For Eq.(3.11), the two terms will be vanished when $t = T_1$. In particular, $\Theta(t-t')$ is a step function, orders the time-dependent event of this system, which is zero when $t - t' < 0$.

We are interested in the remaining ground-state amplitude that is

$$\langle 0_+|0_-\rangle_F = \frac{\langle 0|\psi(T_2)\rangle}{\langle 0|\psi(T_2)\rangle\Big|_{F=0}}, \tag{3.12}$$

which subscribed F is defined that this transition is disturbed by the external force. For $\langle 0|\psi(T_2)\rangle$, it is given as

$$\langle 0|\psi(T_2)\rangle = \langle 0|\exp\left[-\frac{i}{\hbar}\left(\xi(T_2)a^\dagger + \eta(T_2)\right)\right]|0\rangle\,, \tag{3.13}$$

$$= \langle 0|\exp\left[-\frac{i}{\hbar}\eta(T_2)\right]|0\rangle\,, \tag{3.14}$$

where using the fact that $\langle 0|\,a^\dagger = 0$. This result is substituted in Eq.(3.12) then we obtain

$$\langle 0_+|0_-\rangle_F = \frac{\langle 0|\exp\left[-\frac{i}{\hbar}\eta(T_2)\right]|0\rangle}{\langle 0|\exp\left[-\frac{i}{\hbar}\eta(T_2)\right]|0\rangle\Big|_{F=0}}, \tag{3.15}$$

where $\langle 0_+|$ implies the ground-state at a final time.

At last, Eq.(3.15) is immediately rewritten as

$$\langle 0_+|0_-\rangle_F = \exp\left[-\frac{i}{2}\omega(T_2 - T_1)\right]$$
$$\times \exp\left[-\frac{1}{\hbar^2}\int_{-\infty}^{\infty} \mathrm{d}t'' \int_{-\infty}^{\infty} \mathrm{d}t'\, e^{-i\omega(t''-t')} F(t'')\Theta(T_2 - t'')\Theta(t'' - t')F(t')\right], \quad (3.16)$$

when $F = 0$, for a dividing term. It is become equaling to one and $\langle 0|0\rangle$ are cancel each other. After, $\exp\left[-i\omega(T_2 - T_1)/2\right]$ is canceled because of zero point energy, from boundary condition, and $\Theta(T_2 - t'') = 1$ due to $F(t'') = 0$ when $t'' \geqslant T_2$. Thus, we actually have

$$\langle 0_+|0_-\rangle_F = \exp\left[-\frac{1}{\hbar^2}\int_{-\infty}^{\infty} \mathrm{d}t'' \int_{-\infty}^{\infty} \mathrm{d}t'\, e^{-i\omega(t''-t')} F(t'')\Theta(t'' - t')F(t')\right]. \quad (3.17)$$

Therefore, in the case of remaining ground-state probability, it is computed and written as

$$|\langle 0_+|0_-\rangle_F|^2 = \exp\left[-\frac{2}{\hbar^2}\int_{-\infty}^{\infty} \mathrm{d}t'' \int_{-\infty}^{\infty} \mathrm{d}t'\, F(t'')F(t')\cos(\omega(t'' - t'))\Theta(t'' - t')\right]. \quad (3.18)$$

This Eq.(3.18) may be rewritten where we obtain

$$|\langle 0_+|0_-\rangle_F|^2 = \exp\left[-\frac{1}{\hbar}\left|\int_{-\infty}^{\infty} \mathrm{d}t\, e^{-i\omega t} F(t)\right|^2\right] \quad (3.19)$$

and then we use the Fourier transform. So, immediately, Eq.(3.19) is simple written as

$$|\langle 0_+|0_-\rangle_F|^2 = \exp\left[-\frac{|F(\omega)|^2}{\hbar^2}\right], \quad (3.20)$$

where

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t} F(t).$$

Importantly, for all possible states have the probability that is defined as

$$1 = |\langle 0_+|0_-\rangle_F|^2 + |\langle n_+|0_-\rangle_F|^2. \quad (3.21)$$

Thus, we have the excited states, by $F(t)$ source, which is

$$|\langle n_+|0_-\rangle_F|^2 = 1 - |\langle 0_+|0_-\rangle_F|^2$$
$$= 1 - \exp\left[-\frac{|F(\omega)|^2}{\hbar^2}\right]. \quad (3.22)$$

Accordingly, for the transition amplitude, defined as $\langle n_+|0_-\rangle$, we give this source, is the sum of two sources, is

$$F(t) = F_1(t) + F_2(t). \tag{3.23}$$

By setting, $F_1(t)$ is switched on and off then $F_2(t)$ is switched on, immediately.

Consequently, we use Eq.(3.17) to directly obtain

$$\begin{aligned}\langle 0_+|0_-\rangle_F &= \langle 0_+|0_-\rangle_{F_2} \langle 0_+|0_-\rangle_{F_1} \\ &\quad\times \exp\left[-\frac{1}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}t'' \int_{-\infty}^{\infty} \mathrm{d}t'\, e^{-i\omega(t''-t')} F_2(t'')F_1(t')\right],\end{aligned} \tag{3.24}$$

by using the step same as Eq.(3.15), where $\Theta(t''-t) = 1$ and $F_1(t'') = 0$ when $F_2(t'') \neq 0$.

We also follow the previously representation of the Fourier transform and have

$$\langle 0_+|0_-\rangle_F = \langle 0_+|0_-\rangle_{F_2} \langle 0_+|0_-\rangle_{F_1} \exp\left[\frac{iF_2^*(\omega)}{\hbar}\frac{iF_1(\omega)}{\hbar}\right]. \tag{3.25}$$

Using the taylor series, the above equation becomes

$$\langle 0_+|0_-\rangle_F = \sum_{n=0}^{\infty} \langle 0_+|0_-\rangle_{F_2} \frac{[iF_2^*(\omega)/\hbar]^n}{\sqrt{n!}} \frac{[iF_1(\omega)/\hbar]^n}{\sqrt{n!}} \langle 0_+|0_-\rangle_{F_1}. \tag{3.26}$$

Therefore, comparing with the completeness relation, we obtain

$$\langle 0_+|0_-\rangle_F = \sum_{n,m} \langle 0_+|m_-\rangle_{F_2} \langle m|n\rangle_0' \langle n_+|0_-\rangle_{F_1}, \tag{3.27}$$

where $F_1(t)$ is switched on then off therefore the $|0_-\rangle$ state may transit to the $|n_+\rangle$ state that stays in this state until $F_2(t)$ is switched on.

In particular, $\langle m|n\rangle_0'$ is the amplitude of a force-free interval between the two sources $F_1(t)$ and $F_2(t)$. This transition evolves in time with a free Hamiltonian which is

$$\langle m|n\rangle_0' = \exp\left[-i\omega n(t_2 - t_1)\right]\delta_{mn}. \tag{3.28}$$

As a result, Eq.(3.27) becomes

$$\begin{aligned}\langle 0_+|0_-\rangle_F &= \sum_{n,m} \langle 0_+|m_-\rangle_{F_2} \exp\left[-i\omega n(t_2-t_1)\right]\delta_{mn} \langle n_+|0_-\rangle_{F_1}, \\ &= \sum_n \left\langle 0_+\middle|n'_-\right\rangle_{F'} \left[e^{-i\omega(t_2-t_1)}\right]^n \langle n_+|0_-\rangle_F,\end{aligned} \tag{3.29}$$

where m have to equal to n due to the property of Kronecker delta function and we must define the new variables, n' is the states before F_2 is switched on. For F_2 and F_1 are also redefined as F' and F, respectively.

From Eq.(3.26) and Eq.(3.29), so, the transition amplitude, disturbed by the external sources, are meaningful given as

$$\langle n_+|0_-\rangle_F = \frac{[ie^{-i\omega t_2}F(\omega)/\hbar]^n}{\sqrt{n!}} \langle 0_+|0_-\rangle_F \,, \tag{3.30}$$

$$\langle 0_+|n_-\rangle_{F'} = \langle 0_+|0_-\rangle_{F'} \frac{[ie^{i\omega t_1}F'^*(\omega)/\hbar]^n}{\sqrt{n!}}, \tag{3.31}$$

which t_2 represents the time that $F(\omega)$ is switched off and, t_1 denotes the time that $F'(\omega)$ is switched on.

Importantly, thus, the following probability of the transition states from the ground-state $|0_-\rangle$ to an excited state $|n_+\rangle$, from Eq.(3.30), we finally obtain

$$\begin{aligned} |\langle n_+|0_-\rangle_F|^2 &= \frac{[|F(\omega)|^2/\hbar^2]^n}{n!} |\langle 0_+|0_-\rangle_F|^2 \,, \\ &= \frac{[|F(\omega)|^2/\hbar^2]^n}{n!} \exp\left[-\frac{|F(\omega)|^2}{\hbar^2}\right] \end{aligned} \tag{3.32}$$

by using the probability of remaining ground-state from Eq.(3.20).

This result leads to the average of the system at an excited state which starts with

$$\bar{n} = \sum_{i=1}^{\infty} n_i \, |\langle n_+|0_-\rangle_F|^2 \,. \tag{3.33}$$

After, we substitute Eq.(3.32) in an above equation and obtain

$$\bar{n} = \exp\left(-\frac{|F(\omega)|^2}{\hbar^2}\right) \sum_{i=1}^{\infty} \frac{[|F(\omega)|^2/\hbar^2]^{n_i}}{(n_i - 1)!}. \tag{3.34}$$

Then we and change the variables to $l = n_i - 1$ so it is rewritten as

$$\begin{aligned} \bar{n} &= \exp\left(-\frac{|F(\omega)|^2}{\hbar^2}\right) \left(\frac{|F(\omega)|^2}{\hbar^2}\right) \sum_{l=0}^{\infty} \frac{[|F(\omega)|^2/\hbar^2]^l}{l!}, \\ &= \exp\left(-\frac{|F(\omega)|^2}{\hbar^2}\right) \left(\frac{|F(\omega)|^2}{\hbar^2}\right) \exp\left(+\frac{|F(\omega)|^2}{\hbar^2}\right), \end{aligned} \tag{3.35}$$

where using the Taylor series. Finally, the average of system at the excited states is found as

$$\bar{n} = \frac{|F(\omega)|^2}{\hbar^2}, \tag{3.36}$$

after the intervening source is activated.

For obtaining the transition from $|n_-\rangle$ to $|m_+\rangle$, we provide the intervening source, previously, is switch on, therefore, the source is given by

$$F(t) = F_1(t) + F_2(t) + F_3(t), \tag{3.37}$$

where all of these sources are defined a same logic ordering as early in Eq.(3.23), $F_1 \to F_2 \to F_3$. Then, we also imply Eq.(3.17) and immediately get

$$\begin{aligned}\langle 0_+|0_-\rangle_F &= \langle 0_+|0_-\rangle_{F_3}\langle 0_+|0_-\rangle_{F_2}\langle 0_+|0_-\rangle_{F_1}\exp\left[\frac{iF_3^*(\omega)}{\hbar}\frac{iF_2(\omega)}{\hbar}\right] \\ &\quad\times\exp\left[\frac{iF_3^*(\omega)}{\hbar}\frac{iF_1(\omega)}{\hbar}\right]\exp\left[\frac{iF_2^*(\omega)}{\hbar}\frac{iF_1(\omega)}{\hbar}\right] \\ &= \sum_{n,m}\langle 0_+|m_-\rangle_{F_3}\langle m_+|n_-\rangle_{F_2}\langle n_+|0_-\rangle_{F_1}.\end{aligned} \tag{3.38}$$

By using Eq.(3.30) and (3.31), the last line in Eq.(3.38) is rewritten as

$$\begin{aligned}\langle 0_+|0_-\rangle_F = \sum_{n,m}\langle 0_+|0_-\rangle_{F_3}&\frac{\left[\frac{i}{\hbar}e^{i\omega T_2}F_3^*(\omega)\right]^m}{\sqrt{m!}}\langle m_+|n_-\rangle_{F_2} \\ &\times\frac{\left[\frac{i}{\hbar}e^{-i\omega T_1}F_1^*(\omega)\right]^n}{\sqrt{n!}}\langle 0_+|0_-\rangle_{F_1}.\end{aligned} \tag{3.39}$$

Then, we use the identity[9] which is

$$\begin{aligned}&\exp\left[\frac{iF_3^*(\omega)}{\hbar}\frac{iF_2(\omega)}{\hbar}\right]\exp\left[\frac{iF_3^*(\omega)}{\hbar}\frac{iF_1(\omega)}{\hbar}\right]\exp\left[\frac{iF_2^*(\omega)}{\hbar}\frac{iF_1(\omega)}{\hbar}\right] \\ &= \sum_{L,M,N}\frac{\left[\frac{i}{\hbar}F_3^*(\omega)\right]^{L+M}}{L!}\frac{\left[\frac{i}{\hbar}F_2(\omega)\right]^M\left[\frac{i}{\hbar}F_2^*(\omega)\right]^N}{M!}\frac{\left[\frac{i}{\hbar}F_1(\omega)\right]^{L+N}}{N!}.\end{aligned} \tag{3.40}$$

By setting

$$L + M = m, \qquad L + N = n,$$

so Eq.(3.40) becomes

$$\begin{aligned}&\exp\left[\frac{iF_3^*(\omega)}{\hbar}\frac{iF_2(\omega)}{\hbar}\right]\exp\left[\frac{iF_3^*(\omega)}{\hbar}\frac{iF_1(\omega)}{\hbar}\right]\exp\left[\frac{iF_2^*(\omega)}{\hbar}\frac{iF_1(\omega)}{\hbar}\right] \\ &= \sqrt{m!}\sqrt{n!}\frac{\left[\frac{i}{\hbar}F_3^*(\omega)\right]^m}{\sqrt{m!}}\frac{\left[\frac{i}{\hbar}F_2(\omega)\right]^{m-L}\left[\frac{i}{\hbar}F_2^*(\omega)\right]^{n-L}}{L!(m-L)!(n-L)!}\frac{\left[\frac{i}{\hbar}F_1(\omega)\right]^n}{\sqrt{n!}}.\end{aligned} \tag{3.41}$$

After that, we compare Eq.(3.40) to Eq.(3.41), thus, and obtain

$$\langle m_+|n_-\rangle_F = \langle 0_+|0_-\rangle_F \sqrt{m!n!} \sum_{L=0}^{min(m,n)} \frac{\left[\frac{i}{\hbar}F(\omega)\right]^{m-L} e^{-i\omega m T_2} e^{i\omega n T_1} \left[\frac{i}{\hbar}F^*(\omega)\right]^{n-L}}{(m-L)!L!(n-L)!}, \tag{3.42}$$

where changing $F_2 \rightarrow F$. This equation is the transition $|n_-\rangle$ to $|m_+\rangle$ states by the intervening source, $F_2(t)$, is activated.

In a case of non-transition, the state is still at its old state which $|m_+\rangle \rightarrow |n_+\rangle$. At this step, we absolutely obtain

$$\begin{aligned} \langle n_+|n_-\rangle_F &= \langle 0_+|0_-\rangle_F\, n! \sum_{L=0}^{min(n)} \frac{\left[\frac{i}{\hbar}F(\omega)\right]^{n-L} e^{-i\omega n T_2} e^{i\omega n T_1} \left[\frac{i}{\hbar}F^*(\omega)\right]^{n-L}}{[(n-L)!]^2 L!} \\ &= \langle 0_+|0_-\rangle_F\, n! \sum_{L=0}^{min(n)} \frac{\left[-\frac{|F(\omega)|^2}{\hbar^2}\right]^{n-L} e^{-i\omega n(T_2-T_1)}}{[(n-L)!]^2 L!}. \end{aligned} \tag{3.43}$$

And then we set $k = n - L \rightarrow L = n - k$ and $n = k + L$ therefore Eq.(3.43) is rewritten as

$$\langle n_+|n_-\rangle_F = \langle 0_+|0_-\rangle_F\, n! \sum_{k=0}^{n} \frac{\left[-\frac{|F(\omega)|^2}{\hbar^2}\right]^{k}}{(k!)^2(n-k)!} \tag{3.44}$$

where, lastly, exponential term is divided for normalization of $\langle n_+|n_-\rangle_F$. At the result in Eq.(3.42) and (3.44) are the transition of any excited state to other excited state or unchanged state.

The important point is that the external force or source affects the system to be changed reaching an excited state, with initial state. For application of this theory is given in the next section where you will understand how to use it. Moreover, how is it role as the action of dynamical variables to generate the quantum variables.

3.2 The Quantum Dynamical Principle

This section is about how the quantum dynamical principle was proved. At first, let consider the general Hamiltonian that is

$$H(t,\lambda) = H_1(t) + H_2(t,\lambda) \tag{3.45}$$

where $H_1(t), H_2(t,\lambda)$ are time-dependent but $H_2(t,\lambda)$ is added some parameters (λ) such as *masses, coupling constants, prescribed frequencies, external sources*, etc. Furthermore, $H(t,\lambda)$ comes from a priori that given the time-dependent potential or external sources.

For the time evolution operator of Hamiltonian $H(t,\lambda)$ is $U(t,\lambda)$ that is defined as

$$i\hbar\frac{\partial}{\partial t}U(t,\lambda) = H(t,\lambda)U(t,\lambda). \tag{3.46}$$

And, for the specific state a, we have

$$i\hbar\frac{\partial}{\partial t}\langle \mathbf{a}t| = \langle \mathbf{a}t|H(t,\lambda). \tag{3.47}$$

For the Hamiltonian $H_1(t)$, independent of λ, gives the following time evolution operator $U_1(t)$ as

$$i\hbar\frac{\partial}{\partial t}U_1(t) = H_1(t)U_1(t), \tag{3.48}$$

then setting

$$i\hbar\frac{\partial}{\partial t}\,{}_1\langle \mathbf{a}t| = {}_1\langle \mathbf{a}t|H_1(t). \tag{3.49}$$

Clearly, the physical state $\langle \mathbf{a}t|$ which is related to ${}_1\langle \mathbf{a}t|$, becomes

$$\langle \mathbf{a}t| = {}_1\langle \mathbf{a}t|U_1^\dagger(t)U(t,\lambda). \tag{3.50}$$

Introducing, the unitary operator is

$$V(t,\lambda) = U_1^\dagger(t)U(t,\lambda) \tag{3.51}$$

so Eq.(3.50) becomes

$$\langle \mathbf{a}t| = {}_1\langle \mathbf{a}t|V(t,\lambda). \tag{3.52}$$

Thus, Eq.(3.46), by multiplying the $U_1^\dagger(t)$ is

$$i\hbar\frac{\partial}{\partial t}V(t,\lambda) = U_1^\dagger(t)H_2(t,\lambda)U(t,\lambda) \tag{3.53}$$

by using the Eq.(3.51).

To end this, we use the identity which beginning with

$$i\hbar\frac{\partial}{\partial \tau}\left[V(t,\lambda)V^\dagger(\tau,\lambda)V(\tau,\lambda')V^\dagger(t',\lambda')\right] = V(t,\lambda)i\hbar\frac{\partial}{\partial \tau}\left[V^\dagger(\tau,\lambda)V(\tau,\lambda')\right]V^\dagger(t',\lambda'), \tag{3.54}$$

and considering only the function of τ term, so

$$\begin{aligned} i\hbar\frac{\partial}{\partial\tau}\left[V^\dagger(\tau,\lambda)V(\tau,\lambda')\right] &= i\hbar\left[V^\dagger(\tau,\lambda)\frac{\partial}{\partial\tau}V(\tau,\lambda') + \frac{\mathrm{d}}{\mathrm{d}\tau}V^\dagger(\tau,\lambda)V(\tau,\lambda')\right] \\ &= V^\dagger(\tau,\lambda)\left(i\hbar\frac{\partial}{\partial\tau}V(\tau,\lambda')\right) + \left(i\hbar\frac{\partial}{\partial\tau}V^\dagger(\tau,\lambda)\right)V(\tau,\lambda'). \end{aligned} \tag{3.55}$$

From the Eq.(3.53) and using the complex conjugate, we obtain

$$\begin{aligned} (i\hbar V(t,\lambda))^\dagger &= \left(U_1^\dagger(t)H_2(t,\lambda)U(t,\lambda)\right)^\dagger \\ \rightarrow -i\hbar\frac{\partial}{\partial\tau}V^\dagger(\tau,\lambda) &= U^\dagger(\tau,\lambda)H_2^\dagger(\tau,\lambda)U_1(\tau), \end{aligned} \tag{3.56}$$

and Eq.(3.55), by substituting Eq.(3.56), becomes

$$\begin{aligned} & i\hbar\frac{\partial}{\partial\tau}\left[V^\dagger(\tau,\lambda)V(\tau,\lambda')\right] \\ &= V^\dagger(\tau,\lambda)U_1^\dagger(\tau)H_2(\tau,\lambda')U(\tau,\lambda') - U^\dagger(\tau,\lambda)H_2^\dagger(\tau,\lambda)U_1(\tau)V(\tau,\lambda'). \end{aligned} \tag{3.57}$$

From the unitary operator, $V(t,\lambda)$ and $V^\dagger(t,\lambda)$, Eq.(3.57) is rewritten as

$$\begin{aligned} i\hbar\frac{\partial}{\partial\tau}\left[V^\dagger(\tau,\lambda)V(\tau,\lambda')\right] &= U^\dagger(\tau,\lambda)\left(U_1(\tau)U_1^\dagger(\tau)\right)H_2(\tau,\lambda')U(\tau,\lambda') \\ &\quad - U^\dagger(\tau,\lambda)H_2^\dagger(\tau,\lambda)\left(U_1(\tau)U_1^\dagger(\tau)\right)U(\tau,\lambda'), \\ &= U^\dagger(\tau,\lambda)\left[H_2(\tau,\lambda') - H_2(\tau,\lambda)\right]U(\tau,\lambda'), \end{aligned} \tag{3.58}$$

where using the unitary $U(t)U^\dagger(t) = 1$ and the Hermitian operator.

Finally, this Eq.(3.54) becomes

$$\begin{aligned} & i\hbar\frac{\partial}{\partial\tau}\left[V(t,\lambda)V^\dagger(\tau,\lambda)V(\tau,\lambda')V^\dagger(t',\lambda')\right] \\ &= V(t,\lambda)\left[U^\dagger(\tau,\lambda)\left(H_2(\tau,\lambda') - H_2(\tau,\lambda)\right)U(\tau,\lambda')\right]V^\dagger(t',\lambda'), \\ &= V(t,\lambda)\left[U^\dagger(\tau,\lambda)\left(H(\tau,\lambda') - H(\tau,\lambda)\right)U(\tau,\lambda')\right]V^\dagger(t',\lambda'), \end{aligned} \tag{3.59}$$

by using the earlier Hamiltonian equation.

Note that, generally, the λ' and λ are not the same, and the unitary of $V(t,\lambda)$ are

$$V(t,\lambda)V^\dagger(t,\lambda) = 1 \quad , \quad V^\dagger(t,\lambda)V(t,\lambda) = 1. \tag{3.60}$$

Let integrating over τ from t' to t, Eq.(3.59) becomes

$$\begin{aligned}
&\left[V(t,\lambda)V^\dagger(\tau,\lambda)V(\tau,\lambda')V^\dagger(t',\lambda')\right]\Big|_{\tau=t'}^{\tau=t} \\
&= -\frac{i}{\hbar}V(t,\lambda)\left[\int_{t'}^{t}\mathrm{d}\tau U^\dagger(\tau,\lambda)\left(H(\tau,\lambda')-H(\tau,\lambda)\right)U(\tau,\lambda')\right]V^\dagger(t',\lambda'),
\end{aligned} \tag{3.61}$$

and consider only the left hand side, is

$$\begin{aligned}
&= V(t,\lambda)V^\dagger(t,\lambda)V(t,\lambda')V^\dagger(t',\lambda') - V(t,\lambda)V^\dagger(t',\lambda)V(t',\lambda)V^\dagger(t',\lambda') \\
&= V(t,\lambda)\left[V^\dagger(t,\lambda)V(t,\lambda') - V^\dagger(t',\lambda)V(t',\lambda')\right]V^\dagger(t',\lambda').
\end{aligned} \tag{3.62}$$

Rewriting, the two terms in a bracket are

$$\begin{aligned}
V^\dagger(t,\lambda)V(t,\lambda') &= U^\dagger(t,\lambda)U(t,\lambda'), \\
V^\dagger(t',\lambda)V(t',\lambda') &= U^\dagger(t',\lambda)U(t',\lambda'),
\end{aligned} \tag{3.63}$$

then we input Eq.(3.63) into Eq.(3.62). So, we get

$$\begin{aligned}
&= U_1 \dagger (t)U(t,\lambda)\left[U^\dagger(t,\lambda)U(t,\lambda) - U^\dagger(t',\lambda)U(t',\lambda')\right]U^\dagger(t',\lambda')U_1(t') \\
&= \left[U_1^\dagger(t)U(t,\lambda')\right]\left[U^\dagger(t',\lambda')U_1(t')\right] - \left[U_1^\dagger(t)U(t,\lambda)\right]\left[U^\dagger(t',\lambda)U_1(t')\right].
\end{aligned} \tag{3.64}$$

by using the unitary of $U(t,\lambda)$ and a property of the unitary operator $V(t,\lambda)$.

From a mathematical result of the left hand side is

$$\left[V(t,\lambda)V^\dagger(\tau,\lambda)V(\tau,\lambda')V^\dagger(t',\lambda')\right]\Big|_{\tau=t'}^{\tau=t} = V(t,\lambda')V^\dagger(t',\lambda') - V(t,\lambda)V^\dagger(t',\lambda). \tag{3.65}$$

Finally, we obtain

$$\begin{aligned}
&\left[V(t,\lambda')V^\dagger(t',\lambda') - V(t,\lambda)V^\dagger(t',\lambda)\right] \\
&= -\frac{i}{\hbar}V(t,\lambda)\left[\int_{t'}^{t}\mathrm{d}\tau U^\dagger(\tau,\lambda)\left(H(\tau,\lambda')-H(\tau,\lambda)\right)\right]V^\dagger(t',\lambda').
\end{aligned} \tag{3.66}$$

Let setting $\lambda' = \lambda + \delta\lambda$, get the variational of Eq.(3.66) that is

$$\delta\left[V(t,\lambda)V^\dagger(t',\lambda)\right] = -\frac{i}{\hbar}V(t,\lambda)\left[\int_{t'}^{t}\mathrm{d}\tau U^\dagger(\tau,\lambda)\delta H(\tau,\lambda)U(\tau,\lambda)\right]V^\dagger(t',\lambda). \tag{3.67}$$

Using the ($\mathbf{q}$,$\mathbf{p}$) language, the Hamiltonian $H(\tau,\lambda)$ can be rewritten as

$$H(\tau,\lambda) = H(\mathbf{q},\mathbf{p},\tau;\lambda) \tag{3.68}$$

where $\delta H(\mathbf{q},\mathbf{p},\tau;\lambda)$, the λ parameter is changing but $\mathbf{q},\mathbf{p}$ and τ are kept fixed, is presented the change of $H(\mathbf{q},\mathbf{p},\tau;\lambda)$.

The Heisenberg representation of $H(\mathbf{q},\mathbf{p},\tau;\lambda)$ was defined as

$$\mathcal{H}(\tau,\lambda) = U^\dagger(\tau,\lambda)H(\mathbf{q},\mathbf{p},\tau;\lambda)U(\tau,\lambda) \equiv H(\mathbf{q}(\tau),\mathbf{p}(\tau),\tau;\lambda) \tag{3.69}$$

where $\mathbf{q}(\tau)$ and $\mathbf{p}(\tau)$ are given by

$$\mathbf{q}(\tau) = U^\dagger(\tau,\lambda)\mathbf{q}U(\tau,\lambda), \qquad \mathbf{p}(\tau) = U^\dagger(\tau,\lambda)\mathbf{p}U(\tau,\lambda) \tag{3.70}$$

which are the Heisenberg representation of $\mathbf{q}$ and $\mathbf{p}$.

Thus, we can rewrite Eq.(3.67) as

$$\delta\left[V(t,\lambda)V^\dagger(t',\lambda)\right] = -\frac{i}{\hbar}V(t,\lambda)\left[\int_{t'}^{t}\mathrm{d}\tau\delta H\left(\mathbf{q}(\tau),\mathbf{p}(\tau),\tau;\lambda\right)\right]V^\dagger(t',\lambda), \tag{3.71}$$

where it gives us the variation of $H(\mathbf{q}(\tau),\mathbf{p}(\tau),\tau;\lambda)$ that depends on the λ parameter changing, $\mathbf{q}(\tau),\mathbf{p}(\tau)$ and τ are kept fixed. The $\mathbf{q}$ and $\mathbf{p}$ are carried the indices of various degrees of freedom.

Now, we take the matrix element ${}_1\langle\mathbf{a}t|\mathbf{b}t'\rangle_1$ into an Eq.(3.71) and use Eq.(3.52) to obtain

$$\delta\langle\mathbf{a}t|\mathbf{b}t'\rangle = -\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\langle\mathbf{a}t|\delta H\left(\mathbf{q}(\tau),\mathbf{p}(\tau),\tau;\lambda\right)|\mathbf{b}t'\rangle \tag{3.72}$$

The equation above is the celebrated **Schwinger's dynamical (action) principle or the quantum dynamical principle** [5]. This is expression in the physical states $|\mathbf{a}t\rangle$ and $|\mathbf{b}t'\rangle$ which depend on λ. Also, the $\mathbf{a}$, $\mathbf{b}$ are kept fixed as same as $\mathbf{q}(\tau)$, $\mathbf{p}(\tau)$.

The particular transformation functions are $\langle\mathbf{q}t|\mathbf{q}'t'\rangle$, $\langle\mathbf{q}t|\mathbf{p}t'\rangle$ and $\langle\mathbf{p}t|\mathbf{p}'t'\rangle$, given as

$$\delta\langle\mathbf{q}t|\mathbf{q}'t'\rangle = -\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\langle\mathbf{q}t|\delta H(\mathbf{q}(\tau),\mathbf{p}(\tau),\tau;\lambda)|\mathbf{q}'t'\rangle, \tag{3.73}$$

$$\delta\langle\mathbf{q}t|\mathbf{p}t'\rangle = -\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\langle\mathbf{q}t|\delta H(\mathbf{q}(\tau),\mathbf{p}(\tau),\tau;\lambda)|\mathbf{p}t'\rangle, \tag{3.74}$$

and

$$\delta\langle\mathbf{p}t|\mathbf{p}'t'\rangle = -\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\langle\mathbf{p}t|\delta H(\mathbf{q}(\tau),\mathbf{p}(\tau),\tau;\lambda)|\mathbf{p}'t'\rangle. \tag{3.75}$$

The application of all above equations will be given below.

Considering, the Hamiltonian equation as

$$H(\mathbf{q},\mathbf{p},\tau;\mathbf{F}(\tau),\mathbf{S}(\tau)) = H(\mathbf{q},\mathbf{p},\tau) - \mathbf{q}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau), \tag{3.76}$$

where $\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$, the numerical functions of τ, are the *external sources*, $H(\tau, \lambda)$ independents of these sources and a minus sign of $\mathbf{q} \cdot \mathbf{F}(\tau)$ is used for mathematical convenience.

The definition of the functional derivatives, with the meaning that the sources $\mathbf{F}, \mathbf{S}$ will only exist at time t $(\tau = t)$, are

$$\frac{\delta}{\delta \mathbf{F}(t)} \mathbf{F}(\tau) = \delta(t - \tau), \tag{3.77}$$

$$\frac{\delta}{\delta \mathbf{S}(t)} \mathbf{S}(\tau) = \delta(t - \tau), \tag{3.78}$$

and then we obtain, from Eq.(3.76), as

$$\frac{\delta}{\delta \mathbf{F}(t)} H(\mathbf{q}, \mathbf{p}, \tau; \mathbf{F}(\tau), \mathbf{S}(\tau)) = -\mathbf{q}\delta(t - \tau), \tag{3.79}$$

$$\frac{\delta}{\delta \mathbf{S}(t)} H(\mathbf{q}, \mathbf{p}, \tau; \mathbf{F}(\tau), \mathbf{S}(\tau)) = \mathbf{p}\delta(t - \tau), \tag{3.80}$$

where the λ parameter is replaced by the external sources ,$\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$.

The important following result, examples, from the Eq.(3.73)-(3.75) are

$$(-i\hbar) \frac{\delta}{\delta \mathbf{F}(\tau)} \langle \mathbf{q}t | \mathbf{q}'t' \rangle = \langle \mathbf{q}t | \mathbf{q}(\tau) | \mathbf{q}'t' \rangle, \tag{3.81}$$

$$(i\hbar) \frac{\delta}{\delta \mathbf{S}(\tau)} \langle \mathbf{q}t | \mathbf{q}'t' \rangle = \langle \mathbf{q}t | \mathbf{p}(\tau) | \mathbf{q}'t' \rangle, \tag{3.82}$$

where these equations for the matrix elements, in Eq.(3.73), of Heisenberg operators $\mathbf{q}(\tau)$ and $\mathbf{p}(\tau)$, for $t' < \tau < t$, which depend on the $\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$ sources are set to be zero, eventually.

3.2.1 The Arbitrary Function

Example of functional (function of function), we consider

$$G[\mathbf{F}, \mathbf{S}] = \int_{t'}^{t} \mathrm{d}\tau' \int_{t'}^{t} \mathrm{d}\tau'' \mathbf{F}(\tau') A(\tau', \tau'') \mathbf{S}(\tau''), \tag{3.83}$$

where $A(\tau', \tau'')$ is independent of $\mathbf{F}, \mathbf{S}$.

So that,

$$\frac{\delta}{\delta \mathbf{F}(\tau_1)} G[\mathbf{F}, \mathbf{S}] = \int_{t'}^{t} \mathrm{d}\tau'' A(\tau_1, \tau'') \mathbf{S}(\tau''), \tag{3.84}$$

$$\frac{\delta}{\delta \mathbf{S}(\tau_2)} \frac{\delta}{\delta \mathbf{F}(\tau_1)} G[\mathbf{F}, \mathbf{S}] = A(\tau_1, \tau_2), \tag{3.85}$$

where $t' < \tau_1 < t$, $t' < \tau_2 < t$.

Likewise, for $t' < \tau < t$, we get

$$\begin{aligned} &\frac{\delta}{\delta F(\tau)} \int_{t'}^{t} \mathrm{d}\tau' \int_{t'}^{t} \mathrm{d}\tau'' \mathbf{F}(\tau') A(\tau', \tau'') \mathbf{F}(\tau'') \\ &\quad = \int_{t'}^{t} \mathrm{d}\tau'' A(\tau, \tau'') \mathbf{F}(\tau'') + \int_{t'}^{t} \mathrm{d}\tau' \mathbf{F}(\tau') A(\tau', \tau). \end{aligned} \tag{3.86}$$

Now, let consider the arbitrary function, $B(\mathbf{q}, \mathbf{p}, \tau; \lambda)$, with the Heisenberg representation as

$$\mathcal{B}(\tau, \lambda) \equiv B(\mathbf{q}(\tau), \mathbf{p}(\tau), \tau; \lambda) = U^\dagger(\tau, \lambda) B(\mathbf{q}, \mathbf{p}, \tau; \lambda) U(\tau, \lambda). \tag{3.87}$$

Taking the unitary operator to the previous function, we get

$$\begin{aligned} &V(t, \lambda) \mathcal{B}(\tau, \lambda) V^\dagger(t', \lambda) \\ &= V(t, \lambda) V^\dagger(\tau, \lambda) U_1^\dagger(\tau) B(\mathbf{q}, \mathbf{p}, \tau; \lambda) U_1(\tau) V(\tau, \lambda) V^\dagger(t', \lambda), \end{aligned} \tag{3.88}$$

so the functional derivative, the variation of λ, for the above equation is

$$\begin{aligned} &\delta \left[V(t, \lambda) \mathcal{B}(\tau, \lambda) V^\dagger(t', \lambda) \right] \\ &= \delta \left[V(t, \lambda) V^\dagger(\tau, \lambda) U_1^\dagger(\tau) B(\mathbf{q}, \mathbf{p}, \tau; \lambda) U_1(\tau) V(\tau, \lambda) V^\dagger(t', \lambda) \right] \end{aligned} \tag{3.89}$$

$$\begin{aligned} = &\, \delta \left[V(t, \lambda) V^\dagger(\tau, \lambda) \right] U_1^\dagger(\tau) B(\mathbf{q}, \mathbf{p}, \tau; \lambda) U_1(\tau) V(\tau, \lambda) V^\dagger(t', \lambda) \\ &+ V(t, \lambda) \delta \mathcal{B}(\tau, \lambda) V^\dagger(t', \lambda) \\ &+ V(t, \lambda) V^\dagger(\tau, \lambda) U_1^\dagger(\tau) B(\mathbf{q}, \mathbf{p}, \tau; \lambda) U_1(\tau) \delta \left[V(\tau, \lambda) V^\dagger(t', \lambda) \right]. \end{aligned} \tag{3.90}$$

Using the Eq.(3.71), we consider the first term of Eq.(3.90) as

$$\begin{aligned} &\delta \left[V(t, \lambda) V^\dagger(\tau, \lambda) \right] U_1^\dagger(\tau) B(\mathbf{q}, \mathbf{p}, \tau; \lambda) U_1(\tau) V(\tau, \lambda) V^\dagger(t', \lambda) \\ &= -\frac{i}{\hbar} V(t, \lambda) \left[\left(\int_{\tau}^{t} \mathrm{d}\tau' U^\dagger(\tau', \lambda) \delta H(\tau', \lambda) U(\tau, \lambda) \right) \mathcal{B}(\tau, \lambda) \right] V^\dagger(t', \lambda) \\ &= -\frac{i}{\hbar} V(t, \lambda) \left[\int_{\tau}^{t} \mathrm{d}\tau' \delta \mathcal{H}(\tau', \lambda) \mathcal{B}(\tau, \lambda) \right] V^\dagger(t', \lambda), \end{aligned} \tag{3.91}$$

where

$$\begin{aligned} &V^\dagger(\tau, \lambda) U_1^\dagger(\tau) B(\mathbf{q}, \mathbf{p}, \tau; \lambda) U_1(\tau) V(\tau, \lambda) \\ &= U^\dagger(\tau, \lambda) U_1(\tau) U_1^\dagger(\tau) B(\mathbf{q}, \mathbf{p}, \tau; \lambda) U_1(\tau) U_1^\dagger(\tau) U(\tau, \lambda) \\ &= U^\dagger(\tau, \lambda) B(\mathbf{q}, \mathbf{p}, \tau; \lambda) U(\tau, \lambda) \\ &\equiv \mathcal{B}(\tau, \lambda) \end{aligned} \tag{3.92}$$

by using the unitary operator's property. As the same as the third term, is

$$\begin{aligned}&V(t,\lambda)V^{\dagger}(\tau,\lambda)U_1^{\dagger}(\tau)B(\mathbf{q},\mathbf{p},\tau;\lambda)U_1(\tau)\delta\left[V(\tau,\lambda)V^{\dagger}(t',\lambda)\right]\\&=-\frac{i}{\hbar}V(t,\lambda)\left[\int_{t'}^{\tau}\mathrm{d}\tau'\mathcal{B}(\tau,\lambda)\delta\mathcal{H}(\tau',\lambda)\right]V^{\dagger}(t',\lambda).\end{aligned}\tag{3.93}$$

Finally, substituting Eq.(3.91) and Eq.(3.93), we obtain

$$\begin{aligned}\delta\left[V(t,\lambda)\mathcal{B}(\tau,\lambda)V^{\dagger}(t',\lambda)\right]=&-\frac{i}{\hbar}V(t,\lambda)\left[\int_{\tau}^{t}\mathrm{d}\tau'\delta\mathcal{H}(\tau',\lambda)\mathcal{B}(\tau,\lambda)\right]V^{\dagger}(t',\lambda)\\&+V(t\lambda)\delta\mathcal{B}(\tau,\lambda)V^{\dagger}(t',\lambda)\\&-\frac{i}{\hbar}V(t,\lambda)\left[\int_{t'}^{\tau}\mathrm{d}\tau'\mathcal{B}(\tau,\lambda)\delta\mathcal{H}(\tau',\lambda)\right]V^{\dagger}(t',\lambda).\end{aligned}\tag{3.94}$$

Using the definition of the **Chronological Ordering**[6] which is denoted as

$$(H(t)H(t'))_+=(H(t')H(t))_+=H(t)H(t'),\tag{3.95}$$

then we can merge the two term, the first and the third, as

$$\begin{aligned}\delta\left[V(t,\lambda)\mathcal{B}(\tau,\lambda)V^{\dagger}(t',\lambda)\right]=&-\frac{i}{\hbar}V(t,\lambda)\int_{t'}^{t}\mathrm{d}\tau'\left(\mathcal{B}(\tau,\lambda)\delta\mathcal{H}(\tau',\lambda)\right)_+V^{\dagger}(t',\lambda)\\&+V(t,\lambda)\delta\mathcal{B}(\tau,\lambda)V^{\dagger}(t',\lambda).\end{aligned}\tag{3.96}$$

Taking Eq.(3.96) for the matrix element, ${}_1\langle\mathbf{a}t|\mathbf{b}t'\rangle_1$, we get

$$\begin{aligned}&\delta_1\langle at|V(t,\lambda)\mathcal{B}(\tau,\lambda)V^{\dagger}(t',\lambda)|bt'\rangle_1\\&=-\frac{i}{\hbar}{}_1\langle\mathbf{a}t|V(t,\lambda)\int_{t'}^{t}\mathrm{d}\tau'\left(\mathcal{B}(\tau,\lambda)\delta\mathcal{H}(\tau',\lambda)_+\right)V^{\dagger}(t',\lambda)|\mathbf{b}t'\rangle_1\\&+{}_1\langle\mathbf{a}t|V(t,\lambda)\delta\mathcal{B}(\tau,\lambda)V^{\dagger}(t',\lambda)|\mathbf{b}t'\rangle_1.\end{aligned}\tag{3.97}$$

At last, an equation above is

$$\delta\langle\mathbf{a}t|\mathcal{B}(\tau,\lambda)|\mathbf{b}t'\rangle=-\frac{i}{\hbar}\langle\mathbf{a}t|\int_{t'}^{t}\mathrm{d}\tau'\left(\mathcal{B}(\tau,\lambda)\delta\mathcal{H}(\tau',\lambda)\right)_+|\mathbf{b}t'\rangle+\langle\mathbf{a}t|\delta\mathcal{B}(\tau,\lambda)|\mathbf{b}t'\rangle,\tag{3.98}$$

where, similarly, $\mathbf{q}(\tau),\mathbf{p}(\tau)$ and also $\mathbf{a},\mathbf{b}$ are kept fixed.

Eq.(3.98), by replacing $\mathcal{B}(\tau,\lambda)$ by $\mathbf{q}(\tau)$ and using the functional derivative, $\mathbf{F}(\tau')$, leads to

$$\begin{aligned}(-i\hbar)\frac{\delta}{\delta\mathbf{F}(\tau')}\langle\mathbf{q}t|\mathbf{q}(\tau)|\mathbf{q}'t'\rangle=&\frac{\delta}{\delta\mathbf{F}(\tau')}\langle\mathbf{q}t|\int_{t'}^{t}\mathrm{d}\tau'\left(\mathbf{q}(\tau)\delta\mathcal{H}(\tau',\lambda)\right)_+|\mathbf{q}'t'\rangle\\&+\frac{\delta}{\delta\mathbf{F}(\tau')}\langle\mathbf{q}t|\delta\mathbf{q}(\tau)|\mathbf{q}'t'\rangle\\=&\langle\mathbf{q}t|\left(\mathbf{q}(\tau')\mathbf{q}(\tau)\right)_+|\mathbf{q}'t'\rangle\\=&(i\hbar)\frac{\delta}{\delta\mathbf{F}(\tau')}(-i\hbar)\frac{\delta}{\delta\mathbf{F}(\tau)}\langle\mathbf{q}t|\mathbf{q}'t'\rangle,\end{aligned}\tag{3.99}$$

where without the last term, be zero, because of $\mathbf{q}(\tau)$ is kept fixed, and from Eq.(3.81) that given in a last line.

Replacing Eq.(3.99) again, by using Eq.(3.81) and Eq.(3.82), it gives

$$\begin{aligned}(-i\hbar)\frac{\delta}{\delta\mathbf{F}(\tau_1)}...(-i\hbar)\frac{\delta}{\delta\mathbf{F}(\tau)_n}(i\hbar)\frac{\delta}{\delta\mathbf{S}(\tau_1')}...(i\hbar)\frac{\delta}{\delta\mathbf{S}(\tau_m')}\langle\mathbf{q}t|\mathbf{q}'t'\rangle \\ = \langle\mathbf{q}t|\left(\mathbf{q}(\tau_1)...\mathbf{q}(\tau_n)\mathbf{p}(\tau_1')...\mathbf{p}(\tau_m')\right)_+|\mathbf{q}'t'\rangle,\end{aligned} \tag{3.100}$$

where $t' \leq \tau_1, ..., \tau_n$ and $\tau_1', ..., \tau_m' \leq t$. Note that the all functional derivative operators *commute*.

The functional derivative operators are imposed as the **Generating function** that can create the various degrees of freedom, $\mathbf{q}(\tau)$ and $\mathbf{p}(\tau)$.

3.2.2 The Propagator of Harmonic Oscillator

We will give you an easy example to find the propagator, $\langle x_2t_2|x_1t_1\rangle$ (but it does not quite short) that base on the quantum dynamical principle [24] at a below detail.

The propagator satisfies the differential equation that is

$$i\hbar\frac{\partial}{\partial t}K(x_2,t_2;x_1,t_1) = \langle x_2,t_2|\hat{H}|x_1,t_1\rangle, \qquad (t_2 > t_1). \tag{3.101}$$

The first step, starting with the Hamiltonian operator which is

$$\hat{H} = \frac{\hat{P}^2(t)}{2m} + \frac{1}{2}m\omega^2\hat{X}^2(t) \tag{3.102}$$

with the initial and final state are

$$\hat{H} = \frac{\hat{P}^2(t_1)}{2m} + \frac{1}{2}m\omega^2\hat{X}^2(t_1), \tag{3.103}$$

$$\hat{H} = \frac{\hat{P}^2(t_2)}{2m} + \frac{1}{2}m\omega^2\hat{X}^2(t_2). \tag{3.104}$$

The Hamiltonian operator is independent of time but $\hat{P}(t)$ and $\hat{X}(t)$ are time-dependent operators.

Corresponding, the Heisenberg equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{X}(t) = -\frac{i}{\hbar}[\hat{X}(t),\hat{H}] = \frac{\hat{P}(t)}{m}, \tag{3.105}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{P}(t) = -\frac{i}{\hbar}[\hat{P}(t), \hat{H}] = -m\omega^2\hat{X}(t). \tag{3.106}$$

Next, using the normal differential equation method to find the solution of this system that is $X(t) = A\cos\omega t$ and time-splitting then we get

$$X(t) = A\cos(\omega(t - t_1) + \omega t_1).$$

From that solution, the momentum operator is

$$P(t) = mdX(t)/dt = -m\omega A\sin(\omega t).$$

Finally, for the two operators at t_2, we obtain

$$\hat{X}(t_2) = \hat{X}(t_1)\cos(\omega(t_2 - t_1)) + \frac{\hat{P}(t_1)}{m\omega}\sin(\omega(t_2 - t_1)), \tag{3.107}$$

$$\hat{P}(t_2) = -m\omega\hat{X}(t_1)\sin(\omega(t_2 - t_1)) + \hat{P}(t_1)\cos(\omega(t_2 - t_1)). \tag{3.108}$$

Now, rewriting Eq.(3.107) as

$$\hat{P}(t_1) = \frac{m\omega}{\sin(\omega(t_2 - t_1))}\left[\hat{X}(t_2) - \hat{X}(t_2)\cos(\omega(t_2 - t_1))\right], \tag{3.109}$$

and substituting in Eq.(3.103), we get

$$\hat{H}_{ord} = \frac{m\omega^2}{2\sin^2(\omega T)}\left[\hat{X}(t_2) + \hat{X}^2(t_1) - 2\hat{X}(t_2)\hat{X}(t_1)\cos(\omega T)\right] - \frac{i\hbar\omega}{2}\cot(\omega T), \tag{3.110}$$

where $T = t_2 - t_1$ and by using $\left[\hat{X}(t_2), \hat{X}(t_1)\right] = i\hbar\sin(\omega T)/m\omega$, the commuting.

The second step, we find the function $F(x_2, t_2; x_1, t_1)$ which is

$$\begin{aligned} F(x_2, t_2; x_1, t_1) &= \frac{\langle x_2, t_2|\hat{H}(\hat{X}(t_2), \hat{X}(t_1))|x_1, t_1\rangle}{\langle x_2, t_2|x_1, t_1\rangle} \\ &= \frac{m\omega^2}{2}\left[(x_2^2 + x_1^2)\csc^2(\omega T) - 2x_2x_1\cot(\omega T)\csc(\omega T)\right] \\ &\quad - \frac{i\hbar\omega}{2}\cot(\omega T). \end{aligned} \tag{3.111}$$

From Eq.(3.101), we integrate over t time and it gives

$$\langle x_2, t_2|x_1, t_1\rangle = C(x_2, x_1)\exp\left(-\frac{i}{\hbar}\int_{t_1}^{t_2}\mathrm{d}T\, F(x_2, T; x_1, 0)\right). \tag{3.112}$$

By the way, the $F(x_2,t_2;x_1,t_1)$ function in an above equation is replaced by the Eq.(3.111) and we get

$$\begin{aligned}\langle x_2,T|x_1,0\rangle = C(x_2,x_1)\exp\Bigg(&-\frac{i}{\hbar}\int_0^T \mathrm{d}T'\,\frac{m\omega^2}{2}\Big((x_2^2+x_1^2)\csc^2(\omega T') \\ &- 2x_2x_1\cot(\omega T')\csc(\omega T')\Big) + \frac{i}{\hbar}\int_0^T \mathrm{d}T'\,\frac{i\hbar\omega}{2}\cot(\omega T')\Bigg).\end{aligned} \tag{3.113}$$

Finally, we obtain the propagator which is

$$\langle x_2,T|x_1,0\rangle = \frac{C(x_2,x_1)}{\sqrt{\sin(\omega T)}}\exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left[(x_2^2+x_1^2)\cos(\omega T) - 2x_2x_1\right]\right). \tag{3.114}$$

The third step, we prove that $C(x_2,x_1)$ are independent of x_2, x_1 and try to write $\hat{P}(t_2)$ that is a function of $x(t_2)$ and $x(t_1)$, as

$$\begin{aligned}\hat{P}(t_2) &= m\omega\cot(\omega(t_2-t_1))\left[\hat{X}(t_2) - \hat{X}(t_1)\cos(\omega(t_2-t_1))\right] \\ &\quad - m\omega\hat{X}(t_1)\sin(\omega(t_2-t_1)).\end{aligned} \tag{3.115}$$

The conditions are

$$\langle x_2,t_2|\hat{P}(t_2)|x_1,t_1\rangle = -i\hbar\frac{\partial}{\partial x_2}\langle x_2,t_2|x_1,t_1\rangle, \tag{3.116}$$

$$\langle x_2,t_2|\hat{P}(t_1)|x_1,t_1\rangle = i\hbar\frac{\partial}{\partial x_1}\langle x_2,t_2|x_1,t_1\rangle. \tag{3.117}$$

So, we substitute the Eq.(3.109) and Eq.(3.115) in the two above equations and we actually get

$$\frac{\partial C(x_2,x_1)}{\partial x_1} = \frac{\partial C(x_2,x_1)}{\partial x_2} = 0 \tag{3.118}$$

that $C(x_2,x_1)$ is not depend on x_2, x_1.

To evaluate C, we have to take the limit of T to zero of the Eq.(3.114) and use the Gaussian integral which is

$$\begin{aligned}\lim_{T\to 0^+}\langle x_2,T|x_1,0\rangle &= \lim_{T\to 0^+}\frac{C}{\sqrt{\omega T}}\exp\left[\frac{im}{2\hbar T}(x_2-x_1)^2\right] \\ &= C\sqrt{\frac{2\pi i\hbar}{m\omega}}\delta(x_2-x_1),\end{aligned} \tag{3.119}$$

where the value of ωT is small, leads to $\sin(\omega T)\approx\omega T$ and $\cos(\omega T)\approx 1$.

The result of C is given as

$$C = \frac{m\omega}{2\pi i\hbar} \tag{3.120}$$

by the definition of initial,

$$\lim_{T\to 0^+} \langle x_2, T | x_1, 0\rangle = \delta(x_2 - x_1). \tag{3.121}$$

Finally, the Eq.(3.114), the transformation function, becomes

$$\langle x_2, T | x_1, 0\rangle = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega T)}} \exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left[(x_2^2 + x_1^2)\cos(\omega T) - 2x_2x_1\right]\right). \tag{3.122}$$

The truly propagator, includes the time ordering condition, is

$$\begin{aligned} K(x_2, t_2; x_1, t_1) = \Theta(t_2 - t_1)&\sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega T)}} \\ &\times \exp\left(\frac{im\omega}{2\hbar\sin(\omega T)}\left[(x_2^2 + x_1^2)\cos(\omega T) - 2x_2x_1\right]\right). \end{aligned} \tag{3.123}$$

For the applications of the Quantum Dynamical Principle is given in the next chapter that about free particle and the potential scattering.

CHAPTER IV

THE PARTICLE WITH/WITHOUT THE POTENTIAL

In this chapter, we would like to provide a detail about the particle wave travels trough free space or some potential. With the potential, the particle is scattered and detected at last. We apply the **Quantum Dynamical Principle (QDP)** to this situation (system) for finding the propagator. Eventually, we have to find the *Asymptotically Free Green function* which is the time functional, going to infinity, and independent of x for the word "free".

4.1 The Free Particle

For free particle, ($H = \mathbf{p}^2/2m$, with the momentum p) we introduce the Hamiltonian equation, includes the sources term, as

$$H_0 = \xi \frac{\mathbf{p}^2}{2m} - \mathbf{q}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau), \tag{4.1}$$

and from the previous chapter, see Eq.(3.73), we get

$$\frac{\partial}{\partial\xi}\langle \mathbf{q}t|\mathbf{q}'t'\rangle_\xi = -\frac{i}{\hbar}\int_{t'}^{t} \mathrm{d}\tau \langle \mathbf{q}t|H(\mathbf{q}(\tau),\mathbf{p}(\tau),\tau)|\mathbf{q}'t'\rangle_\xi, \tag{4.2}$$

$$\frac{\partial}{\partial\xi}\langle \mathbf{q}t|\mathbf{q}'t'\rangle_\xi = -\frac{i}{\hbar}\int_{t'}^{t} \mathrm{d}\tau H\left(-i\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}, i\hbar\frac{\delta}{\delta\mathbf{S}(\tau)},\tau\right)\langle \mathbf{q}t|\mathbf{q}'t'\rangle_\xi, \tag{4.3}$$

where ξ is the arbitrary parameter which varying and replacing $\mathbf{q}(\tau),\mathbf{p}(\tau)$ by the sources $\mathbf{F}(\tau),\mathbf{S}(\tau)$, the functional derivative that we have been proof before.

We integrate over ξ from $\xi = 0$ to 1 and get

$$\ln\langle \mathbf{q}t|\mathbf{q}'t'\rangle_\xi\Big|_0^1 = -\frac{i}{\hbar}\int_{t'}^{t} \mathrm{d}\tau H\left(-i\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}, i\hbar\frac{\delta}{\delta\mathbf{S}(\tau)},\tau\right), \tag{4.4}$$

$$\langle \mathbf{q}t|\mathbf{q}'t'\rangle = \exp\left[-\frac{i}{\hbar}\int_{t'}^{t} \mathrm{d}\tau H\left(-i\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}, i\hbar\frac{\delta}{\delta\mathbf{S}(\tau)},\tau\right)\right]\langle \mathbf{q}t|\mathbf{q}'t'\rangle_0 \tag{4.5}$$

with $t' \leq \tau < t$ and $\langle \mathbf{q}t|\mathbf{q}'t'\rangle_{\xi=1} = \langle \mathbf{q}t|\mathbf{q}'t'\rangle$. So, for the Hamiltonian equation Eq.(4.1) with the lower script 0, the Eq.(4.5) becomes

$$\langle \mathbf{q}t|\mathbf{q}'t'\rangle^{(0)} = \exp\left[-\frac{i}{2m\hbar}\int_{t'}^{t} \mathrm{d}\tau \left(i\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)^2\right] \langle \mathbf{q}t|\mathbf{q}'t'\rangle_0 \Bigg|_{\mathbf{S}=0,\mathbf{F}=0}, \tag{4.6}$$

where, for the free particle, we have to finally set $\mathbf{F}, \mathbf{S} = 0$.

Next, we have to find $\langle \mathbf{q}t|\mathbf{q}'t'\rangle_0$ which is $\xi = 0$. From the Hamiltonian equation is

$$H = -\mathbf{q}\cdot\mathbf{F}(\tau) + \mathbf{p}\cdot\mathbf{S}(\tau). \tag{4.7}$$

The Heisenberg equations from this Hamiltonian are

$$\dot{\mathbf{q}}(\tau) = \mathbf{S}(\tau), \tag{4.8}$$

$$\dot{\mathbf{p}}(\tau) = \mathbf{F}(\tau). \tag{4.9}$$

These all the above equations are integrated as

$$\mathbf{q}(\tau) = \mathbf{q}(t) - \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau'-\tau)\mathbf{S}(\tau') \qquad ;\tau' < \tau, \tag{4.10}$$

$$\mathbf{p}(\tau) = \mathbf{p}(t') + \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau-\tau')\mathbf{F}(\tau') \qquad ;\tau' > \tau, \tag{4.11}$$

where $\Theta(\tau'-\tau)$ and $\Theta(\tau-\tau')$ are the time ordering which tell us that the existing of sources in time interval. For $\mathbf{q}(\tau)$, we integrate from τ to t because the $\mathbf{S}(\tau)$ exists in that time. Also for the $\mathbf{F}(\tau)$ source, because of $\mathbf{p}(\tau)$, is integrated from t' to τ.

Next, we take the matrix elements $\langle \mathbf{q}t|$ and $|\mathbf{p}t'\rangle$ into the two equations above for $\xi = 0$ and calculate as

$$\begin{aligned}\langle \mathbf{q}t|\mathbf{q}(\tau)|\mathbf{p}t'\rangle_0 &= {}_0\langle \mathbf{q}t|\mathbf{q}(t)|\mathbf{p}t'\rangle_0 - \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau'-\tau)\mathbf{S}(\tau')\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 \\ &= \mathbf{q}\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 - \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau'-\tau)\mathbf{S}(\tau')\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 \\ &= \left[\mathbf{q} - \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau'-\tau)\mathbf{S}(\tau')\right]\langle \mathbf{q}t|\mathbf{p}t'\rangle_0,\end{aligned} \tag{4.12}$$

where using the relation for this equation is ${}_0\langle \mathbf{q}t|\mathbf{q}(t) = (\mathbf{q})_0\langle \mathbf{q}t|$. Then,

$$\begin{aligned}\langle \mathbf{q}t|\mathbf{p}(\tau)|\mathbf{p}t'\rangle_0 &= {}_0\langle \mathbf{q}t|\mathbf{p}(t')|\mathbf{p}t'\rangle_0 + \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau-\tau')\mathbf{F}(\tau')\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 \\ &= \mathbf{p}\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 + \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau-\tau')\mathbf{F}(\tau')\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 \\ &= \left[\mathbf{p} + \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau-\tau')\mathbf{F}(\tau')\right]\langle \mathbf{q}t|\mathbf{p}t'\rangle_0,\end{aligned} \tag{4.13}$$

where this equation with $p(t')|pt'\rangle_0 = (p)|pt'\rangle_0$.

Finally, we get

$$\langle \mathbf{q}t|\mathbf{q}(\tau)|\mathbf{p}t'\rangle_0 = \left[\mathbf{q} - \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau'-\tau)\mathbf{S}(\tau')\right] \langle \mathbf{q}t|\mathbf{p}t'\rangle_0, \tag{4.14}$$

$$\langle \mathbf{q}t|\mathbf{p}(\tau)|\mathbf{p}t'\rangle_0 = \left[\mathbf{p} + \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau-\tau')\mathbf{F}(\tau')\right] \langle \mathbf{q}t|\mathbf{p}t'\rangle_0. \tag{4.15}$$

From the previous chapter ((3.81) and (3.82)), the Eq.(4.14) and Eq.(4.15) are rewritten as

$$-i\hbar\frac{\delta}{\delta \mathbf{F}(\tau)}\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 = \left[\mathbf{q} - \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau'-\tau)\mathbf{S}(\tau')\right] \langle \mathbf{q}t|\mathbf{p}t'\rangle_0, \tag{4.16}$$

$$i\hbar\frac{\delta}{\delta \mathbf{S}(\tau)}\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 = \left[\mathbf{p} + \int_{t'}^{t} \mathrm{d}\tau'\Theta(\tau-\tau')\mathbf{F}(\tau')\right] \langle \mathbf{q}t|\mathbf{p}t'\rangle_0. \tag{4.17}$$

The Eq.(4.16) and Eq.(4.17) are integrated as

$$\begin{aligned}\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 = \exp\left[\frac{i}{\hbar}\mathbf{q}\int_{t'}^{t} \mathrm{d}\tau\mathbf{F}(\tau)\right] \exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\mathbf{F}(\tau)\Theta(\tau'-\tau)\mathbf{S}(\tau')\right] \\ \times \exp\left[\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{p}\right]\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 = \exp\left[-\frac{i}{\hbar}\mathbf{p}\int_{t'}^{t} \mathrm{d}\tau\mathbf{S}(\tau)\right] \exp\left[\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\mathbf{S}(\tau)\Theta(\tau-\tau')\mathbf{F}(\tau')\right] \\ \times \exp\left[\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{p}\right],\end{aligned} \tag{4.19}$$

where $\exp(i\mathbf{q}\cdot\mathbf{p}/\hbar) = \langle \mathbf{q}t'|\mathbf{p}t'\rangle$ is satisfied the boundary condition for $\mathbf{F} = 0$ and $\mathbf{S} = 0$.

The final result is

$$\begin{aligned}\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 = \exp\left[\frac{i}{\hbar}\mathbf{q}\int_{t'}^{t} \mathrm{d}\tau\mathbf{F}(\tau)\right] \exp\left[-\frac{i}{\hbar}\mathbf{p}\int_{t'}^{t} \mathrm{d}\tau\mathbf{S}(\tau)\right] \exp\left(\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{p}\right) \\ \times \exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau'\mathbf{S}(\tau)\Theta(\tau-\tau')\mathbf{F}(\tau')\right].\end{aligned} \tag{4.20}$$

For the other application, immediately, we multiply Eq.(4.20) by $\langle \mathbf{p}t'|\mathbf{q}'t'\rangle = \exp(-i\mathbf{q}'\cdot\mathbf{p}/\hbar)$ and integrate over $\mathbf{p}$ by $\mathrm{d}\mathbf{p}/2\pi\hbar$

$$\begin{aligned}\langle \mathbf{q}t|\mathbf{p}t'\rangle\langle \mathbf{p}t'|\mathbf{q}'t'\rangle_0 = \int \frac{\mathrm{d}\mathbf{p}}{2\pi\hbar} \exp\left[-\frac{i}{\hbar}\mathbf{p}\int_{t'}^{t} \mathrm{d}\tau\mathbf{S}(\tau)\right] \exp\left(\frac{i}{\hbar}(\mathbf{q}-\mathbf{q}')\cdot\mathbf{p}\right) \\ \times \exp\left[\frac{i}{\hbar}\mathbf{q}\int_{t'}^{t} \mathrm{d}\tau\mathbf{F}(\tau)\right] \exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\mathbf{S}(\tau)\Theta(\tau-\tau')\mathbf{F}(\tau')\right]\end{aligned} \tag{4.21}$$

then it was rewriting as

$$\langle \mathbf{q}t|\mathbf{q}'t'\rangle_0 = \int \frac{\mathrm{d}\mathbf{p}}{2\pi\hbar} \exp\left[\frac{i}{\hbar}\left(\mathbf{q} - \mathbf{q}' - \int_{t'}^{t} \mathrm{d}\tau \mathbf{S}(\tau)\right)\mathbf{p}\right] \exp\left[\frac{i}{\hbar}\mathbf{q}\int_{t'}^{t} \mathrm{d}\tau \mathbf{F}(\tau)\right]$$
$$\times \exp\left[-\frac{i}{\hbar}\int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau' \mathbf{S}(\tau)\Theta(\tau - \tau')\mathbf{F}(\tau')\right]. \tag{4.22}$$

At last, we get

$$\langle \mathbf{q}t|\mathbf{q}'t'\rangle_0 = \delta\left(\mathbf{q} - \mathbf{q}' - \int_{t'}^{t} \mathrm{d}\tau \mathbf{S}(\tau)\right) \exp\left[\frac{i}{\hbar}\mathbf{q}\int_{t'}^{t} \mathrm{d}\tau \mathbf{F}(\tau)\right]$$
$$\times \exp\left[-\frac{i}{\hbar}\int_{t'}^{t} \mathrm{d}\tau \int_{t'}^{t} \mathrm{d}\tau' \mathbf{S}(\tau)\Theta(\tau - \tau')\mathbf{F}(\tau')\right], \tag{4.23}$$

where using the Dirac delta function, is

$$\delta(\mathbf{q} - \mathbf{q}' - \int_{t'}^{t} \mathrm{d}\tau \mathbf{S}(\tau)) = \int \frac{\mathrm{d}\mathbf{p}}{2\pi\hbar} \exp\left[\frac{i\mathbf{p}}{\hbar}\left(\mathbf{q} - \mathbf{q}' - \int_{t'}^{t} \mathrm{d}\tau \mathbf{S}(\tau)\right)\right].$$

The above transformation function, Eq.(4.23), is very useful for the path integral.

We use the result in Eq.(4.20) for the Eq.(4.6) and set

$$\int_{t'}^{t} \mathrm{d}\tau' \Theta(\tau - \tau')\mathbf{F}(\tau') = \hat{\mathbf{F}}(\tau). \tag{4.24}$$

For Eq.(4.20) is rewritten as

$$\langle \mathbf{q}t|\mathbf{p}t'\rangle_0 = \exp\left[\frac{i}{\hbar}\mathbf{q}\int_{t'}^{t} \mathrm{d}\tau \mathbf{F}(\tau)\right] \exp\left(\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{p}\right) \exp\left[-\frac{i}{\hbar}\int_{t'}^{t} \mathrm{d}\tau \mathbf{S}(\tau)\left(\mathbf{p} + \hat{\mathbf{F}}(\tau)\right)\right] \tag{4.25}$$

and, in above equation, the last exponential term, after using the variation of $\mathbf{S}(\tau)$, generates $\left[-i(\mathbf{p} + \hat{\mathbf{F}}(\tau))/\hbar\right]$ when the limit is $\mathbf{S}(\tau) \to 0$.
Thus, $(\delta/\delta\mathbf{S}(\tau))^2$ can be replacing by $\left[-i(\mathbf{p} + \hat{\mathbf{F}}(\tau))/\hbar\right]^2$ and from Eq.(4.6), we obtain

$$\langle \mathbf{q}t|\mathbf{p}t'\rangle^{(0)} = \exp\left[-\frac{i}{2\pi\hbar}\int_{t'}^{t} \mathrm{d}\tau \left(\mathbf{p} + \hat{\mathbf{F}}(\tau)\right)^2\right] \exp\left[\frac{i}{\hbar}\mathbf{q}\int_{t'}^{t} \mathrm{d}\tau F(\tau)\right]$$
$$\times \exp\left(\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{p}\right) \exp\left[-\frac{i}{\hbar}\int_{t'}^{t} \mathrm{d}\tau \mathbf{S}(\tau)(\mathbf{p} + \hat{\mathbf{F}}(\tau))\right]\Bigg|_{\mathbf{S}=0}, \tag{4.26}$$

where associates with the Hamiltonian, for $\mathbf{S}(\tau) = 0$, is

$$H_{(0)} = \frac{\mathbf{p}^2}{2m} - \mathbf{q}\cdot\mathbf{F}(\tau). \tag{4.27}$$

Finally, Eq.(4.26) becomes

$$\langle \mathbf{q}t|\mathbf{p}t'\rangle^{(0)}\Big|_{\mathbf{S}=0} = \exp\left[\frac{i}{\hbar}\left(\mathbf{q}\cdot\mathbf{p} - \frac{\mathbf{p}^2}{2m}(t-t')\right)\right]\exp\left[\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\mathbf{F}(\tau)\left(\mathbf{q} - \frac{\mathbf{p}}{2m}(t-\tau)\right)\right]$$
$$\times\exp\left[-\frac{i}{2\pi\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\mathbf{F}(\tau)(t-\tau_>)\mathbf{F}(\tau')\right], \tag{4.28}$$

where $\tau_> = max(\tau,\tau')$. This equation apply to next section for finding the scattering propagator.

Multiplying Eq.(4.28) by $\langle \mathbf{p}t'|\mathbf{q}'t'\rangle = \exp(-i\mathbf{q}'\cdot\mathbf{p}/\hbar)$ and, similar Eq.(4.23), integrating over $\mathbf{p}$ with $\mathrm{d}\mathbf{p}/2\pi\hbar$, we get

$$\begin{aligned}\langle \mathbf{q}t|\mathbf{q}'t'\rangle^{(0)} = &\int\frac{\mathrm{d}\mathbf{p}}{2\pi\hbar}\exp\left[\frac{i}{\hbar}\left(\mathbf{q}\cdot\mathbf{p} - \mathbf{q}'\cdot\mathbf{p} - \frac{\mathbf{p}^2}{2m}(t-t')\right)\right]\\ &\times\exp\left[\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\mathbf{F}(\tau)\left(\mathbf{q} - \frac{\mathbf{p}}{m}(t-\tau)\right)\right]\\ &\times\exp\left[-\frac{i}{2m\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\mathbf{F}(\tau)(t-\tau_>)\mathbf{F}(\tau')\right].\end{aligned} \tag{4.29}$$

Then, by using the Gaussian integral that is denoted as

$$\int \mathrm{d}\,xe^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}+c}, \tag{4.30}$$

we obtain **The Free Particle** which is expressed as

$$\begin{aligned}\langle \mathbf{q}t|\mathbf{q}'t'\rangle^{(0)} = &\sqrt{\frac{m}{2\pi i\hbar T}}\exp\left[\frac{im}{2\hbar T}(\mathbf{q}-\mathbf{q}')^2\right]\\ &\times\exp\left[\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\mathbf{F}(\tau)\left(\mathbf{q}' + \frac{(\mathbf{q}-\mathbf{q}')}{T}(\tau-t')\right)\right]\\ &\times\exp\left[-\frac{i}{m\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\mathbf{F}(\tau)\frac{(t-\tau)\Theta(\tau-\tau')(\tau'-t')}{T}\mathbf{F}(\tau')\right],\end{aligned} \tag{4.31}$$

where, after, setting $\mathbf{F} = 0$ which generates $\frac{i}{\hbar}(\mathbf{q}' + (\mathbf{q}-\mathbf{q}')(\tau-t')/T)$ that can replace $(\delta/\delta\mathbf{F}(\tau))$.

At last, this method gives us the above equation is the transformation function, for the free particle from the $\mathbf{q}' \to \mathbf{q}$, which is very useful for many application. For the next section, you will understand this method.

4.2 The Functional Treatment for Quantum Scattering

In this case, we pleased to apply the functional method (QDP) [5, 7] to find the propagator of this general potential. We would like to start with the given Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}), \tag{4.32}$$

where the particle mass is m that interact with the potential $V(\mathbf{x})$. This section, we use the $\mathbf{q}, \mathbf{p}$ language as $\mathbf{x}, \mathbf{p}$.

Introducing, the new Hamiltonian associates with the external sources $\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$ as

$$H'(\lambda, \tau) = \frac{\mathbf{p}^2}{2m} + \lambda V(\mathbf{x}) - \mathbf{x} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau), \tag{4.33}$$

where λ, the arbitrary parameter, will be set equal to one.

From the QDP, Eq.(4.3), the variation of the transformation function of the Hamiltonian $H'(\lambda, \tau)$ which respect to the λ parameter, is

$$\delta\langle \mathbf{x}t|\mathbf{p}t'\rangle = -\frac{i}{\hbar}\int_{t'}^{t} \mathrm{d}\tau \delta\left[\lambda V\left(-i\hbar\frac{\delta}{\delta \mathbf{F}(\tau)}\right)\right]\langle \mathbf{x}t|\mathbf{p}t'\rangle, \tag{4.34}$$

where $V(\mathbf{x}) = V(-i\hbar\delta/\delta\mathbf{F}(\tau))$ with $\mathbf{x}$ is replaced by $-i\hbar\delta/\delta\mathbf{F}(\tau)$.

The Eq.(4.34) above can be immediately integrated over λ from 0 to 1 and $\mathbf{F}(\tau), \mathbf{S}(\tau)$ are set equal to zero, so, it becomes

$$\langle \mathbf{x}t|\mathbf{p}t'\rangle = \exp\left[-\frac{i}{\hbar}\int_{t'}^{t} \mathrm{d}\tau V\left(-i\hbar\frac{\delta}{\delta \mathbf{F}(\tau)}\right)\right]\langle \mathbf{x}t|\mathbf{p}t'\rangle^{(0)}\Big|_{\mathbf{F},\mathbf{S}=0} \tag{4.35}$$

which here the $\langle \mathbf{x}t|\mathbf{p}t'\rangle$ is governed by the Hamiltonian in Eq.(4.32).

The Hamiltonian for the transformation function $\langle \mathbf{x}t|\mathbf{p}t'\rangle^{(0)}$ which $\lambda = 0$, is

$$H'(0, \tau) = \frac{\mathbf{p}^2}{2m} - \mathbf{x} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau) \tag{4.36}$$

and, for the free particle which depend on the Hamiltonian in Eq. (4.1), we get

$$\langle \mathbf{x}t|\mathbf{p}t'\rangle^{(0)} = \exp\left[-\frac{i}{2m\hbar}\int_{t'}^{t} \mathrm{d}\tau \left(i\hbar\frac{\delta}{\delta \mathbf{S}(\tau)}\right)^2\right]\langle \mathbf{x}t|\mathbf{p}t'\rangle_0 \tag{4.37}$$

by using a same step as integration over ξ from $0 \to 1$ in Eq.(4.6).
The transformation function $\langle xt|pt'\rangle_0$ associates to

$$\hat{H}(\tau) = -\mathbf{x} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau) \tag{4.38}$$

and is token to calculate, finally, as

$$\langle \mathbf{x}t|\mathbf{p}t'\rangle_0 = \exp\left[\frac{i}{\hbar}\mathbf{x}\cdot\left(\mathbf{p}+\int_{t'}^{t}\mathrm{d}\tau\mathbf{F}(\tau)\right)\right]\exp\left[-\frac{i}{\hbar}\mathbf{p}\int_{t'}^{t}\mathrm{d}\tau\mathbf{S}(\tau)\right]$$
$$\times\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\mathbf{S}(\tau)\Theta(\tau-\tau')\mathbf{F}(\tau')\right]. \tag{4.39}$$

Next, we set $\mathbf{S}(\tau)=0$ that give the boundary condition is $\langle\mathbf{x}t|\mathbf{p}t\rangle = \exp(i\mathbf{x}\cdot\mathbf{p}/\hbar)$ and set the same as Eq.(4.24) (You can turn pages back to reread this method again).

Finally, this process likes the previous deriving Eq.(4.28) that is

$$\langle\mathbf{x}t|\mathbf{p}t'\rangle^{(0)}\Big|_{\mathbf{S}=0} = \exp\left[\tfrac{i}{\hbar}\left(\mathbf{x}\cdot\mathbf{p}-\tfrac{\mathbf{p}^2}{2m}(t-t')\right)\right]\exp\left[\tfrac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\mathbf{F}(\tau)\left(\mathbf{x}-\frac{\mathbf{p}}{2m}(t-\tau)\right)\right]$$
$$\times\exp\left[-\tfrac{i}{2\pi\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\mathbf{F}(\tau)(t-\tau_>)\mathbf{F}(\tau')\right],$$

and substitute this equation back to Eq.(4.35). We obtain

$$\langle\mathbf{x}t|\mathbf{p}t'\rangle = \exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau V\left(-i\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}\right)\right]\exp\left[\frac{i}{\hbar}\left(\mathbf{x}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}(t-t')\right)\right]$$
$$\times\exp\left[\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\mathbf{F}(\tau)\left(\mathbf{x}-\frac{\mathbf{p}}{m}(t-\tau)\right)\right] \tag{4.40}$$
$$\times\exp\left[-\frac{i}{2m\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\mathbf{F}(\tau)(t-\tau_>)\mathbf{F}(\tau')\right],$$

where we just set $\mathbf{S}(\tau)=0$ and, next, set $\mathbf{F}(\tau)=0$ that give $\delta/\delta\mathbf{F}(\tau)$ can be replaced by $\frac{i}{\hbar}(\mathbf{x}-\mathbf{p}(t-\tau)/m)$. Also, for the $\mathbf{F}(\tau)$ source is replaced by $-i\hbar\delta/\delta\mathbf{F}(\tau)$.

At last, it is given us the important result as

$$\langle\mathbf{x}t|\mathbf{p}t'\rangle = \exp\left[\frac{i}{\hbar}\left(\mathbf{x}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}(t-t')\right)\right]$$
$$\times\exp\left[\frac{i\hbar}{2m}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'(t-\tau_>)\frac{\delta}{\delta\mathbf{F}(\tau)}\frac{\delta}{\delta\mathbf{F}(\tau')}\right] \tag{4.41}$$
$$\times\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\, V\left(\mathbf{x}-\frac{\mathbf{p}}{m}(t-\tau)+\mathbf{F}(\tau)\right)\right]\Bigg|_{\mathbf{F}=0},$$

which is **the Translational Invariant in Time** from this theory.

For the Green function, $t>t'$, is defined as

$$\langle\mathbf{x}t|\mathbf{x}'t'\rangle = G_+(\mathbf{x}t,\mathbf{x}'t') \tag{4.42}$$

with the condition is $G_+(\mathbf{x}t, \mathbf{x}'t') = 0$ for $t < t'$.

We introduce the Fourier transform that are

$$\langle \mathbf{x}t|\mathbf{p}t'\rangle = G_+(\mathbf{x}t, \mathbf{p}t') = \int \mathrm{d}^3\mathbf{x}'\, e^{i\mathbf{p}\cdot\mathbf{x}'/\hbar} G_+(\mathbf{x}t, \mathbf{x}'t'), \tag{4.43}$$

and

$$G_+(\mathbf{p}, \mathbf{p}'; p^0) = -\frac{i}{\hbar}\frac{1}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}\,T e^{i(p^0+i\epsilon)T/\hbar} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}\langle \mathbf{x}T|\mathbf{p}'0\rangle, \tag{4.44}$$

where $t - t' \equiv T$ and $\epsilon \to +0$.

We rewrite the Eq.(4.41) for $\langle \mathbf{x}T|\mathbf{p}'0\rangle$, becomes

$$\begin{aligned}\langle \mathbf{x}T|\mathbf{p}'0\rangle = \exp&\left[\frac{i}{\hbar}\left(\mathbf{x}\cdot\mathbf{p}' - \frac{\mathbf{p}'^2}{m}T\right)\right] \\ &\times \exp\left[\frac{i\hbar}{2m}\int_0^T \mathrm{d}\tau \int_0^T \mathrm{d}\tau'(t-\tau_>)\frac{\delta}{\delta\mathbf{F}(\tau)}\frac{\delta}{\delta\mathbf{F}(\tau')}\right] \\ &\times \exp\left[-\frac{i}{\hbar}\int_0^T \mathrm{d}\tau\, V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}(t-\tau) + \mathbf{F}(\tau)\right)\right]\Bigg|_{\mathbf{F}=0},\end{aligned} \tag{4.45}$$

and substitute back into the Eq.(4.44), we obtain

$$G_+(\mathbf{p}, \mathbf{p}'; p^0) = -\frac{i}{\hbar}\frac{1}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}\alpha e^{i(p^0 - E(\mathbf{p}')+i\epsilon)\alpha/\hbar} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} K(\mathbf{x}, \mathbf{p}'; \alpha), \tag{4.46}$$

where $E(\mathbf{p}') = \mathbf{p}^2/2m$, $p^0 = mc^2$ is rest mass energy. For the last function term is denoted as

$$\begin{aligned}K(\mathbf{x}, \mathbf{p}'; \alpha) = \exp&\left[\frac{i\hbar}{2m}\int_{t'}^t \mathrm{d}\tau \int_{t'}^t \mathrm{d}\tau'(t-\tau_>)\frac{\delta}{\delta\mathbf{F}(\tau)}\frac{\delta}{\delta\mathbf{F}(\tau')}\right] \\ &\times \exp\left[-\frac{i}{\hbar}\int_{t'}^t \mathrm{d}\tau V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}(t-\tau) + \mathbf{F}(\tau)\right)\right]\Bigg|_{\mathbf{F}=0},\end{aligned} \tag{4.47}$$

for above equation, we just replace T by α, $t - t' \equiv \alpha$, which α is often used in field theory [5, 7].

The physical meaning of the $K(\mathbf{x}, \mathbf{p}'; \alpha)$ is the term of source which reacts to the particle with depends on the time parameter. Moreover, the $[\mathbf{x} - \mathbf{p}'(t-\tau)/m]$ functional of $V(\mathbf{x})$ is the trajectory of this particle after scattering that respect to $\mathbf{F}(\tau)$.

Note that, for the α-integrand in Eq.(4.46), the $[p^0 - E(\mathbf{p}) + i\epsilon]$ term is the inverse of the free Green function in the energy-momentum representation.

Next, the scattering amplitude $f(\mathbf{p},\mathbf{p}')$ with initial $\mathbf{p}$ momentum and final $\mathbf{p}'$ momenta, is

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{p}''\, V(\mathbf{p}-\mathbf{p}'')G_+(\mathbf{p}'',\mathbf{p}';p^0)[p^0 - E(\mathbf{p}')]\Big|_{p^0=E(\mathbf{p}')}\quad , \tag{4.48}$$

where here the Fourier transform for the potential is

$$V(\mathbf{p}) = \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot\mathbf{p}/\hbar}V(\mathbf{x}). \tag{4.49}$$

Let consider to find the two last terms in Eq.(4.48), by multiplying the Eq.(4.45) with $[p^0 - E(\mathbf{p}')]$, we calculate

$$\begin{aligned}
G_+(\mathbf{p},\mathbf{p}';p^0)[p^0 - E(\mathbf{p}')] &= -\frac{i}{\hbar}\frac{[p^0 - E(\mathbf{p}') + i\epsilon]}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}\alpha e^{i\alpha[p^0-E(\mathbf{p}')+i\epsilon]/\hbar} \\
&\quad\times\int \mathrm{d}^3\mathbf{x}\, e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha), \\
&= -\frac{1}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}\alpha\left(\frac{\partial}{\partial\alpha}e^{i\alpha[p^0-E(\mathbf{p}')_i\epsilon]/\hbar}\right) \\
&\quad\times\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha).
\end{aligned} \tag{4.50}$$

We integrate the above equation over α from 0 to ∞ with the integration by parts.

Before integrating, we have to determine the $K(\mathbf{x},\mathbf{p}';\alpha)$ for each boundary condition, after, is set $p^0 = E(\mathbf{p}')$, that are

For $\alpha = 0$;

$$K(\mathbf{x},\mathbf{p}';0) = \exp[0]\exp[0] = 1, \tag{4.51}$$

where $t - t' \equiv \alpha = 0$ cause the integrate boundaries similarly equal to zero.

For $\alpha \to \infty$;

$$\lim_{\alpha\to\infty}\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha), \tag{4.52}$$

if this case *exists*, for $\epsilon > 0$, it will imply that

$$\lim_{\alpha\to\infty} e^{(-\epsilon\alpha/\hbar)}\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha) = 0. \tag{4.53}$$

Starting integrate, from Eq.(4.50), we separate computing, without constant, that is

$$\int_0^\infty \mathrm{d}\alpha\left(\frac{\partial e^{(-\epsilon\alpha/\hbar)}}{\partial\alpha}\right)\left[\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(\mathbf{x},\mathbf{p}';\alpha)\right]. \tag{4.54}$$

Next, we use the integration by parts method to consider this case, beginning with

$$\int udv = uv| - \int vdu, \tag{4.55}$$

where we set

$$\begin{aligned} u &= \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} K(\mathbf{x},\mathbf{p}';\alpha), \\ \mathrm{d}\, u &= \left[\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} \frac{\partial K(\mathbf{x},\mathbf{p}';\alpha)}{\partial\alpha}\right] \mathrm{d}\alpha, \end{aligned} \tag{4.56}$$

and

$$\begin{aligned} \mathrm{d}\, v &= \left(\frac{\partial e^{(-\epsilon\alpha/\hbar)}}{\partial\alpha}\right), \\ v &= e^{(-\epsilon\alpha/\hbar)}. \end{aligned} \tag{4.57}$$

We immediately input the results of Eq.(4.56) and Eq.(4.57) into Eq.(4.55) and obtain

$$\begin{aligned} &\int_0^\infty \mathrm{d}\alpha \left(\frac{\partial e^{(-\epsilon\alpha/\hbar)}}{\partial\alpha}\right) \left[\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} K(\mathbf{x},\mathbf{p}';\alpha)\right] \\ &= - \lim_{\alpha\to\infty} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} K(\mathbf{x},\mathbf{p}';\alpha). \end{aligned} \tag{4.58}$$

Finally, the Eq.(4.50) becomes

$$G_+(\mathbf{p},\mathbf{p}';p^0)[p^0 - E(\mathbf{p}')]\Big|_{p^0=E(\mathbf{p}')} = \lim_{\alpha\to\infty} \frac{1}{(2\pi\hbar)^3} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} K(\mathbf{x},\mathbf{p}';\alpha) \tag{4.59}$$

with $p^0 = E(\mathbf{p}')$ means that the scattering occurs on the energy shell.

The scattering amplitude in Eq.(4.48) can be rewritten as

$$\begin{aligned} f(\mathbf{p},\mathbf{p}') = &-\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3\mathbf{p}'' \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} V(\mathbf{x}) \\ &\times \lim_{\alpha\to\infty} \frac{1}{(2\pi\hbar)^3} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}''-\mathbf{p}')/\hbar} K(\mathbf{x},\mathbf{p}';\alpha), \end{aligned} \tag{4.60}$$

where using the Fourier transform, Eq.(4.49).

After, integrating over $\mathbf{p}''$, we take the Dirac delta function to evaluate the last form which Eq.(4.60) is

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2} \lim_{\alpha\to\infty} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} V(\mathbf{x}) K(\mathbf{x},\mathbf{p}';\alpha). \tag{4.61}$$

In the Born approximation for the scattering amplitude is given as

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')}V(\mathbf{x}) \tag{4.62}$$

with $K(\mathbf{x},\mathbf{p}';\alpha) = 1$.

Previously, in Eq.(4.47), $V[\mathbf{x}-\mathbf{p}'(t-\tau)/m+\mathbf{F}(\tau)]$ is a term with the functional differentiations that respect to $\mathbf{F}(\tau)$ which is created by the potential. This $\mathbf{F}(\tau)$ is the fluctuation or derivations of dynamics from straight line trajectory.

When we ignore all functional differentiations with respect to $\mathbf{F}(\tau)$, approximately, setting to be zero. We get the scattering amplitude $f(\mathbf{p},\mathbf{p}')$, expressed as

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}V(\mathbf{x})\exp\left[-\frac{i}{\hbar}\int_0^\infty \mathrm{d}\alpha V\left(\mathbf{x}-\frac{\mathbf{p}'}{m}\alpha\right)\right]. \tag{4.63}$$

This equation is **the Modifies of Born approximation** with additional phase factor in the last integrand, depends on the potential. For Eq.(4.61), when the scattering is small deflections at high energies, called eikonal approximation, where obtain from the straight line trajectory approximation with including the functional differential operation.

The result of this section is very powerful for applying to many potentials of the scattering problems that we will give you a detail in next section.

4.3 The Coulomb Potential

The Coulomb potential is the interaction between charge particles, for scattering, where the incident particle straight travels to the target, own potential, and is deflected then becomes the outgoing particle.

The Coulomb potential's problem is about that it is a long range potential when $\alpha \to \infty$, it increases with no bound. In case of Eq.(4.52) does not *exist* where it cannot be integrated by parts.

Therefore, the asymptotically free Green function is used to find the propagator;

4.3.1 Asymptotically Free Green function

In a case of the $\alpha \to \infty$ in Eq.(4.52) does not exist. Thus, we consider the $G_+(\mathbf{p}.\mathbf{p}'; p^0)$ near the energy shell, $p^0 \simeq \mathbf{p}'^2/2m$.

We introduce the new integral variable, is

$$z = \frac{\alpha}{\hbar}\left[p^0 - E(\mathbf{p}')\right]. \tag{4.64}$$

Therefore, the Eq.(4.46) becomes

$$\begin{aligned} G_+(\mathbf{p}, \mathbf{p}'; p^0)\left[p^0 - E(\mathbf{p}')\right] &= -\frac{i}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}z\, e^{iz(1+i\epsilon)} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} \\ &\quad \times K\left(\mathbf{x}, \mathbf{p}'; \frac{z\hbar}{p^0 - E(\mathbf{p}')}\right). \end{aligned} \tag{4.65}$$

Next, we separately consider the $K(\mathbf{x}, \mathbf{p}'; z\hbar/(p^0 - E(\mathbf{p}')))$ term, $\mathbf{F}(\tau)$ is neglected. So, we get

$$K\left(\mathbf{x}, \mathbf{p}'; \frac{z\hbar}{p^0 - E(\mathbf{p}')}\right) \approx \exp\left[-\frac{i}{\hbar}\int_0^{z\hbar/(p^0 - E(p'))} \mathrm{d}\alpha V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right)\right]. \tag{4.66}$$

Following result, Eq.(4.65) is rewritten as

$$\begin{aligned} G_+(\mathbf{p}, \mathbf{p}'; p^0)\left[p^0 - E(\mathbf{p}')\right] &= -\frac{i}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}z\, e^{iz(1+i\epsilon)} \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} \\ &\quad \times \exp\left[-\frac{i}{\hbar}\int_0^{z\hbar/(p^0 - E(\mathbf{p}'))} \mathrm{d}\alpha V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right)\right]. \end{aligned} \tag{4.67}$$

Using the inverse Fourier transform of $G_+(\mathbf{p}, \mathbf{p}'; p^0)$, for $\mathbf{x}$ 3-dimensions exchanging to $\mathbf{p}$ momenta space with the definition as

$$\psi(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3} e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}\psi(\mathbf{p}), \tag{4.68}$$

we obtain

$$\begin{aligned} \int \mathrm{d}^3\mathbf{p}\, e^{i\mathbf{x}\cdot\mathbf{p}/\hbar} G_+(\mathbf{p}, \mathbf{p}'; p^0) &\approx \frac{-ie^{i\mathbf{x}\cdot\mathbf{p}/\hbar}}{[p^0 - E(\mathbf{p}') + i\epsilon]}\int_0^\infty dze^{iz(1+i\epsilon)} \\ &\quad \times \exp\left[-\frac{i}{\hbar}\int_0^{z\hbar/(p^0 - E(\mathbf{p}'))} \mathrm{d}\alpha V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right)\right], \end{aligned} \tag{4.69}$$

where always recall $\epsilon \to +0$.

Paying more attention here this is important and many mathematical step, for a case of the Coulomb potential $V(x) = \lambda/|x|$, we get

$$\begin{aligned}&\int_0^{z\hbar/(p^0-E(p'))} d\alpha V\left(x - \frac{p'}{m}\alpha\right)\\&= \int_0^{z\hbar/(p^0-E(p'))} d\alpha \frac{\lambda}{\sqrt{|x|^2 + |p'|^2\frac{\alpha^2}{m^2} - 2\frac{\alpha}{m}|p'||x|\cos\theta}},\end{aligned} \tag{4.70}$$

where $\mathbf{x}$ and $\mathbf{p}$ are vector variable, for the absolute, we have to calculate it in term of vector magnitude.

For integrating, we set a new integral variable which is

$$\begin{aligned}u &= \sqrt{|\mathbf{x}|^2 + |\mathbf{p}'|^2\frac{\alpha^2}{m^2} - 2\frac{\alpha}{m}|\mathbf{p}'||\mathbf{x}|\cos\theta}\\ \mathrm{d}\alpha &= \frac{m}{|\mathbf{p}'|}\frac{udu}{\sqrt{u^2 - |\mathbf{x}|^2(1-\cos^2\theta)}},\end{aligned} \tag{4.71}$$

and then Eq.(4.70) is rewritten as

$$\frac{\lambda m}{|\mathbf{p}'|}\int \frac{du}{\sqrt{u^2 - |\mathbf{x}|^2\sin^2\theta}}. \tag{4.72}$$

After this step, we will ignore a boundary condition as well as a constant to simplify integrating for a while.

Next, we set $a^2 = |\mathbf{x}|^2\sin^2\theta$ and let $u = a\sec\phi$ that leads to $\mathrm{d}\,u = a\sec\phi\tan\phi\mathrm{d}\phi$ thus the above equation becomes

$$\begin{aligned}\int\frac{\mathrm{d}\,u}{\sqrt{u^2-a^2}} &= \int\sec\phi = \ln|\sec\phi + \tan\phi|\\ &= \ln|u + \sqrt{u^2-a^2}|.\end{aligned} \tag{4.73}$$

Taking the integral variable back to get the result that we want, therefore, it is

$$\begin{aligned}&\ln\left|\frac{|\mathbf{p}'|\alpha}{m} - |\mathbf{x}|\cos\theta + \sqrt{|\mathbf{x}|^2 + \frac{|\mathbf{p}'|^2\alpha^2}{m^2} - 2\frac{\alpha}{m}|\mathbf{p}'||\mathbf{x}|\cos\theta}\right|\Bigg|_0^{z\hbar/(p^0-E(\mathbf{p}'))}\\&= \ln\left|\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))} - |\mathbf{x}|\cos\theta\right.\\&\left.+\sqrt{|\mathbf{x}|^2(1-\cos^2\theta) + \left(\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}')))} - |\mathbf{x}|\cos\theta\right)^2}\right| - \ln\Big||\mathbf{x}|(1-\cos\theta)\Big|,\end{aligned} \tag{4.74}$$

where we just give a boundary condition back.

For $\alpha = z\hbar/[p^0 - E(\mathbf{p}')]$, we early set $(\alpha \to \infty)$ thus we actually get

$$\begin{aligned} \ln\left|\frac{|\mathbf{p}'|\alpha}{m} - |\mathbf{x}|\cos\theta + \sqrt{|\mathbf{x}|^2 + \frac{|\mathbf{p}'|^2\alpha^2}{m^2} - 2\frac{\alpha}{m}|\mathbf{p}'||\mathbf{x}|\cos\theta}\right|\Bigg|_0^{z\hbar/(p^0-E(\mathbf{p}'))} \\ \approx \ln\left|\frac{2|\mathbf{p}'|z\hbar}{m(p^0 - E(\mathbf{p}'))}\right| - \ln\Big||\mathbf{x}|(1-\cos\theta)\Big| \\ \approx \ln\left|\frac{2|\mathbf{p}'|z\hbar}{m(p^0 - E(\mathbf{p}'))|\mathbf{x}|(1-\cos\theta)}\right|. \end{aligned} \tag{4.75}$$

Finally, for Eq.(4.70), the following result is

$$\int_0^{z\hbar/(p^0-E(\mathbf{p}'))} \mathrm{d}\alpha V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right) \approx \frac{\lambda m}{|\mathbf{p}'|}\ln\left[\frac{2|\mathbf{p}'|z\hbar}{m(p^0 - E(\mathbf{p}'))|\mathbf{x}|(1-\cos\theta)}\right], \tag{4.76}$$

where, from vector dot product, $\cos\theta = \mathbf{p}'\cdot\mathbf{x}/|\mathbf{p}'||\mathbf{x}|$.

Therefore, the last exponential term in Eq.(4.69) is given as

$$\begin{aligned} &\exp\left[-\frac{i}{\hbar}\int_0^{z\hbar/(p^0-E(\mathbf{p}'))} \mathrm{d}\alpha V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right)\right] \\ &\approx \frac{1}{[p^0 - E(\mathbf{p}') + i\epsilon]^{-i\gamma}}\exp\left[-i\gamma\ln\left(\frac{2|\mathbf{p}'|^2 z\hbar}{m(|\mathbf{p}'||\mathbf{x}| - \mathbf{p}'\cdot\mathbf{x})}\right)\right], \end{aligned} \tag{4.77}$$

where $\gamma = \lambda m/\hbar|\mathbf{p}'|$.

The above result leads Eq.(4.69) to

$$\begin{aligned} \int \mathrm{d}^3\mathbf{p}\, e^{ip\cdot x/\hbar}G_+(\mathbf{p},\mathbf{p}';p^0) &\approx \frac{-ie^{i\mathbf{x}\cdot\mathbf{p}'/\hbar}}{[p^0 - E(\mathbf{p}') + i\epsilon]^{1-i\gamma}}\int_0^\infty \mathrm{d}z\, e^{iz(1+i\epsilon)}z^{-i\gamma} \\ &\times \exp\Big[-i\gamma\ln\Big(2|\mathbf{p}'|^2/m\Big)\Big]\exp\left[i\gamma\ln\left(\frac{|\mathbf{p}'||\mathbf{x}| - \mathbf{p}'\cdot\mathbf{x}}{\hbar}\right)\right], \end{aligned} \tag{4.78}$$

and, for only the z-functional term, we use the complex analysis to integrate this term, is

$$\int_0^\infty dz e^{iz(1+i\epsilon)}z^{-i\gamma}. \tag{4.79}$$

Now, we change the integral variable as

$$\begin{aligned} u &= -iz(1+i\epsilon), \\ \mathrm{d}z &= -\frac{\mathrm{d}u}{i(1+i\epsilon)}, \end{aligned} \tag{4.80}$$

so, we obtain

$$\int_0^\infty \mathrm{d}z\, e^{iz(1+i\epsilon)} z^{-i\gamma} = \frac{i}{1+i\epsilon}\left(\frac{i}{1+i\epsilon}\right)^{-i\gamma} \int_0^\infty e^{-u} u^{-i\gamma}\,\mathrm{d}u\,. \tag{4.81}$$

The last integrating term in Eq.(4.81) can be used the Gamma function to evaluate as

$$\int_0^\infty e^{-u} u^{-i\gamma}\,\mathrm{d}u = \Gamma(1-i\gamma). \tag{4.82}$$

Next, we evaluate a complex value constant in Eq.(4.81), for $\epsilon \to +0$, which is

$$\left(\frac{i}{1+i\epsilon}\right)^{1-i\gamma} \quad\to\quad \frac{i}{i^{i\gamma}}\,. \tag{4.83}$$

Using the exponential properties to calculate this $i^{i\gamma}$, is expressed as

$$i^{i\gamma} = e^{i\gamma \log i}, \tag{4.84}$$

where

$$\log i = \ln(1) + i\left(\frac{\pi}{2} + 2n\pi\right) = 0 + \left(2n + \frac{1}{2}\right)\pi i \qquad ; n \in \mathcal{I}. \tag{4.85}$$

Therefore, we get

$$i^{i\gamma} = e^{i\gamma\left(2n+\frac{1}{2}\right)\pi i}. \tag{4.86}$$

For $n = 0$, it gives Eq.(4.84) as

$$i^{i\gamma} = \exp\left[-\frac{\pi\gamma}{2}\right], \tag{4.87}$$

and, certainly, Eq.(4.83) is

$$\left(\frac{i}{1+i\epsilon}\right)^{1-i\gamma} = i\exp\left[\frac{\pi\gamma}{2}\right]. \tag{4.88}$$

Finally, Eq.(4.79) becomes

$$\int_0^\infty \mathrm{d}z\, e^{iz(1+i\epsilon)} z^{-i\gamma} = i e^{\pi\gamma/2}\Gamma(1-i\gamma), \tag{4.89}$$

and we obtain

$$\int \mathrm{d}^3\mathbf{p}\, e^{i\mathbf{x}\cdot\mathbf{p}/\hbar} G_+(\mathbf{p},\mathbf{p}';p^0) \simeq e^{i\mathbf{x}\cdot\mathbf{p}'/\hbar} \frac{e^{-i\gamma\ln\left(2|\mathbf{p}'|^2/m\right)}}{\left[p^0 - E(\mathbf{p}') + i\epsilon\right]^{1-i\gamma}} \times \exp\left[i\gamma\ln\left(\frac{|\mathbf{p}'||\mathbf{x}| - \mathbf{p}'\cdot\mathbf{x}}{\hbar}\right)\right] e^{\pi\gamma/2}\Gamma(1-i\gamma). \tag{4.90}$$

When we separate a **x**-independent part, eventually, we obtain

$$G^0_{+C}(\mathbf{p}) = \frac{e^{i\gamma \ln\left(2|\mathbf{p}'|^2/m\right)}}{[p^0 - E(\mathbf{p}') + i\epsilon]^{1-i\gamma}} e^{\pi\gamma/2}\Gamma(1-i\gamma), \tag{4.91}$$

which is **the Asymptotically free Green function**, in the energy-momentum representation, that can be plot below as Figure 8.

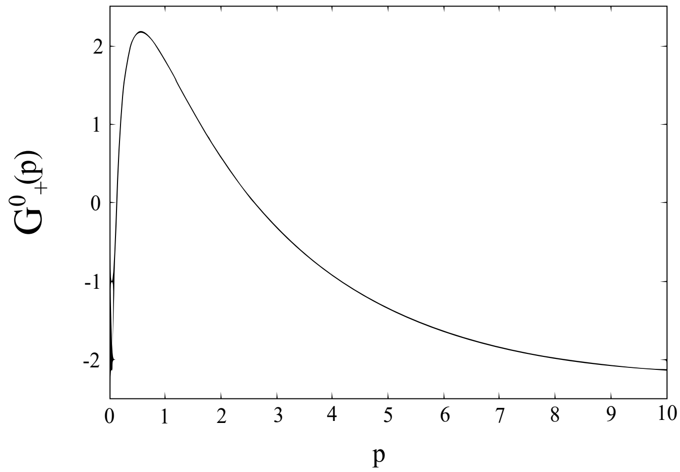

Figure 8 The Asymptotically free Green function of Coulomb scattering.

4.3.2 Scattering Amplitude

For this step, we have to find the scattering amplitude so we recall the Eq.(4.48) as

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{p}''\, V(\mathbf{p}-\mathbf{p}'')G_+(\mathbf{p}'',\mathbf{p}';p^0)[p^0 - E(\mathbf{p}')]\Big|_{p^0=E(\mathbf{p}')}, \tag{4.92}$$

and then we use the solution from Eq.(4.90) that is given as

$$\int \mathrm{d}^3\mathbf{p}\, e^{i\mathbf{x}\cdot\mathbf{p}/\hbar}G_+(\mathbf{p},\mathbf{p}';p^0) \simeq e^{-i\mathbf{p}'\cdot\mathbf{x}/\hbar}\exp\left[i\gamma\ln\left(\frac{|\mathbf{p}'||\mathbf{x}| - \mathbf{p}'\cdot\mathbf{x}}{\hbar}\right)\right] G^0_{+C}(\mathbf{p}'), \tag{4.93}$$

where $G^0_{+C}(\mathbf{p}')$ is previously defined in Eq.(4.91).

After using the Fourier transform, $V(\mathbf{p}) = \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot\mathbf{p}/\hbar}V(\mathbf{x})$, thus, Eq.(4.92) is rewritten as

$$\begin{aligned} f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{p}''\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}'')/\hbar}V(\mathbf{x}) \\ \times G_+(\mathbf{p}'',\mathbf{p}';p^0)\left[p^0 - E(\mathbf{p}')\right]\Big|_{p^0=E(\mathbf{p}')}. \end{aligned} \tag{4.94}$$

Eventually, Eq.(4.93) is substituted in the Eq.(4.92) for the scattering amplitude, becomes

$$\begin{aligned} f(\mathbf{p},\mathbf{p}') = &-\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{p}''\, e^{-i\mathbf{x}\cdot\mathbf{p}''/\hbar}G_+(\mathbf{p}'',\mathbf{p}';p^0) \\ &\times e^{-i\mathbf{x}\cdot\mathbf{p}/\hbar}V(\mathbf{x})\left[p^0 - E(\mathbf{p}')\right]\Big|_{p^0=E(\mathbf{p}')}, \end{aligned} \tag{4.95}$$

$$\begin{aligned} = &-\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{x}\,\frac{e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}V(\mathbf{x})}{[p^0 - E(\mathbf{p}') + i\epsilon]^{-i\gamma}}\exp\left[i\gamma\ln\left(\frac{|\mathbf{p}'||\mathbf{x}| - \mathbf{p}'\cdot\mathbf{x}}{\hbar}\right)\right] \\ &\times e^{-i\gamma\ln\left(2\mathbf{p}'^2/m\right)}e^{\pi\gamma/2}\Gamma(1-i\gamma)\Big|_{p^0=E(\mathbf{p}')}. \end{aligned} \tag{4.96}$$

Next, we use the Fourier transform for $\mathbf{x}$ variables, so, we immediately get

$$\begin{aligned} f(\mathbf{p},\mathbf{p}') = &-\frac{m}{2\pi\hbar^2}V(\mathbf{p}-\mathbf{p}')\exp\left[i\gamma\ln\left(\frac{|\mathbf{p}'||\mathbf{p}-\mathbf{p}'| - \mathbf{p}'\cdot(\mathbf{p}-\mathbf{p}')}{\hbar}\right)\right] \\ &\times \frac{e^{-i\gamma\ln\left(2\mathbf{p}'^2/m\right)}}{[p^0 - E(\mathbf{p}') + i\epsilon]^{-i\gamma}}e^{\pi\gamma/2}\Gamma(1-i\gamma)\Big|_{p^0=E(\mathbf{p}')}. \end{aligned} \tag{4.97}$$

Using the Coulomb potential in the momenta space which is denoted as

$$V(\mathbf{p}-\mathbf{p}') = \frac{4\pi\lambda}{(\mathbf{p}-\mathbf{p}')^2}, \tag{4.98}$$

therefore, Eq.(4.97) is

$$\begin{aligned} f(\mathbf{p},\mathbf{p}') = &-\frac{m}{2\pi\hbar^2}\frac{4\pi\lambda}{(\mathbf{p}-\mathbf{p}')^2}\left(\frac{|\mathbf{p}'|||\mathbf{p}-\mathbf{p}'|| - \mathbf{p}'\cdot\mathbf{p}-\mathbf{p}'}{\hbar}\right)^{i\gamma} \\ &\times \frac{e^{-i\gamma\ln\left(2\mathbf{p}'^2/m\right)}}{[p^0 - E(\mathbf{p}') + i\epsilon]^{-i\gamma}}e^{\pi\gamma/2}\Gamma(1-i\gamma)\Big|_{p^0=E(\mathbf{p}')}. \end{aligned} \tag{4.99}$$

Finally, we get the **Coulomb scattering amplitude** that is

$$\begin{aligned} f_C(\mathbf{p},\mathbf{p}') = &-\frac{m\lambda}{\hbar^2(\mathbf{p}-\mathbf{p}')^2}\left(\frac{|\mathbf{p}'||\mathbf{p}-\mathbf{p}'| - \mathbf{p}'\cdot\mathbf{p}-\mathbf{p}'}{\hbar}\right)^{i\gamma} \\ &\times \frac{e^{i\gamma\ln\left(2|\mathbf{p}'|^2/m\right)}}{[p^0 - E(\mathbf{p}') + i\epsilon]^{-i\gamma}}e^{\pi\gamma/2}\Gamma(1-i\gamma)\Big|_{p^0=E(\mathbf{p}')}. \end{aligned} \tag{4.100}$$

4.3.3 Cross Section

Next, we can calculate the cross section for this situation that is early defined as

$$D(\theta) = \frac{d\sigma}{d\Omega} = |f(\mathbf{p},\mathbf{p}')|^2. \tag{4.101}$$

So, the following result ,by taking Eq.(4.100) into an above equation, is

$$|f(\mathbf{p},\mathbf{p}')|^2 = \left(-\frac{m\lambda}{\hbar^2(\mathbf{p}-\mathbf{p}')^2}\right)^2 e^{\pi\gamma}\Gamma(1-i\gamma)\Gamma^*(1-i\gamma), \tag{4.102}$$

where, for the Gamma function, it is approximated as $\Gamma(1-i\gamma)\Gamma^*(1-i\gamma) \approx 1$ in case of γ is small. So, the previous equation becomes

$$|f(p,p')|^2 = \frac{4m^2\lambda^2 e^{\pi\gamma}}{\hbar^4(p-p')^4}. \tag{4.103}$$

By defining $\mathbf{p} = |\mathbf{p}'|n$ and $\mathbf{p}\cdot\mathbf{p}' = |\mathbf{p}'|^2\cos\theta$ for the angle is θ, it gives

$$(\mathbf{p}-\mathbf{p}')^2 = 4|\mathbf{p}'|^2\sin^2(\frac{\theta}{2}). \tag{4.104}$$

At last, the differential cross section is imposed as

$$D_C(\theta) = \frac{d\sigma}{d\Omega} = \frac{m^2\lambda^2 e^{\pi\gamma}}{4\hbar^4|\mathbf{p}'|^4\sin^4(\theta/2)}, \tag{4.105}$$

where $\gamma = \lambda m/\hbar|\mathbf{p}'|$.

This is, may be called, the **modified Coulomb differential cross section** that different from the Classical cross section. Therefore, we can illustrate a graph of this differential cross section as below, Figure 9,

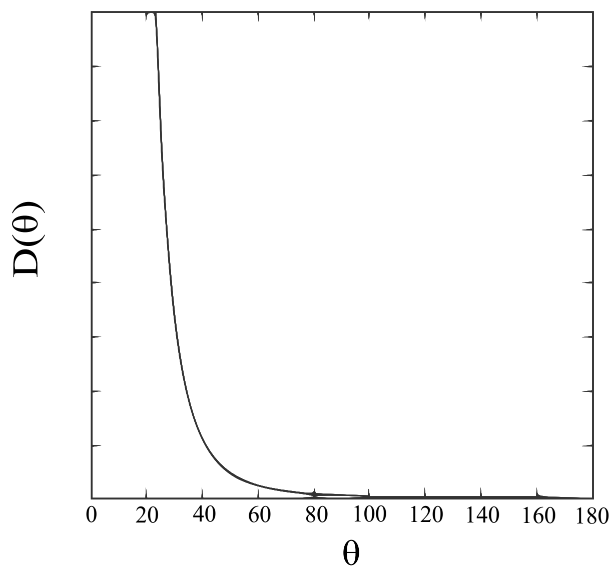

Figure 9 The Coulomb differential cross section of scattering angle θ (degrees).

CHAPTER V

YUKAWA SCATTERING BY DYNAMICAL PRINCIPLE

The most important part of this project, we will talk about the procedure of the Quantum Dynamical Principle which is applied for the case of Yukawa potential.

5.1 The Functional Treatment Method (QDP)

From the early proving in the fourth chapter, we start with the **Asymptotically free Green function** which is in case of $\alpha \to \infty$.

Therefore, we directly recall the Eq.(4.65) that is

$$G_+(\mathbf{p}, \mathbf{p}'; p^0)[p^0 - E(\mathbf{p}')] = -\frac{i}{(2\pi\hbar)^3} \int_0^\infty \mathrm{d}z\, e^{iz(1+i\epsilon)} \times \int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} K\left(\mathbf{x}, \mathbf{p}'; \frac{z\hbar}{p^0 - E(\mathbf{p}')}\right), \tag{5.1}$$

where $z = \alpha/\hbar[p^0 - E(\mathbf{p}')]$ and $\epsilon \to +0$.

The first step, we consider only K term for nearing the energy shell, $p^0 \simeq E(\mathbf{p}')$, so it becomes

$$K\left(\mathbf{x}, \mathbf{p}'; \frac{z\hbar}{(p^0 - E(\mathbf{p}'))}\right) \simeq \exp\left[-\frac{i}{\hbar} \int_0^{z\hbar/(p^0-E(\mathbf{p}'))} \mathrm{d}\alpha V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right)\right], \tag{5.2}$$

when we set $\mathbf{F} = 0$. To end this, we have to find the propagator by finishing of K term that is substituted for the Yukawa potential which is

$$V(\mathbf{x}) = \lambda \frac{e^{-kM|\mathbf{x}|}}{|\mathbf{x}|}, \tag{5.3}$$

where k is scaling parameter and M, target mass, is a mass of particle mediating the force.

Thus, it is evaluated as

$$\int_0^{z\hbar/(p^0-E(\mathbf{p}'))} \mathrm{d}\alpha V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right) \simeq \lambda \int_0^{z\hbar/(p^0-E(p'))} \mathrm{d}\alpha \frac{\exp\left(-kM|\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha|\right)}{\left|\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right|}, \tag{5.4}$$

and changed the integral variable by setting

$$
\begin{aligned}
u &= \sqrt{|\mathbf{x}|^2 + \frac{|\mathbf{p}'|^2\alpha^2}{m^2} - 2\frac{\alpha}{m}|\mathbf{p}'||\mathbf{x}|\cos\theta}, \\
\mathrm{d}\,\alpha &= \frac{m}{|\mathbf{p}'|}\frac{u\mathrm{d}\,u}{\sqrt{u^2-a^2}} \qquad ; \quad a = x\sin\theta,
\end{aligned} \tag{5.5}
$$

therefore, it becomes

$$
\frac{\lambda k m M}{|\mathbf{p}'|}\int \mathrm{d}\,u \frac{e^{-u}}{\sqrt{u^2-a^2}}. \tag{5.6}
$$

The terrible problem of this function, $e^{-u}/\sqrt{u^2-a^2}$, is the integrating diverges. So, lets approximate the exponential function, by Taylor series, as

$$
e^{-u} \simeq 1 - u + \frac{u^2}{2!} - \frac{u^3}{3!} + \frac{u^4}{4!} \pm ... \tag{5.7}
$$

Separating integration, for each part, we get

$$
\begin{aligned}
&1) \quad \int \frac{\mathrm{d}u}{\sqrt{u^2-a^2}} \simeq \ln\left(\frac{2|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))|x|(1-\cos\theta)}\right), \\
&2) \quad \int \frac{u\,\mathrm{d}u}{\sqrt{u^2-a^2}} \simeq \left(\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))}\right) \qquad , \\
&3) \quad \int \frac{u^2\,\mathrm{d}u}{\sqrt{u^2-a^2}} \simeq \frac{1}{2}\left(\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))}\right)^2 \qquad , \\
&4) \quad \int \frac{u^3\,\mathrm{d}u}{\sqrt{u^2-a^2}} \simeq \frac{1}{3}\left(\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))}\right)^3 \qquad , \\
&5) \quad \int \frac{u^4\,\mathrm{d}u}{\sqrt{u^2-a^2}} \simeq \frac{1}{4}\left(\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))}\right)^4 \qquad , \\
&\qquad \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned} \tag{5.8}
$$

after input the boundary condition and z is very large value.

Next, we sum all parts from above equations thus the following result is

$$\begin{aligned}\frac{\lambda kmM}{|\mathbf{p}'|}\int \mathrm{d}u\,\frac{e^{-u}}{\sqrt{u^2-a^2}} \simeq \frac{\lambda kmM}{|\mathbf{p}'|}\Bigg[&\ln\left(\frac{2|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}')|\mathbf{x}|(1-\cos\theta)}\right)\\ &-\left(\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))}\right)+\frac{1}{2!2}\left(\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))}\right)^2\\ &-\frac{1}{3!3}\left(\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))}\right)^3+\frac{1}{4!4}\left(\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))}\right)^4\\ &-\frac{1}{5!5}\left(\frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))}\right)^5 \pm \dots \quad\Bigg].\end{aligned} \tag{5.9}$$

We can take the summation form for the second term and so on where it becomes

$$\begin{aligned}\frac{\lambda kmM}{|\mathbf{p}'|}\int \mathrm{d}\,u\frac{e^{-u}}{\sqrt{u^2-a^2}} \simeq \eta\Bigg[&\ln\left(\frac{2|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))|x|(1-\cos\theta)}\right)\\ &+\sum_{n=1}^{\infty}\frac{(-1)^n S^n}{n!n}\Bigg].\end{aligned} \tag{5.10}$$

Simplify, we set

$$\eta = \frac{\lambda kmM}{|\mathbf{p}'|}, \qquad \text{and} \qquad S = \frac{|\mathbf{p}'|z\hbar}{m(p^0-E(\mathbf{p}'))}. \tag{5.11}$$

For the term of summation, it can be evaluated by the incomplete Gamma function which is explain below as

$$\Gamma(a,z) = \Gamma(a) - \gamma(a,z), \tag{5.12}$$

where $\Gamma(a,z)$ is the upper incomplete Gamma function, $\Gamma(a)$ is the Gamma function and $\gamma(a,z)$ is the lower incomplete Gamma function.

We start with the lower incomplete Gamma function, it take in term of

$$\begin{aligned}\gamma(a,z) &= \int_0^z t^{a-1}e^{-t}\mathrm{d}\,t,\\ &= \int_0^z \sum_{n=0}^{\infty}(-1)^n\frac{t^{a+n-1}}{n!}\mathrm{d}\,t,\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n z^{a+n}}{n!(a+n)},\\ &= z^a\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!(a+n)}.\end{aligned} \tag{5.13}$$

Adding $(-1/a)$ for both side of equation, we get

$$\gamma(a,z) - \frac{1}{a} = -\frac{1}{a} + z^a \sum_{n=0}^{\infty} \frac{(-z)^n}{n!n} = \frac{z^a - 1}{a} + z^a \sum_{n=1}^{\infty} \frac{(-z)^n}{n!n}. \tag{5.14}$$

Next, we take the limit that $a \to 0$ for Eq.(5.12) and adding $(-1/a)$ so it becomes

$$\begin{aligned}
\lim_{a\to 0} \Gamma(a,z) &= \lim_{a\to 0} \left(\Gamma(z) - \gamma(a,z)\right), \\
&= \lim_{a\to 0} \left(\Gamma(z) - \frac{1}{a} - \left(\gamma(a,z) - \frac{1}{a}\right)\right), \\
&= \lim_{a\to 0} \left(\Gamma(z) - \frac{1}{a}\right) - \lim_{a\to 0} \left(\gamma(a,z) - \frac{1}{a}\right), \\
&= -\gamma - \lim_{a\to} \left(\frac{z^a - 1}{a} + z^a \sum_{n=1}^{\infty} \frac{(-z)^n}{n!n}\right) \\
\Gamma(0,z) &= -\gamma - \ln(z) - \sum_{n=1}^{\infty} \frac{(-z)^n}{n!n}.
\end{aligned} \tag{5.15}$$

The final result is

$$\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!n} = -\ln(z) - \Gamma(0,z) - \gamma\ , \tag{5.16}$$

where $\gamma = 0.577$, is the Euler–Mascheroni constant.

Comparing, for the summation in Eq.(5.10), it equals to

$$\sum_{n=1}^{\infty} \frac{(-1)^n S^n}{n!n} = -\ln(S) - \Gamma(0,S) - \gamma\ , \tag{5.17}$$

and, for the upper incomplete Gamma function in case of $S \to \infty$, very large, we get

$$\lim_{S\to\infty} \Gamma(0,S) = 0. \tag{5.18}$$

Then, substituting it back and replacing for S to the Eq.(5.10), we obtain

$$\begin{aligned}
\eta \int_0^{z\hbar/(p^0 - E(\mathbf{p}'))} \mathrm{d}\alpha V\left(\mathbf{x} - \frac{\mathbf{p}'}{m}\alpha\right) &\simeq \eta \left[\ln\left(\frac{2|\mathbf{p}'|z\hbar}{m(p^0 - E(\mathbf{p}'))|\mathbf{x}|(1-\cos\theta)}\right)\right. \\
&\quad \left. - \ln\left(\frac{|\mathbf{p}'|z\hbar}{m(p^0 - E(\mathbf{p}'))}\right) - \gamma\right], \\
&\simeq \eta\left[\ln\left(\frac{2}{|\mathbf{x}|(1-\cos\theta)}\right) - \gamma\right].
\end{aligned} \tag{5.19}$$

The second step, for the Eq.(5.2), it can be rewritten as

$$\begin{aligned} K\left(\mathbf{x},\mathbf{p}';\frac{z\hbar}{(p^0-E(\mathbf{p}'))}\right) &\simeq \exp\left[-\frac{i}{\hbar}\frac{\lambda k m^2}{|\mathbf{p}'|}\left[\ln\left(\frac{2}{|\mathbf{x}|(1-\cos\theta)}\right)-\gamma\right]\right] \\ &\simeq e^{i\beta\gamma}\exp\left[-i\beta\ln\left(\frac{2|\mathbf{p}'|}{|\mathbf{p}'||\mathbf{x}|-\mathbf{p}'\cdot\mathbf{x}}\right)\right], \end{aligned} \tag{5.20}$$

where $\beta = \lambda kmM/\hbar|\mathbf{p}'|$ and $\cos\theta = \mathbf{p}'\cdot\mathbf{x}/|\mathbf{p}'||\mathbf{x}|$.

Eventually, from the Eq.(5.1), we obtain

$$\begin{aligned} G_+(\mathbf{p},\mathbf{p}';p^0) \simeq \frac{1}{(2\pi\hbar)^3}&\frac{-ie^{-i\beta\ln(2|\mathbf{p}'|)}e^{i\beta\gamma}}{[p^0-E(\mathbf{p}')+i\epsilon]}\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar} \\ &\times\exp\left[i\beta\ln(|\mathbf{p}'||\mathbf{x}|-\mathbf{p}'\cdot\mathbf{x})\right]\int_0^\infty \mathrm{d}\, z e^{iz(1+i\epsilon)}. \end{aligned} \tag{5.21}$$

And we consider the last term that is z functional, as

$$\begin{aligned} \int_0^\infty \mathrm{d}\, z e^{i(1+i\epsilon)z} &= \left.\frac{e^{i(1+i\epsilon)z}}{i(1+i\epsilon)}\right|_0^\infty \\ &= \frac{1}{i(1+i\epsilon)}\left[\lim_{z\to\infty} e^{i(1+i\epsilon)z}-1\right] \\ &= \frac{-1}{i(1+i\epsilon)}, \end{aligned} \tag{5.22}$$

where we thank to the epsilon for easiest integration of this case.

The following result is

$$\begin{aligned} \int \mathrm{d}^3\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}G_+(\mathbf{p},\mathbf{p}';p^0) \simeq e^{i\mathbf{x}\cdot\mathbf{p}'/\hbar}&\frac{e^{-i\beta\ln(2|\mathbf{p}'|)}e^{i\beta\gamma}}{[p^0-E(\mathbf{p}')+i\epsilon]} \\ &\times\exp\left[i\beta\ln(|\mathbf{p}'||\mathbf{x}|-\mathbf{p}'\cdot\mathbf{x})\right], \end{aligned} \tag{5.23}$$

and, for the asymptotically free Green function, without the x-dependent, it is finally expressed as

$$G^0_{+Y}(\mathbf{p}) = \frac{e^{i\beta\gamma}e^{-i\beta\ln(2|\mathbf{p}|)}}{[p^0-E(\mathbf{p})+i\epsilon]}, \tag{5.24}$$

which is in the term of the energy-momentum representation. This Green function can be plotted as below.

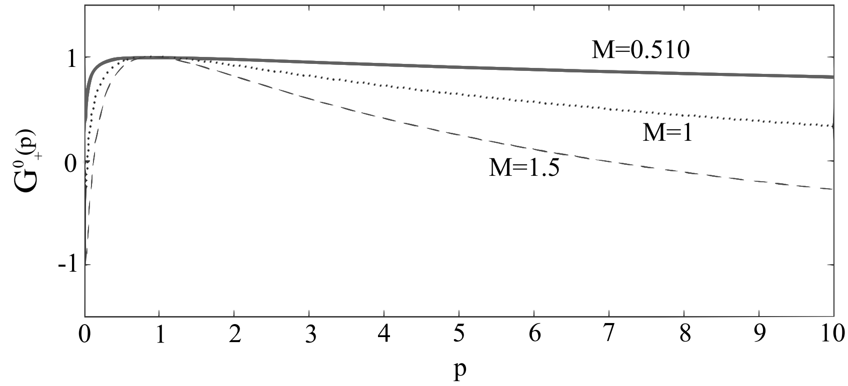

Figure 10 The Asymptotically free Green function of Yukawa scattering.

5.2 Scattering Amplitude and Cross section

5.2.1 Scattering Amplitude

We recall the Eq.(4.48) again which is

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{p}''\, V(\mathbf{p}-\mathbf{p}'')G_+(\mathbf{p}'',\mathbf{p}';p^0)[p^0 - E(\mathbf{p}')]\bigg|_{p^0=E(\mathbf{p}')}, \quad (5.25)$$

and use the Fourier transform for $V(\mathbf{p}-\mathbf{p}'')$ for an above equation. So, we get

$$\begin{aligned} f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{p}''\, e^{i\mathbf{p}''\cdot\mathbf{x}/\hbar}G_+(\mathbf{p}'',\mathbf{p}';p^0) \\ \times\; e^{-i\mathbf{x}\cdot\mathbf{p}/\hbar}V(\mathbf{x})[p^0 - E(\mathbf{p}')]\bigg|_{p^0=E(\mathbf{p}')}. \end{aligned} \quad (5.26)$$

After inputting the result from Eq.(5.23), we obtain

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}e^{i\beta\ln(|\mathbf{p}'||\mathbf{x}|-\mathbf{p}'\cdot\mathbf{x})}e^{i\beta\gamma}e^{-i\beta\ln(2|\mathbf{p}'|)}, \quad (5.27)$$

and, eventually, by using the Fourier transform for $\mathbf{x}$ variables, also get

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}V(\mathbf{p}-\mathbf{p}')e^{i\beta\ln(|\mathbf{p}'||\mathbf{p}-\mathbf{p}'|-\mathbf{p}'\cdot(\mathbf{p}-\mathbf{p}'))}e^{i\beta\gamma}e^{-i\beta\ln(2|\mathbf{p}'|)}. \quad (5.28)$$

Next, for the Yukawa potential in the momentum representation is

$$V(\mathbf{p}-\mathbf{p}') = \frac{4\pi\lambda}{(\mathbf{p}-\mathbf{p}')^2 + (kM)^2}, \quad (5.29)$$

so the Eq.(5.28) becomes

$$f_Y(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\frac{4\pi\lambda}{(\mathbf{p}-\mathbf{p}')^2+(kM)^2} \\ \times; e^{i\beta\ln(|\mathbf{p}'||\mathbf{p}-\mathbf{p}'|-\mathbf{p}'\cdot(\mathbf{p}-\mathbf{p}'))}e^{i\beta\gamma}e^{-i\beta\ln(2|\mathbf{p}'|)}\Big|_{p^0-E(\mathbf{p}')}. \tag{5.30}$$

So, finally, the above equation becomes

$$f(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\frac{4\pi\lambda}{(\mathbf{p}-\mathbf{p}')^2+(kM)^2} \\ \times\left(|\mathbf{p}'||\mathbf{p}-\mathbf{p}'|-\mathbf{p}'\cdot(\mathbf{p}-\mathbf{p}')\right)^{i\beta}e^{i\beta\gamma}e^{-i\beta\ln(2|\mathbf{p}'|)}. \tag{5.31}$$

which is the **Yukawa scattering amplitude**.

5.2.2 Cross Section

We have to find the differential cross section that is defined as

$$D(\theta) = \frac{d\sigma}{d\Omega} = |f(\mathbf{p},\mathbf{p}')|^2, \tag{5.32}$$

and calculate the absolute that is

$$|f(\mathbf{p},\mathbf{p}')|^2 = \left(-\frac{m}{2\pi\hbar^2}\right)^2\left(\frac{4\pi\lambda}{(\mathbf{p}-\mathbf{p}')^2+(kM)^2}\right). \tag{5.33}$$

Where we use the similar condition as Eq.(4.104), $(\mathbf{p}-\mathbf{p}')^2 = 4|\mathbf{p}'|^2\sin^2(\theta/2)$, then we obtain the differential cross section as

$$D_Y(\theta) = \frac{d\sigma}{d\Omega} = \frac{4m^2\lambda^2}{\hbar^4(4|\mathbf{p}'|^2\sin^2(\theta/2)+(kM)^2)}, \tag{5.34}$$

where is the **Yukawa differential cross section**, is illustrated as below.

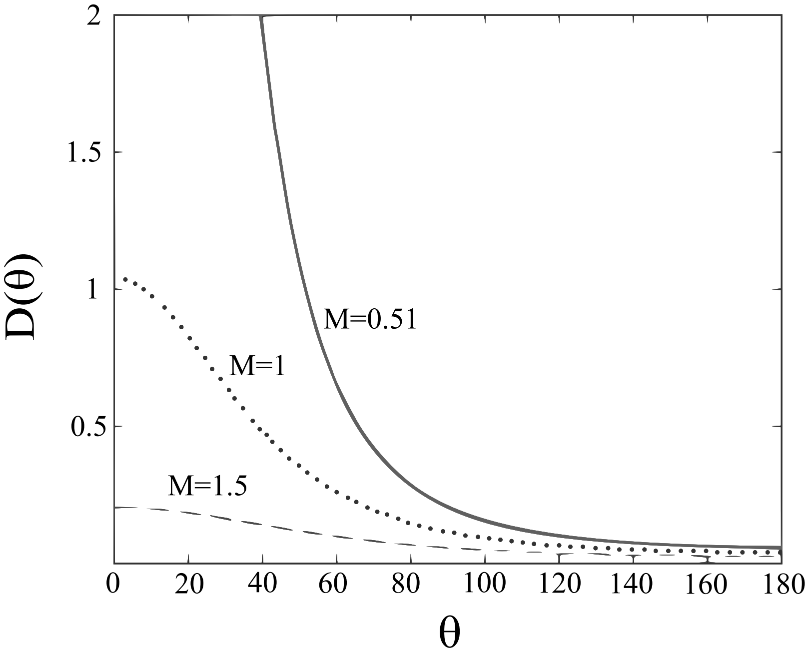

Figure 11 The Differential Cross section of Yukawa scattering.

CHAPTER VI

CONCLUSION

The conclusion, after we apply the **Quantum Dynamical Principle** to the specific case of Yukawa potential where the incident particles are scattered. Consequently, all details is given and described below.

From many step of mathematical deriving, we obtain the final result of the **Asymptotically Free Green function** of this scattering, can be expressed in term of the energy-momentum representation as

$$G^0_{+Y}(\mathbf{p}) = \frac{e^{i\beta\gamma} e^{-i\beta \ln(2|\mathbf{p}|)}}{[p^0 - E(\mathbf{p}) + i\epsilon]}, \tag{6.1}$$

where $\beta = \lambda k m M/\hbar|\mathbf{p}|$ and $E(\mathbf{p}) = \mathbf{p}^2/2m$.

For this asymptotically free Green function, we can repeat it in the graph below, where varying the parameter of mass.

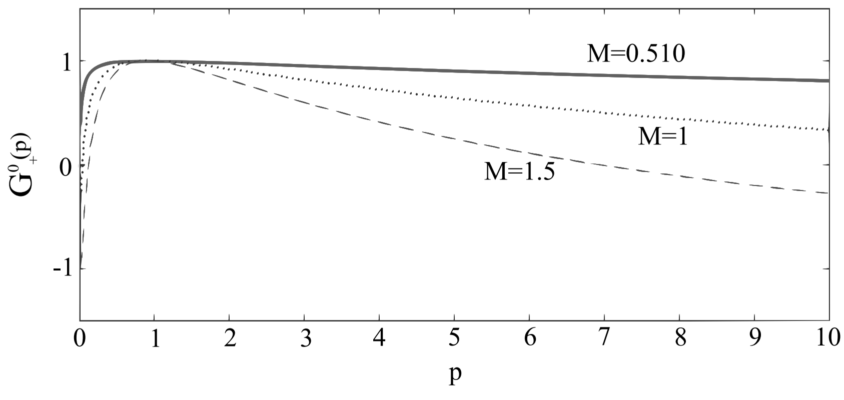

Figure 12 The Asymptotically free Green function of Yukawa scattering.

According to this graph, Figure 12, when the incident wave, traveling with the form of sine, cosine or some kind of periodic waves, is coming to the target and then it is scattered. The consequent result is when the parameter, mass, of the Yukawa potentail is increasing cause a range of potential is shorten. Therefore, for the dot-line and dash-line maintain the form of periodic wave in a short range.

In a contrast, for a lowest parameter, $M = 0$, it seem to be the Coulomb asymptotic free propagator but it is not, and for a case of parameter M is 1.5, an effect of this potential, is stronger than others higher values parameters. In particular, the influence is powerful so the incident wave is lost its periodic.

Next, for the **Scattering Amplitude**, defined in Eq.(4.48), we obtain

$$f_Y(\mathbf{p},\mathbf{p}') = -\frac{m}{2\pi\hbar^2}\frac{4\pi\lambda}{(\mathbf{p}-\mathbf{p}')^2+(kM)^2}\left(|\mathbf{p}'||\mathbf{x}| - \mathbf{p}'\cdot\mathbf{x}\right)^{i\beta} e^{i\beta\gamma}e^{-i\beta\ln(2|\mathbf{p}'|)}, \qquad (6.2)$$

which leads to the **Differential Cross section**. It is expressed as

$$D_Y(\theta) = \frac{d\sigma}{d\Omega} = \frac{4m^2\lambda^2}{\hbar^4(4|\mathbf{p}'|^2\sin^2(\theta/2)+(kM)^2)}. \qquad (6.3)$$

Finally, we present the graph of the differential cross section again which is given below.

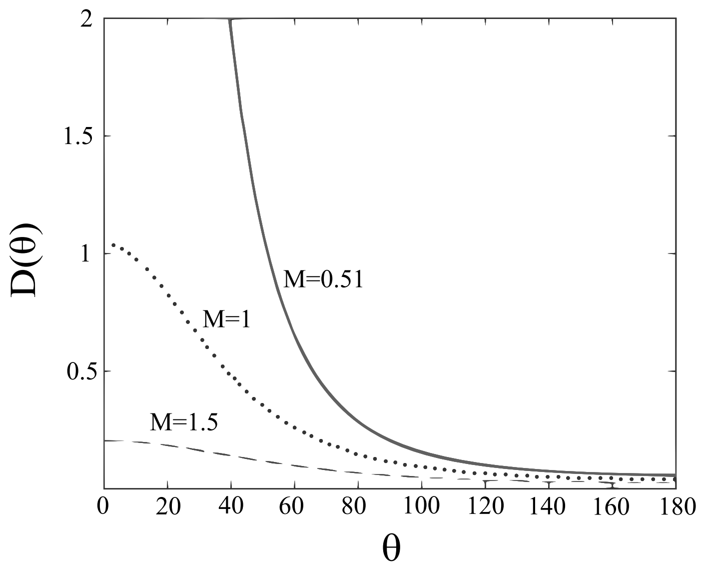

Figure 13 The Differential Cross section of Yukawa scattering.

According to the graph above, Figure 13, comparing the M mass parameters of Yukawa potential, we can say that when the mass parameter is increased the curves of amplitude are lowered and more expanded along the angle degrees. For $m \to 0$, this differential cross section of Yukawa scattering approaches to the differential cross section of Coulomb potential. A physical meaning of these line is a probability of detecting the outgoing particles wave at any angle.

Certainly, this method, quantum dynamical principle, provides the Green function or transformation functions for the scattering case, which leads to the theoretical foundation of the particles physics. Especially, the external sources affect the system from the ground-state to the excited states and also generates the quantum variables. Summary, you will found that this method is very powerful and useful.

REFERENCES

REFERENCES

[1] Griffths, D. J. (2005). *Introduction to Quantum Mechanics*, USA: Pearson Education, Inc.

[2] Griffths, D. J. (2008). *Introduction to Elementary Particles*, Germany: Wiley-VCH Verlag GmbH& Co.KGaA, Weinheim.

[3] Arfken, G. B. and Weber, H. J. (2005). *Mathematical Methods for Physicists*, London: Elsevier Inc.

[4] Cahill, K. (2013). *Physical Mathematics*, New York: Cambridge University press.

[5] Manoukian, E. B. (2006). *Quantum Theory: A wide spectrum*, Dordrecht: Springer.

[6] Joachain, C. J. (1975). *Quantum Collision Theory*, Scotland: North-Holland Publishing Company.

[7] Manoukian, E. B. and Sukkhasena, S. (2006). Functional treatment of quantum scattering via the dynamical principle. *Prog. Theor. Phys.* 116, 795.

[8] Manoukian, E. B. (1985). Quantum Action Principle and Path Integrals for Long-Range interactions. *Nuovo Cimento A.* 90, 295.

[9] Manoukian, E. B. (1986). Exact Solutions for Stimulated Emissions by External Sources. *Int. J. Theor. Phys.* 25, 147.

[10] Manoukian, E. B. (1988). Functional Approach to Scattering in Quantum Field Theory. *Int. J. Theor. Phys.* 27, 401.

[11] Bogdanski, P. and Ouerdane, H. (2006). Scattering states of coupled valence-band holes in point defect potential derived from variable phase theory. *Phys. Rev.* 74, 085210.

[12] Efimov, G. V. (2014). Elastic Scattering and The Path Integral. *Theor. Math. Phys.* 179(3), 695.

[13] Feynman, R. P. (1946). Space-Time Approach to Quantum Electrodynamics. *Phys. Rev.* 76, 769.

[14] Feynman, R. P. (1946). The Theory of Positrons. *Phys. Rev.* 76, 749.

[15] Feynman, R. P. (1948). Space-Time Approach to Non-Relativistic Quantum Mechanics. *Rev. Mod. Phys.* 76, 749.

[16] Schwinger, J. (1951). The Theory of Quantized Fields. I. *Phys. Rev.* 82, 914.

[17] Schwinger, J. (1951). On The Green's Functions of Quantized Fields. I. *Proc. Natl. Acad. Sci. USA.* 37, 452.

[18] Schwinger, J. (1953). The Theory of Quantized Fields. II. *Phys. Rev.* 91, 713.

[19] Schwinger, J. (1960). Unitary Transformations and The Action Principle. *Proc. Natl. Acad. Sci. USA.* 46, 883.

[20] Schwinger, J. (1960). The Special Canonical Group. *Proc. Natl. Acad. Sci. USA.* 46, 1401.

[21] Schwinger, J. (1961). Quantum Variables and the Action Principle. I. *Proc. Natl. Acad. Sci. USA.* 47, 1075.

[22] Schwinger, J. (1962). Exterior Algebra and the Action Principle. I. *Proc. Natl. Acad. Sci. USA.* 48, 603.

[23] Schwinger, J. (1966). Particles and Sources. *Proc. Natl. Acad. Sci. USA.* 48, 663.

[24] Barone, F. A., Boschi-Filho, H. and Farina, C. (2002). Three methods for calculating the Feynman propagator. *Am. J. Phys.* 71(5), 483.

APPENDIX

APPENDIX A COMPLEX ANALYSIS

Recalling Eq.(2.24), as

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{s}\cdot\mathbf{r}}\frac{1}{(k^2-s^2)}\mathrm{d}^3\mathbf{s}, \tag{A.1}$$

so, we take this integrand in spherical coordinates that the derivative is

$$\mathrm{d}^3\mathbf{s} = s^2\sin\theta\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi\,. \tag{A.2}$$

Next, we change Eq.(A.1) to become

$$G(\mathbf{r}) = -\frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{e^{isr\cos\theta}}{(s^2-k^2)}s^2\sin\theta\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi\,. \tag{A.3}$$

Considering only integrand that is

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{e^{isr\cos\theta}}{(s^2-k^2)}s^2\sin\theta\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\phi = 2\pi\int_0^{\pi}\int_0^{\infty}\frac{s^2e^{isr\cos\theta}}{(s^2-k^2)}\sin\theta\,\mathrm{d}s\,\mathrm{d}\theta \tag{A.4}$$

and changing the integral variable as

$$u = isr\cos\theta \quad \text{and} \quad \mathrm{d}u = -isr\sin\theta\,\mathrm{d}\theta\,. \tag{A.5}$$

Integrating over θ, it is

$$-\int \mathrm{d}u\,\frac{se^u}{ir\,(k^2-s^2)} = -\left.\frac{se^{isr\cos\theta}}{ir\,(k^2-s^2)}\right|_0^{\pi} = \frac{2s\sin(sr)}{r\,(k^2-s^2)}, \tag{A.6}$$

then we get

$$G(\mathbf{r}) = -\frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\frac{s\sin(sr)\,\mathrm{d}s}{(k-s)(k+s)}. \tag{A.7}$$

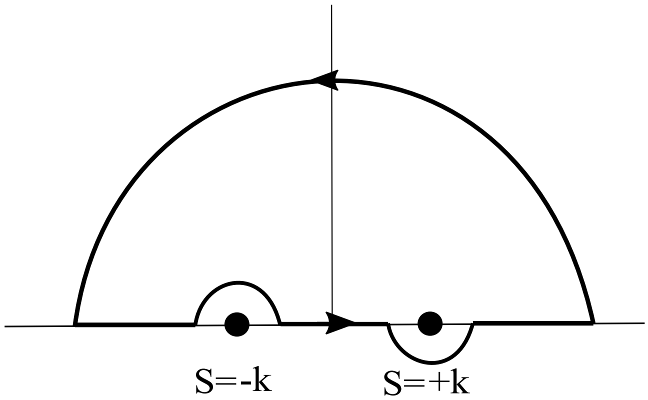

Figure 14 The contour integral, avoids the poles.

Using the **Cauchy's integral**, the determining function as

$$\oint_C \mathrm{d}z\, \frac{ze^{izr}}{(z^2-k^2)}, \tag{A.8}$$

which is in Fig.14.

We consider two path integration

$$\oint_C \mathrm{d}z\, \frac{ze^{izr}}{(z^2-k^2)} = I_C = I_1 + I \tag{A.9}$$

where I_C is the contour integral. To end this, I_1 is calculated as

$$I_1 = \int \mathrm{d}z\, \frac{ze^{izr}}{z^2-k^2} = i\int \mathrm{d}\theta\, \frac{e^{iRr\cos\theta}}{\left(1-\frac{k^2}{R^2e^{i2\theta}}\right)e^{Rr\sin\theta}}, \tag{A.10}$$

when using $z = Re^{i\theta}$ and $\mathrm{d}z = iRe^{i\theta}\,\mathrm{d}\theta$ and let $R\to\infty$, we finally get

$$I_1 = \lim_{R\to\infty} i\int \mathrm{d}\theta\, \frac{e^{iRr\cos\theta}}{\left(1-\frac{k^2}{R^2e^{i2\theta}}\right)e^{Rr\sin\theta}} = 0 \tag{A.11}$$

where $\exp(Rr\sin\theta)\to\infty$.

For I_C, we use the **residue theorem** which the poles are in this contour integral, $z_0 = k, -k$, thus, it is

$$I_C = 2\pi\, i \sum Res\{z_0\}. \tag{A.12}$$

Where $z_0 = k$, its residue is

$$Res\{k\} = \left.\frac{ze^{izr}(z-k)}{(z-k)(z+k)}\right|_{z=k} = \frac{e^{ikr}}{2}, \tag{A.13}$$

and for $z_0 = -k$, its residue is

$$Res\{-k\} = \left.\frac{ze^{izr}(z+k)}{(z-k)(z+k)}\right|_{z=k} = \frac{e^{-ikr}}{2}. \tag{A.14}$$

So, we get

$$I_C = i\pi\left(e^{ikr} + e^{-ikr}\right) = i\pi\, e^{ikr} \tag{A.15}$$

where $r \to \infty$ lead to $\exp(-ikr) = 0$. Recall Eq.(A.7), for sin function, we want only an imaginary part, so, we get

$$G(\mathbf{r}) = -\frac{1}{4\pi^2 r}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{se^{isr}\,\mathrm{d}s}{(s^2-k^2)} = -\frac{1}{4\pi^2 r}\,\mathrm{Im}\left[i\pi\, e^{ikr}\right]. \tag{A.16}$$

At last, it becomes

$$G(\mathbf{r}) = -\frac{e^{ikr}}{4\pi\, r}. \tag{A.17}$$

BIOGRAPHY

BIOGRAPHY

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