PERTURBATIVE ANALYSIS OF UNCERTAINTY IN COHERENT STATES OF NEWTON-EQUIVALENT QUANTUM HARMONIC OSCILLATOR

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Abstract

Coherent states are quantum states which are importance for both theoretical and experimental sides. Their theoretical constructions in quantum harmonic oscillator are simple, but they possess many important properties. In this work, we study coherent states for Newton-equivalent quantum harmonic oscillator, which are models of a one-parameter family of Hamiltonians alternative to the standard quantum harmonic oscillator. When the value of the parameter tends to zero, one recovers the standard quantum harmonic oscillator. The coherent states are perturbatively constructed up to the second order in the parameter. Then the uncertainty relations for some of the states are considered. In particular, we consider the case $|\alpha| \leq 2$, where α is the complex number characterising the coherent states. The results suggest that the product of the uncertainty in measuring the position and the momentum takes, to a good approximation, the minimal value. This behaviour agrees with that of the coherent states for the standard quantum harmonic oscillator.

Keywords: Newton-equivalent quantum harmonic oscillator, coherent state, perturbation, uncertainty relation

Introduction

Coherent states are quantum states of theoretical importance and with wide range of applications in physics. The simplest examples of coherent states are those coming from quantum harmonic oscillator (QHO). By definition, a coherent state is an eigenstate of the lowering operator. The corresponding eigenvalue is a complex number. This kind of states has various important properties. For example, they can be thought of as being a

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ground state which is shifted in the phase space in such a way that the real part of the eigenvalue is proportional to the displacement along the *x*-direction, whereas the imaginary part is proportional to the displacement along the *p*-direction. Another example is that the measurements of position and momentum in these states satisfy the minimal value of uncertainty. So these states can be thought to be the most classical.

Coherent states also arise in the context of quantum optics. The quantisation of the oscillation of electromagnetic wave gives rise to operators which create or destroy one photon in a photon number state. In this context, coherent states are certain superpositions of photon number states in such a way that they are eigenstates to the annihilation operators. The probability of detecting a number of photons in a coherent state satisfies Poisson distribution. As for applications of coherent states, the basic example is that a laser beam is a coherent state. Laser beams are stable thanks to the property of coherent states. This is in the sense that even after one photon is detected (theoretically, this is by acting on the coherent state with an annihilation operator), the coherent state remains the same.

For more details on theoretical importance and applications of coherent states, see for example (Glauber, 1963; Klauder and Skagerstam, 1985; Gazeau, 2009; Combescure and Robert, 2012). For us, however, we will focus on the issue closely related to QHO coherent states and their uncertainty relations.

In Hamiltonian mechanics, one considers a Hamiltonian which is a classical function of position and momentum. The Hamilton's equations of the Hamiltonian govern the dynamics of the classical system. For a particle moving in one-dimension, one may derive Newton's equations from an appropriately given Hamiltonian. It turns out however that different Hamiltonians can give rise to the same Newton's equations. This poses no problem in classical physics. After quantisation, however, these Hamiltonians may lead to different physics from one another. It is therefore important to be able to distinguish different physical phenomena corresponding to these Hamiltonians.

For quantum harmonic oscillator, the reference (Degasperis and Ruijsenaars, 2001) gives a one-parameter family of Hamiltonians all of whose classical version give rise to the same Newton's equation for simple harmonic oscillator. The energy spectrum and the corresponding eigenstates are derived. This is a first step to study physical phenomena relating to these Hamiltonians. It turns out, however, that not much has been done along this direction. As far as we are aware, there has only been several related works. For example, a mathematical extension to this work is given in (Odake and Sasaki, 2011), the explanations of related systems are given in (Calogero and Degasperis, 2004; Tita and Vanichchapongjaroen, 2018; Janaun and Vanichchapongjaroen, 2019).

To distinguish from the quantum harmonic oscillator with standard Hamiltonian and those with the one-parameter family in (Degasperis and Ruijsenaars, 2001), let us keep labelling the quantum harmonic oscillator with standard Hamiltonian as QHO. But for the quantum harmonic oscillator with the one-parameter family of Hamiltonians, we will call them as Newton-equivalent quantum harmonic oscillator (NEQHO). We will give more details about the NEQHO in the main sections.

In this paper, we present our investigation of coherent states corresponding to NEQHO Hamiltonians. In particular, we attempt to work out the uncertainty relations of these coherent states and argue whether it is possible to distinguish them from the coherent states corresponding to QHO Hamiltonians.

Theoretical Background

Coherent States of Quantum Harmonic Oscillator

The Hamiltonian of one-dimensional QHO is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \qquad (1)$$

where $\hat{x}$ and $\hat{p}$ are position and momentum operators respectively. These operators can be given in coordinate space representation as $\hat{x} = x$ and $\hat{p} = -i\hbar\partial_x$ respectively. The time independent Schrödinger's equation

$$\hat{H}\phi_n(y) = E_n\phi_n(y), \tag{2}$$

can be solved by using lowering and raising operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{y} + i\hat{p}_y\right), \tag{3}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{y} - i\hat{p}_y\right), \tag{4}$$

where $y = \sqrt{m\omega/\hbar}\, x$, and correspondingly,

$$\hat{y} = \sqrt{\frac{m\omega}{\hbar}}\hat{x},\ \ \hat{p}_y = \frac{1}{\sqrt{m\omega\hbar}}\hat{p}. \tag{5}$$

By following the standard procedure, this gives the eigenenergies

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \qquad \text{for } n = 1,2,3,..., \tag{6}$$

and the corresponding normalised eigenstates

$$\phi_n(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{(-1)^n e^{\frac{y^2}{2}}}{\sqrt{2^n n!}}\frac{d^n e^{-y^2}}{dy^n}. \tag{7}$$

Coherent states are eigenstates to the annihilation operator. The eigenvalues are complex numbers labelled as α. Therefore, the corresponding eigenvalue equation is given by

$$\hat{a}\,\psi_\alpha(y) = \alpha\,\psi_\alpha(y), \tag{8}$$

whose normalised eigenfunction is

$$\psi_\alpha(y) = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\phi_n(y). \tag{9}$$

One of the special features of coherent states is that the measurements of position and momentum satisfies the minimal value of uncertainty. That is, the coherent state wave functions in Equation (9) all satisfy

$$(\Delta y)_\alpha(\Delta p_y)_\alpha = \frac{1}{2}, \tag{10}$$

where

$$(\Delta y)_\alpha = \sqrt{\langle\hat{y}^2\rangle_\alpha - \langle\hat{y}\rangle_\alpha^2}, \tag{11}$$

$$(\Delta p_y)_\alpha = \sqrt{\left\langle\hat{p}_y^2\right\rangle_\alpha - \left\langle\hat{p}_y\right\rangle_\alpha^2}, \tag{12}$$

with

$$\left\langle\hat{O}\right\rangle_\alpha \equiv \sqrt{\frac{\hbar}{m\omega}}\int_{-\infty}^{\infty} dy\psi_\alpha^*(y)\hat{O}\psi_\alpha(y). \tag{13}$$

Newton-Equivalent Hamiltonians for Quantum Harmonic Oscillator

In the study of Hamiltonian mechanics, one starts with a Hamiltonian of the system of interest, then one uses Hamilton's equations to obtain the equations of motion. Conversely, given the equations of motion, one may ask for the Hamiltonian which gives rise to these equations. In general, the Hamiltonians are not unique. This non-uniqueness poses no problem in classical mechanics. This is because, it is the equations of motion that determines the physics of the system.

The situation is different in the case of quantum mechanics. The existence of multiple possible Hamiltonians for a single system would lead to a problem. After all, physics in quantum mechanical systems heavily rely on Hamiltonians. In principle, different Hamiltonians would lead to different eigenenergies and the corresponding eigenstate wave functions.

The reference (Degasperis and Ruijsenaars, 2001) starts by giving alternative forms of Hamiltonian for a particle of mass m moving in one dimension under a potential $V(x)$. By substituting Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial x} \tag{14}$$

into the Newton's equation

$$m\ddot{x} + V'(x) = 0, \tag{15}$$

they obtained

$$\frac{\partial^2 H}{\partial x\partial p}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial p^2}\frac{\partial H}{\partial x} + \frac{1}{m}\frac{dV}{dx} = 0. \tag{16}$$

One of the important forms of the Hamiltonians satisfying Equation (16) is given by

$$H_\beta = \frac{1}{m\beta^2} cosh(\beta p) \left(1 + 2m\beta^2 V(x)\right)^{\frac{1}{2}} - \frac{1}{m\beta^2},$$
$$\beta \in (-\infty, \infty). \quad (17)$$

In the limit $\beta \to 0$, one recovers the standard form of the Hamiltonian:

$$\lim_{\beta \to 0} H_\beta = \frac{p^2}{2m} + V(x) \quad (18)$$

The quantum version of Equation (17) for the case of simple harmonic oscillator are also constructed in (Degasperis and Ruijsenaars, 2001). In our notation and after appropriate shift, the Hamiltonians are given in the coordinate space representation as

$$\hat{H}_\lambda = \frac{\hbar\omega}{2\lambda^2}\Big((1 + i\lambda y)^{\frac{1}{2}} exp(-i\lambda\partial_y)\,(1 - i\lambda y)^{\frac{1}{2}} + c.c.\Big) - \frac{\hbar\omega}{\lambda^2} + \frac{1}{2}\hbar\omega, \quad (19)$$

where

$$\lambda = \beta\sqrt{\hbar m\omega}\,. \quad (20)$$

Let us call these Hamiltonians as NEQHO Hamiltonians. The energy spectrum and the corresponding eigenstate wavefunctions are obtained by using ladder operators. These operators no longer take the same form as their counterparts in the standard QHO Hamiltonian. Instead, the lowering operator is given by

$$\hat{A}_\lambda = \frac{1}{\sqrt{2}}\left(y + \frac{i}{2\lambda}\Big((1 + i\lambda y)^{\frac{1}{2}} exp(-i\lambda\partial_y)\,(1 - i\lambda y)^{\frac{1}{2}} - c.c.\Big)\right), \quad (21)$$

and the raising operator is

$$\hat{A}_\lambda^\dagger = \frac{1}{\sqrt{2}}\left(y - \frac{i}{2\lambda}\Big((1 + i\lambda y)^{\frac{1}{2}} exp(-i\lambda\partial_y)\,(1 - i\lambda y)^{\frac{1}{2}} - c.c.\Big)\right). \quad (22)$$

To justify that they are indeed ladder operators, it can be checked that

$$\left[\hat{H}_\lambda, \hat{A}_\lambda\right] = -\hbar\omega\hat{A}_\lambda,\ \left[\hat{H}_\lambda, \hat{A}_\lambda^\dagger\right] = \hbar\omega\hat{A}_\lambda^\dagger. \quad (23)$$

The eigenvalue equation for NEQHO Hamiltonian is given by

$$\hat{H}_\lambda \Phi_n^{(\lambda)}(y) = E_n^{(\lambda)}\Phi_n^{(\lambda)}(y). \quad (24)$$

By using ladder operators, it can be shown that the eigenenergies are

$$E_n^{(\lambda)} = \left(n + \frac{1}{2}\right)\hbar\omega, n = 0,1,2, ..., \quad (25)$$

which agree with the results from standard QHO Hamiltonian. As for the eigenstate wave functions, however, they do not agree with their counterparts from the standard QHO Hamiltonian. The wave functions can be obtained by first solving the equation $\hat{A}_\lambda\Phi_0^{(\lambda)} = 0$ to obtain the ground state wave function, then repeatedly applying the raising operator on $\Phi_0^{(\lambda)}(x)$ to obtain the *n*th state wave function. Let us however not quote the form of $\Phi_n^{(\lambda)}(x)$ as it is too complicated for our purpose.

An alternative way to obtain the eigenenergies and eigenstate wave functions is by using perturbation theory. This way of study is carried out in (Janaun and Vanichchapongjaroen, 2019). This method is particularly useful when one wishes to study NEQHO Hamiltonians with small values of λ. Furthermore, it is explicitly checked up to order λ^{10} that the result agrees with that of the reference (Degasperis and Ruijsenaars, 2001).

Coherent State Wave Functions for NEQHO with Small λ

The fact that NEQHO Hamiltonian tends to the standard QHO Hamiltonian in the limit $\lambda \to 0$ gives rise to a natural question. The standard QHO Hamiltonian has been extensively used in order to describe phenomena relating to quantum harmonic oscillators. We would like to investigate whether it would be possible that these phenomena are in fact described by NEQHO Hamiltonian with small λ. In particular, we would like to investigate the uncertainty relation of coherent states corresponding to NEQHO Hamiltonian with small λ.

Therefore, the first task is to obtain coherent state wave functions. One may argue that based on the commutation relations in Equation (23) the relationship between the ladder operators and the Hamiltonian are the same as their counterparts in the standard QHO. So one expects that the coherent state wave function should be of the form

$$\Psi_\alpha^{(\lambda)}(y) \stackrel{?}{=} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \Phi_n^{(\lambda)}(y). \qquad (26)$$

It turns out, however, that this is not the case. Even perturbatively, the equation $\hat{A}_\lambda \Psi_\alpha^{(\lambda)}(y) = \alpha \Psi_\alpha^{(\lambda)}(y)$ is not satisfied by Equation (26).

Let us propose a way to obtain the coherent state wave function. We first write the lowering operator to the second order in λ. This gives

$$\hat{A}_\lambda = \hat{a} + \frac{i\lambda^2}{6\sqrt{2}}(\hat{p}^3 - 3\hat{p} - 3i\hat{y} + 3\hat{y}^2\hat{p}) + O(\lambda^3). \quad (27)$$

Let us express $\Psi_\alpha^{(\lambda)}(y)$ as a linear combination of $\phi_n(y)$ as

$$\Psi_\alpha^{(\lambda)}(y) = \sum_{n=0}^{\infty} c_n^{(\lambda)}(\alpha)\phi_n(y). \qquad (28)$$

In practice, it is useful to truncate the linear combination up to $n = N$ and see if the result converges as N increases. So let us consider

$$\Psi_\alpha^{(\lambda,N)}(y) = \sum_{n=0}^{N} c_n^{(\lambda,N)}(\alpha)\phi_n(y), \qquad (29)$$

where we expect that

$$\lim_{N\to\infty} \Psi_\alpha^{(\lambda,N)}(y) \to \Psi_\alpha^{(\lambda)}(y). \qquad (30)$$

It can be seen that

$$\hat{A}_\lambda \Psi_\alpha^{(\lambda,N)}(y) - \alpha \Psi_\alpha^{(\lambda,N)}(y) = \sum_{n=0}^{N+3} B_n(\alpha)\phi_n(y) + O(\lambda^3). \qquad (31)$$

If we demand the right-hand side to vanish, this would give rise to N+4 conditions: $B_n(\alpha) = 0$, $n = 0,1,2,\ldots, N+3$. But we only have N+1 unknowns: $c_n^{(\lambda,N)}$, $n = 0,1,2,\ldots, N$. So, let us relax the conditions to $B_n(\alpha) = 0$, $n = 0,1,2,\ldots, N-1$, which gives rise to N conditions for N+1 unknowns. The other condition will come from normalisation requirement.

The removal of 4 conditions as explained above might seem arbitrary. But we expect that as N increases, our method would not lead to any problem. Instead, Equation (30) should be satisfied. We will verify this by presenting the result in the next section.

After obtaining the truncated wave function, the next step is to compute

$$(\Delta y)_\alpha^{(\lambda,N)} = \sqrt{\langle \hat{y}^2 \rangle_\alpha^{(\lambda,N)} - \left(\langle \hat{y} \rangle_\alpha^{(\lambda,N)}\right)^2}, \qquad (32)$$

$$(\Delta p_y)_\alpha^{(\lambda,N)} = \sqrt{\langle \hat{p}_y^2 \rangle_\alpha^{(\lambda,N)} - \left(\langle \hat{p}_y \rangle_\alpha^{(\lambda,N)}\right)^2}, \qquad (33)$$

with

$$\langle \hat{O} \rangle_\alpha \equiv \sqrt{\frac{\hbar}{m\omega}} \int_{-\infty}^{\infty} dy\, \Psi_\alpha^{(\lambda,N)*}(y) \hat{O} \Psi_\alpha^{(\lambda,N)}(y). \quad (34)$$

Then we may compute $(\Delta y)_\alpha^{(\lambda,N)} (\Delta p_y)_\alpha^{(\lambda,N)}$, and see to what value it converges as N increases. Then compare with the result from the standard QHO.

Results and Discussions

By following the algorithm outlined in the previous section, we have computed $(\Delta y)_\alpha^{(\lambda,N)}$, $(\Delta p_y)_\alpha^{(\lambda,N)}$, and $(\Delta y)_\alpha^{(\lambda,N)} (\Delta p_y)_\alpha^{(\lambda,N)}$ for various α and N. For any fixed N, these quantities are expressible as Taylor series expansion in λ up to the order of λ^2 such that each coefficient is a function of α. These coefficients, however, are too lengthy to be of use. So we do not present them. Instead, it is better to demonstrate the results numerically. After substituting in a numerical value for α, these coefficients are given by some numerical values. Since these coefficients are mathematical functions, their values can be given precisely (i.e. up to any significant figures), as long as we give a precise value of α. Correspondingly, the programming language that we have used in the calculation allows arbitrary precision, and we have made the full use of this capability.

As an example result, consider the case where N = 2 and α = 0.5+0.7i. We obtain, presenting up to 4 significant figures,

$$(\Delta y)^{(\lambda,2)}_{0.5+0.7i} = 0.8043 - 0.02622\lambda^2 + O(\lambda^3), \quad (35)$$

$$(\Delta p_y)^{(\lambda,2)}_{0.5+0.7i} = 0.7259 - 0.04001\lambda^2 + O(\lambda^3). \quad (36)$$

Let us also present the results for α = 0.5+0.7i. with N = 4,6,…,20. These results can be read off from Table 1 by starting from the values when N = 20, which are

$$(\Delta y)^{(\lambda,20)}_{0.5+0.7i} = 0.70710678118654752440 + 0.26162950903902257180\lambda^2 + O(\lambda^3), \quad (37)$$

$$(\Delta p_y)^{(\lambda,20)}_{0.5+0.7i} = 0.70710678118654752440 - 0.26162950903902257557\lambda^2 + O(\lambda^3), \quad (38)$$

then work out from the bottom of the table to the top. From the results, it can also be seen that the values of $(\Delta y)^{(\lambda,N)}_{0.5+0.7i}$ and $(\Delta p_y)^{(\lambda,N)}_{0.5+0.7i}$ converge as N is increased. Furthermore, using the results read off from Table 1, the values of $(\Delta y)^{(\lambda,N)}_{0.5+0.7i}(\Delta p_y)^{(\lambda,N)}_{0.5+0.7i}$ can be worked out, and are presented in Table 2.

We have checked that for other values of α with $|\alpha| \leq 1$, the behaviours are also qualitatively the same. That is, when N is large enough the values of $(\Delta y)^{(\lambda,N)}_{\alpha}$ and $(\Delta p_y)^{(\lambda,N)}_{\alpha}$ converge. We suppose that other values of α also share this behaviour.

We therefore choose the value of N sufficiently large enough. In particular, we choose N = 25 which is useful to study coherent state wave functions with $|\alpha| \leq 2$. Larger values of α can also be studied accurately, provided that we increase N to an appropriate value. It is a well-known result that coherent states for standard QHO has the minimal value of uncertainty. See Equation (10). As for the coherent states of NEQHO, however, it is non-trivial whether these states give the minimal value of uncertainty. So we study the values $(\Delta y)^{(\lambda,N)}_{\alpha}(\Delta p_y)^{(\lambda,N)}_{\alpha}$ for N = 25, $|\alpha| \leq 2$, and compare with the minimal value of uncertainty, which is 0.5. In particular, Table 3 demonstrates the difference in each of the cases where α = 0,0.2,0.4,…2.

The differences for other cases with $|\alpha| \leq 2$ (recall that α is a complex number) also share the same feature as those presented in Table 3. That is, the coefficients of the 0th and 2nd order of λ are very small.

Conclusions

We may conclude that up to the second order in λ, the uncertainties of NEQHO coherent states with $|\alpha| \leq 2$ are approximately equal to the minimal value. So by using this consideration, it is not a simple matter to

Table 1. The values of $(\Delta y)^{(\lambda,N)}_{\alpha} - (\Delta y)^{(\lambda,N-2)}_{\alpha}$ and $(\Delta p_y)^{(\lambda,N)}_{\alpha} - (\Delta p_y)^{(\lambda,N-2)}_{\alpha}$ for $\alpha = 0.5 + 0.7i$ with various values of N

N	$(\Delta y)^{(\lambda,N)}_{\alpha} - (\Delta y)^{(\lambda,N-2)}_{\alpha}$	$(\Delta p_y)^{(\lambda,N)}_{\alpha} - (\Delta p_y)^{(\lambda,N-2)}_{\alpha}$
4	$-8.97 \times 10^{-2} + 1.65 \times 10^{-1}\lambda^2 + O(\lambda^3)$	$-2.02 \times 10^{-2} - 1.41 \times 10^{-1}\lambda^2 + O(\lambda^3)$
6	$-7.34 \times 10^{-3} + 1.14 \times 10^{-1}\lambda^2 + O(\lambda^3)$	$1.29 \times 10^{-3} - 7.66 \times 10^{-2}\lambda^2 + O(\lambda^3)$
8	$-1.86 \times 10^{-4} + 8.58 \times 10^{-3}\lambda^2 + O(\lambda^3)$	$7.31 \times 10^{-5} - 4.30 \times 10^{-3}\lambda^2 + O(\lambda^3)$
10	$-2.31 \times 10^{-6} + 2.62 \times 10^{-4}\lambda^2 + O(\lambda^3)$	$1.20 \times 10^{-6} - 1.08 \times 10^{-4}\lambda^2 + O(\lambda^3)$
12	$-1.71 \times 10^{-8} + 4.41 \times 10^{-6}\lambda^2 + O(\lambda^3)$	$1.03 \times 10^{-8} - 1.86 \times 10^{-6}\lambda^2 + O(\lambda^3)$
14	$-8.31 \times 10^{-11} + 4.54 \times 10^{-8}\lambda^2 + O(\lambda^3)$	$5.51 \times 10^{-11} - 2.18 \times 10^{-8}\lambda^2 + O(\lambda^3)$
16	$-2.87 \times 10^{-13} + 3.05 \times 10^{-10}\lambda^2 + O(\lambda^3)$	$2.03 \times 10^{-13} - 1.66 \times 10^{-10}\lambda^2 + O(\lambda^3)$
18	$-7.40 \times 10^{-16} + 1.42 \times 10^{-12}\lambda^2 + O(\lambda^3)$	$5.48 \times 10^{-16} - 8.56 \times 10^{-13}\lambda^2 + O(\lambda^3)$
20	$-1.48 \times 10^{-18} + 4.79 \times 10^{-15}\lambda^2 + O(\lambda^3)$	$1.13 \times 10^{-18} - 3.12 \times 10^{-15}\lambda^2 + O(\lambda^3)$

Table 2. **The values of $(\Delta y)_{\alpha}^{(\lambda,N)}(\Delta p_y)_{\alpha}^{(\lambda,N)}$ for α = 0.5 + 0.7i with various values of N**

N	$(\Delta y)_{\alpha}^{(\lambda,N)}(\Delta p_y)_{\alpha}^{(\lambda,N)}$
2	$0.584 - 5.12 \times 10^{-2}\lambda^2 + O(\lambda^3)$
4	$0.504 - 3.10 \times 10^{-2}\lambda^2 + O(\lambda^3)$
6	$0.500 - 3.20 \times 10^{-3}\lambda^2 + O(\lambda^3)$
8	$0.500 - 1.12 \times 10^{-4}\lambda^2 + O(\lambda^3)$
10	$0.500 - 1.83 \times 10^{-6}\lambda^2 + O(\lambda^3)$
12	$0.500 - 1.68 \times 10^{-8}\lambda^2 + O(\lambda^3)$
14	$0.500 - 9.88 \times 10^{-11}\lambda^2 + O(\lambda^3)$
16	$0.500 - 4.00 \times 10^{-13}\lambda^2 + O(\lambda^3)$
18	$0.500 - 1.18 \times 10^{-15}\lambda^2 + O(\lambda^3)$
20	$0.500 - 2.67 \times 10^{-18}\lambda^2 + O(\lambda^3)$

Table 3. **The difference between the uncertainty $(\Delta y)_{\alpha}^{(\lambda,N)}(\Delta p_y)_{\alpha}^{(\lambda,N)}$ with N = 25, α = 0,0.2,0.4,…,2, and the minimal value 0.5 of uncertainty**

α	$(\Delta y)_{\alpha}^{(\lambda,25)}(\Delta p_y)_{\alpha}^{(\lambda,25)} - 0.5$
0.0	0
0.2	$2.79 \times 10^{-62} + 2.22 \times 10^{-55}\lambda^2 + O(\lambda^3)$
0.4	$1.11 \times 10^{-46} + 5.66 \times 10^{-41}\lambda^2 + O(\lambda^3)$
0.6	$1.31 \times 10^{-37} + 1.36 \times 10^{-32}\lambda^2 + O(\lambda^3)$
0.8	$3.11 \times 10^{-31} + 1.07 \times 10^{-26}\lambda^2 + O(\lambda^3)$
1.0	$2.37 \times 10^{-26} + 3.51 \times 10^{-22}\lambda^2 + O(\lambda^3)$
1.2	$2.00 \times 10^{-22} + 1.51 \times 10^{-18}\lambda^2 + O(\lambda^3)$
1.4	$3.60 \times 10^{-19} + 1.56 \times 10^{-15}\lambda^2 + O(\lambda^3)$
1.6	$2.05 \times 10^{-16} + 5.52 \times 10^{-13}\lambda^2 + O(\lambda^3)$
1.8	$4.75 \times 10^{-14} + 8.43 \times 10^{-11}\lambda^2 + O(\lambda^3)$
2.0	$5.32 \times 10^{-12} + 6.51 \times 10^{-9}\lambda^2 + O(\lambda^3)$

distinguish them from the coherent states for QHO.

We expect that even for $|\alpha| > 2$, the uncertainties of NEQHO coherent states are still approximately equal to the minimal value. Nevertheless, this should be still be investigated. We leave this verification to future works.

It is also interesting to go beyond the second order in λ to see whether the uncertainties of NEQHO coherent states still have the minimal value.

Furthermore, other physical phenomena relating to coherent states should also be investigated to see whether it is possible to distinguish NEQHO coherent states from their QHO counterparts. One of the phenomena to be checked is the noise in the time evolution of coherent states.

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