Scaling Solutions in Scalar Field Cosmology

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Abstract

The Friedmann-Robertson-Walker universe containing non-relativistic matter and a canonical scalar field is considered here. We assumed scaling solution of a scalar field density and dust matter density, neglecting radiation components, so that we can obtain the corresponding scalar field potential, with attractor behaviors. Two cases of the potential occur. These are an exponential potential and a negative power-law potential. This work is a re-investigation of literatures by Liddle and Scherrer (1998) and Rubano and Barrow (2001).

***Keywords*:** Friedmann-Robertson-Walker universe, dark energy, scalar field, scaling solution

Introduction

Our present universe is in an accelerating expansion phase of the cause is still unknown. Our existing physics is not sufficient to explain it. We describe it in form of dark energy [1]. There are several models of dark energy, such as cosmological constant and scalar field. The widely acceptable one is the dark energy in the form of a canonical scalar field because it can solve the fine-tuning problem [1] of the cosmological constant. The scalar field density ρ_ϕ can evolve in time.

In this work we assume scaling solutions [2,3] as which the scalar field energy and matter densities [3] decrease like an inverse power function of scale factor: $\rho_\phi \propto a^{-n}$ and $\rho_m \propto a^{-m}$. We find an exact form of the potential, the case where $m = n$ is an exponential potential [5,6] and case $n < m$ is produced by a negative power-law potential [6,7]. The exact solution at late time is also analyzed.

Scaling solutions

A flat universe contains a homogeneous and isotropic barotropic perfect fluid and a scalar field ϕ, with the potential $V(\phi)$, energy density, ρ, and pressure p, therefore

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_m\right], \quad (1)$$

$$\dot{\rho}_m = -3H(\rho_m + p_m), \quad (2)$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{dV}{d\phi}, \quad (3)$$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad (4)$$

where H is the Hubble parameter. Scalar field dynamics is governed by Eq. (3) and Eq. (4) is the total scalar field energy density. We has the equation of state $p = (\gamma - 1)\rho$, which implies

$$\rho_\phi = Ka^{-n}, \qquad \rho_m = Da^{-m}, \quad (5)$$

where m and n are exponents of the scale factor of matter ($m = 3\gamma$) and of scalar field. We assume in first part of this paper that $\rho_\phi \ll \rho_m$ at earlier time. The constant D and K are their density values at present time. We let $n < m$ so that the scalar field can dominate at late time. From fluid equation, the field density rate of change is

$$\dot{\rho}_\phi = -3H\dot{\phi}^2. \quad (6)$$

Dividing the fluid equation of the scalar field by ρ_ϕ hence $\dot{\rho}_\phi / \rho_\phi = -n(\dot{a}/a)$ and

$$\frac{\dot{\phi}^2/2}{\rho_\phi} = \frac{n}{6}. \quad (7)$$

We see that for a scaling assumption, the ratio between the scalar field kinetic energy density and its total energy density is constant. If the kinetic energy dominates, then $n=6$ or if the field kinetic term is negligible, $n=0$ and field density is constant. Hence scaling behavior for the scalar field energy density lies in a range, $0\le n\le 6$.

Potential Construction

With the scaling-solution assumed ρ_ϕ and ρ_m, we substitute them into Friedmann equation. Let the perfect fluid with $\rho_m \propto a^{-m}$ dominate, then

$$a \propto t^{2/m}. \qquad (8)$$

Hence, the fluid equation can be written as

$$\ddot{\phi} = -\frac{6}{m}\frac{1}{t}\dot{\phi} - \frac{dV}{d\phi}. \qquad (9)$$

Considering scaling behavior of ρ_ϕ, substitute it into Eq. (7), therefore

$$\dot{\phi} = At^{-n/m}. \qquad (10)$$

When $m=n$, solution of Eq. (10) is $\phi = A\ln(t)+\phi_0$. Hence $t=\exp[(\phi-\phi_0)/A]$. Using Eq. (9), hence we obtain the field potential

$$V(\phi) = \frac{2}{\lambda^2}\left(\frac{6}{m}-1\right)\exp[-\lambda(\phi-\phi_0)], \qquad (11)$$

where $\lambda=(2/A)$ and $(2/\lambda^2)=(A^2/2)$. The potential has an exponential form as in [5] and can be comparable to the potential found in Lucchin and Matarrese [4],

$$V(\phi) = \left(\frac{\sigma}{t_0}\right)^2\left(\frac{3p-1}{2}\right)\exp\left[\frac{-2}{\sigma}(\phi-\phi_0)\right]$$

as we express, $\lambda = 2/\sigma$, $3p=6/m$ and $t_0\equiv 1$. Note that λ can be either positive or negative.

For the case $m\ne n$, we integrate Eq. (10) to yield

$$\phi = \frac{A}{1-(n/m)}t^{(1-n/m)}+\phi_0. \qquad (12)$$

The field acceleration is $\ddot{\phi} = A(-n/m)t^{-1-(n/m)}$. Substituting it into Eq. (9), one can find

$$V(\phi) = A^2\left(\frac{3}{n}-\frac{1}{2}\right)\left[\left(1-\frac{n}{m}\right)\left(\frac{\phi-\phi_0}{A}\right)\right]^{\frac{2n}{n-m}}+V_0. \qquad (13)$$

The potential has a power-law form. The fact that $n<m$ makes the exponent negative and the scalar field grows with time.

Potential Reconstruction without Pre-assumption of Perfect Fluid Domination at Early Time

It is possible to reconstruct the scalar field potential in the case of $n<m$. We can find the constant D from $\Omega_{m,0}=\rho_m/\rho_c$. We used the present value of the scale factor $a_0=1$ and the present density parameter $\Omega_{m,0}$, we obtain

$$D = \frac{3H_0^2\Omega_{m,0}}{8\pi G}. \qquad (14)$$

Since $\Omega_{m,0}+\Omega_{\phi,0}=1$, hence

$$K = \left(1-\Omega_{m,0}\right)\frac{3H_0^2}{8\pi G}. \qquad (15)$$

From (7), we can directly write down

$$\dot{\phi}^2 = \frac{Kn}{3}a^{-n}. \qquad (16)$$

The time derivative of the field is

$$\frac{d\phi}{dt} = Ha\frac{d\phi}{da}. \qquad (17)$$

We then have

$$\left(\frac{d\varphi}{da}\right)^2 = \frac{1}{(Ha)^2}(\dot{\phi}^2). \qquad (18)$$

Substituting ρ_m and ρ_ϕ into the Friedmann equation, gives

$$H^2 = \left(H_0^2\Omega_{m,0}a^{-m}+H_0^2(1-\Omega_{m,0})a^{-n}\right). \qquad (19)$$

Substituting Eq. (19) and Eq. (16) into Eq. (18), we obtain

$$\left(\frac{d\phi}{da}\right)^2 = \frac{K}{3H_0^2}\frac{n}{\Omega_{m,0}a^{n-m+2}+(1-\Omega_{m,0})a^2}. \qquad (20)$$

Integrating Eq. (20) yields

$$d\phi = \int \sqrt{\frac{1-\Omega_{m,0}}{8\pi G}} \frac{\sqrt{n}}{\sqrt{\Omega_{m,0} a^{n-m+2} + (1-\Omega_{m,0}) a^2}} da$$

$$= \frac{2\sqrt{n}}{\sqrt{8\pi G}(m-n)} \operatorname{arcsin} hyperbolic \left(\sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}} a^{\frac{m-n}{2}} \right) + \phi_0 \ . \quad (21)$$

From Eq. (4), (15) and Eq. (16)

$$V = \frac{3H_0^2}{8\pi G}(1-\Omega_{m,0})\left(1-\frac{n}{6}\right)a^{-n}. \qquad (22)$$

We substitute scale factor in Eq. (22), giving ultimately

$$V(\phi) = \frac{3H_0^2}{8\pi G}(1-\Omega_{m,0})\left(1-\frac{n}{6}\right)\left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)^{\frac{n}{m-n}}$$

$$\times \left(\sinh\left(\sqrt{8\pi G}\frac{m-n}{\sqrt{n}}(\phi-\phi_0) \right)\right)^{-\frac{2n}{m-n}} . \quad (23)$$

The slope of potential for a tracker condition is

$$\Gamma = \frac{V''V}{(V')^2} . \qquad (24)$$

For the potential in Eq. (23), it is

$$\Gamma = 1 + \frac{m-n}{2n}\left(\sec hyperbolic \left(\frac{m-n}{\sqrt{n}}\phi \right)\right)^2 > 1 \cdot \quad (25)$$

When n = 6, the potential is zero. The slope and the amplitude of $V(\phi)$ is ϕ - dependent. When different barotropic fluid dominates, potential acquires different slopes. We should include three fluids and the same time in the analysis. That is

$$\phi = \int \sqrt{\frac{\Omega_{\phi,0}}{8\pi G}} \frac{\sqrt{n} da}{\sqrt{\Omega_{r,0} a^{n-2} + \Omega_{d,0} a^{n-1} + \Omega_{\phi,0} a^2}} + \phi_0 , \quad (26)$$

$$V = \frac{3H_0^2}{8\pi G}\Omega_{\phi,0}\left(1-\frac{n}{6}\right)a^{-n} , \qquad (27)$$

which is not possible find the solution and analytically. One needs to employ numerical integration for the potential.

Conclusions

The cosmological density solutions are assumed here in such the way that the scalar field density scales with the barotropic density. Our demanded cosmic solution is that the scalar field must be sub-dominant at early time for structure to form and dominant at late time for describing late acceleration. The exact scalar field potentials are found analytically assuming barotropic density domination at early time. These are in form of exponential function and inverse-power law. The case when the assumption of early barotropic domination is turn off, the potential can be found analytically as function of observational parameters and of the field. If we consider more realistic situation of which there are dust and radiation in coexistence, the potential can not be found analytically as function of field but it needs to be done numerically. In this work, apart from reinvestigation of the work by [2] and [3], we correct typos of the results therein.

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