Research Signpost
37/661 (2), Fort P.O., Trivandrum-695 023, Kerala, India

Dark Energy—Current Advances and Ideas, 2009: 99-116 ISBN: 978-81-308-0341-8
Editor: Jeong Ryeol Choi

4 Scalar field cosmology: Its non-linear Schrödinger-type formulation

Burin Gumjudpai[1,2]
[1]Centre for Theoretical Cosmology, DAMTP, University of Cambridge CMS, Wilberforce Road, Cambridge CB3 0WA, United Kingdom
[2]Fundamental Physics & Cosmology Research Unit, The Tah Poe Academia Institute (TPTP) Department of Physics, Naresuan University, Phitsanulok 65000, Thailand†

Scalar field cosmology is a model for dark energy and inflation. It has been recently found that the standard Friedmann formulation of the scalar field cosmology can be expressed in a nonlinear Schrödinger-type equation. The new mathematical formulation is hence called non-linear Schrödinger (NLS) formulation which is suitable for a FRLW cosmological system with non-negligible barotropic fluid density. Its major features are reviewed here.

Correspondence/Reprint request: Dr. Burin Gumjudpai, Centre for Theoretical Cosmology, DAMTP University of Cambridge, CMS, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom
E-mails: B.Gumjudpai@damtp.cam.ac.uk or buring@nu.ac.th

1. Introduction

The present universe is under accelerating expansion. This is convinced by many present observational data from cosmic microwave background [1], large scale structure surveys [2] and supernovae type Ia [3-5]. There are many ideas to explain such an expanding state, mainly it can be classified into three types: braneworlds and modification of gravitational theory (e.g. [6]), backreaction effect from inhomogeneity [7] and dark energy (for review, see [8]). Dark energy is a type of cosmological fluid appearing in the matter term of the Einstein equation with equation of state $w_{\rm D.E.} < -1/3$ so that it can generate repulsive gravity and therefore accelerating the universe. The simplest dark energy model is just a cosmological constant with $w_\Lambda = -1$. However the cosmological constant suffers from fine-tuning problem. Observational data suggests that the present value of $w_{\rm D.E.}$ is very close to -1 and it also allows possibility that dark energy could be dynamical. Therefore scalar field model of dark energy became interesting topic in cosmology since time-evolving behavior of the scalar field gives hope for resolving the fine-tuning problem. Although the scalar field has not yet been observed, it is motivated from many ideas in high energy physics and quantum gravities. Theoretical predictions of its existence at TeV scale could be tested at LHC and Tevatron in very near future. Phenomenologically the scalar field is also motivated in model building of inflation where super-fast acceleration happens in the early universe [9]. Cosmic microwave background data combined with other results allows possibility that the scalar field could be phantom, i.e. having equation of state coefficient $w_\phi < -1$ [10]. The phantom equation of state is attained from negative kinetic energy term in its Lagrangian density [11,12]. The most recent five-year WMAP result [13] combined with Baryon Acoustic Oscillation of large scale structure survey from SDSS and 2dFGRS [14] and type Ia supernovae data from HST [4], SNLS [5] and ESSENCE [15] assuming dynamical w with flat universe yields $-1.38 < w_{\phi,0} < -0.86$ at 95% CL and $w_{\phi,0} = -1.12 \pm 0.13$ at 68% CL. With additional BBN constraint of limit of expansion rate [16,17], $w_{\phi,0} = -1.09 \pm 0.12$ at 68% CL. The phantom field will finally dominate the universe in future, leading to Big Rip singularity [18]. There have been many attempts to resolve the singularity from both phenomenological and fundamental inspirations [19]. However fundamental physics of the phantom field is still incomplete due to severe UV instability of the field's quantum vacuum state [20].

This review interests in non-linear Schrödinger-type formulation of scalar field cosmology. We shall call the formulation, NLS formulation. In our NLS system, cosmological ingredients are scalar field and a barotropic fluid with constant equation of state, $p_\gamma = w_\gamma \rho_\gamma$. We also have non-zero spatial curvature. This is a system resembling of our present universe filled with scalar field dark energy and barotropic cold dark matter or of the early inflationary universe in presence of inflaton and other fields behaving barotropic-like considered in e.g. [32]. In such a model, the scale invariant spectrum in the cosmic microwave background was claimed to be generated not only from fluctuation of scalar field alone but rather from both scalar field and interaction between gravity to other gauge fields such as Dirac and gauge vector fields.

Not long ago, mathematical alternatives to the standard Friedmann canonical scalar field cosmology with barotropic perfect fluid, was proposed e.g. non-linear Ermakov-Pinney equation [21,22]. Expressing standard cosmology with $k > 0$ in Ermakov equation system yields a system similar to Bose-Einstein condensates [23]. Another example is a connection from a generalized Ermakov-Pinney equation with perturbative scheme to a generalized WKB method of comparison equation [24]. It was then realized that solutions of the generalized Ermakov-Pinney equation are correspondent to solutions of a non-linear Schrödinger-type equation, and then the NLS version of the Friedmannıtre-Robertson-Walker (FLRW) cosmology was formulated [22]. In the NLS framework, the system of FLRW cosmological equations: Friedmann equation, acceleration equation and fluid equation are written in a single nonlinear Schrödinger-type equation. We will not prove it here but instead, referring to Ref. [25]. Few recent applications [26-29] of the NLS formulation have been made and this review intends to conclude its major aspects.

2. Scalar field cosmology

A. Friedman formulation

We set up major concepts in this section before considering its application later. In the Friedmann system, barotropic fluid has pressure p_γ and density ρ_γ with an equation of state, $p_\gamma = w_\gamma \rho_\gamma = [(n-3)/3]\rho_\gamma$ where $n = 3(1+w_\gamma)$. Scalar field pressure obeys $p_\phi = w_\phi \rho_\phi$. To sum up, $\rho_{\rm tot} = \rho_\gamma + \rho_\varphi$ and $p_{\rm tot} = p_\gamma + p_\phi$. Therefore $n = 0$ means $w_\gamma = -1$. The others are: $n = 2$ for $w_\gamma = -1/3$; $n = 3$ for $w_\gamma = 0$; $n = 4$ for $w_\gamma = 1/3$; $n = 6$ for $w_\gamma = 1$. Barotropic fluid and scalar fluid are conserved separately. Dynamics of the barotropic is governed by fluid equation, $\dot{\rho}_\gamma = -nH\rho_\gamma$ with solution, $\rho_\gamma = D/a^n$, where a is scale factor. The dot denotes time derivative. $H = \dot{a}/a$ is Hubble parameter and

$D \geq 0$ is a proportional constant. Scalar field is minimally-coupled to gravity with Lagrangian density, $\mathcal{L} = (1/2)\varepsilon\dot{\phi}^2 - V(\varphi)$ and is homogenously spread all over the universe. The scalar field density and pressure are

$$\rho_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi)\,, \qquad p_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi)\,. \tag{1}$$

The branch $\varepsilon = 1$ is for non-phantom field case and $\in = -1$ is for phantom field case. Dynamics of the scalar field is controlled by conservation equation, $\varepsilon(\ddot{\phi} + 3H\dot{\phi}) = -\mathrm{d}V/\mathrm{d}\phi$, in which the spatial expansion H of the universe sources friction to dynamics of the field. The Hubble parameter is governed by Friedmann equation, $H^2 = (\kappa^2/3)\rho_{\rm tot} - k/a^2$, and by acceleration equation, $\ddot{a}/a = -(\kappa^2/6)(\rho_{\rm tot} + 3p_{\rm tot})$, which gives acceleration condition $p_{\rm tot} < -\rho_{\rm tot}/3$. Here $p_{\rm tot} = w_{\rm eff}\,\rho_{\rm tot}$, $\kappa^2 \equiv 8\pi G = 1/M_{\rm P}^2$. G is Newton's gravitational constant. $M_{\rm P}$ is reduced Planck mass. k is spatial curvature and

$$w_{\rm eff} = \frac{\rho_\phi w_\phi + \rho_\gamma w_\gamma}{\rho_{\rm tot}}\,. \tag{2}$$

If we express the field speed and the field potential in term of $a(t)$ and time derivative of $a(t)$, then

$$\epsilon\dot{\phi}(t)^2 = -\frac{2}{\kappa^2}\left[\dot{H} - \frac{k}{a^2}\right] - \frac{nD}{3a^n} \quad \text{and} \quad V(\phi) = \frac{3}{\kappa^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \left(\frac{n-6}{6}\right)\frac{D}{a^n}\,. \tag{3}$$

B. NLS formulation

NLS formulation is a mathematical alternative to the standard Friedmann formulation with hope that the new formulation might suggest some new mathematical tackling to problems in scalar field cosmology. In the NLS formulation, there is no such an analogous equation to Friedmann equation or fluid equation. Instead both of them combine in single non-linear Schrödinger-type equation,

$$u''(x) + [E - P(x)]\,u(x) = -\frac{nk}{2}u(x)^{(4-n)/n}\,. \tag{4}$$

The links to cosmology are valid as one defines NLS quantities [25],

$$u(x) \equiv a(t)^{-n/2}, \quad E \equiv -\frac{\kappa^2 n^2}{12} D, \quad P(x) \equiv \frac{\kappa^2 n}{4} a(t)^n \epsilon \dot{\phi}(t)^2 \,. \tag{5}$$

where ‘ denotes d/dx. Independent variable t is scaled to NLS independent variable x as $x = \sigma(t)$, such that

$$\dot{x}(t) = u(x) \quad \text{and} \quad \phi(t) = \psi(x)\,, \tag{6}$$

which gives $\epsilon\dot{\phi}(t)^2 = \epsilon \dot{x}^2\,\psi'(x)^2$. Hence $\epsilon\,\psi'(x)^2 = (4/\kappa^2 n)\,P(x)$, and

$$\psi(x) = \pm \frac{2}{\kappa\sqrt{n}} \int \sqrt{\frac{P(x)}{\epsilon}}\,\mathrm{d}x \,. \tag{7}$$

Inverse function $\psi^{-1}(x)$ exists for $P(x) \neq 0$ and $n \neq 0$. In this circumstance, $x(t) = \psi^{-1} \circ \phi(t)$ and the scalar field potential, $V \circ \sigma^{-1}(x)$ and $\epsilon\dot{\phi}(t)^2$ can be expressed in NLS formulation as

$$\epsilon\dot{\phi}(x)^2 = \frac{4}{\kappa^2 n} u u'' + \frac{2k}{\kappa^2} u^{4/n} + \frac{4E}{\kappa^2 n} u^2\,, \tag{8}$$

and

$$V(x) = \frac{12}{\kappa^2 n^2}(u')^2 - \frac{2P}{\kappa^2 n} u^2 + \frac{12E}{\kappa^2 n^2} u^2 + \frac{3k}{\kappa^2} u^{4/n}\,. \tag{9}$$

The other equations are

$$\rho_\phi = \frac{12}{\kappa^2 n^2}(u')^2 + \frac{12E}{\kappa^2 n^2} u^2 + \frac{3k}{\kappa^2} u^{4/n}\,; \tag{10}$$

$$p_\phi = -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4P}{\kappa^2 n} u^2 - \frac{12E}{\kappa^2 n^2} u^2 - \frac{3k}{\kappa^2} u^{4/n}, \tag{11}$$

$$\rho_{\text{tot}} = \frac{12}{\kappa^2 n^2}(u')^2 + \frac{3k}{\kappa^2} u^{4/n}\,, \tag{12}$$

$$p_{\text{tot}} = -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4}{\kappa^2 n} u u'' - \frac{k}{\kappa^2} u^{4/n}\,. \tag{13}$$

$$H = -\frac{2}{n}u', \qquad \dot{H} = -\frac{2}{n}uu'', \tag{14}$$

$$\ddot{\phi} = \frac{2Puu' + P'u^2}{\kappa\sqrt{P\epsilon n}}, \quad 3H\dot{\phi} = -\frac{12u'u}{n\kappa}\sqrt{\frac{P}{\epsilon n}}. \tag{15}$$

We shall see later examples that the program of NLS formulation must start from presuming the "wave function", $u(x) \equiv a^{-n/2} = \dot{x}(t)$, before proceeding to calculate the other quantities. We know that normalization condition for a wave function is $\int_{-\infty}^{\infty} |u(x)|^2 \mathrm{d}x = 1$. If applying this to our NLS wave function, then $\int_{-\infty}^{\infty} \dot{x}^2 \mathrm{d}x = 1$. In order to satisfy the condition, x must be constant (hence so is t) with an integrating constant = 1. In connecting Friedmann formulation to NLS formulation, we are forced to have $u(x) = \dot{x}(t)$. Therefore $u(x)$ is, in general, non-normalizable.

3. Slow-roll conditions

A. Slow-roll conditions: Flat geometry and scalar field domination

In flat universe with scalar field domination, $\dot{H} = -(\kappa^2/2)\dot{\phi}^2\epsilon$. Hencefor ε = -1(phantom field),

$$0 < aH^2 < \ddot{a}, \tag{16}$$

i.e. the acceleration is greater than speed of expansion per Hubble radius, $\dot{a}/cH^{-1}$ and for $\in$ = 1 (non-phantom field),

$$0 < \ddot{a} < aH^2. \tag{17}$$

Slow-roll condition [30,31] assumes negligible kinetic term, i.e. $|\epsilon\dot{\phi}^2/2| \ll V(\phi)$ which makes an approximation $H^2 \simeq \kappa^2 V/3$. This results in a condition $\dot{H}| \ll H^2$. Slow-roll parameter, $\varepsilon \equiv -\dot{H}/H^2$ is hence defined from this relation. The condition $|\epsilon\dot{\phi}^2/2| \ll V(\phi)$ is then equivalent to $|\varepsilon| << 1$, i.e. $-1 << \varepsilon < 0$ for phantom field case and $0 < \varepsilon << 1$ for non-phantom field case. Considering $\dot{H} \simeq 0$ implying approximative constancy in H during the slow-rolling regime. For non-phantom field, this condition is necessary for inflation to happen (though not sufficient) [31] however, for phantom field case, the negative kinetic term always results in acceleration with $w_\phi \leq -1$ then it does not need the slow-roll approximation. Another slow-roll parameter can be defined when the friction term dominates $|\ddot{\phi}| \ll |3H\dot{\phi}|$. This gives the second parameter, $\eta \equiv -\ddot{\phi}/H\dot{\phi}$ and the approximation is made to $|\eta| << 1$ [31]. The field fluid equation is then $\dot{\phi} \simeq -V_\phi/3\epsilon H$ which implies that if $\varepsilon = -1$, the field can roll up the hill. With all assumptions imposed here, i.e. $k = 0$, $\rho_\gamma = 0$, $|\epsilon\dot{\phi}^2/2| \ll V$ and $|\ddot{\phi}| \ll |3H\dot{\phi}|$, one can derive $\varepsilon = (1/2\kappa^2\varepsilon)\,(V_\phi/V)^2$ and $\eta = (1/\kappa^2)\,(V_{\phi\phi}/V)$ as known where the subscript ϕ denotes $d/d\phi$.

B. Slow-roll conditions: Non-flat geometry and non-negligible barotropic density

There are also inflationary models in presence of other field behaving barotropic-like apart from having only single scalar fluid [32]. The scale invariant spectrum in the cosmic microwave background was claimed to be generated not only from fluctuation of scalar field alone but rather from both scalar field and interaction between gravity to other gauge fields. Assuming this scenario with $k \neq 0$ and $\rho_\gamma = 0$, then

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2\epsilon + \frac{k}{a^2} - \frac{n\kappa^2}{6}\frac{D}{a^n}\,. \tag{18}$$

The slow-roll condition becomes $\kappa^2\epsilon\dot{\phi}^2/6| \ll (\kappa^2 V/3) - (k/a^2) + (\kappa^2 D/3a^n)$ hence

$$H^2 \simeq -\frac{\dot{H}}{3} + \frac{k}{3a^2} - \frac{n\kappa^2}{18}\frac{D}{a^n} + H^2\,, \tag{19}$$

implying $|-(\dot{H}/3) + (k/3a^2) - (n\kappa^2 D/18a^n)| \ll H^2$. We can reexpress this slow-roll condition as

$$|\,\varepsilon + \varepsilon_k + \varepsilon_D\,| << 1, \tag{20}$$

where $\varepsilon_k \equiv k/\,a^2H^2$ and $\varepsilon_D \equiv -n\kappa^2\,D/6a^nH^2$. Another slow-roll parameter η is defined as $\eta \equiv -\ddot{\phi}/H\dot{\phi}$, i.e. the same as the flat scalar field dominated case since the condition $|\ddot{\phi}| \ll |3H\dot{\phi}|$ is independent of k and ρ_γ.

Writing the condition $\epsilon\dot{\phi}^2/2| \ll V$ in NLS form using Eqs. (5) and (9),

$$|P(x)| \ll \frac{3}{n}\left[\left(\frac{u'}{u}\right)^2 + E\right] + \frac{3}{4}k\,n\,u^{(4-2n)/n}\,. \tag{21}$$

If the absolute sign is not used, the condition is then $\epsilon\dot{\phi}^2/2 \ll V$, allowing fast-roll negative kinetic energy. Then Eq. (21), when combined with the NLS equation (4), yields

$$u'' \ll \frac{3}{n}\frac{u'^2}{u} + \left(\frac{3}{n} - 1\right)Eu + \frac{kn}{4}u^{(4-n)/n}\,. \tag{22}$$

Friedman analog of this condition can be obtained simply by using Eq. (3) in the condition. Using Eq. (15), the second slow-roll condition, $|\ddot{\phi}| \ll |3H\dot{\phi}|$ in the NLS form is written as,

$$\left|\frac{P'}{P}\right| \ll \left|-2\left(\frac{6+n}{n}\right)\frac{u'}{u}\right|\,. \tag{23}$$

This condition yields the approximation $3H\epsilon\dot{\phi}^2 \simeq -\mathrm{d}V/\mathrm{d}\phi$ which, in NLS form, is

$$\frac{P'}{P} \simeq -\frac{2u'}{u} = nHa^{n/2}\,. \tag{24}$$

The slow-roll parameters ε, ε_k and ε_D, in NLS form, are

$$\varepsilon = \frac{nuu''}{2u'^2}\,, \quad \varepsilon_k = \frac{n^2ku^{4/n}}{4u'^2}\,, \quad \varepsilon_D = \frac{nE}{2}\left(\frac{u}{u'}\right)^2\,, \tag{25}$$

therefore

$$\varepsilon_{\mathrm{tot}} = \varepsilon + \varepsilon_k + \varepsilon_D = \frac{n}{2}\left(\frac{u}{u'}\right)^2 P(x)\,. \tag{26}$$

Hence the slow-rollcondition, $|\varepsilon_{\mathrm{tot}}| << 1$, is just

$$\left|\left(\frac{u}{u'}\right)^2 P(x)\right| \ll 1\,. \tag{27}$$

Another slow-roll parameter $\eta = -\ddot{\phi}/H\dot{\phi}$ can be found as follow. First considering $\psi(x) = \phi(t)$ and Eq. (7), using relation $\mathrm{d}/\mathrm{d}t = \dot{x}\,\mathrm{d}/\mathrm{d}x$, one can obtain

$$\eta = \frac{n}{2}\left(\frac{u}{u'}\frac{\psi''}{\psi'}+1\right) = \frac{n}{2}\left(\frac{u}{u'}\frac{P'}{2P}+1\right). \quad (28)$$

The slow-roll condition $|\eta| << 1$ in NLS form is just

$$\left|\frac{u}{u'}\frac{P'}{2P}+1\right| \ll 1. \quad (29)$$

4. Acceleration condition

For the phantom field, since its kinetic term is always negative and could take any large negative values, the slow-roll condition is not needed. The acceleration equation is taken as acceleration condition straightforwardly, i.e. $\ddot{a} > 0$ hence

$$\epsilon\dot{\phi}(x)^2 \;<\; -\left(\frac{n-2}{2}\right)\frac{D}{a^n}+V. \quad (30)$$

This, in NLS-type form, is equivalent to

$$E-P \;>\; -\frac{2}{n}\left(\frac{u'}{u}\right)^2 - \frac{nk}{2}\left(\frac{u^{2/n}}{u}\right)^2, \quad (31)$$

which is reduced to

$$u'' < \frac{2}{n}\frac{u'^2}{u}. \quad (32)$$

with help of the Eq. (4). Using Eqs. (14), the acceleration condition is just $\varepsilon < 1$.

5. Power-law cosmology

The power-law expansion $a(t) = t^q$, with $q > 1$ is assumed here as the first step of calculation. In some high-energy physics models, during inflation, flat geometry and scalar field domination are assumed. The universe was driven by an exponential potential $V(\phi) = [q(3q - 1)/(\kappa^2 t_0^2)]$ exp $\left\{-\kappa\sqrt{2/q}\,[\phi(t)-\phi(t_0)]\right\}$ [33]. Also, at late time with dark matter component, the expansion could be power-law. Recent results from X-Ray gas of galaxy clusters put a constraint of $q \sim 2.3$ for $k = 0$, $q \sim 1.14$ for $k = -1$ and $q \sim 0.95$ for $k = 1$ [34]. For a flat universe, the power law expansion,

$a = t^q$, is attained when $-1 < w_{\rm eff} < -1/3$ where $q = 2/[3(1 + w_{\rm eff})]$. If using $q = 2.3$ as above, it gives $w_{\rm eff} = -0.71$ (only flat case). Latest combined WMAP5 results with SNI and BAO yield $-0.0175 < \Omega_k < 0.0085$ at 95% maximum likelihood [13]. The mean is Ω_k = -0.0045 corresponding to closed universe with $q = 0.986$ [35]. Assuming power-law expansion, the Schrödinger wave function is [26]

$$u(x) = \dot{x}(t) = t^{-qn/2}\,. \tag{33}$$

Integrating the equation above so that the Schrödinger scale, x is related to cosmic time scale, t as

$$x = x(t) = -\frac{t^{-\beta}}{\beta} + x_0, \quad \text{and} \quad t(x) = \frac{1}{[-\beta(x-x_0)]^{1/\beta}}\,, \tag{34}$$

where $\beta \equiv (qn - 2)/2$ and x_0 is an integrating constant. The parameters x and t have the same dimension since β is only a number. Then the wave function is

$$u(x) = \left[\left(-\frac{1}{2}qn + 1\right)(x - x_0)\right]^{qn/(qn-2)}\,, \tag{35}$$

which depends on only q and n. Wave functions for a range of barotropic fluids are presented in Fig. 1. The result is confirmed by substituting Eq. (35) into Eq. (4). The field speed and scalar potential are:

$$\epsilon\dot{\phi}(t)^2 = \frac{2q}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} - \frac{nD}{3t^{qn}} \quad \text{and} \quad V(t) = \frac{q(3q-1)}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} + \left(\frac{n-6}{6}\right)\frac{D}{t^{qn}}\,. \tag{36}$$

From Eq. (5), therefore the Schrödinger potential is found to be

$$P(x) \;=\; \frac{2qn}{(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2}\left[\frac{-2}{(qn-2)(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}\,. \tag{37}$$

With $E = -\kappa^2 n^2 D/12$, the Schrödinger kinetic energy is

$$T(x) \;=\; -\frac{2qn}{(qn-2)^2}\frac{1}{(x-x_0)^2} - \frac{kn}{2}\left[\frac{-2}{(qn-2)(x-x_0)}\right]^{2q(n-2)/(qn-2)}\,. \tag{38}$$

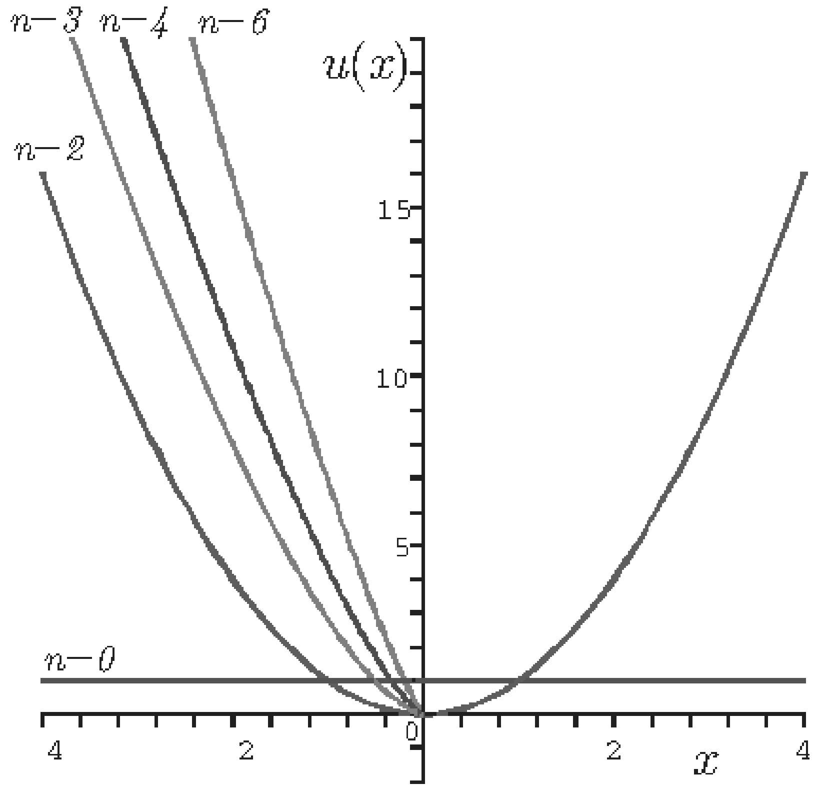

Figure 1. $u(x)$ versus x for power-law cosmology with $q = 2$. We set $x_0 = 0$. There is no real-value wave function for $n = 3$, $n = 4$ and $n = 6$ unless $x < 0$.

A disadvantage of Eq. (37) is that we can not use it in the case of scalar field domination. Dropping D term in Eq. (37) can not be considered as scalar field domination case since the barotropic fluid coefficient n still appears in the other terms. The non-linear Schrödinger-type formulation is therefore suitable when there are both scalar field and a barotropic fluid together such as the situation when dark matter and scalar field dark energy live together in the late universe or in the inflationary models in presence of other fields behaving barotropic-like and single scalar fluid [32]. $P(x)$ is plotted versus x for power-law expansion with $q = 2$ in closed, flat and open universe in Fig. 2. One can check that the acceleration condition (32) for the power-law case is just $q > 1$.

There is application of the NLS scalar field function ψ in Eq. (7) to solve for scalar field exact solutions in power-law, phantom expansion ($a \sim (t_a - t)^q$, $q < 0$) and exponential (de Sitter) expansion $a \sim \exp(t/\tau)$ [27,28]. For example in power-law case:

$$\psi(x) = \frac{\pm 2}{\kappa\sqrt{n}} \times \int \sqrt{\frac{2qn}{\epsilon(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2\epsilon}\left[\frac{-2}{(qn-2)}\frac{1}{(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12\epsilon}}\, dx. \quad (39)$$

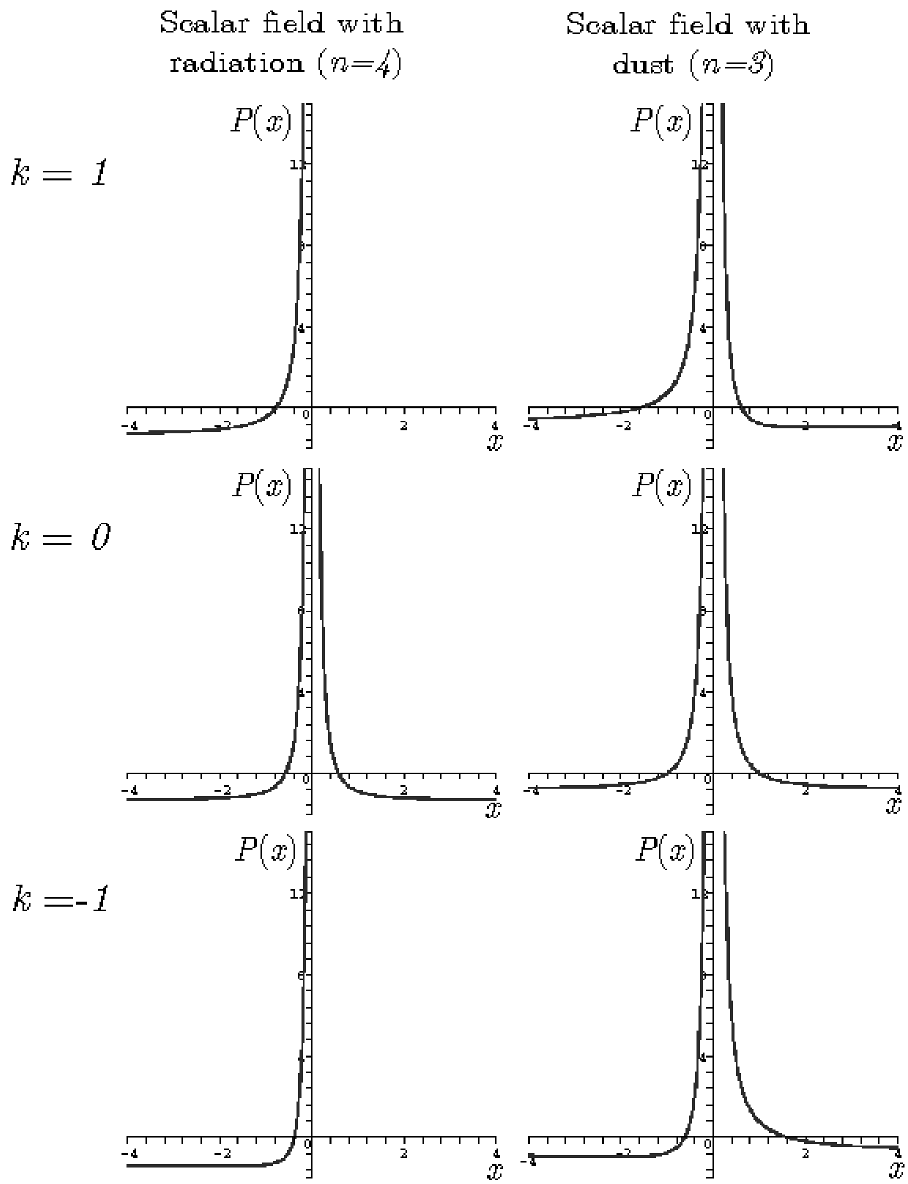

Figure 2. $P(x)$ plotted versus x for power-law expansion. We set $q = 2$, $\kappa = 1$, $D = 1$ and x_0. There is only a real-value $P(x)$ for the cases $k = \pm 1$ with $n = 4$ because, when $x > 0$, $P(x)$ becomes imaginary in these cases. The physical value is when $x < 0$ since t has a reverse sign of x.

The solution can be found only when assuming $k = 0$,

$$\phi(t) = \pm\frac{1}{qn-2}\sqrt{\frac{2q}{\epsilon\kappa^2}}\left\{\ln\left[\frac{t^{-qn+2}}{\left(1+\sqrt{1-(nD\kappa^2/6q)\,t^{-qn+2}}\right)^2}\right]+2\sqrt{1-\left(\frac{nD\kappa^2}{6q}\right)t^{-qn+2}}+\ln\left(\frac{qn-2}{2qn}\right)^2\right\}+\phi_0 \qquad (40)$$

When $q = 2/n$ or $n = 0$, the field has infinite value. q and ε must have the same sign for the solution to be real. The last logarithmic term does not restrict sign of q. This is unlike the solution obtained from Friedmann formulation which requires $q < 0$ which violates power-law expansion condition ($q > 1$). Working in neither of them can obtain exact solution with $k \neq 0$. In NLS formulation, we can not set D to zero while n is multiplied to the other terms then it can not be reduced to the scalar dominant case. This is a weak aspect. Obviously, the most difficult case is when $k \neq 0$ with $D \neq 0$. This case can not be integrated out in both frameworks unless assuming $n = 2$ (equivalent to $w_\gamma = -1/3$) which is not physical.

There are other good aspects of the NLS formulation. Since transforming standard Friedmann formulation (t as independent variable) to NLS formulation (x as independent variable) makes n appear in all terms of the integrand and also changes fluid density term D from time-dependent term to a constant E, therefore the number of x (or equivalently t)-dependent terms is reduced by one and hence simplifying the integral (7). In the case of exponential (de Sitter) expansion using NLS formulation, the solution when $k \neq 0$ and $D \neq 0$ can be obtained without assuming n value but $n = 0, 2, 3, 4, 6$ must be given if working within Friedmann formulation. The phantom expansion case is very similar to the power-law case but only with different sign (see Ref. [28]).

6. Phantom cosmology and big rip singularity

If we assume the expansion to a form, $a(t) \sim (t_a - t)^q$ with a finite time t_a, one can see that $q = 2/3(1 + w_{\rm eff}) < 0$ (for a flat universe). This corresponds to $w_{\rm eff} < -1$. Such equation of state is called phantom. The Schrödinger scale, x is related to cosmic time scale, t as

$$x(t) = \frac{1}{\beta}\left[(t_{\rm a} - t)^{-\beta}\right] + x_0\,, \qquad (41)$$

and the wave function is

$$u(x) = [\beta(x\text{-}x_0)]^{qn/(qn\text{-}2)} \qquad (42)$$

which is plotted in Fig. 3 with various types of barotropic fluid [28].
Therefore

$$P(x) \;=\; \frac{2qn}{(qn-2)^2}\frac{1}{(x-x_0)^2} + \frac{kn}{2}\left[\frac{2}{(qn-2)(x-x_0)}\right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}. \qquad (43)$$

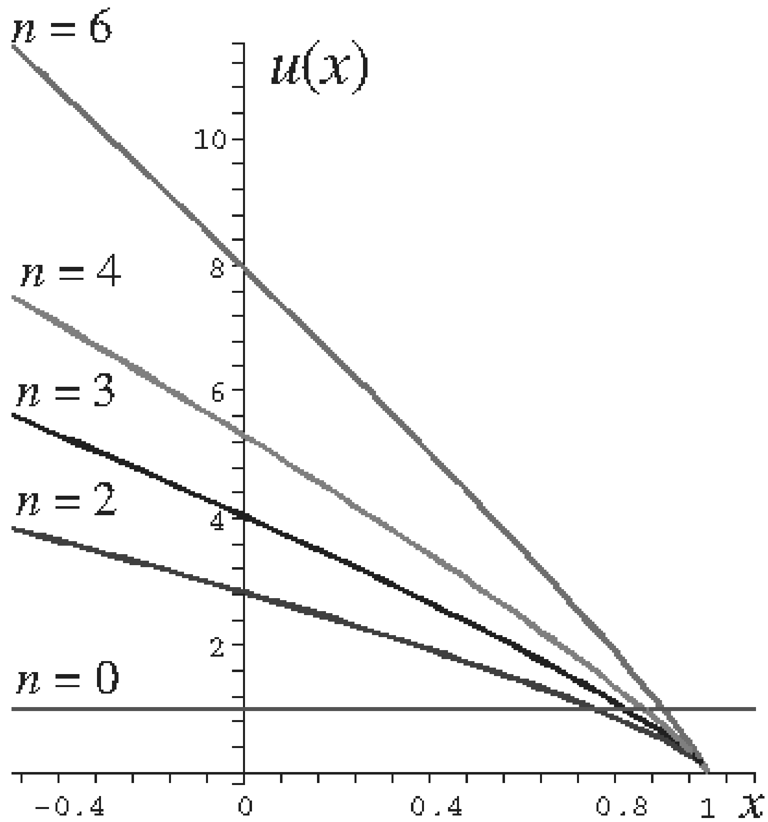

Figure 3. Schrödinger wave function, $u(x)$ when assuming phantom expansion. $u(x)$ depends on only q, n and t_a. Here we set $t_a = 1.0$ and $q = -6.666$. If $k = 0$, $q = -6.666$ corresponds to $w_{\rm eff} = -1.1$.

Fig. 4 shows $P(x)$ plots for three cases of k with dust and radiation. $P(x)$ goes to negative infinity at $x = x_0 = 1$. Expansion of the form, $a(t) \sim (t_a - t)^q$ leads to unwanted future Big Rip singularity [11]. The Big Rip conditions are that $(a, \rho_{\rm tot}, |p_{\rm tot}|) \to \infty$ which happen when $t \to t_a^{-n}$ in finite future time. Written in NLS language, if $a \to \infty$, $u \to 0^+$ and then $u \to 0$ (see Fig. 3). Considering also Eqs. (12) and (13), hence conditions of the Big Rip singularity are [29]

$$
\begin{aligned}
t \to t_{\rm a}^- \quad &\Leftrightarrow \quad x \to x_0^- \\
a \to \infty \quad &\Leftrightarrow \quad u(x) \to 0^+ \\
\rho_{\rm tot} \to \infty \quad &\Leftrightarrow \quad u'(x) \to \infty \\
|p_{\rm tot}| \to \infty \quad &\Leftrightarrow \quad u'(x) \to \infty\,.
\end{aligned}
\tag{44}
$$

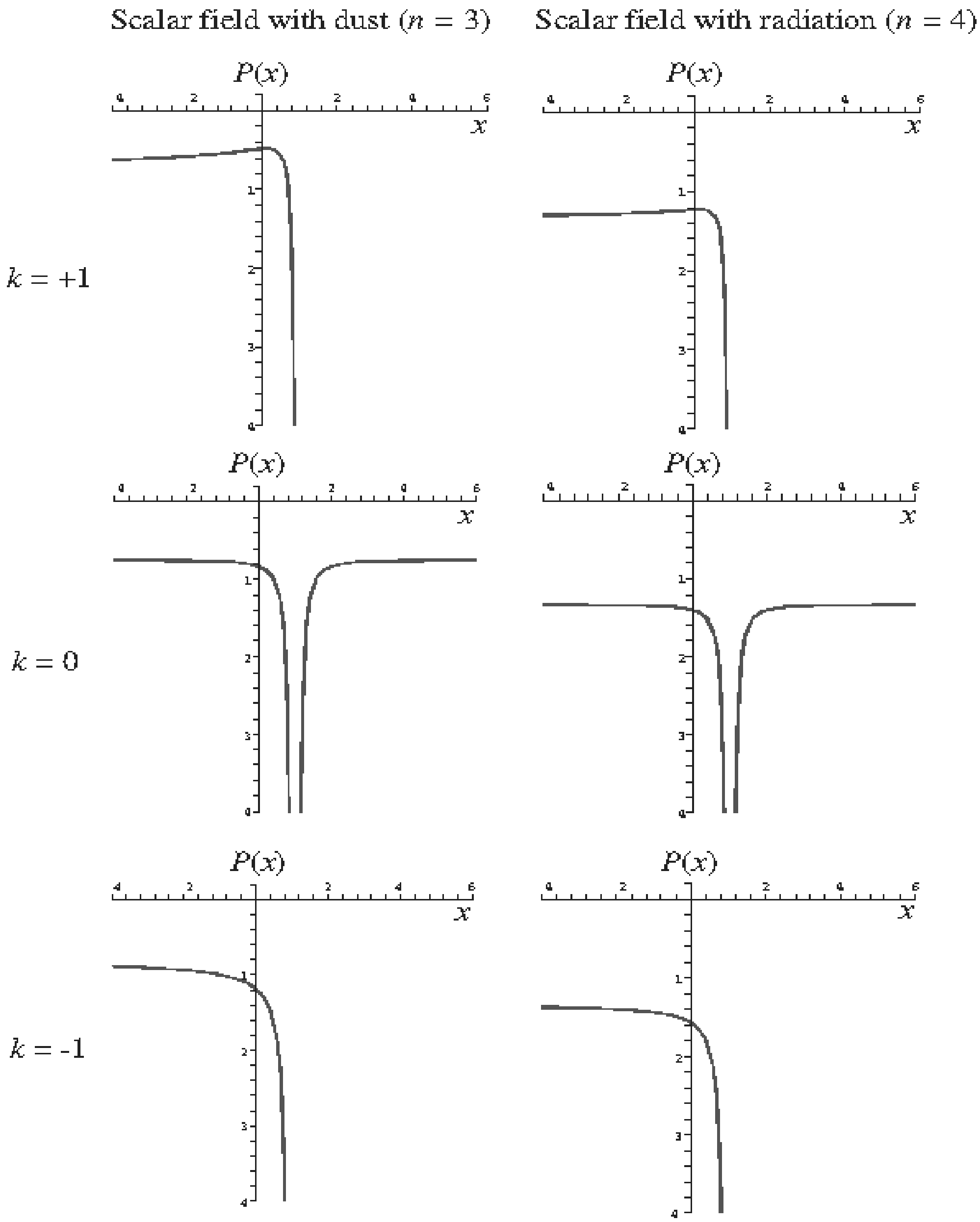

Figure 4. Schrödinger potential in phantom expansion case for dust and radiation fluids with $k = 0, \pm 1$. Numerical parameters are as in the $u(x)$ plots (Fig. 3). x_0 is set to 1. For non-zero k, there is only one real branch of $P(x)$.

We have one less infinite value in NLS Big Rip condition, i.e. $u(x)$ goes to zero. The NLS effective equation of state $w_{\text{eff}} = p_{\text{tot}} / \rho_{\text{tot}}$ can be expressed using Eqs. (12) and (13). Approaching the Big Rip, $x \to x^-_0$, $u \to 0^+$, then $w_{\text{eff}} \to -1 + 2/3q$, where $q < 0$ is a constant. This limit is the same as the effective phantom equation of state in the case $k = 0$. It is important to note that scalar field potential here is built phenomenologically based on expansion function, not on fundamental physics.

7. WKB approximation

WKB approximation in quantum mechanics is a tool to obtain wave function. However, in NLS formulation of scalar field cosmology, the wave function is first presumed before working out the shape of $P(x)$. Procedure is opposite to that of quantum mechanics. Hence the WKB approximation might not be needed at all for the NLS. Anyway, if one wants to test the WKB approximation in the NLS formulation, these below are some results. The WKB are valid when the coefficient of highest-order derivative term in the Schrödinger equation is small or when the potential is very slowly-varying. Consider linear case of Eq. (4), ($k = 0$),

$$-u'' + [P(x) - E]\, u = 0. \qquad (45)$$

In Figs 2 and 4, the left-hand side of $P(x)$ is physical since it corresponds to positive time. In most regions, there are ranges of slowly varying $P(x)$ at large value of $|x|$, in which the WKB is valid. The approximation gives

$$a \sim A \exp\left[\pm(2/n)i \int_{x_1}^{x_2} \sqrt{E - P(x)}\,\mathrm{d}x\right], \qquad (46)$$

where A is a constant.

8. Conclusions

Here we conclude aspects of NLS-type formulation of scalar field cosmology. The NLS-type formulation is well-applicable in presence of barotropic fluid and a canonical scalar field. There are few advantages of the NLS formulation as well as disadvantages to the conventional Friedmann formulation. With hope that some more interesting and useful features could be revealed in future.

Acknowledgements

The author thanks Jeong Ryeol Choi for invitation to write this review. The author is a TRF Research Scholar under a TRF-CHE Research Career Development Grant of the Thailand Research Fund. In Cambridge, the author is supported by Naresuan University's Overseas Visiting Postdoctoral

Research Fellowship and the Centre for Theoretical Cosmology, D.A.M.T.P., University of Cambridge at which very much gratitude is expressed to Anne-Christine Davis, Stephen Hawking and Neil Turok for providing the centre's subsistence.

References

1. Masi, S. et al. 2002, Prog. Part. Nucl. Phys., 48, 243.
2. Scranton, R. et al. [SDSS Collaboration] 2003, arXiv:astro-ph/0307335.
3. Riess, A.G. et al. [Supernova Search Team Collaboration] 1998, Astron. J.,116,1009; Perlmutter, S. et al. [Supernova Cosmology Project Collaboration] 1999, Astrophys. J., 517,565; Riess, A.G. 1999, arXiv:astro-ph/9908237; Goldhaber, G. et al. [The Supernova Cosmology Project Collaboration] 2001, Astrophys. J., 558,359; Tonry, J.L. et al. [Supernova Search Team Collaboration] 2003, Astrophys. J., 594,1.
4. Riess, A.G. et al. [Supernova Search Team Collaboration] 2004, Astrophys. J., 607,665; Riess, A.G. et al. 2007, Astrophys. J., 659, 98;
5. Astier, P. et al. [SNLS Collaboration] 2006, Astron. Astrophys., 447, 31.
6. Dvali, G.R., Gabadadze, G., and Porrati, M. 2000, Phys. Lett. B,485,208; Gumjudpai, B. 2004, Gen. Rel. Grav., 36,747; Maartens, R. 2004, Living Rev. Rel., 7,7; Nojiri, S., and Odintsov, S.D. 2007, Int. J. Geom. Meth. Mod. Phys., 4,115.
7. Kolb, E.W., Matarrese, S., Notari, A., and Riotto, A. 2005, arXiv:hep-th/0503117; Kolb, E.W., Matarrese, S., and Riotto, A. 2006, New J. Phys., 8,322.
8. Padmanabhan, T. 2005, Curr. Sci., 88,1057; Copeland, E.J., Sami, M., and Tsujikawa, S. 2006, Int. J. Mod. Phys. D, 15,1753; Padmanabhan, T. 2006, AIP Conf. Proc., 861,179.
9. Kazanas, D. 1980, Astrophys. J., 241, L59; Starobinsky, A.A. 1980, Phys. Lett. B 91,99; Guth, A.H. 1981, Phys. Rev. D, 23,347; Sato, K., 1981, Mon. Not. Roy. Astro. Soc. 195,467; Albrecht, A., and Steinhardt, P.J. 1982, Phys. Rev. Lett., 48,1220; Linde, A.D. 1982, Phys. Lett. B, 108,389.
10. Spergel, D.N., et al. [WMAP Collaboration] 2007, Astrophys. J. Suppl., 170,377.
11. Caldwell, R.R. 2002, Phys. Lett. B, 545,23; Gibbons, G.W. 2003, arXiv:hep-th/0302199; Nojiri, S., and Odintsov, S.D. 2003, Phys. Lett. B,562,147.
12. Melchiorri, A., Mersini-Houghton, L., Odman, C.J., and Trodden, M. 2003, Phys. Rev. D, 68,043509; Corasaniti, P.S., Kunz, M., Parkinson, D., Copeland, E.J., and Bassett, B.A. 2004, Phys. Rev. D, 70,083006; Alam, U., Sahni, V., Saini, T.D., and Starobinsky, A.A. 2004, Mon. Not. Roy. Astron. Soc., 354,275.
13. Hinshaw, G. et al. [WMAP Collaboration] 2008, arXiv:astro-ph/0803.0732; Dunkley, J., et al. [WMAP Collaboration] 2008, arXiv:astro-ph/0803.0586; Komatsu, E., et al. [WMAP Collaboration] 2008, arXiv:astro-ph/0803.0547.
14. Percival, W.J., Cole, S., Eisenstein, D.J., Nichol, R.C., Peacock, J.A., Pope, A.C., and Szalay, A.S. 2007, Mon. Not. Roy. Astron. Soc., 81, 053.
15. Wood-Vasey, W.M. et al. [ESSENCE Collaboration] 2007, Astrophys. J., 666,694.

16. Steigman, G. 2007, Ann. Rev. Nucl. Part. Sci., 57,463.
17. Wright, E.L. 2007, Astrophys. J., 664,633.
18. Caldwell, R.R., Kamionkowski, M., and Weinberg, N.N. 2003, Phys. Rev. Lett., 91,071301; Nesseris, S., and Perivolaropoulos, L. 2004, Phys. Rev. D, 70,123529; Hao, J.g., and Li, X.z. 2003, Phys. Rev. D,67,107303; Li, X.z., and Hao, J.g. 2004, Phys. Rev. D, 69,107303; Hao, J.G., and Li, X.z. 2004, Phys. Rev. D, 70,043529; Sami, M., and Toporensky, A. 2004, Mod. Phys. Lett. A, 19,1509; Nojiri, S., Odintsov, S.D., and Tsujikawa, S. 2005, Phys. Rev. D, 71,063004; Gumjudpai, B., Naskar, T., Sami, M., and Tsujikawa, S. 2005, J. Cosmol. Astropart. Phys., 0506, 007; Urena-Lopez, L.A. 2005, J. Cosmol. Astropart. Phys., 0509, 013.
19. Singh, P., Sami, M., and Dadhich, N. 2003, Phys. Rev. D, 68,023522; Nojiri, S., Odintsov, S.D., and Sasaki, M. 2005, Phys. Rev. D, 71,123509; Sami, M., Toporensky, A., Tretjakov, P.V., and Tsujikawa, S. 2005, Phys. Lett. B, 619,193; Calcagni, G., Tsujikawa, S., and Sami, M. 2005, Class. Quant. Grav., 22,3977; Wei, H., and Cai, R.G. 2005, Phys. Rev. D, 72,123507; Koivisto, T., and Mota, D.F. 2007, Phys. Lett. B, 644,104; Koivisto, T., and Mota, D.F. 2007, Phys. Rev. D, 75,023518; Leith, B.M., and Neupane, I.P. 2007, J. Cosmol. Astropart. Phys., 0705, 019; Samart, D., and Gumjudpai, B. 2007, Phys. Rev. D, 76,043514; Naskar, T., and Ward, J. 2007, Phys. Rev. D, 76,063514; Gumjudpai, B. 2007, Thai J. Phys. Series 3: Proc. of the SIAM Phys. Cong. [arXiv:gr-qc/0706.3467].
20. Carroll, S.M., Hoffman, M., and Trodden, M. 2003, Phys. Rev. D, 68, 23509.
21. Hawkins, R.M., and Lidsey, J.E. 2002, Phys. Rev. D, 66,023523; Williams, F.L., and Kevrekidis, P.G. 2003, Class. Quant. Grav., 20, L177; Williams, F.L., Kevrekidis, P.G., Christodoulakis, T., Helias, C., Papadopoulos, G.O., and Grammenos, T., 2006, Trends in General Relativity and Quantum Cosmology, C.V. Benton (Ed.), Nova Science Pub., New York, 37-48.
22. Williams, F.L. 2005, Int. J. Mod. Phys. A, 20,2481.
23. Lidsey, J.E. 2004, Class. Quant. Grav., 21,777.
24. Kamenshchik, A., Luzzi, M., and Venturi, G. 2006, arXiv:math-ph/506017.
25. D'Ambroise, J., and Williams, F.L. 2007, Int. J. Pure Appl. Maths., 34, 117.
26. Gumjudpai, B. Astropart. Phys. 30, 186 (2008).
27. Gumjudpai, B. Gen. Rel. Grav. 41, 249 (2009).
28. Phetnora, T., Sooksan, R., and Gumjudpai, B. 2008, arXiv:gr-qc/0805. 3794.
29. Gumjudpai, J. Cosmol. Astropart. Phys. 0809, 028 (2008).
30. Liddle, A.R., and Lyth, D.H. 2000, Cosmological Inflation and Large-Scale Structure, Cambridge University Press, Cambridge.
31. Liddle, A.R., and Lyth, D.H. 1992, Phys. Lett. B, 291,391.
32. Chaicherdsakul, K. 2007, Phys. Rev. D, 75,063522.
33. Lucchin, F., and Matarrese, S. 1985, Phys. Rev. D, 32,1316.
34. Zhu, Z.H., Hu, M., Alcaniz, J.S., and Liu, Y.X. 2008, Astron. and Astrophys., 483,15.
35. Tepsuriya, K., and Gumjudpai, B., in preparation.