Aspects of phantom cosmology in NLS formulation of scalar field cosmology

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To my parents and my dreams

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Declaration

The presentation in this project is partly based on collaboration with my supervisor, Dr. Burin Gumjudpai (the Tah Poe Academia Institute for Theoretical Physics & Cosmology and Department of Physics of Naresuan University) and Mr. Roongtum Sooksan (Chiang Mai Rajabhat University, previously at the Tah Poe Academia Institute for Theoretical Physics & Cosmology). In particular, Chapter 5 and Appendix A. are partially based on [1],

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I hereby declare that this report or dissertation has not been submitted, either in the similar or different forms, to this or any other academic institutions for a degree or diploma. This report or dissertation represents my own work except where references to other works are given.

Theerakarn Phetnora
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Abstract

We use non-linear Schrödinger-type (NLS) formulation to describe non-flat standard Friedmann cosmology which features a barotopic perfect fluid and a canonical scalar field. All cosmological dynamical quantities are expressed in term of Schrödinger quantities as in time-independent non-relativistic quantum mechanics. We assume the expansion of universe to be phantom type. We report all Schrödinger-analogous quantities of the scalar field cosmology and we analyze the effective equation of state coefficient. We show that in a non-flat universe, assuming $a = (t_{\rm a} - t)^q$, $q < 0$, there is no fixed $w_{\rm eff}$ value for the phantom divide. In a closed universe, the phantom expansion could happen with $w_{\rm eff} < -1/3$ while in open universe, it could happen even with $w_{\rm eff} > -1/3$.

Contents

Chapter 1

Introduction

1.1 Background

The present universe is in accelerating phase of expansion. This is confirmed via several astrophysical observations and the contemporary modern cosmology now is built based on this fact. Scalar field is believed to cause the acceleration. Although, it is predicted by high energy physics theories, its existence has not yet been revealed by either observations or experiments. The accelerating expansion rate, with data from many sources, renders the field possessing phantom equation of state which gives phantom expansion function $a \propto (t_\mathrm{a} - t)^q$, where $q < 0$ in flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe. The non-linear Schrödinger (NLS)-type formulation cosmology is an alternative form of cosmological equations. There have been a few works on using power-law and de-Sitter expansion in the NLS-type context [2, 3]. Here we will investigate the phantom expansion in the context of NLS-type formulation.

1.2 Objectives

- To study aspects of phantom expansion and phantom field
- To study the NLS formulation
- To express phantom expansion in both Friedmann and NLS formulations
- To analyse of effective Equation of state in both Friedmann and NLS formulations

1.3 Frameworks

- The universe is assumed to be maximally symmetric.
- The expansion is assumed to be phantom, i.e. $a \sim (t_{\rm a} - t)^q, \;\; q < 0$.
- Fluid components are canonical scalar field and cold dark matter.
- Non-zero curvature is allowed.

1.4 Expected use of the project

- Obtaining NLS - quantities for phantom expansion in FLRW universe.
- Obtaining Schrödinger wave function $u(x)$ and Schrödinger potential $P(x)$ for phantom expansion.
- Obtaining the effective equation of state coefficient value for phantom expansion in presence of barotropic fluid matter.
- Knowing effect of spatial curvature to the effective equation of state in phantom expansion universe.

1.5 Tools

- A high efficiency personal computer.
- Software e.g. LaTeX2e, WinEdit, Maple, and Photoshop.
- High-speed internet.
- Pre-print research article database (http://arxiv.org).

1.6 Procedure

- Studying concept of FLRW cosmology and topics on phantom field.
- Surveying research literatures on NLS-formulation of scalar field cosmology and its applications.
- Re-expressing the equation of state coefficient w in NLS form.

- Applying $a = (t_{\mathrm{a}} - t)^q$ in the equation of state coefficient.
- Analyzing the equation of state coefficient value.

1.7 Outcome

- Obtaining the wave function of NLS-type equation for the phantom expansion.
- Knowing behavior of the effective equation of state coefficient for the phantom expansion.

Chapter 2

Standard FLRW cosmology

2.1 Introduction

After Hubble's observation in 1920s, we know that our universe are expanding. Nowadays, major models of universe have been being developed from general relativistic theory. We assume that the universe are both isotropic and homogeneous, which are criteria of cosmological principles. Then, the Friedmann equation and acceleration equation describe our universe from these assumptions.

2.2 Hubble's law

In 1929, Edwin Hubble found that our galaxies is not alone but one of hundreds which nowadays known to be billions. Observations show that distant galaxies's spectra are redshifted. He announced that almost all galaxies appear to be moving away from us, and he found an empirical law,

$$v = H_0 R. \tag{2.1}$$

This law is known as Hubble's law. It says that our universe is expanding. H_0 is Hubble constant at the present time (t_0), and Hubble parameter H depends on scale factor $a(t)$ which implies relative size of the universe. H is time-dependent, i.e.

$$H = \frac{\dot{a}(t)}{a(t)} \,. \tag{2.2}$$

2.3 Cosmological principle

Large-scale observations today support the idea that the universe is both isotropic and homogeneous. Both facts are linked to what is called the cosmological principle which is general version of the Copernican principle but the cosmological principle has no foundation in any particular physical model or theory, i.e. it can not be 'proved' in a mathematical sense. However, it has been supported by numerous observations and has great value from purely empirical grounds.

The homogeneity states that conditions of universe and the laws of physics are universal. The same physical laws valid here on Earth also works at distant stars, galaxies, and all parts of the universe. Assuming that physical constants are also unchange from place to place within the universe, and are unchange over time, implying the 'perfect' cosmological principle. The laws of nature are unchange and things we observe in the past are controlled under the same physics as things today. The isotropic universe looks the same in all directions and its energy is uniformly distributed in space. The isotropy is true only on very large scales because on smaller scales the universe is non-uniform.

Evidence for the isotropy is the measurements of cosmic microwave background (CMB). The CMB is smooth up to level of 10^{-5}K, therefore supporting isotropy of space.

2.4 Cosmological equations

2.4.1 Friedmann equation

We describe the universe in terms of four-dimensional spacetime. The geometric structure of spacetimes is described by the FLRW metric which is both homogeneous and isotropic. The time evolutions of spacetime is given by a scale factor a which evolves with time according to the Friedmann equation,

$$\left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}. \tag{2.3}$$

Here ρ is the density of the universe. G is the gravitational constant. c is the speed of light. k is the curvature constant that appears in FLRW metric. Defining $H = \dot{a}/a$, where $\dot{}$ denotes time-derivative, therefore,

$$H^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}. \tag{2.4}$$

The Friedmann equation is conservation equation of energy describing evolution of the universe according to its matter-energy contents. The terms on the right-hand side of Friedmann equation represent density and curvature effects on the expansion.

2.4.2 Acceleration equation

The Einstein field equation with FLRW metric also give the fluid equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(3p + \rho\right). \tag{2.5}$$

This equation is called the acceleration equation. It tells us how the rate of expansion of the universe changes. Since k does not appear in this equation, therefore acceleration equation is independent of curvature.

2.4.3 Fluid equation

The energy-momentum tensor in the FLRW universe is conserved. Then, we can find the equation,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0\,. \tag{2.6}$$

Here p is pressure. The fluids are assumed to be perfect, i.e. no heat transfer and isotropic pressure. The equation of state of the perfect fluids is,

$$p = \rho c^2 w. \tag{2.7}$$

Here w is the equation of state coefficient, $w = 0$ is for dust, $w = 1/3$ is for radiation, and $w < -1/3$ is for dark energy[1]. We set $c = 1$, the fluid equation can be rewritten as,

$$\dot{\rho} + 3H\rho(1 + w) = 0. \tag{2.8}$$

The fluid equation is, in fact, the conservation equation of energy and matter. The first term shows the rate of change in density, and the second term is the kinetic energy change due to gravitational potential energy.

For the barotropic fluid, its equation of state coefficient $w_{\rm m}$ is written in term of n. We define $w_{\rm m} \equiv (n-3)/3$ so that $n = 3(1 + w_{\rm m})$. The fluid equation of the barotropic fluid is written as

$$\dot{\rho}_{\rm m} = -nH\rho_{\rm m}, \tag{2.9}$$

and the solution is

$$\rho_{\rm m} = \frac{D}{a^n}\,. \tag{2.10}$$

The barotropic pressure is then

$$p_{\rm m} = \frac{(n-3)D}{3a^n}. \tag{2.11}$$

[1]for cosmological constant, $w_\Lambda = -1$

2.5 Types of energy and matter

The value of the equation of state coefficient represents how pressure depends on density and it represents type of energy and matter in the universe, i.e. dust, radiation and dark energy. we call the matters that are non-relativistic, dust, e.g. dark matter, atom, stars and others. Dust possesses zero w, while the equation of state coefficient of radiation is $1/3$. Dark energy is a hypothetical form of energy that permeates all over space. It increases with rate of expansion of the universe, and its equation of state coefficient is less than $-1/3$. In the universe, only about 4% of the total energy density can be observed directly. About 23% is thought to be composed of dark matter. The remaining 73% is thought to consist of dark energy.

Chapter 3

Scalar field dark energy

3.1 Acceleration universe

Supernovae type Ia data and cosmic microwave background observation show strong evidence of present acceleration phase of the universe [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] while inflation model of the early universe is widely accepted in cosmology [16, 17, 18, 19, 20, 21]. Causes of acceleration and of inflation are both believed to result from scalar field with time-dependent equation of state coefficient $w_\phi(t) < -1/3$ or cosmological constant with $w_\Lambda = -1$

3.2 Phantom field

Phantom field was first introduced in Hoyle's version of steady state theory [22, 23]. In Hoyle and Narlikar theory of gravitation, the action of the phantom field is given by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}(\nabla\phi)^2 - V(\phi)\right]. \tag{3.1}$$

Where $(\nabla\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and $V(\phi)$ is potential of the field. In flat FLRW spacetime, variation of the action with respect to ϕ gives

$$\epsilon\left[\ddot{\phi} + 3H\dot{\phi}\right] + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,. \tag{3.2}$$

When $\epsilon = -1$, we call the scalar field *phantom*. We obtain energy density and pressure density of the scalar field as

$$\rho_\phi = \epsilon\frac{\dot{\phi}^2}{2} + V(\phi)\,, \quad p_\phi = \epsilon\frac{\dot{\phi}^2}{2} - V(\phi)\,. \tag{3.3}$$

We can find the equation of state of phantom field as

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)} \,. \tag{3.4}$$

Then we obtain $w_\phi < -1$ for $\dot{\phi}^2 < 2V(\phi)$.

3.3 Cosmology with canonical scalar field and barotropic perfect fluid

We consider that ingredients of the universe are barotropic fluid and scalar field. The equation of state of barotropic fluid is given by

$$p_{\rm m} = w_{\rm m}\rho_{\rm m} \,, \tag{3.5}$$

and

$$p_\phi = w_\phi \rho_\phi \,, \tag{3.6}$$

for scalar field. Total density and total pressure are

$$\rho_{\rm tot} = \rho_{\rm m} + \rho_\phi \,, \tag{3.7}$$

and

$$p_{\rm tot} = p_{\rm m} + p_\phi \,. \tag{3.8}$$

The effective equation of state can be written as

$$w_{\rm eff} = \frac{\rho_\phi w_\phi + \rho_{\rm m} w_{\rm m}}{\rho_{\rm tot}} \,. \tag{3.9}$$

The Friedmann equation and the acceleration equation in FLRW universe are then given by

$$H^2 = \frac{\kappa^2}{3}\rho_{\rm tot} - \frac{k}{a^2} \,, \tag{3.10}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\rho_{\rm tot}(1 + 3w_{\rm eff}) \,, \tag{3.11}$$

where $\kappa^2 \equiv 8\pi G = 1/M_{\rm P}^2$, G is Newton's gravitational constant, $M_{\rm P}$ is reduced Planck mass, and k is spatial curvature. The energy densities $\rho_{\rm m}$ and ρ_ϕ satisfy

$$\dot{\rho_\phi} + 3H(1 + w_\phi)\rho_\phi = 0 \,, \tag{3.12}$$

$$\dot{\rho_{\rm m}} + 3H(1 + w_{\rm m})\rho_{\rm m} = 0 \,. \tag{3.13}$$

The equation of state coefficient of barotropic fluid is constant. We use equations (2.10), (3.3), and (3.5) to calculate the scalar field potential. From the acceleration equation, (2.5) we get

$$\dot{H} = -\frac{\kappa^2}{2}(p_{\mathrm{m}} + \rho_{\mathrm{m}} + p_\phi + \rho_\phi) + \frac{k}{a^2}\,. \tag{3.14}$$

Inserting density and pressure into equation (3.14), then

$$\dot{H} = -\frac{\kappa^2}{2}\left[\epsilon\dot{\phi}^2 + \frac{D}{a^n} + \left(\frac{n-3}{3}\right)\frac{D}{a^n}\right] + \frac{k}{a^2}\,. \tag{3.15}$$

The equation (3.15) is rewritten as

$$\epsilon\dot{\phi}(t)^2 = -\frac{2}{\kappa^2}\left[\dot{H} - \frac{k}{a^2}\right] - \frac{nD}{3a^n}\,. \tag{3.16}$$

From the Friedmann equation (3.10),

$$H^2 = \frac{\kappa^2}{3}(\rho_{\mathrm{m}} + \rho_\phi) - \frac{k}{a^2}\,, \tag{3.17}$$

we substitute density into equation (3.17), then

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi) + \frac{D}{a^n}\right] - \frac{k}{a^2}\,. \tag{3.18}$$

Using equation (3.16) into equation (3.18) gives

$$H^2 = \frac{\kappa^2}{3}\left\{\frac{1}{2}\left[-\frac{2}{\kappa^2}\left(\dot{H} - \frac{k}{a^2}\right) - \frac{nD}{3a^n}\right] + V(\phi) + \frac{D}{a^n}\right\} - \frac{k}{a^2}\,. \tag{3.19}$$

The potential $V(\phi)$, then, can be expressed as

$$V(\phi) = \frac{3}{\kappa^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \left(\frac{n-6}{6}\right)\frac{D}{a^n}\,. \tag{3.20}$$

Chapter 4

NLS - formulation of scalar field cosmology

Formulation of canonical scalar field cosmology with barotropic perfect fluid, can be expressed as non-linear Ermakov-Milne-Pinney equation as shown recently [24, 25, 26, 27, 28]. However, non-Ermakov-Milne-Pinney equation (EMP) for such system can be written in form of a non-linear Schrödinger type equation (NLS). The solution of the NLS-type equation corresponds to solutions of the generalized EMP equation of scalar field cosmology [29, 30]. The NLS-type formulation was concluded and shown in case of power-law expansion in Ref. [2] where all Schrödinger-type quantities corresponding to scalar field cosmology are worked out. NLS-type formulation also provides an alternative way of solving for the scalar field exact solution in various cases even with non-zero curvature [3]. The field equations (2.4) and (2.5) of canonical scalar field with barotropic perfect fluid corresponding to non-linear Schrödinger-type equation,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + [E - P(x)]\,u(x) = -\frac{nk}{2}u(x)^{(4-n)/n}\,. \tag{4.1}$$

The solution for this NLS-equation is related to scale factor in cosmology and it is satisfied for

$$u(x) \equiv a(t)^{-\frac{n}{2}}\,, \tag{4.2}$$

$$E \equiv -\frac{\kappa^2 n^2}{12}D\,, \tag{4.3}$$

$$P(x) \equiv \frac{n\kappa^2}{4}a(t)^n\dot{\phi}(t)^2\,. \tag{4.4}$$

The mapping from cosmic time t to the variable x is via,

$$x = \sigma(t), \tag{4.5}$$

such that [1-2],

$$\dot{\sigma}(t) = u(x)\,, \tag{4.6}$$

$$\phi(t) = \psi(x) = \frac{\pm 2}{\kappa\sqrt{n}} \int \sqrt{\frac{P(x)}{\epsilon}} \mathrm{d}x. \tag{4.7}$$

If $P(x) = 0$ and $n = 0$, the $\psi(x)$ in equation (4.7) does not exist. Hence, $P(x)$ and n must not be zero at the same time. If $P(x) \neq 0$ and $n \neq 0$, then inverse function of $\psi(x)$ exists as $\psi^{-1}(x)$. Therefore $x(t) = \psi^{-1} \circ \phi(t)$ and the scalar field potential, $V \circ \sigma^{-1}(x)$ can be written as

$$V(t) = \frac{12}{\kappa^2 n^2} \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 - \frac{2u^2}{\kappa^2 n} P(x) + \frac{12u^2}{\kappa^2 n^2} E + \frac{3ku^{4/n}}{\kappa^2}. \tag{4.8}$$

Chapter 5

NLS-formulation with phantom expansion

5.1 Phantom expansion

The phantom expansion are the expansion with $a \sim (t_\mathrm{a} - t)^q$ when $q < 0$ in flat universe. The Schrödinger wave function $u(x)$ are related to standard cosmological quantity by using equation (4.2), (4.5) and (4.6).

$$u(x) = \dot{x}(t) = (t_\mathrm{a} - t)^{-qn/2}. \tag{5.1}$$

Simple integration of (5.1), we found relation between x and cosmic time t as,

$$\begin{aligned} x(t) &= \int (t_\mathrm{a} - t)^{-qn/2} \mathrm{d}t\,, \\ &= -\int (t_\mathrm{a} - t)^{-qn/2} \mathrm{d}(t_\mathrm{a} - t)\,, \\ &= \left(\frac{2}{qn-2}\right)(t_\mathrm{a} - t)^{(2-qn)/2}\,. \end{aligned}$$

x_0 is intergrating constant and we define $\beta \equiv (qn-2)/2$, then

$$x(t) = \frac{1}{\beta}(t_\mathrm{a} - t)^{-\beta} + x_0. \tag{5.2}$$

Then, we rewrite the cosmic time, $t(x)$. From (5.2),

$$\begin{aligned} x(t) &= \frac{1}{\beta}(t_{\mathrm{a}} - t)^{-\beta} + x_0 \,, \\ (x - x_0) &= \frac{1}{\beta}(t_{\mathrm{a}} - t)^{-\beta} \,, \\ \beta(x - x_0) &= (t_{\mathrm{a}} - t)^{-\beta} \,, \\ [\beta(x - x_0)]^{-1/\beta} &= (t_{\mathrm{a}} - t) \,, \\ t_{\mathrm{a}} - [\beta(x - x_0)]^{-1/\beta} &= t(x). \end{aligned}$$

We have the cosmic time in form of x as,

$$t(x) = t_{\mathrm{a}} - \frac{1}{[\beta(x - x_0)]^{\frac{1}{\beta}}}. \tag{5.3}$$

Now, the Schrödinger wave function of NLS-type equation, $u(x)$ in equation (5.1) can be written in term of x by using equation (5.3),

$$\begin{aligned} u(x) &= (t_{\mathrm{a}} - t)^{-qn/2} \,, \\ &= \left[t_{\mathrm{a}} - \left(t_{\mathrm{a}} - \frac{1}{[\beta(x - x_0)]^{\frac{1}{\beta}}} \right) \right]^{-qn/2} \,, \\ u(x) &= [\beta(x - x_0)]^{qn/qn-2}. \end{aligned} \tag{5.4}$$

For the phantom expansion, $\epsilon\dot{\phi}(t)^2$ in equation (3.16) can also be rewritten in term of cosmic time, t,

$$\begin{aligned} \epsilon\dot{\phi}(t)^2 &= -\frac{2}{\kappa^2}\left[\dot{H} - \frac{k}{a^2} \right] - \frac{nD}{3a^n} \,, \\ &= -\frac{2}{\kappa^2}\left[\frac{a\ddot{a} - \dot{a}^2}{a^2} - \frac{k}{a^2} \right] - \frac{nD}{3a^n} \,, \\ &= -\frac{2}{\kappa^2}\left[\left(\frac{\ddot{a}}{a} \right) - \left(\frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} \right] - \frac{nD}{3a^n} \,, \\ &= -\frac{2}{\kappa^2}\left[\left(\frac{q(q-1)(t_{\mathrm{a}} - t)^{q-2}}{(t_{\mathrm{a}} - t)^q} \right) - \left(\frac{q(t_{\mathrm{a}} - t)^{q-1}}{(t_{\mathrm{a}} - t)^q} \right)^2 - \frac{k}{(t_{\mathrm{a}} - t)^{2q}} \right] - \frac{nD}{3(t_{\mathrm{a}} - t)^{nq}} \,, \\ \epsilon\dot{\phi}(t)^2 &= \frac{2q}{\kappa^2(t_{\mathrm{a}} - t)^2} + \frac{2k}{\kappa^2(t_{\mathrm{a}} - t)^{2q}} - \frac{nD}{3(t_{\mathrm{a}} - t)^{qn}}. \end{aligned} \tag{5.5}$$

$P(t)$ in equation (4.4) is found by using equation (5.5),

$$\begin{aligned} P(t) &= \frac{n\kappa^2}{4} a(t)^n \epsilon \dot{\phi}(t)^2 \,, \\ &= \frac{n\kappa^2}{4}(t_{\rm a}-t)^{nq}\left[\frac{2q}{\kappa^2(t_{\rm a}-t)^2}+\frac{2k}{\kappa^2(t_{\rm a}-t)^{2q}}-\frac{nD}{3(t_{\rm a}-t)^{qn}}\right] \,, \\ P(t) &= \frac{qn}{2}(t_{\rm a}-t)^{nq-2}+\frac{kn}{2}(t_{\rm a}-t)^{q(n-2)}-\frac{\kappa^2 n^2 D}{12}. \end{aligned} \tag{5.6}$$

Then, using equation (5.3) and (5.6) to find $P(x)$,

$$\begin{aligned} P(x) &= \frac{qn}{2}(t_{\rm a}-t)^{nq-2}+\frac{kn}{2}(t_{\rm a}-t)^{q(n-2)}-\frac{\kappa^2 n^2 D}{12} \,, \\ &= \frac{qn}{2}[\beta(x-x_0)]^{-\frac{nq-2}{\beta}}+\frac{kn}{2}[\beta(x-x_0)]^{-\frac{q(n-2)}{\beta}}-\frac{\kappa^2 n^2 D}{12} \,, \\ P(x) &= \frac{2qn}{(qn-2)^2(x-x_0)^2}+\frac{kn}{2}\left[\frac{2}{(qn-2)(x-x_0)}\right]^{\frac{2q(n-2)}{(qn-2)}}-\frac{\kappa^2 n^2 D}{12}. \end{aligned} \tag{5.7}$$

The scalar field potential can be obtained from equation (3.20),

$$\begin{aligned} V(\phi) &= \frac{3}{\kappa^2}\left[H^2+\frac{\dot{H}}{3}+\frac{2k}{3a^2}\right]+\left(\frac{n-6}{6}\right)\frac{D}{a^n} \,, \\ &= \frac{3}{\kappa^2}\left[\frac{\dot{a}^2}{a^2}+\frac{1}{3}\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)+\frac{2k}{3a^2}\right]+\left(\frac{n-6}{6}\right)\frac{D}{a^n} \,, \\ &= \frac{3}{\kappa^2}\left(\frac{2\dot{a}^2}{3a^2}+\frac{\ddot{a}}{3a}+\frac{2k}{3a^2}\right)+\left(\frac{n-6}{6}\right)\frac{D}{a^n} \,, \\ &= \frac{3}{\kappa^2}\left[\frac{2q(t_{\rm a}-t)^{2q-2}}{3(t_{\rm a}-t)^{2q}}+\frac{q(q-1)(t_{\rm a}-t)^{q-2}}{3(t_{\rm a}-t)^{q}}+\frac{2k}{3(t_{\rm a}-t)^{2q}}\right]+\left(\frac{n-6}{6}\right)\frac{D}{(t_{\rm a}-t)^{nq}} \,, \\ V(\phi) &= \frac{q(3q-1)}{\kappa^2(t_{\rm a}-t)^2}+\frac{2k}{\kappa^2(t_{\rm a}-t)^{2q}}+\left(\frac{n-6}{6}\right)\frac{D}{(t_{\rm a}-t)^{qn}}. \end{aligned} \tag{5.8}$$

Wave function of the NLS-formulation is found to be non-normalizable [2] as seen in Fig. 5.1 for case of phantom expansion with various types of barotropic fluid. Here q is chosen to -6.666. In flat universe $q = -6.666$ can be attained when $w_{\rm eff} = -1.1$. Fig. 5.2 shows $P(x)$ plot for three cases of k with dust and radiation. In there $x_0 = 1$, therefore $P(x)$ goes to negative infinity at $x = 1$.

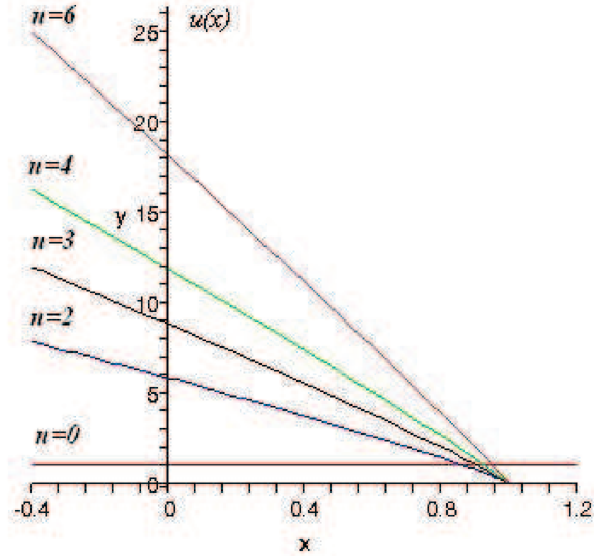

Figure 5.1: Schrödinger wave function, $u(x)$ when assuming phantom expansion. $u(x)$ depends on only q, n and t_{a} but does not depend on k. Here we set $t_{\mathrm{a}} = 1.0$ and $q = -6.666$. If $k = 0$, $q = -6.666$ corresponds to $w_{\mathrm{eff}} = -1.1$

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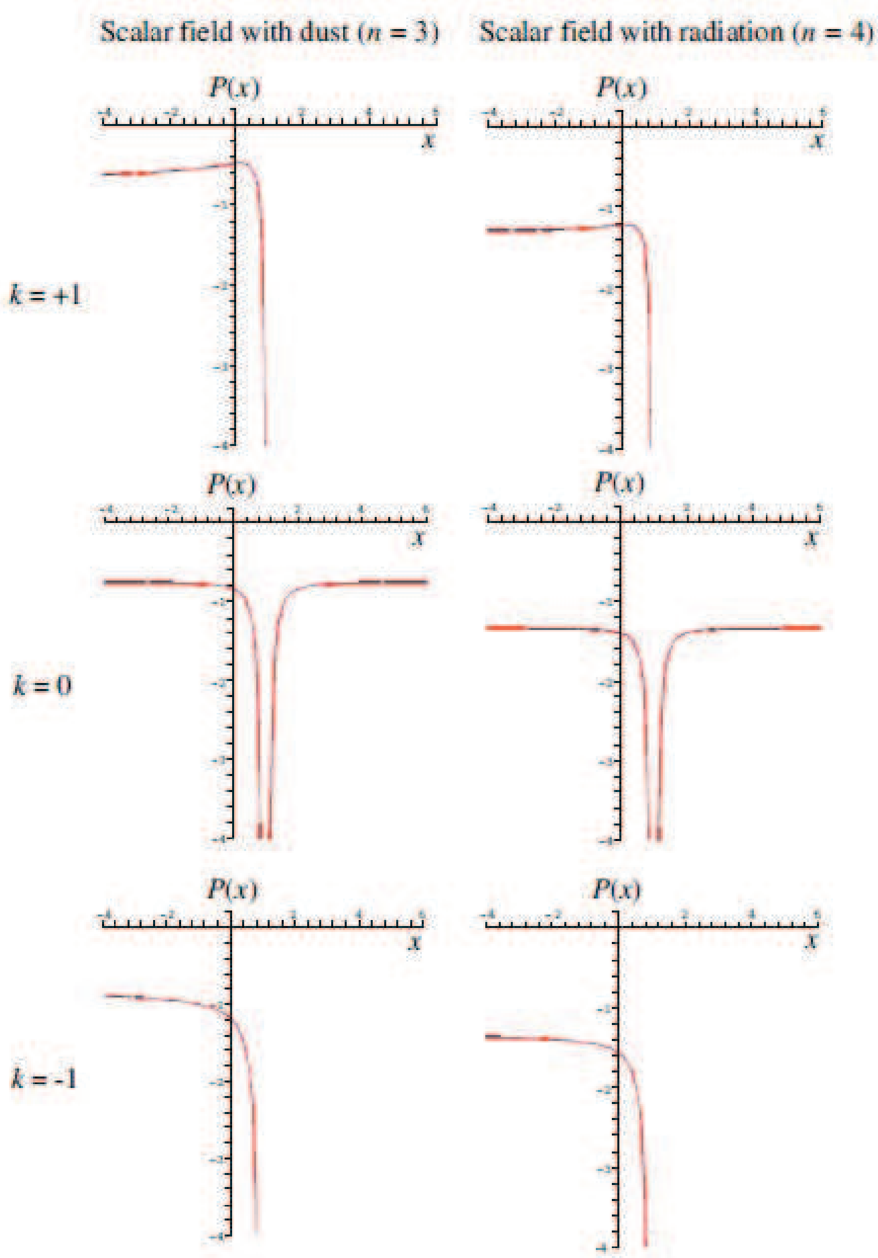

Figure 5.2: Schrödinger potential in phantom expansion case for dust and radiation fluids with $k = 0, \pm 1$. Numerical parameters are as in the $u(x)$ plots (Fig. 5.1). x_0 is set to 1. For non-zero k, there is only on real branch of $P(x)$.

5.2 The effective equation of state coefficient

The effective equation of state coefficient, $w_{\rm eff}$ in equation (3.9), can be written in from of the cosmic time by using the phantom formulation. Starting with rewriting the effective equation of state coefficient in equation (3.9) as

$$w_{\rm eff} = \frac{p_\phi + p_{\rm m}}{\rho_\phi + \rho_{\rm m}}. \tag{5.9}$$

Then, inserting the density and pressure from equation (2.10), (2.11) and (3.3) into equation (5.9) and using equation (5.5) and (5.8) to find effective equation of state coefficient,

$$\begin{aligned}
w_{\rm eff} &= \frac{\frac{\dot\phi^2}{2} - V(\phi) + \frac{(n-3)D}{3a^n}}{\frac{\dot\phi^2}{2} + V(\phi) + \frac{D}{a^n}} \\
&= \frac{\left[\frac{q}{\kappa^2(t_{\rm a}-t)^2} + \frac{k}{\kappa^2(t_{\rm a}-t)^{2q}} - \frac{nD}{6(t_{\rm a}-t)^{nq}}\right] - \left[\frac{q(q-1)}{\kappa^2(t_{\rm a}-t)^2} + \frac{2k}{\kappa^2(t_{\rm a}-t)^2} + \left(\frac{n-6}{6}\right)\frac{D}{(t_{\rm a}-t)^{nq}}\right] + \left(\frac{n-3}{3}\right)\frac{D}{(t_{\rm a}-t)^{nq}}}{\left[\frac{q}{\kappa^2(t_{\rm a}-t)^2} + \frac{k}{\kappa^2(t_{\rm a}-t)^{2q}} - \frac{nD}{6(t_{\rm a}-t)^{nq}}\right] + \left[\frac{q(q-1)}{\kappa^2(t_{\rm a}-t)^2} + \frac{2k}{\kappa^2(t_{\rm a}-t)^2} + \left(\frac{n-6}{6}\right)\frac{D}{(t_{\rm a}-t)^{nq}}\right] + \frac{D}{(t_{\rm a}-t)^{nq}}} \\
&= \frac{\frac{q-q(3q-1)}{\kappa^2(t_{\rm a}-t)^2} + \frac{k-2k}{\kappa^2(t_{\rm a}-t)^{2q}} + \left(\frac{-n}{6} - \frac{n-6}{6} + \frac{n-3}{3}\right)\frac{D}{(t_{\rm a}-t)^{nq}}}{\frac{q+q(3q-1)}{\kappa^2(t_{\rm a}-t)^2} + \frac{k+2k}{\kappa^2(t_{\rm a}-t)^{2q}} + \left(\frac{-n}{6} + \frac{n-6}{6} + 1\right)\frac{D}{(t_{\rm a}-t)^{nq}}} \\
&= \frac{\frac{-3q^2+2q}{\kappa^2(t_{\rm a}-t)^2} + \frac{k}{\kappa^2(t_{\rm a}-t)^{2q}}}{\frac{3q}{\kappa^2(t_{\rm a}-t)^2} + \frac{3k}{\kappa^2(t_{\rm a}-t)^{2q}}} \\
&= \frac{(-3q^2+2q)(t_{\rm a}-t)^{2q-2} - k}{3q(t_{\rm a}-t)^{2q-2} + 3k}.
\end{aligned} \tag{5.10}$$

There is a locus,

$$t = t_{\rm a} - \left(\frac{-k}{q^2}\right)^{1/(2q-2)}, \tag{5.11}$$

where $w_{\rm eff}$ becomes infinite along the locus. Hence for $k = -1$ the locus is $t = t_{\rm a} - q^{-1/(q-1)}$. Hence for $k = 0$, the coefficient $w_{\rm eff}$ is infinite at $q = 0$ or $t = t_{\rm a}$. From the equation above, $w_{\rm eff}$ does not depend on property, n or amount of the barotropic fluid, D. If $k = 0$, the equation (5.10) reduces to

$$q = \frac{2}{3(1+w_{\rm eff})}, \tag{5.12}$$

and therefore the phantom condition $w_{\rm eff} < -1$ implies $q < 0$ as it is known. This corresponds to a condition,

$$w_\phi < -1 - (1+w_{\rm m})\frac{\rho_{\rm m}}{\rho_\phi}. \tag{5.13}$$

Therefore for a fluid with $w_{\rm m} > -1$, w_ϕ is always less than -1 in a flat universe. In order to have the expansion $a \sim (t_{\rm a}-t)^q$ in $k=0$ universe, we must have $w_{\rm eff} < -1$. We find $\rho_{\rm m}/\rho_\phi$ in term of t,

$$
\begin{aligned}
\frac{\rho_{\rm m}}{\rho_\phi} &= \frac{\frac{D}{(t_{\rm a}-t)^{nq}}}{\frac{\epsilon\dot{\phi}^2}{2}+V(\phi)}\,, \\
&= \frac{\frac{D}{(t_{\rm a}-t)^{nq}}}{\left[\frac{q}{\kappa^2(t_{\rm a}-t)^2}+\frac{k}{\kappa^2(t_{\rm a}-t)^{2q}}-\frac{nD}{6(t_{\rm a}-t)^{nq}}\right]+\left[\frac{q(3q-1)}{\kappa^2(t_{\rm a}-t)^2}+\frac{2k}{\kappa^2(t_{\rm a}-t)^{2q}}+\left(\frac{n-6}{6}\right)\frac{D}{(t_{\rm a}-t)^{nq}}\right]}\,, \\
&= \frac{\frac{D}{(t_{\rm a}-t)^{nq}}}{\frac{3q^2}{\kappa^2(t_{\rm a}-t)^2}+\frac{3k}{\kappa^2(t_{\rm a}-t)^{2q}}-\frac{D}{(t_{\rm a}-t)^{nq}}}\,, \\
\frac{\rho_{\rm m}}{\rho_\phi} &= \frac{1}{\frac{3q^2}{\kappa^2 D}(t_{\rm a}-t)^{nq-2}+\frac{3k}{D\kappa^2}(t_{\rm a}-t)^{(n-2)q}-1}\,. \qquad (5.14)
\end{aligned}
$$

Using the equation (5.14) into (5.9), we write w_ϕ in term of $w_{\rm eff}$

$$
\begin{aligned}
w_\phi &= \left(1+\frac{\rho_\phi}{\rho_{\rm m}}\right)w_{\rm eff}-\frac{\rho_\phi}{\rho_{\rm m}}w_{\rm m}\,, \\
&= \left[1+\frac{1}{\left(\frac{3q^2}{D\kappa^2}(t_{\rm a}-t)^{nq-2}+\frac{3k}{D\kappa^2}(t_{\rm a}-t)^{(n-2)q}-1\right)}\right]w_{\rm eff} \\
&\quad -\frac{w_{\rm m}}{\left(\frac{3q^2}{D\kappa^2}(t_{\rm a}-t)^{nq-2}+\frac{3k}{D\kappa^2}(t_{\rm a}-t)^{(n-2)q}-1\right)}\,, \\
&= \frac{\left(\frac{3q^2}{D\kappa^2}(t_{\rm a}-t)^{nq-2}+\frac{3k}{D\kappa^2}(t_{\rm a}-t)^{(n-2)q}\right)w_{\rm eff}-\left(\frac{n-3}{3}\right)}{\left(\frac{3q^2}{D\kappa^2}(t_{\rm a}-t)^{nq-2}+\frac{3k}{D\kappa^2}(t_{\rm a}-t)^{(n-2)q}-1\right)}\,, \\
w_\phi &= \frac{\left(\frac{3q^2}{\kappa^2}(t_{\rm a}-t)^{-2}+\frac{3k}{\kappa^2}(t_{\rm a}-t)^{2q}\right)w_{\rm eff}-\left(\frac{n-3}{3}\right)D(t_{\rm a}-t)^{-qn}}{\left(\frac{3q^2}{\kappa^2}(t_{\rm a}-t)^{-2}+\frac{3k}{\kappa^2}(t_{\rm a}-t)^{-2q}-D(t_{\rm a}-t)^{-qn}\right)}\,. \qquad (5.15)
\end{aligned}
$$

For equation (5.15), when $D = 0$ and $k = 0$, $w_\phi = w_{\rm eff}$. Although we set only $D = 0$, it gives the same result since w_ϕ is independent of geometrical background. However, since the expansion law is fixed, w_ϕ is tied up with D implicitly via equation (3.9). Note that w_ϕ has value in range $(-\infty, -1]$ and $[1, \infty)$ so that the phantom crossing can not happen when the scalar field is dominant. However, presence of the dust barotropic fluid in the system gives a multiplication factor that is less than 1 to the equation of state, i.e.

$$w_{\rm eff} = \left(\frac{\rho_\phi}{\rho_\phi + \rho_{\rm m}} \right) w_\phi \, . \tag{5.16}$$

We can see that the phantom crossing from $w_{\rm eff} > -1$ to $w_{\rm eff} < -1$ can happen in this situation. Fig. 5.3 presents parametric plots of the $(w_{\rm eff}, q, t)$ diagram for various k values. From the figure, we see the locus in equation (5.11) where $w_{\rm eff}$ blows up. In the parametric plots, the value of $w_{\rm eff}$ at any instance can be obtained if we know the value of q. We need to know q from observation in order to know the realistic value of $w_{\rm eff}$ or the other way around. Fig. 5.4 plotted from equation (5.7) setting $t_{\rm a} = 1$ and $t = 0.7$ shows that if $k = \pm 1$, q could be negative, i.e. phantom accelerating expansion, ever when $w_{\rm eff} > -1$. Regardless of $t_{\rm a}$ and t,

$$\lim_{q \to -\infty} w_{\rm eff}(q) = -1 \quad \text{and} \quad \lim_{q \to +\infty} w_{\rm eff}(q) = -\frac{1}{3}, \tag{5.17}$$

for phantom expansion. In particular, for $k = -1$, $w_{\rm eff} > 0$ could give $q < 0$ and $w_{\rm eff}$ is infinite when $\ln q / \ln(t_{\rm a} - t) + q = 1$

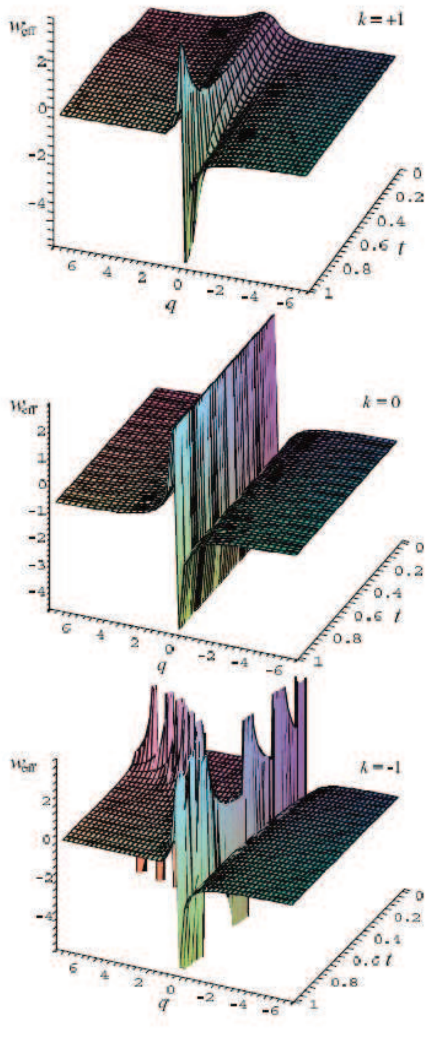

Figure 5.3: Parametric plots of $w_{\rm eff}$ for the phantom expansion $a \sim (t_{\rm a} - t)^q$ in closed, flat and open universe. Here $t_{\rm a}$ is set 1.

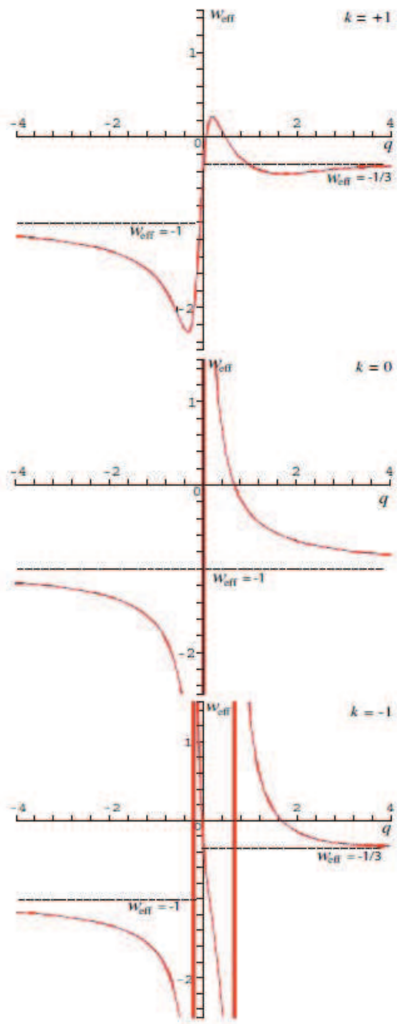

Figure 5.4: $w_{\rm eff}$ for the expansion $a \sim (t_{\rm a} - t)^q$ in closed, flat and open universe. Here $t_{\rm a}$ is set to 1 and t is 0.7

Chapter 6

Conclusion

We consider a system of FLRW cosmology of scalar field and barrotropic fluid assuming phantom acceleration. We have worked out cosmological quantities in the NLS-formulation of the system for flat and non-flat curvature. The Schrödinger wave functions are illustrated in Fig. 5.1 for various types of barotropic fluid. These wave functions are non-normalizable. We show Schrödinger potential plots for dust and radiation cases in closed, flat and open universe. The procedure considered here is reverse to a problem solving in quantum mechanics in which the Schrödinger potential must be known before solving for wave function. In NLS formulation, the Schrödinger equation is non-linear and the wave function is expressed first by the expansion function, $a(t)$. Afterward, the Schrödinger potential is worked out based on expansion function assumed. Moreover, the NLS total energy E is negative. We also perform analysis on effective equation of state. We express w_{eff} in term of q and k. In flat universe, there is no fixed w_{eff} value for the phantom divide. We show this by analyzing equation (5.10) and by presenting illustrations in Fig. 5.3 and Fig. 5.4. In these plots, even $w_{\text{eff}} > -1$, the expansion can still be phantom, i.e. q can be negative. Especially, in $k = -1$ case, positive w_{eff} could also give $q < 0$. The value of w_{eff} approaches -1 when $q \to (-\infty)$ and -1/3 when $q \to (+\infty)$. In open universe, w_{eff} blows up when $\ln q / \ln(t_{\text{a}} - t) + q = 1$.

Appendix A

Maple code

A.1 Maple codes for Fig. 5.1

In Fig. 5.1, we use Maple 9 program to plot $u(x)$ in there cases of curvature, $k = 0, \pm 1$, by using following code to plot. First, we defined $u(x)$ from equation (5.4). Observation suggests that without assuming flat universe $w_\phi = -1.06$ then we assume $w_{\text{eff}} = -1.1$ giving $q = -6.66$. We set $q = -6.6666666666$ and set $x_0 = 1.0000000000$. Then, we compute $u(x)$ for each n value, $n = 0, 2, 3, 4, 6$. Finally, we plot all $u(x)$ together.

```
> restart;

> u(x):=(((n*q-2)/2)*(x-x0))^(n*q/(n*q-2));


> q:=-6.6666666666;

  x0:=1.0000000000;


> n:=0;

> u0(x):=(((n*q-2)/2)*(x-x0))^(n*q/(n*q-2));


> n:=2;

> u2(x):=(((n*q-2)/2)*(x-x0))^(n*q/(n*q-2));


> n:=3;

> u3(x):=(((n*q-2)/2)*(x-x0))^(n*q/(n*q-2));
```

```
> n:=4;
> u4(x):=(((n*q-2)/2)*(x-x0))^(n*q/(n*q-2));

> n:=6;
> u6(x):=(((n*q-2)/2)*(x-x0))^(n*q/(n*q-2));

> plot([u0(x),u2(x),u3(x),u4(x),u6(x)], x =-0.5..1.1, y=-0.2..11.5,
        colour=[red,blue,black,green,orange]);
```

A.2 Maple codes for Fig. 5.2

In Fig. 5.2, we shown the plots of $P(x)$ for radiation ($n = 4$) and dust ($n = 3$) with $k = 0, \pm 1$. First, we define $P(x)$ from equation (5.7). Then, we set $q = -6.6666666666$, $x_0 = 1.000000000000$, $D = 1.0000000000$ and $\kappa = 1.000000000$. We define $P(x)$ for radiation and dust with each k and plot them for each case.

```
> restart;
> P(x):= ((n*q/2)*(((n*q-2)/2)*(x-x0))^(-2))+((n*k/2)*(((n*q-2)/2)*(x-x0))
          ^((-2*q*(n-2))/(n*q-2)))-(n*kappa)^2*d/12;

> q:=-6.6666666666;
  x0:=1.0000000000;
  d:=1.0000000000;
  kappa:=1.000000000;

> k:=0.0; n:=3.0;
> P(x):= ((n*q/2)*(((n*q-2)/2)*(x-x0))^(-2))+((n*k/2)*(((n*q-2)/2)*(x-x0))
        ^((-2*q*(n-2))/(n*q-2)))-(n*kappa)^2*d/12;
> plot(P(x),x=-4..6,y=0.2..-4.0);

> k:=0.0; n:=4.0;
> P(x):= ((n*q/2)*(((n*q-2)/2)*(x-x0))^(-2))+((n*k/2)*(((n*q-2)/2)*(x-x0))
        ^((-2*q*(n-2))/(n*q-2)))-(n*kappa)^2*d/12;
> plot(P(x),x=-4..6,y=0.2..-4.0);
```

```
> k:=1.0; n:=3.0;
> P(x):= ((n*q/2)*(((n*q-2)/2)*(x-x0))^(-2))+((n*k/2)*(((n*q-2)/2)*(x-x0))
        ^((-2*q*(n-2))/(n*q-2)))-(n*kappa)^2*d/12;
> plot(P(x),x=-4..6,y=0.2..-4.0);

> k:=1.0; n:=4.0;
> P(x):= ((n*q/2)*(((n*q-2)/2)*(x-x0))^(-2))+((n*k/2)*(((n*q-2)/2)*(x-x0))
        ^((-2*q*(n-2))/(n*q-2)))-(n*kappa)^2*d/12;
> plot(P(x),x=-4..6,y=0.2..-4.0);

> k:=-1.0; n:=3.0;
> P(x):= ((n*q/2)*(((n*q-2)/2)*(x-x0))^(-2))+((n*k/2)*(((n*q-2)/2)*(x-x0))
        ^((-2*q*(n-2))/(n*q-2)))-(n*kappa)^2*d/12;
> plot(P(x),x=-4..6,y=0.2..-4.0);

> k:=-1.0; n:=4.0;
> P(x):= ((n*q/2)*(((n*q-2)/2)*(x-x0))^(-2))+((n*k/2)*(((n*q-2)/2)*(x-x0))
        ^((-2*q*(n-2))/(n*q-2)))-(n*kappa)^2*d/12;
> plot(P(x),x=-4..6,y=0.2..-4.0);
```

A.3 Maple codes for Fig. 5.3

We show 3-dimension plots of the effective equation of state coefficient for phantom expansion case in open, flat and close universe. First, we define the effective equation of state coefficient from equation (5.10) and set $t_{\mathrm{a}} = 1.0000000000$. Then, we set w_{eff} for each curvature, $k = 0, \pm 1$, and plot them for each case.

```
> restart;

> weff(q,t):=((-3*q^2+2*q)*(ta-t)^(2*q-2)-k)/((3*q^2)*(ta-t)^(2*q-2)+3*k);

> ta:=1.0000000000;

> k:=0;

> weff(q,t):=((-3*q^2+2*q)*(ta-t)^(2*q-2)-k)/((3*q^2)*(ta-t)^(2*q-2)+3*k);

> plot3d(weff(q,t), t=0..1, q=-6..6,numpoints=2000);

> k:=-1;

> weff(q,t):=((-3*q^2+2*q)*(ta-t)^(2*q-2)-k)/((3*q^2)*(ta-t)^(2*q-2)+3*k);

> plot3d(weff(q,t), t=0..1, q=-6..6,numpoints=2000);

> k:=1;

> weff(q,t):=((-3*q^2+2*q)*(ta-t)^(2*q-2)-k)/((3*q^2)*(ta-t)^(2*q-2)+3*k);

> plot3d(weff(q,t), t=0..1, q=-6..6,numpoints=2000);
```

A.4 Maple codes for Fig. 5.4

We show 2-dimension plots of effective equation of state coefficient for phantom expansion case in open, flat and close universe with $t_{\rm a} = 0.7$. First, we define the effective equation of state coefficient from equation (5.10) and set $t_{\rm a} = 0.7$ and $t_{\rm a} = 1.000000000$. Then, we set $w_{\rm eff}$ in each curvature, $k = 0, \pm 1$, and plot them for each case.

```
> restart;

> weff(q,t):=((-3*q^2+2*q)*(ta-t)^(2*q-2)-k)/((3*q^2)*(ta-t)^(2*q-2)+3*k);

> ta:=1.0;

> t:=0.7;


> k:=1;

> weff(q,t):=((-3*q^2+2*q)*(ta-t)^(2*q-2)-k)/((3*q^2)*(ta-t)^(2*q-2)+3*k);

> plot(weff(q,t),q=-4..4,numpoints=2000);


> k:=0;

> weff(q,t):=((-3*q^2+2*q)*(ta-t)^(2*q-2)-k)/((3*q^2)*(ta-t)^(2*q-2)+3*k);

> plot(weff(q,t),q=-4..4,numpoints=2000);


> k:=-1;

> weff(q,t):=((-3*q^2+2*q)*(ta-t)^(2*q-2)-k)/((3*q^2)*(ta-t)^(2*q-2)+3*k);

> plot(weff(q,t),q=-4..4,numpoints=2000);
```

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