Dynamics of scalar field with cosmological barotropic fluid under exponential potential

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To My Parents and My Dreams

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The candidate has passed the viva voce examination by the examination panel. This report has been accepted to the panel as partially fulfilment of the course 261493 Independent Study.

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Abstract

To explain acceleration of the universe observed recently, ideas of dark energy in form of scalar field draws attention to cosmology. We study in detail the work of Copeland, Liddle and Wands [1]. The canonical self-interacting scalar field under exponential potential, $V = V_0 \exp(-\lambda\kappa\phi)$ in existence of cosmological barotropic fluid is considered here. We perform dynamical analysis in detail and interpret cosmological consequence. Five fixed points are found. The scaling solution is found when $\Omega_\phi = 3\gamma/\lambda^2$.

Contents

Chapter 1

Introduction

1.1 Background

The large-scale universe, galaxies are assumed to be fluid particles. The barotropic fluid does not completely dominate the universe. From recent observational data we found that the universe is expanding with acceleration. One possible cause of the expansion could be dark energy in form of scalar field which dominate other component todays. Scalar field is also believed to be responsible for inflation. Investigation of scalar field dynamics in existence of barotropic fluid is therefore an interesting topic.

1.2 Objectives

- Studying Friedmann-Robertson-Walker cosmology in domination of canonical scalar field and barotropic fluid under exponential potential following the paper of Copeland, Liddle and Wands [1].
- Dynamical analysis of autonomous system in existence of the scalar field and barotropic fluid under the exponential potential.
- Interpreting scaling solution obtaining from fixed points.
- Proving power-law solution of exponential potential.

1.3 Frameworks

- The universe is assumed to be maximally symmetric e.g. Friedmann-Robertson-Walker cosmology.
- A universe contains a barotropic fluid with scalar field under exponential potential $V = V_0 \exp(-\lambda\kappa\phi)$.
- The space is flat.

1.4 Expected use

- Learning original motivation of scalar field in cosmology.
- Attaining ability of analyzing dynamical system.
- Understanding late time behavior of scalar field under exponential potential and its effects to cosmology.
- Obtaining proof of power-law solution.

1.5 Tools

- Text books in physics and mathematics.
- A high efficiency personal computer.
- Software e.g. LaTeX, WinEdit, Maple, Mayura Draw and Photoshop.

1.6 Procedure

- Studying second-order autonomous system of differential equations in [2].
- Studying autonomous system in existence of the scalar field and barotropic fluid under the exponential potential.
- Analyzing fixed points.
- Studying Friedmann-Robertson-Walker cosmology in domination of canonical scalar field and barotropic fluid under exponential potential following paper of Copeland, Liddle and Wands [1].

- Studying interpreting scaling solution obtaining from fixed points.
- Proof of power-law expansion $a \propto t^p$.
- Making conclusion.

1.7 Outcome

- Obtaining full derivation of cosmological dynamical analysis of scalar field and barotropic fluid under exponential potential.
- Full proof of power-law solution of scalar field dominate universe under exponential potential.

Chapter 2

Standard cosmology, dark energy and inflation

2.1 Introduction

Begin with Hubble's studies of galaxies in the 1920's it became clear that the universe is not static but expanding. Due to expansion, matter and radiation are diluted. The universe becomes cold and empty as we observe at present from our location on scales larger than cluster of galaxies. Down cooling of the universe allows galaxies and planets to form, thus creating eventually life of which we ourselves are a part of. From ancient time till today, our civilization has been gradually adding many small pieces of jigsaw, either on mythological, philosophical or scientific aspects to our picture of the universe. Different viewpoints of our universe vary from culture to culture. In science, cosmology bases on a belief that our physical laws applied on earth can also be applied in any other parts of the universe. We are not special and we are not living at a special place in the universe. Physics that works on earth must be the same physics that works everywhere in the universe.

Cosmology is concerned with the study of the universe at a large scale. We are not primarily interested in subjects like stellar formation, galaxy dynamics or even clusters of galaxies. We rather look at distance that are comparable to the size of the visible universe, that is, distances of order of billions of light years. At such large distances, the exact nature of the constituent particles (galaxies) that make up the universe is no longer relevant. We may treat the energy content of the universe as a continuous distribution. This concept is an excellent approximation in description of the universe.

The main objective of cosmology is to understand nature of the universe as a whole: why does the universe look like what we see it today? Looking far into the universe is to look into the past. This automatically leads us to the study of the evolution of our universe. The universe we see is, to large extent, isotropic (provided we look on scales large enough).

2.1.1 Cosmological principle

The cosmological principle is not a principle, but rather an assumption or axiom which is a more general version of the Copernican principle. It follows from observation that universe at large scales is homogeneous and isotropic. The Copernican principle states that the Earth is not the center of the universe and we are not living at a special location in the universe. *Homogeneity* of the universe means that the universe has the same property at any regions from point to point. *Isotropy* means that the universe looks the same from all directions. These two properties are therefore separate and we need both of them for the cosmological principle.

We indeed do know that at small scales the universe is not homogenous and not isotropic otherwise any structures e.g. galaxies, stars, planets and humans would not even exist. However, provided that we consider the universe on average at large scales, it looks approximately homogenous and isotropic. Strong evidence for this is the cosmic microwave background (CMB) observed in 1992 by the COBE mission [3].

2.2 Standard cosmological equations

2.2.1 Hubble's law

In 1920s, Edwin Hubble's observations of variable stars in spiral nebulae enabled him to calculate the distances to these objects. These objects were discovered to be at distances beyond the Milky Way. He could summarize that universe is not static but expanding. This was noticed by observing that distant galaxies' spectra are redshifted. He found relation

$$\mathbf{v} = H_0 \mathbf{R} \,. \tag{2.1}$$

This is *Hubble's law*, a simple rule that tells us that the further galaxies move away faster than the nearer ones. H_0 is Hubble constant at present time (t_0). Hubble parameter (H) is related to scale factor $a(t)$, the relative size of space, as

$$H = \frac{\dot{a}(t)}{a(t)} \,. \tag{2.2}$$

2.2.2 Friedmann equation

The equation

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} \tag{2.3}$$

is known as the *Friedmann equation.* According to the cosmological principle, the equation can be applied to any large-scale regions. Where ρ is total density, k is curvature of space which describes geometry of the universe.

2.2.3 Fluid equation

Fluid equation is expressed as

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0\,. \tag{2.4}$$

The equation of state can be written as

$$p = \rho c^2 w \tag{2.5}$$

where $w = 0$ and $1/3$ for dust and radiation, while $w < -1/3$ for dark energy. From hereafter, we set $c \equiv 1$, the fluid equation is then simply

$$\dot{\rho} + 3H\rho(1 + w) = 0\,. \tag{2.6}$$

The fluid equation, like Friedmann equation, is in fact energy conservation law. The first term $\dot{\rho}$ tells us how fast the density changes (e.g. dilutes) and the second term is the lost of kinetic energy from fluid to gravitational potential energy.

2.2.4 Acceleration equation

The acceleration equation tells us how fast rate of expansion of the universe could change, i.e. slowing down or speeding up. The equation is a mixture of Friedmann and fluid equations. These two equations are in fact energy conservation law in classical mechanics and thermodynamics respectively. After differentiating the Friedmann equation with respect to time and using the fluid equation, we finally obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(1 + 3w)\,. \tag{2.7}$$

The good feature of the acceleration equation is that it does not contain k and we can use this equation regardless of the geometry of the universe. From the equation it seems that the universe is decelerating. When we let the universe be dominated by $w < -1/3$ fluid (dark energy) with $p < -\rho/3$, it makes $\ddot{a}$ positive and accelerates the expansion. Indeed recent observations from Type Ia Supernovae strongly support the accelerating expansion [4]. In high energy physics viewpoint, we can have dark energy in the form of scalar field that can yield negative pressure and hence accelerate the universe. These scalar fields are sometimes called *quintessence*, the name of the fifth element in the ancient Greek.

2.3 Dark energy and inflation

2.3.1 The implications of flatness and origin of dark energy

Before discussion about what drives the expansion of the universe, we must briefly consider the implications of the observations supporting the value $k = 0$, i.e. the flatness of our universe. Consider density evolution of the universe in terms of a dimensionless density parameter

$$\Omega = \frac{8\pi G\rho}{3H^2} \equiv \frac{\rho}{\rho_c}, \tag{2.8}$$

where $\rho_c = 3H^2/8\pi G$ is the critical density of the universe. Comparing to equation (2.3), we see that the critical density is that of a flat universe. In term of Ω, the equation (2.3) is written as

$$\Omega = 1 + \frac{k}{a^2H^2}\,. \tag{2.9}$$

If $k = 0$, Ω remains 1 at all times. This value agrees well with observations. In the year 2003, observational data yields evidence for the flat universe, $\Omega_{\rm tot} = 1.00^{+0.11}_{-0.12}$ [5] by comparison with BOOMERANG [6, 7, 8], MAXIMA [9, 10] and DASI data [11].

Today we found that the total density in baryonic and dark matter of around $\Omega_{\rm baryons} + \Omega_{\rm dark\ matter} \approx 0.3$. The missing density is *dark energy* [12]. In cosmology, dark energy has negative pressure and permeates all over space. While the theory of relativity, the effect of such negative pressure is qualitatively similar to a force acting as repulsive gravity at large scales. This effect agrees with observation that the universe is expanding acceleratingly. A form of dark energy that we study in this thesis is scalar field whose density varies in time and space.

2.3.2 Inflationary universe

To explain the observed isotropy and homogeneity of the universe, in particular the CMB, it is proposed that the universe expands so fast that light can not escape from a region of fixed size in co-moving coordinates. The idea of inflation is that at one stage in the evolution of the universe, before the standard big bang, there has been a period of accelerated expansion. The inflation of the universe is equivalent to

$$\ddot{a} > 0 \tag{2.10}$$

so that $\dot{a}$ increases during the inflation phase, and the comoving Hubble length $(aH)^{-1}$ must be decreasing in this phase, i.e.

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{aH}\right) < 0\,. \tag{2.11}$$

The acceleration equation (2.7) requires that

$$\rho + 3p \;<\; 0 \tag{2.12}$$

$$p \;<\; -\frac{\rho}{3}\,. \tag{2.13}$$

Inflation is able to solve the problems of the big bang, which is shown in section 2.4. It does not substitute the big bang idea but instead it adds on some ideas and also modifies the big bang model. Particle physics model suggests existence of scalar field $\phi(t)$ driving inflation. The field is therefore called *inflaton field*. The energy and pressure of the inflaton field are given by

$$\rho_\phi \;=\; \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2.14}$$

$$p_\phi \;=\; \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.15}$$

where $V(\phi)$ is potential energy of the scalar field. The scalar field dominated Friedmann equation is

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]\,. \tag{2.16}$$

If we use the energy density and pressure of scalar field in the fluid equation (2.6), we get

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\mathrm{d}}{\mathrm{d}\phi}V(\phi) \tag{2.17}$$

This equation dominates the dynamics and energy conservation of the scalar field. Using equation (2.14) and (2.15) in equation (2.13), we get

$$\dot{\phi} < \sqrt{V}\,. \tag{2.18}$$

It implies that inflation can be sustained when the field moves very slowly. This is called the *slow-roll condition*.

2.4 Problem of big bang model and solving with the concept of inflation

Evidence that support the big bang model are redshift of distant galaxies, primodial nucleosynthesis and detection of CMB. However, although big bang model can explain all of evidence, but there are still some puzzle that can not be explained in the big bang framework. These problems and solutions are as follows.

2.4.1 Flatness problem

In the big bang model with $\ddot{a} < 0$, the $(aH)^2$ term in equation (2.9) always decreases. This show that Ω shift away from one with expansion of the universe. However, present observations suggest that Ω is the order of one [13], Ω needs to be very close to one in the past. For example, we want $|\Omega - 1|$ less than order 10^{-16} at the age of nucleosynthesis [14, 15]. This is an extreme fine-tuning of initial conditions.
Solution:
Since the $(aH)^2$ term in equation (2.9) increase during inflation, Ω quickly approaches one. After the inflation ends, the evolution of the universe is followed by the conventional big bang phase and $|\Omega - 1|$ begins to increase. In the present age, Ω stays in order of one.

2.4.2 Horizon problem

Horizon problem is about causal connection between different areas in the universe. From concept of universe with limited age, distance that light can travel is limited as well. An important properties of CMB is isotropy with temperature 2.7 K. Equal temperature is the specific properties of thermal equilibrium. The light has traveled to us since decoupling epoch with CMB isotropy. Thus the universe might be in thermal equilibrium before decoupling. However distance that light can travel from big bang to decoupling is very short. We should not see same temperature from different part of the universe
Solution:
Inflation enlarges space, meanwhile Hubble scale (cH^{-1}) is fixed. A small area is enlarged to bigger than size of universe at present. Thus CMB from different parts of the universe can be in thermal equilibrium.

2.4.3 Magnetic monopole problem

Theory of particle physics predicts that strange particles might be created in early epoch, e.g. magnetic monopoles, graviton, moduli fields, and others in higher dimension theory e.g. cosmic string, domain wall, and texture. Where are them?

Solution:

Inflation can dilute strange particles away.

2.4.4 Origin of structure problem

CMB anisotropy has been discovered by COBE since 1992. It can not be explained within the big bang framework. The anisotropy in large scales was created during decoupling. Big bang can not explain how the anisotropy is created.

Solution:

The scalar field possesses quantum fluctuation behavior. The fluctuation is enlarged to form origin of structure.

Chapter 3

Autonomous system of scalar-field model

3.1 Introduction to autonomous system

When we study mathematical model or physical model, the important step is to define the system. System is things or set of things that we want to study. For example, the system could be a physical situation such as the motion of a stone thrown through a window, the motion of the planets around the sun or the temperature of a cooling body.

The first step in describing any system mathematically is to find a mathematical model (or models) consisting of a set of assumptions and simplifications with one or two variables that represent features we are interested in. In this section the mathematical model describing each system will be a first order differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v(x, t) \tag{3.1}$$

where x is dependent variable and t is independent variable, usually representing time. The function $v(x, t)$ is called the *velocity field* of the system and its value for particular values of x and t is the *phase velocity*. In modeling problems in mechanics we must be very careful with the interpretation of the velocity field and phase velocity.

If the velocity field does not depend explicitly on time, so that $v(x, t)$ is a function of x only, then the system is called an *autonomous system*, otherwise the system is non-autonomous. To describe the state of an autonomous dynamical system over an interval of time we need to solve the equation $\mathrm{d}x/\mathrm{d}t = v(x)$ to find a solution $x(t)$. The state of the system at some given time t_0 is given by $x(t_0)$ which can be represented by a point on the x-axis, called the *phase point*. As time increases, the state of the dynamical system changes and the point representing the system moves along the x-axis with velocity $v(x)$. Thus the changes of the system are represented by the motion of the phase point along the x-axis which is often called the *phase line*. The point that gives $v(x_0) = 0$ which is called the *fixed point*.

In the previous paragraph we introduce the idea of a fixed point defined as a value of $x = x_\mathrm{f}$ for which $v(x_\mathrm{f}) = 0$. At a fixed point, $\mathrm{d}x/\mathrm{d}t = 0$ and a system initially having $v(x) = 0$ remains there for all time. At a fixed point, a system does not change, so that when modeling real systems, fixed point are an important part of the description. For example, consider a fluid obeying Newton's law of cooling;

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - 20)\,, \tag{3.2}$$

where $T = T(t)$ is a time dependence function of temperature and k is a cooling coefficient. The fixed point is $T = 20^\circ$ C. If the fluid is at this temperature, it neither warms up nor cools down. Therefore it is in equilibrium. This fixed point is stable. This description is typical features of a system that we are interested in. These are

1.Where are the fixed points?

2.What happens if the system is displaced from equilibrium?

3.What is the long term behavior of the system?

The phase portrait of a system is an important part of these questions. To present the velocity field of an autonomous system, at a selected number of points on the x-axis, we draw arrows with the following properties:
1. the length of the arrow is proportional to $|v(x)|$;
2. the direction of the arrow is to the right (x increasing) if $v(x) > 0$ and to the left (x decreasing) if $v(x) < 0$.

The fixed points are clearly important in describing the motion of the system. If the initial state is a fixed point then the system does not change. However, if the system is displaced slightly from a fixed point, its motion depends on the velocity field near the fixed point. There are essentially four types of fixed points shown in Figure 3.1 (a) If the system is given a small displacement from the fixed point $x = x_\mathrm{f}$ and it tends to return to the fixed point because the direction field is towards $x = x_\mathrm{f}$ on each side. The fixed point is said to be *stable*.

(b) If the system is given a small displacement from the fixed point $x = x_\mathrm{f}$ and it tends to move away from the fixed point. The fixed point is said to be *unstable*.

For types (c) and (d), the system as neither stable nor unstable.

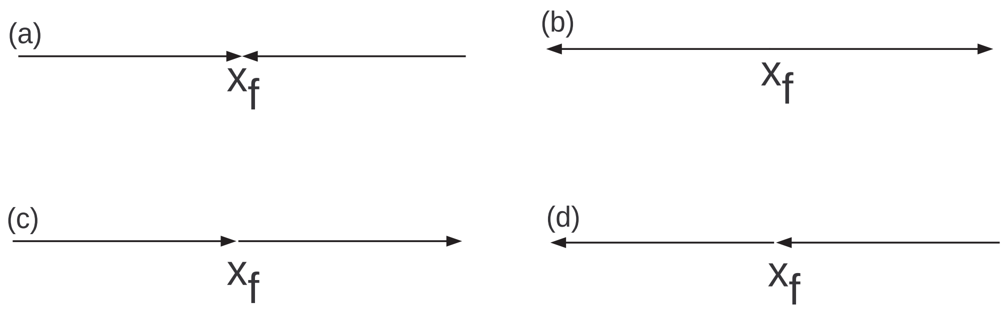

Figure 3.1: The four possible phase portraits associated with a fixed point at $x = x_{\rm f}$

3.2 Second order autonomous system

In the previous section we investigated the motion of first order systems which are modeled by the first order differential equation. We shall see that second order autonomous systems show more interesting behavior.

A dynamical system that plays important role in cosmology is autonomous system [1, 2, 16]. Coupled differential equation is defined by two time-dependent variables x and y:

$$\frac{{\rm d}x}{{\rm d}t} = f(x, y, t)\,, \qquad \frac{{\rm d}y}{{\rm d}t} = g(x, y, t)\,. \tag{3.3}$$

The system (3.3) is autonomous if f and g do not contain explicit time-dependent terms. A point $(x_{\rm c}\,,\, y_{\rm c})$ is fixed point of the autonomous system if

$$(f, g)|_{(x_{\rm c},\, y_{\rm c})} = 0\,. \tag{3.4}$$

A fixed point $(x_{\rm c}, y_{\rm c})$ is an stable node when it satisfies condition

$$(x(t), y(t)) \to (x_{\rm c}, y_{\rm c}) \;\; \text{for} \;\; t \to \infty\,. \tag{3.5}$$

We can find whether autonomous system approaches one of the fixed points or not by studying stability around the fixed points. Consider small perturbations (displacement) δx and δy about the fixed point $(x_{\rm c}, y_{\rm c})$;

$$x = x_{\rm c} + \delta x, \quad y = y_{\rm c} + \delta y\,. \tag{3.6}$$

Then inserting equation (3.6) into equation (3.3), we get first-order differential equations:

$$\frac{{\rm d}}{{\rm d}N} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}\,. \tag{3.7}$$

Where N is e-folding number which tells us size of expansion in logarithmic scale. e-folding number is very useful when we consider inflation of the universe. The perturbation matrix $\mathcal{M}$ depends on x_c and y_c , and is given by

$$\mathcal{M} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x=x_c, y=y_c)} \tag{3.8}$$

where $f \equiv \mathrm{d}x/\mathrm{d}N$ and $g \equiv \mathrm{d}y/\mathrm{d}N$.
There are two eigenvalues μ_1 and μ_2 for this system. General solution for the evolution of linear perturbations is written as

$$\delta x = C_1 \mathrm{e}^{\mu_1 N} + C_2 \mathrm{e}^{\mu_2 N}, \tag{3.9}$$

$$\delta y = C_3 \mathrm{e}^{\mu_1 N} + C_4 \mathrm{e}^{\mu_2 N}, \tag{3.10}$$

where C_1, C_2, C_3, C_4 are integration constants. Thus the stability of the fixed points depends on nature of the eigenvalues. They are generally classified as [1]:

- (i) Stable node for $\mu_1 < 0$ and $\mu_2 < 0$.
- (ii) Unstable node for $\mu_1 > 0$ and $\mu_2 > 0$.
- (iii) Saddle point for $\mu_1 < 0$ and $\mu_2 > 0$ (or $\mu_1 > 0$ and $\mu_2 < 0$).
- (iv) Stable spiral for $\det\mathcal{M} < 0$ and the real parts of μ_1 and μ_2 are negative.

A fixed point is an stable node in the cases (i) and (iv), but it is not so in the cases (ii) and (iii).

Chapter 4

Autonomous phase-plane for cosmological equations

We will consider a scalar field with an exponential potential energy density $V = V_0 \exp(-\lambda\kappa\phi)$ evolving in a spatially-flat Friedmann-Robertson-Walker (FRW) universe containing a fluid with a constant γ, $0 \leq \gamma \leq 2$, such as radiation ($\gamma = 4/3$) or dust ($\gamma = 1$). The evolution equations for a spatially-flat FRW model Hubble parameter H are

$$H^2 = \frac{\kappa^2}{3}(\rho_\gamma + \rho_\phi) \tag{4.1}$$

where $\kappa^2 = 8\pi G$, ρ_γ is density of dust and radiation, and ρ_ϕ is density of scalar field. Differentiate equation (4.1) with respect to time, we obtain

$$2H\dot{H} = \frac{\kappa^2}{3}(\dot{\rho_\gamma} + \dot{\rho_\phi}) \,.$$

The conservation equations of each component are

$$\begin{aligned} \dot{\rho_\gamma} &= -3H(\rho_\gamma + p_\gamma), & (4.2) \\ \dot{\rho_\phi} &= -3H(\rho_\phi + p_\phi) \,. & (4.3) \end{aligned}$$

Therefore

$$\begin{aligned} 2H\dot{H} &= \frac{\kappa^2}{3}[-3H(\rho_\gamma + p_\gamma) - 3H(\rho_\phi + p_\phi)] \\ \dot{H} &= -\frac{\kappa^2}{2}(\rho_\gamma + p_\gamma + \rho_\phi + p_\phi) \,. & (4.4) \end{aligned}$$

Substituting density (2.14) and pressure density (2.15) into equation (4.4),

$$\begin{aligned} \dot{H} &= -\frac{\kappa^2}{2}(\rho_\gamma + p_\gamma + \dot{\phi}^2) \qquad (4.5) \\ &= -\frac{\kappa^2}{2}[\rho_\gamma(1+w_\gamma) + \dot{\phi}^2] \\ &= -\frac{\kappa^2}{2}[\rho_\gamma(1+\gamma-1) + \dot{\phi}^2] \\ &= -\frac{\kappa^2}{2}[\gamma\rho_\gamma + \dot{\phi}^2] \, . \qquad (4.6) \end{aligned}$$

Consider equations (2.14) and (2.15), fluid equation of scalar field equation (4.3)

$$\dot{\rho_\phi} = -3H\dot{\phi}^2 \, . \qquad (4.7)$$

Substituting equation (2.14) in left hand side of equation (4.7)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) &= -3H\dot{\phi}^2 \\ \dot{\phi}\ddot{\phi} + \dot{V}(\phi) &= -3H\dot{\phi}^2 \\ \dot{\phi}\ddot{\phi} + \left(\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right) &= -3H\dot{\phi}^2 \\ \ddot{\phi} &= -3H\dot{\phi} - \frac{\mathrm{d}V}{\mathrm{d}\phi} \, . \qquad (4.8) \end{aligned}$$

Therefore autonomous system is equations (4.2), (4.5) and (4.8). Subjected to the Friedmann constraint

$$H^2 = \frac{\kappa^2}{3}\left(\rho_\gamma + \frac{1}{2}\dot{\phi}^2 + V\right) \, . \qquad (4.9)$$

The total energy density of a homogeneous scalar field is $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$.

We define

$$x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H}; \quad y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H} \qquad (4.10)$$

We can find Friedmann constraint in new variables by using equation (4.9)

$$\begin{aligned} H^2 &= \frac{\kappa^2}{3}\rho_\gamma + \frac{\kappa^2}{6}\dot{\phi}^2 + \frac{\kappa^2}{3}V \\ 1 &= \frac{\kappa^2\rho_\gamma}{3H^2} + \frac{\kappa^2\dot{\phi}^2}{6H^2} + \frac{\kappa^2 V}{3H^2} \\ 1 &= \frac{\kappa^2\rho_\gamma}{3H^2} + x^2 + y^2 \, . \qquad (4.11) \end{aligned}$$

The evolution equation can then be written as a plane-autonomous system in x' (defined by $\mathrm{d}x/\mathrm{d}N$) and y' (defined by $\mathrm{d}y/\mathrm{d}N$). We will define derivative with respect to N as

$$\frac{\mathrm{d}}{\mathrm{d}N} = \frac{\mathrm{d}}{\mathrm{d}[\ln(a/a_0)]} \,. \tag{4.12}$$

Consider derivative with respect to time of $\ln(a/a_0)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\left(\frac{a}{a_0}\right)\right] &= \frac{a_0}{a}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{a_0}\right) \\
&= \frac{a_0}{a}\left(\frac{a_0\dot{a}}{a_0^2}\right) \\
&= \frac{\dot{a}}{a} \\
\mathrm{d}\left[\ln\left(\frac{a}{a_0}\right)\right] &= H\mathrm{d}t \\
\frac{\mathrm{d}}{\mathrm{d}N} &= H^{-1}\frac{\mathrm{d}}{\mathrm{d}t} \qquad (4.13)
\end{aligned}$$

From $x \equiv \kappa\dot{\phi}/\sqrt{6}H$

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}N} &= \frac{\mathrm{d}}{\mathrm{d}N}\left(\frac{\kappa\dot{\phi}}{\sqrt{6}H}\right) \\
&= \frac{\kappa}{\sqrt{6}}\left[\left(\frac{\mathrm{d}\dot{\phi}}{\mathrm{d}N}\right)\frac{1}{H} + \dot{\phi}\frac{\mathrm{d}}{\mathrm{d}N}(H^{-1})\right] \qquad (4.14)
\end{aligned}$$

substituting equation (4.13) into equation (4.14)

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}N} &= \frac{\kappa}{\sqrt{6}}\left(H^{-2}\frac{\mathrm{d}}{\mathrm{d}t}\dot{\phi} + \dot{\phi}H^{-1}\frac{\mathrm{d}}{\mathrm{d}t}H^{-1}\right) \\
&= \frac{\kappa}{\sqrt{6}}\left(\frac{\ddot{\phi}}{H^2} - \frac{\dot{\phi}\dot{H}}{H^3}\right) \\
&= \frac{\kappa}{\sqrt{6}}\frac{\ddot{\phi}}{H^2} - \left(\frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}}{H}\right)\frac{\dot{H}}{H^2} \\
&= \frac{\kappa}{\sqrt{6}}\frac{\ddot{\phi}}{H^2} - x\frac{\dot{H}}{H^2}
\end{aligned}$$

where

$$\begin{aligned}\dot{H} &= -\frac{\kappa^2}{2}(\rho_\gamma + p_\gamma) - \frac{\kappa^2}{2}\dot{\phi}^2 \\ &= -\frac{\kappa^2}{2}(\rho_\gamma + p_\gamma) - \left(\frac{\kappa^2\dot{\phi}^2}{6H^2}\right)\left(\frac{6H^2}{2}\right) \\ &= -\frac{\kappa^2}{2}(\rho_\gamma + p_\gamma) - 3x^2H^2\,,\end{aligned}$$

then

$$\begin{aligned}x' &= \frac{\kappa\ddot{\phi}}{\sqrt{6}H^2} - \frac{x}{H^2}\left[3x^2H^2 - \frac{\kappa^2}{2}(\rho_\gamma + p_\gamma)\right] \\ &= \frac{\kappa}{\sqrt{6}H^2}\left(-3H\dot{\phi} - \frac{\mathrm{d}V}{\mathrm{d}\phi}\right) - 3x^3 + \frac{x\kappa^2}{2H^2}(\rho_\gamma + p_\gamma) \\ &= -3x - \frac{\kappa}{\sqrt{6}H^2}\frac{\mathrm{d}V_0e^{-\lambda\kappa\phi}}{\mathrm{d}\phi} - 3x^3 + \frac{x\kappa^2}{2H^2}(\rho_\gamma + p_\gamma) \\ &= -3x + \frac{3\kappa^2V\lambda}{(\sqrt{3})^2\sqrt{6}H^2} - 3x^3 + \frac{x\kappa^2\rho_\gamma\gamma}{2H^2} \\ &= -3x + \frac{3y^2\lambda}{\sqrt{6}} - 3x^3 + \frac{x\kappa^2\rho_\gamma\gamma}{2H^2}\,.\end{aligned}$$

Substituting new variables into Friedmann constraint (equation (4.11)), then

$$\begin{aligned}x' &= -3x + \lambda\sqrt{\frac{3}{2}}y^2 - 3x^3 + \left(\frac{\kappa^2\rho_\gamma}{3H^2}\right)\frac{3x\gamma}{2} \\ &= -3x + \lambda\sqrt{\frac{3}{2}}y^2 - 3x^3 + \frac{3x\gamma}{2}(1 - x^2 - y^2) \\ &= -3x + \lambda\sqrt{\frac{3}{2}}y^2 + \frac{3}{2}x\left[2x^2 + \gamma(1 - x^2 - y^2)\right]\,. \qquad (4.15)\end{aligned}$$

From $y \equiv \kappa\sqrt{V}/\sqrt{3}H$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}N} &= \frac{\mathrm{d}}{\mathrm{d}N}\left(\frac{\kappa\sqrt{V}}{\sqrt{3}H}\right) \\ &= \frac{\kappa}{\sqrt{3}H}\frac{\mathrm{d}\sqrt{V}}{\mathrm{d}N} + \frac{\kappa\sqrt{V}}{\sqrt{3}}\frac{\mathrm{d}(H^{-1})}{\mathrm{d}N}\,, \qquad (4.16)\end{aligned}$$

substituting equation (4.13) into equation (4.16)

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}N} &= \frac{\kappa H^{-1}}{\sqrt{3}H}\frac{\mathrm{d}\sqrt{V}}{\mathrm{d}t} + \frac{\kappa\sqrt{V}H^{-1}}{\sqrt{3}}\frac{\mathrm{d}H^{-1}}{\mathrm{d}t} \\
&= \frac{\kappa\dot{V}}{2\sqrt{3}H^2\sqrt{V}} - \kappa\frac{\sqrt{V}\dot{H}}{\sqrt{3}H^3} \\
&= \frac{\kappa\dot{V}}{2\sqrt{3}H^2\sqrt{V}} - y\frac{\dot{H}}{H^2} \\
&= \frac{\kappa}{2\sqrt{3}H^2\sqrt{V}}\frac{\mathrm{d}V}{\mathrm{d}t} - \frac{y}{H^2}\left[-3x^2H^2 - \frac{\kappa}{2}(\rho_\gamma + p_\gamma)\right] \\
&= \frac{\kappa}{2\sqrt{3}H^2\sqrt{V}}\frac{\mathrm{d}}{\mathrm{d}t}(V_0\mathrm{e}^{-\lambda\kappa\phi}) + 3x^2y + \frac{\kappa^2 y}{2H^2}(\rho_\gamma + p_\gamma) \\
&= -\frac{\kappa V_0\lambda\kappa\dot{\phi}\mathrm{e}^{-\lambda\kappa\phi}}{2\sqrt{3}H^2\sqrt{V}} + 3x^2y + \frac{\kappa^2 y}{2H^2}(\rho_\gamma + p_\gamma) \\
&= -\frac{\kappa^2\lambda\dot{\phi}\sqrt{V}}{2\sqrt{3}H^2} + 3x^2y + \frac{\kappa^2 y}{2H^2}\left[\rho_\gamma + (\gamma - 1)\rho_\gamma\right] \\
&= -\frac{\lambda\kappa\sqrt{V}\sqrt{6}x}{2\sqrt{3}H} + 3x^2y + \frac{3\gamma y}{2}(1 - x^2 - y^2)
\end{aligned}
$$

then

$$
\begin{aligned}
y' &= -\lambda xy\sqrt{\frac{3}{2}} + \frac{3}{2}y\left[2x^2 + \gamma(1 - x^2 - y^2)\right] \\
y' &= -\lambda\sqrt{\frac{3}{2}}xy + \frac{3}{2}y\left[2x^2 + \gamma(1 - x^2 - y^2)\right] \,. \qquad (4.17)
\end{aligned}
$$

Both equations (4.15) and (4.17) are plane-autonomous system. Note that from Friedmann constraint (equation (4.11)), we have

$$
\Omega_\phi \equiv \frac{\kappa^2\rho_\phi}{3H^2} = x^2 + y^2 \,. \qquad (4.18)
$$

This is bounded, $0 \leq x^2 + y^2 \leq 1$, for a non-negative fluid density, $\rho_\gamma \geq 0$.

The effective equation of state for the scalar field at any point is given by

$$
\gamma_\phi \equiv \frac{\rho_\phi + p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2}{V + \dot{\phi}^2/2} = \frac{2x^2}{x^2 + y^2} \,. \qquad (4.19)
$$

At fixed points $x' = 0$ and $y' = 0$,

$$
\begin{aligned}
-3x + \lambda\sqrt{\frac{3}{2}}y^2 + \frac{3}{2}x\left[2x^2 + \gamma(1 - x^2 - y^2)\right] &= 0 \qquad (4.20) \\
-\lambda\sqrt{\frac{3}{2}}xy + \frac{3}{2}y\left[2x^2 + \gamma(1 - x^2 - y^2)\right] &= 0 \,. \qquad (4.21)
\end{aligned}
$$

Case I: $y = 0$
Consider equation (4.20) given $y = 0$

$$\begin{aligned} -3x + \frac{3}{2}x\left[2x^2 + \gamma(1 - x^2)\right] &= 0 \\ 3x^3 - \frac{3}{2}x^3\gamma + \frac{3}{2}\gamma x - 3x &= 0 \\ \left(3 - \frac{3}{2}\gamma\right)x^3 + \left(\frac{3}{2}\gamma - 3\right)x &= 0 \qquad (4.22) \\ x &= 0, \pm 1\,. \qquad (4.23) \end{aligned}$$

Also, giving $y = 0$ in the equation (4.21)

$$\begin{aligned} -\lambda\sqrt{\frac{3}{2}}x(0) + \frac{3}{2}(0)\left[2x^2 + \gamma(1 - x^2 - 0)\right] &= 0 \\ 0 &= 0 \end{aligned}$$

Therefore the points (0, 0), (±1, 0) are fixed points.

Case II: $y \neq 0$
From equation (4.21),

$$y^2 = -\frac{2}{3}\frac{\lambda}{\gamma}\sqrt{\frac{3}{2}}x + \frac{2}{\gamma}x^2 + 1 - x^2, \qquad (4.24)$$

substituting equation (4.24) into equation (4.20) then

$$\begin{aligned} -3x + \lambda\sqrt{\frac{3}{2}}\left(-\frac{2}{3}\frac{\lambda}{\gamma}\sqrt{\frac{3}{2}}x + \frac{2}{\gamma}x^2 + 1 - x^2\right) & \\ +\frac{3}{2}x\left[2x^2 + \gamma\left(1 - x^2 + \frac{2}{3}\frac{\lambda}{\gamma}\sqrt{\frac{3}{2}}x - \frac{2}{\gamma}x^2 - 1 + x^2\right)\right] &= 0 \\ \frac{\lambda\sqrt{6}}{\gamma}x^2 - \left(3 + \frac{\lambda^2}{\gamma}\right)x + \lambda\sqrt{\frac{3}{2}} &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{(3 + \frac{\lambda^2}{\gamma}) \pm \sqrt{(3 + \frac{\lambda^2}{\gamma})^2 - 4\lambda\frac{\sqrt{6}}{\gamma}(\sqrt{\frac{3}{2}}\lambda)}}{2\sqrt{6}\frac{\lambda}{\gamma}} \\ x &= \frac{\lambda}{\sqrt{6}}, \sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}\,. \end{aligned}$$

Substituting $x = \lambda/\sqrt{6}$ into equation (4.21)

$$\begin{aligned}
-\lambda\sqrt{\frac{3}{2}}\frac{\lambda}{\sqrt{6}}y + \frac{3}{2}y\left[2\left(\frac{\lambda^2}{6}\right) + \gamma\left(1 - \frac{\lambda^2}{6} - y^2\right)\right] &= 0 \\
\frac{3}{2}y\gamma\left(1 - \frac{\lambda^2}{6} - y^2\right) &= 0 \\
\frac{3}{2} - \frac{\lambda^2}{4} - \frac{3}{2}y^2 &= 0 \\
y &= \left(1 - \frac{\lambda^2}{6}\right)^{\frac{1}{2}} . \qquad (4.25)
\end{aligned}$$

Then the point $(\lambda/\sqrt{6},\ \sqrt{1-\lambda^2/6})$ is a fixed point.

Using $x = \sqrt{3}\gamma/\sqrt{2}\lambda$ to equation (4.21)

$$\begin{aligned}
-\frac{3}{2}\gamma y + \frac{3}{2}y\left[2\left(\frac{3}{2}\right)\frac{\gamma^2}{\lambda^2} + \gamma\left(1 - \frac{3}{2}\frac{\gamma^2}{\lambda^2} - y^2\right)\right] &= 0 \\
\frac{3\gamma}{\lambda^2} - \frac{3\gamma^2}{2\lambda^2} - y^2 &= 0 \\
y &= \left[\frac{3(2-\gamma)\gamma}{2\lambda^2}\right]^{\frac{1}{2}} . \qquad (4.26)
\end{aligned}$$

Then the point $(\sqrt{3}\gamma/\sqrt{2}\lambda,\ \sqrt{3(2-\gamma)\gamma/2\lambda^2})$ is a fixed point.

Conclude all fixed points of the autonomous system as listed below:

- Point (1) : $(\ 0,\ \ 0)$ (4.27)
- Point (2) : $(\ 1,\ \ 0)$ (4.28)
- Point (3) : $(-1,\ \ 0)$ (4.29)
- Point (4) : $\left(\ \frac{\lambda}{\sqrt{6}},\ \ \left(1 - \frac{\lambda^2}{6}\right)^{\frac{1}{2}}\right)$ (4.30)
- Point (5) : $\left(\ \sqrt{\frac{3}{2}}\frac{\gamma}{\lambda},\ \ \left(\frac{3(2-\gamma)\gamma}{2\lambda^2}\right)^{\frac{1}{2}}\right)$. (4.31)

Next we investigate the stabilities of these fixed points. Let us consider small perturbations $\delta x, \delta y$ around (x_c, y_c) in (3.6). The left-hand side of the equation (3.7) can be explained to

$$
\begin{aligned}
x' &= -3x + \lambda\sqrt{\frac{3}{2}}y^2 + \frac{3}{2}x\left[2x^2 + \gamma(1 - x^2 - y^2)\right] \\
&= -3x + \lambda\sqrt{\frac{3}{2}}y^2 + 3x^3 + \frac{3}{2}x\gamma - \frac{3}{2}x^3\gamma - \frac{3}{2}x\gamma y^2 \\
x' + \delta x' &= -3(x + \delta x) + \lambda\sqrt{\frac{3}{2}}(y + \delta y)^2 + 3(x + \delta x)^3 + \frac{3}{2}\gamma(x + \delta x) - \frac{3}{2}\gamma(x + \delta x)^3 \\
&\quad -\frac{3}{2}\gamma(x + \delta x)(y + \delta y)^2 \\
&= -3x - 3\delta x + \lambda\sqrt{\frac{3}{2}}(y^2 + 2y\delta y + \delta^2 y) + 3(x^3 + 3x^2\delta x + 3x\delta^2 x + \delta^3 x) + \frac{3}{2}\gamma x \\
&\quad +\frac{3}{2}\gamma\delta x - \frac{3}{2}\gamma(x^3 + 3x^2\delta x + 3x\delta^2 x + \delta^3 x) - \frac{3}{2}\gamma(x + \delta x)(y^2 + 2y\delta y + \delta^2 y) \\
&= -3x - 3\delta x + \lambda\sqrt{\frac{3}{2}}y^2 + \sqrt{6}\lambda y\delta y + \lambda\sqrt{\frac{3}{2}}\delta^2 y + 3x^3 + 9x^2\delta x + 9x\delta^2 x \\
&\quad +3\delta^3 x + \frac{3}{2}\gamma x + \frac{3}{2}\gamma\delta x - \frac{3}{2}\gamma x^3 - \frac{9}{2}\gamma x^2\delta x - \frac{9}{2}\gamma x\delta^2 x - \frac{3}{2}\gamma\delta^3 x - \frac{3}{2}\gamma x y^2 \\
&\quad -3\gamma x y\delta y - \frac{3}{2}\gamma x\delta^2 y - \frac{3}{2}\gamma\delta x y^2 - 3\gamma y\delta x\delta y - \frac{3}{2}\gamma\delta x\delta^2 y
\end{aligned}
$$

Neglecting higher order term of perturbation,

$$
\begin{aligned}
x' + \delta x' &= (-3x + \lambda\sqrt{\frac{3}{2}}y^2 + 3x^3 + \frac{3}{2}\gamma x - \frac{3}{2}\gamma x^3 - \frac{3}{2}\gamma x y^2) \\
&\quad +(-3\delta x + \sqrt{6}\lambda y\delta y + 9x^2\delta x + \frac{3}{2}\gamma\delta x - \frac{9}{2}\gamma x^2\delta x - 3\gamma x y\delta y - \frac{3}{2}\gamma\delta x y^2)\,.
\end{aligned}
$$

From condition $x' = 0$ and equation (4.20) then

$$
\delta x' = -3\delta x + \sqrt{6}\lambda y\delta y + 9x^2\delta x + \frac{3}{2}\gamma\delta x - \frac{9}{2}\gamma x^2\delta x - 3\gamma x y\delta y - \frac{3}{2}\gamma y^2\delta x\,. \qquad (4.32)
$$

Next we show in detail for matrix $\mathcal{M}$ following equation (3.8)

$$f(x,y) = -3x + \lambda\sqrt{\frac{3}{2}}y^2 + 3x^3 + \frac{3}{2}x\gamma - \frac{3}{2}\gamma x^3 - \frac{3}{2}xy^2\gamma \tag{4.33}$$

$$g(x,y) = -\lambda\sqrt{\frac{3}{2}}xy + 3x^2y + \frac{3}{2}\gamma y - \frac{3}{2}\gamma x^2 y - \frac{3}{2}\gamma y^3 \tag{4.34}$$

$$\frac{\partial f}{\partial x} = -3 + 9x^2 + \frac{3}{2}\gamma - \frac{9}{2}\gamma x^2 - \frac{3}{2}\gamma y^2 \tag{4.35}$$

$$\frac{\partial f}{\partial y} = 2\lambda\sqrt{\frac{3}{2}}y - 3\gamma xy \tag{4.36}$$

$$\frac{\partial g}{\partial x} = -\lambda\sqrt{\frac{3}{2}}y + 6xy - 3\gamma xy \tag{4.37}$$

$$\frac{\partial g}{\partial y} = -\sqrt{\frac{3}{2}}\lambda x + 3x^2 + \frac{3}{2}\gamma - \frac{3}{2}\gamma x^2 - \frac{9}{2}\gamma y^2\ . \tag{4.38}$$

Thus our autonomous system of equations the perturbation matrix $\mathcal{M}$ is

$$\mathcal{M} = \begin{pmatrix} -3 + 9x^2 + \frac{3}{2}\gamma - \frac{9}{2}\gamma x^2 - \frac{3}{2}\gamma y^2 & 2\sqrt{\frac{3}{2}}\lambda y - 3\gamma xy \\ -\lambda\sqrt{\frac{3}{2}}y + 6xy - 3\gamma xy & -\lambda\sqrt{\frac{3}{2}}x + 3x^2 + \frac{3}{2}\gamma - \frac{3}{2}\gamma x^2 - \frac{9}{2}\gamma y^2 \end{pmatrix}_{(x=x_c, y=y_c)} \tag{4.39}$$

$$\mathcal{M}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \left(-3 + 9x^2 + \frac{3}{2}\gamma - \frac{9}{2}\gamma x^2 - \frac{3}{2}\gamma y^2\right)\delta x + \left(2\sqrt{\frac{3}{2}}\lambda y - 3\gamma xy\right)\delta y \\ \left(-\lambda\sqrt{\frac{3}{2}}y + 6xy - 3\gamma xy\right)\delta x + \left(-\lambda\sqrt{\frac{3}{2}}x + 3x^2 + \frac{3}{2}\gamma - \frac{3}{2}\gamma x^2 - \frac{9}{2}\gamma y^2\right)\delta y \end{pmatrix}$$

$$= \begin{pmatrix} -3\delta x + 9x^2\delta x + \frac{3}{2}\gamma\delta x - \frac{9}{2}\gamma x^2\delta x - \frac{3}{2}\gamma y^2\delta x + 2\sqrt{\frac{3}{2}}\lambda y\delta y - 3\gamma xy\delta y \\ -\lambda\sqrt{\frac{3}{2}}y\delta x + 6xy\delta x - 3\gamma xy\delta x + -\lambda\sqrt{\frac{3}{2}}x\delta y + 3x^2\delta y + \frac{3}{2}\gamma\delta y - \frac{3}{2}\gamma x^2\delta y - \frac{9}{2}\gamma y^2\delta y \end{pmatrix} \tag{4.40}$$

We found equation (4.32) and component 11 of (4.40) are alike. Verifying the equation (3.7) and proofs in case of component 12 can be found similarly. To do stability analysis we need to find eigenvalues of the matrix $\mathcal{M}$ from

$$\det(\mu I - \mathcal{M}) = 0 \tag{4.41}$$

where I is identity matrix. This yields

$$\left(\mu + 3 - 9x^2 - \frac{3}{2}\gamma + \frac{9}{2}\gamma x^2 + \frac{3}{2}y^2\gamma\right)\left(\mu + \lambda\sqrt{\frac{3}{2}}x - 3x^2 - \frac{3}{2}\gamma + \frac{3}{2}x^2\gamma + \frac{9}{2}y^2\gamma\right)$$
$$- \left(2\lambda\sqrt{\frac{3}{2}}y - 3\gamma xy\right)\left(-\lambda\sqrt{\frac{3}{2}}y + 6xy - 3\gamma xy\right) = 0$$

$$
\begin{aligned}
\mu^2 + \sqrt{\frac{3}{2}}\mu\lambda x - 3\mu x^2 - \frac{3}{2}\gamma\mu + \frac{3}{2}\gamma\mu x^2 + \frac{9}{2}\gamma\mu y^2 + 3\mu + 3\sqrt{\frac{3}{2}}\lambda x \\
-9x^2 - \frac{9}{2}\gamma + \frac{9}{2}\gamma x^2 + \frac{27}{2}\gamma y^2 - 9\mu x^2 - 9\sqrt{\frac{3}{2}}\lambda x^3 + 27x^4 + \frac{27}{2}\gamma x^2 \\
-\frac{27}{2}\gamma x^4 - \frac{81}{2}\gamma x^2 y^2 - \frac{3}{2}\gamma\mu - \frac{3}{2}\sqrt{\frac{3}{2}}\gamma\lambda x + \frac{9}{2}\gamma x^2 + \frac{9}{4}\gamma^2 - \frac{9}{4}\gamma^2 x^2 \\
-\frac{27}{4}\gamma^2 y^2 + \frac{9}{2}\gamma\mu x^2 + \frac{9}{2}\sqrt{\frac{3}{2}}\gamma\lambda x^3 - \frac{27}{2}\gamma x^4 - \frac{27}{4}\gamma^2 x^2 + \frac{27}{4}\gamma^2 x^4 \\
+\frac{81}{4}\gamma^2 x^2 y^2 + \frac{3}{2}\gamma\mu y^2 + \frac{3}{2}\sqrt{\frac{3}{2}}\gamma\lambda x y^2 - \frac{9}{2}\gamma x^2 y^2 - \frac{9}{4}\gamma^2 y^2 + \frac{9}{4}\gamma^2 x^2 y^2 \\
+\frac{27}{4}\gamma^2 y^4 + 3\lambda^2 y^2 - 12\sqrt{\frac{3}{2}}\lambda x y^2 + 6\sqrt{\frac{3}{2}}\gamma\lambda x y^2 - 3\sqrt{\frac{3}{2}}\gamma\lambda x y^2 \\
+18\gamma x^2 y^2 - 9\gamma^2 x^2 y^2 &= 0\,. \quad (4.42)
\end{aligned}
$$

$$
\begin{aligned}
\mu^2 + 27x^4 - \frac{9}{2}\gamma - 27x^4\gamma - 9\gamma^2 x^2 - 9\gamma^2 y^2 + \frac{27}{4}\gamma^2 x^4 \\
+\frac{27}{4}\gamma^2 y^4 + 3\lambda^2 y^2 - 27\gamma x^2 y^2 + 6\gamma\mu x^2 + 6\gamma\mu y^2 - 3\gamma\mu \\
-12\mu x^2 + \frac{45}{2}\gamma x^2 + \frac{27}{2}\gamma y^2 + \frac{27}{2}\gamma^2 x^2 y^2 + 3\mu - \frac{3}{4}\sqrt{6}\gamma\lambda x \\
-6\sqrt{6}\lambda x y^2 + \frac{9}{4}\sqrt{6}\gamma\lambda x^3 + \frac{1}{2}\sqrt{6}\mu\lambda x - 9x^2 + \frac{9}{4}\gamma^2 + \frac{3}{2}\sqrt{6}\lambda x \\
-\frac{9}{2}\sqrt{6}\lambda x^3 + \frac{9}{4}\sqrt{6}\lambda\gamma x y^2 &= 0\,. \qquad (4.43)
\end{aligned}
$$

We can find eigenvalues of the perturbation matrix $\mathcal{M}$ at each points:
the first example, point (2): using $x = 1, y = 0$ in equation (4.43)

$$
\mu^2 + 3\gamma\mu - 9\mu + \frac{\sqrt{6}}{2}\mu\lambda - 9\gamma + \frac{3}{2}\sqrt{6}\gamma\lambda - 3\sqrt{6}\lambda + 18 = 0
$$

$$
\mu_1 = \sqrt{\frac{3}{2}}(\sqrt{6} - \lambda)\ , \quad \mu_2 = 3(2 - \gamma)\,. \qquad (4.44)
$$

The second example, point (4): using $x = \lambda/\sqrt{6}$, $y = (1 - \lambda^2/6)^{1/2}$ in equation (4.43)

$$
\mu^2 + 3\mu + 3\gamma\mu - \frac{3}{2}\lambda^2\mu + \frac{1}{2}\lambda^4 + 9\gamma - 3\lambda^2 - \frac{3}{2}\gamma\lambda^2 = 0
$$

$$
\mu_1 = \frac{\lambda^2 - 6}{2}\ , \quad \mu_2 = \lambda^2 - 3\gamma\,. \qquad (4.45)
$$

Eigenvalues of the other points can be found similarly.

- At point (1):

$$\mu_1 = -\frac{3(2-\gamma)}{2} \ , \quad \mu_2 = \frac{3\gamma}{2} \, . \tag{4.46}$$

- At point (2):

$$\mu_1 = \sqrt{\frac{3}{2}}(\sqrt{6}-\lambda) \ , \quad \mu_2 = 3(2-\gamma) \, . \tag{4.47}$$

- At point (3):

$$\mu_1 = \sqrt{\frac{3}{2}}(\sqrt{6}+\lambda) \ , \quad \mu_2 = 3(2-\gamma) \, . \tag{4.48}$$

- At point (4):

$$\mu_1 = \frac{\lambda^2-6}{2} \ , \quad \mu_2 = \lambda^2 - 3\gamma \, . \tag{4.49}$$

- At point (5):

$$\begin{aligned} \mu_1 &= -\frac{3(2-\gamma)}{4}\left(1+\sqrt{1-\frac{8\gamma(\lambda^2-3\gamma)}{\lambda^2(2-\gamma)}}\right) \\ \mu_2 &= -\frac{3(2-\gamma)}{4}\left(1-\sqrt{1-\frac{8\gamma(\lambda^2-3\gamma)}{\lambda^2(2-\gamma)}}\right) \, . \end{aligned} \tag{4.50}$$

Full analysis all of Ω_ϕ and γ_ϕ for each fixed points can be seen in chapter 5 (*Qualitative analysis*). Where we analyze the properties of the five fixed points and its cosmological consequences given in table 4.1. Two of the fixed points ($x = \pm 1, y = 0$) correspond to solutions where the constraint equation (4.9) is dominated by the kinetic energy of the scalar field with $\gamma_\phi = 2$. As expected these solutions are *unstable* and are only expected to be relevant at early times.

We find that the barotropic fluid dominated solution ($x = 0, y = 0$) where $\Omega_\phi = 0$ is *unstable* for all values of $\gamma > 0$. But for any $\gamma > 0$, and however steep the potential, the energy density of the scalar field never vanishes with respect to the other matter in the universe.

Scalar field dominate solution ($\Omega = 1$) which exists for sufficiently flat potentials, $\lambda^2 < 6$. The scalar field has an effective barotropic index $\gamma_\phi = \lambda^2/3$ giving rise to a power-law inflationary expansion ($\ddot{a} > 0$) for $\lambda^2 < 2$, a full analysis is given in appendix A. We have shown that this scalar field dominated solution is a late-time attractor in the presence of a barotropic fluid when we have $\lambda^2 < 3\gamma$.

x	y	Existence	Stability	Ω_ϕ	γ_ϕ
0	0	All λ and γ	Saddle point for $0 < \gamma < 2$	0	undefined
1	0	All λ and γ	Unstable node for $\lambda < \sqrt{6}$ Saddle point for $\lambda > \sqrt{6}$	1	2
-1	0	All λ and γ	Unstable node for $\lambda > -\sqrt{6}$ Saddle point for $\lambda < -\sqrt{6}$	1	2
$\lambda/\sqrt{6}$	$[1 - \lambda^2/6]^{1/2}$	$\lambda^2 < 6$	Stable node for $\lambda^2 < 3\gamma$ Saddle point for $3\gamma < \lambda^2 < 6$	1	$\lambda^2/3$
$(3/2)^{1/2}\gamma/\lambda$	$[3(2-\gamma)\gamma/2\lambda^2]^{1/2}$	$\lambda^2 > 3\gamma$	Stable node for $3\gamma < \lambda^2 < 24\gamma^2/(9\gamma - 2)$ Stable spiral for $\lambda^2 > 24\gamma^2/(9\gamma - 2)$	$3\gamma/\lambda^2$	γ

Table 4.1: The properties of the critical points.

However for $\lambda^2 > 3\gamma$ we find a differential late-time attractor where neither the scalar-field nor the barotropic fluid entirely dominates the evolution. In stead we have a scaling solution where the energy density of the scalar field remains proportional to that of the barotropic fluid with $\Omega_\phi = 3\gamma/\lambda^2$.

Fixed points at finite values of x and y in the phase-plane agree with solutions where the scalar field has a barotropic equation of state and the scale factor of the universe evolves as $a \propto t^p$ where $p = (2/3)\gamma_\phi$. A full analysis is given in appendix B.

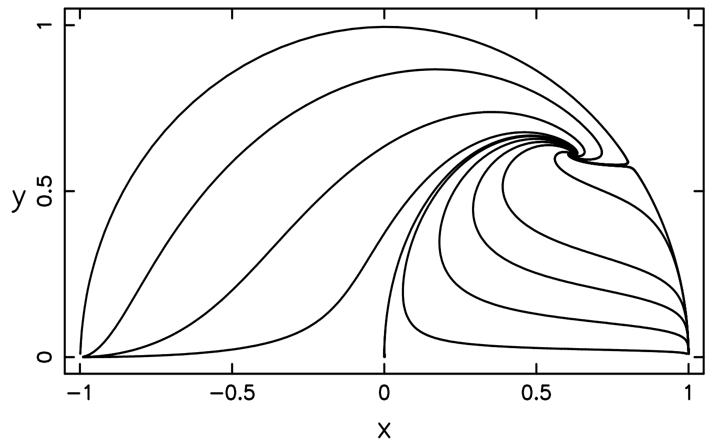

Figure 4.1: The phase plane for $\lambda = 2$, $\gamma = 1$.

In figure 4.1 we show that phase plot for $\lambda = 2$ and $\gamma = 1$. We note that orbit are confined inside circle $x^2 + y^2 = 1$ with $y \geq 0$. In this case the point (4) is saddle point, but the point (5) is a stable spiral. Hence the late-time attractor is the scaling solution following point (5) with $x = y = \sqrt{3/8}$. This behavior is clearly seen in figure 4.1.

Chapter 5

Qualitative analysis

For the fixed points listed in table 4.1 we find:

Fluid-dominated solution, $(x_{\rm c} = 0, y_{\rm c} = 0)$;

$$\mu_1 = -\frac{3(2-\gamma)}{2} \ , \quad \mu_2 = \frac{3\gamma}{2} \, .$$

This point is saddle point for $0 < \gamma < 2$, because $\mu_1 < 0$, and $\mu_2 > 0$. Substituting x and y into equation (4.18), and equation (4.19), we get

$$\Omega_\phi \ = \ 0 \qquad (5.1)$$

$$\gamma_\phi \ \text{is} \ \text{undefined} \, . \qquad (5.2)$$

Kinetic-dominated solution, $(x_c = 1, y_c = 0)$;

$$\mu_1 = \sqrt{\frac{3}{2}}(\sqrt{6} - \lambda), \quad \mu_2 = 3(2-\gamma) \, .$$

This point is unstable node for $\lambda < \sqrt{6}$, because $\mu_1 > 0$. Where γ is a constant, $0 \leq \gamma \leq 2$. Then $\mu_2 > 0$ as well. The point is saddle point for $\lambda > \sqrt{6}$, because $\mu_1 < 0$, and $\mu_2 > 0$. Substituting x and y into equation (4.18) and equation (4.19) we get

$$\Omega_\phi \ = \ 1 \qquad (5.3)$$

$$\gamma_\phi \ = \ 2 \, . \qquad (5.4)$$

Kinetic-dominated solution, $(x_c = -1, y_c = 0)$;

$$\mu_1 = \sqrt{\frac{3}{2}}(\sqrt{6} + \lambda), \quad \mu_2 = 3(2-\gamma) \, .$$

This point is unstable node for $\lambda > -\sqrt{6}$, because $\mu_1 > 0$. Where γ is a constant, $0 \le \gamma \le 2$. Then $\mu_2 > 0$ as well. The point is saddle point for $\lambda < -\sqrt{6}$, because $\mu_1 < 0$, and $\mu_2 > 0$. Substituting x and y into equation (4.18) and equation (4.19) we get

$$\Omega_\phi = 1 \tag{5.5}$$

$$\gamma_\phi = 2\,. \tag{5.6}$$

Scalar field dominated solution, ($x_c = \lambda/\sqrt{6},\ y_c = (1-\lambda^2/6)^{1/2}$)

$$\mu_1 = -\frac{\lambda^2 - 6}{2}, \quad \mu_2 = \lambda^2 - 3\gamma$$

This point is stable node for $\lambda < 3\gamma$, because $\mu_1 < 0$ and $\mu_2 < 0$. It is saddle point for $3\gamma < \lambda^2 < 6$, because $\mu_1 < 0$ and $\mu_2 > 0$. Substituting x and y into equation (4.18) and (4.19) we get

$$\Omega_\phi = 1 \tag{5.7}$$

$$\gamma_\phi = \frac{\lambda^2}{3}\,. \tag{5.8}$$

Scaling solution, ($x_c = \sqrt{3/2}(\gamma/\lambda),\ y_c = [(3\gamma(2-\gamma))/(2\lambda^2)]^{1/2}$);

$$\mu_1 = -\frac{3(2-\gamma)}{4}\left[1 + \sqrt{1 - \frac{8\gamma(\lambda^2 - 3\gamma)}{\lambda^2(2-\gamma)}}\right], \quad \mu_2 = -\frac{3(2-\gamma)}{4}\left[1 - \sqrt{1 - \frac{8\gamma(\lambda^2 - 3\gamma)}{\lambda^2(2-\gamma)}}\right].$$

The point is stable node for $3\gamma < \lambda^2 < (24\gamma^2)/(9\gamma - 2)$, because $\mu_1 < 0$ and $\mu_2 < 0$. The determinant of the perturbation matrix is negative and the real parts of μ_1 and μ_2 are negative. Substituting x and y into equation (4.18) and (4.19) we get

$$\Omega_\phi = \frac{3}{2}\frac{\gamma^2}{\lambda^2} + \frac{3(2-\gamma)\gamma}{2\lambda^2} = \frac{3\gamma}{\lambda^2} \tag{5.9}$$

$$\gamma_\phi = \frac{2x^2}{x^2 + y^2} = \gamma\,. \tag{5.10}$$

Because γ_ϕ at this point is γ of barotropic fluid, energy density of the scalar field remains proportional to that of the barotropic fluid with $\Omega_\phi = 3\gamma/\lambda^2$. The eigenvalue at this point is *scaling solution.* A full analysis is giving in appendix C.

Chapter 6

Conclusion and further study

The analysis of the fixed points shows that one can get an accelerated expansion provided that the solutions approach the fixed points (4) with $\lambda^2 < 2$, in which case the final state of the universe is the scalar field dominated one ($\Omega_\phi = 1$). The scaling solution is not viable to explain a late-time acceleration. However this can be used to provide the cosmological evolution in which the energy density of the scalar field decreases proportional to energy density of the background fluid in either a radiation or matter dominated era. If the slope of the exponential potential becomes shallow enough to satisfy $\lambda^2 < 2$ near to the present. The universe exits from the scaling regime and approaches the fixed point (4) giving rise to an accelerated expansion.

We have presented a phase-plane analysis of the evolution of a spatially-flat FRW universe containing a barotropic fluid plus a scalar field with an exponential potential $V(\phi) = V_0 \exp(-\lambda\kappa\phi)$. We have shown that the energy density of the scalar field dominates at late times for $\lambda^2 < 3\gamma$. For $\lambda^2 > 3\gamma$ we find that the barotropic fluid does not completely dominate and the energy density of the scalar field remains a fixed fraction of the total density at late times.

We can use skills from this project to study different cases e.g. extension to other potential or extension to non-canonical case such as tachyonic and phantom fields.

Appendix A

Proofs of inflationary expansion

Consider in case of scalar field dominate, equation (4.6) is then

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2, \tag{A.1}$$

and equation (4.9) is

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right). \tag{A.2}$$

From inflationary expansion condition ($\ddot{a} > 0$),

$$\ddot{a} = a\dot{H} + aH^2 \tag{A.3}$$

$$\dot{H} + H^2 \quad > \quad 0. \tag{A.4}$$

Substituting equations (A.1) and (A.2) into equation (A.4),

$$\begin{aligned} -\frac{\kappa^2}{2}\dot{\phi}^2 + \frac{\kappa^2}{6}\dot{\phi}^2 + \frac{\kappa^2}{3}V \quad &> \quad 0 \\ -\frac{\kappa^2}{2}\dot{\phi}^2\left(\frac{6H^2}{6H^2}\right) + \frac{\kappa^2}{6}\dot{\phi}^2\left(\frac{H^2}{H^2}\right) + \frac{\kappa^2}{3}V\left(\frac{H^2}{H^2}\right) \quad &> \quad 0 \\ -3\left(\frac{\kappa\dot{\phi}}{\sqrt{6}H}\right)^2 + \left(\frac{\kappa\dot{\phi}}{\sqrt{6}H}\right)^2 + \left(\frac{\kappa\sqrt{V}}{\sqrt{3}H}\right)^2 \quad &> \quad 0 \\ -3x^2 + x^2 + y^2 \quad &> \quad 0\,. \end{aligned} \tag{A.5}$$

Substituting coordinate of the fixed point (4) into equation (A.5),

$$\begin{aligned} -3\left(\frac{\lambda^2}{6}\right)+\frac{\lambda^2}{6}+\left(1-\frac{\lambda^2}{6}\right) &> 0 \\ -\frac{\lambda^2}{2}+\frac{\lambda^2}{6}+1-\frac{\lambda^2}{6} &> 0 \\ -\frac{\lambda^2}{2}+1 &> 0 \\ \lambda^2 &< 2\,. \end{aligned} \tag{A.6}$$

Appendix B

Proofs of power-law solution

In this model, we consider the evolution of the universe during inflation driven by the time-varying vacuum expectation value of some scalar field. The time evolution of the model is determined by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] = -3H\dot{\phi}^2 - \delta\,, \tag{B.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_r = -4H\rho_r + \delta\,, \tag{B.2}$$

$$H^2 = \frac{\kappa^2}{3}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2 + \rho_\gamma\right]\,, \tag{B.3}$$

The quantity δ accounts for the creation of the ultrarelativistic particle due to the time variation of ϕ. Equations (B.1)-(B.3) represent respectively the energy conservation equation for ϕ, the energy conservation equation for radiation, and the Friedmann equation. For the term δ we assumed

$$\delta = \Gamma\dot{\phi}^2, \tag{B.4}$$

where the constant quantity Γ^{-1} representing characteristic time for particle creation by ϕ, depends on interactions of ϕ with other fields.
Form (B.1) we get

$$\ddot{\phi}\dot{\phi} + \dot{V} = -3H\dot{\phi}^2 - \delta\,, \tag{B.5}$$

and from (B.2) we get

$$\delta = \dot{\rho_\gamma} + 4H\rho_\gamma\,,$$

thus equation (B.5) becomes

$$\ddot{\phi}\dot{\phi} + \dot{V} + \dot{\rho_\gamma} = -3H\dot{\phi}^2 - 4H\rho_\gamma\,.$$

Differentiating equation (B.3),

$$\begin{aligned} 2H\dot{H} &= \frac{\kappa^2}{3}\left[\dot{V} + \dot{\phi}\ddot{\phi} + \dot{\rho_\gamma}\right] \\ 2H\dot{H} &= \frac{\kappa^2}{3}\left[-3H\dot{\phi}^2 - 4H\rho_\gamma\right] \\ \dot{\phi}^2 &= -\frac{2}{\kappa^2}\dot{H} - \frac{4}{3}\rho_\gamma\,. \end{aligned} \tag{B.6}$$

Inserting equation (B.6) into equation (B.3) , we get

$$\begin{aligned} \dot{H}^2 &= \frac{\kappa^2}{3}\left[V(\phi) + \frac{1}{2}\left(-\frac{2}{\kappa^2}\dot{H} - \frac{4}{3}\rho_\gamma\right) + \rho_\gamma\right] \\ V(\phi) &= \frac{1}{\kappa^2}(3H^2 + \dot{H}) - \frac{1}{3}\rho_\gamma\,. \end{aligned} \tag{B.7}$$

From equation (B.2) we get

$$\begin{aligned} \dot{\rho_\gamma} &= -4H\rho_\gamma + \Gamma\dot{\phi}^2 \\ &= -4H\rho_\gamma + \Gamma\left(-\frac{2}{\kappa^2}\dot{H} - \frac{4}{3}\rho_\gamma\right) \\ \dot{\rho_\gamma} &= -\frac{4}{3}\left(3H + \Gamma\right)\rho_\gamma - \frac{2}{\kappa^2}\Gamma\dot{H}\,. \end{aligned} \tag{B.8}$$

It is then clear that when $a(t)$ is given, the equations (B.6), (B.7), and (B.8) allows us to determine $\rho_\gamma(t)$, $\phi(t)$, and $V(\phi)$, provided $V(\phi)$ depends on t only through ϕ.

We can assume that ρ_γ is negligible with respect to both the kinetic and potential contributions to ρ, since a short period of inflation is enough to depress it. From now on we will only consider the solution of (B.6), (B.7), and (B.8) under the assumption

$$a = a_0\left(\frac{t}{t_0}\right)^p \tag{B.9}$$

$$\dot{a} = \frac{a_0 p}{t_0}\left(\frac{t}{t_0}\right)^{p-1} \tag{B.10}$$

$$H = \frac{\dot{a}}{a} = pt^{-1} \tag{B.11}$$

$$\dot{H} = -pt^{-2}\,. \tag{B.12}$$

Using x and y from equation (4.10) into equation (4.19) we get

$$\begin{aligned}
\gamma_\phi &= \frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2V} \\
&= \frac{2pt^{-2}}{pt^{-2} + 3H^2 + \dot{H}} \\
&= \frac{2pt^{-2}}{pt^{-2} + 3(pt^{-1})^2 - pt^{-2}} \\
\gamma_\phi &= \frac{2}{3}p\,. \qquad \text{(B.13)}
\end{aligned}$$

In the same way from (B.6) we found that

$$\begin{aligned}
\dot{\phi}^2 &\cong \frac{2}{\kappa^2}pt^{-2} \\
\dot{\phi} &\cong \pm\sqrt{\frac{2p}{\kappa^2}}t^{-1} \\
\phi(t) - \phi_0 &\cong \pm\sqrt{\frac{2p}{\kappa^2}}\ln\left(\frac{t}{t_0}\right) \qquad \text{(B.14)}
\end{aligned}$$

which gives

$$\phi(t) \simeq \phi_0 \pm \sigma \ln\left(\frac{t}{t_0}\right)\,, \qquad \text{(B.15)}$$

where

$$\sigma = \left[\frac{2p}{\kappa^2}\right]^{1/2}.$$

We can find potential in term ϕ by using the solution with the plus sign

$$\begin{aligned}
\phi - \phi_0 &= \sigma \ln\left(\frac{t}{t_0}\right) \\
\exp\left(\frac{\phi - \phi_0}{\sigma}\right) &= \frac{t}{t_0}\,. \qquad \text{(B.16)}
\end{aligned}$$

From equation (B.11)

$$\begin{aligned}
H &= \frac{p}{t_0}\left(\frac{t}{t_0}\right)^{-1} \qquad \text{(B.17)} \\
H &= \frac{p}{t_0}\exp\left(-\frac{\phi - \phi_0}{\sigma}\right) \qquad \text{(B.18)} \\
\dot{H} &= -\frac{p}{t_0}\exp\left(-\frac{\phi - \phi_0}{\sigma}\right)\frac{\dot{\phi}}{\sigma} \qquad \text{(B.19)}
\end{aligned}$$

Differentiate equation (B.15) with the plus sign which gives

$$\begin{aligned}\dot{\phi} &= \frac{\sigma}{t} \\ \dot{\phi} &= \frac{\sigma}{t_0} \exp\left(-\frac{\phi - \phi_0}{\sigma}\right) .\end{aligned} \tag{B.20}$$

Using equation (B.20) into equation (B.19) which gives

$$\dot{H} = -\frac{p}{t_0^2} \exp\left(-2\frac{\phi - \phi_0}{\sigma}\right) . \tag{B.21}$$

Take square of equation (B.18) we get

$$H^2 = \frac{p^2}{t_0^2} \exp\left(-2\frac{\phi - \phi_0}{\sigma}\right) . \tag{B.22}$$

Using equation (B.21) and (B.22) into equation (B.7) with $\rho_\gamma = 0$ we get

$$V(\phi) = \left(\frac{\sigma}{t_0}\right)^2 \left(\frac{3p - 1}{2}\right) \exp\left(-2\frac{\phi - \phi_0}{\sigma}\right) . \tag{B.23}$$

As proved in [17].

Appendix C

Proofs of cosmological scaling solutions

In constructing viable models of dark energy, it is convenient if we know cosmological scaling solutions. The eigenvalue possessing scaling solution satisfies condition

$$\rho_\phi \propto \rho_\gamma \,. \tag{C.1}$$

From

$$\rho_\phi = \frac{\rho_{\phi 0}}{a^3(1+w_\phi)} \tag{C.2}$$

$$\rho_\gamma = \frac{\rho_{\gamma 0}}{a^3(1+w_\gamma)} \tag{C.3}$$

and

$$w_\phi = \gamma_\phi - 1 \tag{C.4}$$

$$w_\gamma = \gamma - 1 \,, \tag{C.5}$$

substituting equations (C.4) and (C.5) into equations (C.2) and (C.3) respectively,

$$\rho_\phi = \frac{\rho_{\phi 0}}{a^{3\gamma_\phi}} \tag{C.6}$$

$$\rho_\gamma = \frac{\rho_{\gamma 0}}{a^{3\gamma}} . \tag{C.7}$$

From table 4.1, we found that γ_ϕ at point (5) equal to γ, then

$$\frac{\rho_\phi}{\rho_\gamma} = \frac{\rho_{\phi 0}}{\rho_{\gamma 0}} = \text{constant}, \tag{C.8}$$

as required in (C.1). The point (5) is therefore scaling solution.

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