A Study of Hyperfine Splitting
in Ground State of H - atom

Phatchaya Maneekhum
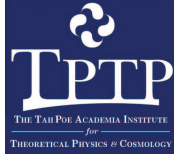
Submitted in partial fulfilment of
the requirements for the award of the degree of

Bachelor of Science in Physics
B.S.(Physics)

To My Parents and My Dreams

A Study of Hyperfine Splitting
in Ground State of H - atom

Phatchaya Maneekhum

The candidate has passed the viva voce examination by the examination panel. This report has been accepted to the panel as partially fulfilment of the course 261493 Independent Study.

....   Supervisor
Dr. Nattapong Yongram, BSc MSc PhD

....   Member
Dr. Amornrat Ungwerojwit, BSc MS PhD

....   Member
Dr. Burin Gumjudpai, BS MSc PhD AMInstP FRAS

Acknowledgements

Title: A Study of Hyperfine Splitting
              in Ground State of H - atom

Candidate: Miss Phatchaya Maneekhum

Supervisor: Dr. Nattapong Yongram

Degree: Bachelor of Science Programme in Physics

Academic Year: 2008

Abstract

We study hyperfine structure of atomic hydrogen in context of electrodynamics and quantum theory to explain this system. We found that the energy of parallel spins (or the magnetic dipoles are antiparallel), so called triplet state, is higher than the energy of the spins that are antiparallel (or the magnetic dipoles are parallel), so called singlet state. The difference of the energy (ΔE) is 5.884×10^{-6} eV. And the interaction energy varies according to the relative orientation of their dipole moments.

Contents

Chapter 1

Introduction

1.1   Background

The electron and the proton in atomic hydrogen constitute tiny magnetic dipoles, whose interaction energy varies according to the relative orientation of their dipole moment. If the spins are parallel (or, more precisely,if they are in the triplet state), the energy is somewhat higher than it is when the spins are antiparallel (the single state). The difference is not large, amounting to a mere 6×10^{-6} eV, as compared with a binding energy of 13.6 eV and typical fine structure splitting on the order of 10^{-4} eV[1]. Nevertheless, this hyperfine splitting is of substantial interest - indeed,before the discovery of the 3° K cosmic background radiation[2], the 21 - cm line resulting from hyperfine transitions in atomic hydrogen was widely regarded as the most pervasive and distinctive radiation in the universe[3].

Hyperfine structure is not usually seem in elementary quantum mechanics text. Although the calculation is quite simple and more accurately than fine structure. The reason for avoiding it probably has to do with a rather subtle point in classical electrodynamics, to wit, the calculation of energy of interaction between two magnetic dipoles.Therefore undergraduate students can't accessible in this part. A study of hyperfine structure in detail serves as a nice application both of electrodynamics and of quantum theory[4].

1.1.1   The hydrogen atom

The hydrogen atom consists of a heavy, essentially motionless proton (we may put it at the origin), of charges $+e$, together with a much lighter electron (charges $-e$) that orbits around it, bound by the mutual attraction of opposite charges (see
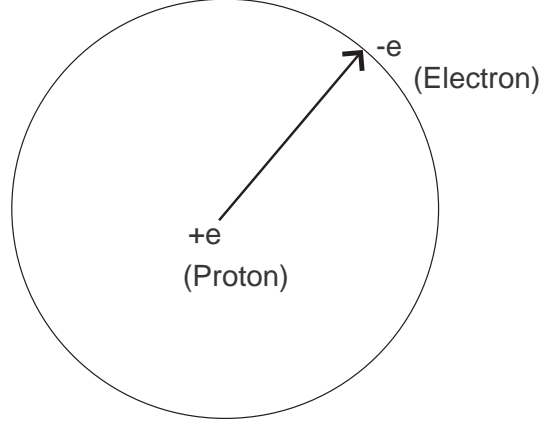
Figure 1.1: The hydrogen atom

Fig. 1.1) from Coulomb's law, the potential energy is

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \tag{1.1}$$

where r is the hydrogen atom radius and e is the electron charge.

The time -independent Schrödinger equation for the hydrogen atom is written as

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \tag{1.2}$$

where E is eigenvalue or energy. With separating variable, we find that the angular dependence of ψ is the spherical harmonics $Y_\ell^m(\theta,\phi)$. The radial part, $R(r)$, satisfies the equation

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{e^2}{4\pi\epsilon_0 r}R + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}R = ER \tag{1.3}$$

For bound states, $R \to 0$ as $r \to \infty$, and R is finite at the origin, $r = 0$. We do not consider continuum states with positive energy. Only when the latter are included do hydrogen wave functions from a complete set.

By use of the abbreviations (resulting from recalling r in the dimensionless radial variable ρ)

$$\rho = \alpha r \qquad \text{with} \quad \alpha^2 = -\frac{8me}{\hbar^2} \; , E < 0 \quad , \quad \lambda = \frac{me^2}{2\pi\epsilon_0\alpha\hbar^2} \tag{1.4}$$

where m is an electron mass, $\hbar = 1.05457168 \times 10^{-34}$ J·s is a Planck's constant Eq.(1.3) becomes

$$\frac{1}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho\frac{\mathrm{d}\chi(\rho)}{\mathrm{d}\rho}\right) + \left(\frac{\lambda}{\rho} - \frac{1}{4} - \frac{\ell(\ell+1)}{\rho^2}\right)\chi(\rho) = 0 \tag{1.5}$$

where $\chi(\rho) = R(\rho/\alpha)$. A comparison with the associated Laguerre equation

$$\frac{d^2\Phi_n^k(x)}{dx^2} + \left(-\frac{1}{4} + \frac{2n+k+1}{2x} - \frac{k^2-1}{4x^2}\right)\Phi_n^k(x) = 0 \tag{1.6}$$

for $\Phi_n^k(x) = e^{-x/2}x^{(k+1)/2}L_n^k$, where $L_n^k = \dfrac{e^x x^{-k}}{n!}\dfrac{d^n(e^{-x}x^{n+k})}{dx^n}$ denoting a Rodrigues representation of the associated Laguerre polynomial , shows that Eq.(1.5) is satisfied by

$$\rho\chi(\rho) = e^{-\rho/2}\rho^{\ell+1}L_{\lambda-\ell-1}^{2\ell+1}(\rho) \tag{1.7}$$

in which k is replaced by $2\ell + 1$ and n by $\lambda - \ell - 1$, upon using

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d\chi}{d\rho}\right) = \frac{1}{\rho}\frac{d^2}{d\rho^2}(\rho\chi)$$

We must restrict the parameter λ by requiring it to be an integer n, $n = 1, 2, 3, ...$ This is necessary because the Laguerre function of nonintegral n would diverge as $\rho^n e^\rho$, which is unacceptable for our physical problem, in which

$$\lim_{r\to\infty} R(r) = 0$$

This restriction on λ, imposed by our boundary condition, has the effect of quantizing the energy,

$$E_n = -\frac{m}{2n^2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \tag{1.8}$$

the negative sign reflects the fact that we are dealing here with bound states $(E < 0)$, corresponding to an electron that is unable to escape to infinity, where the coulomb potential goes to zero. Using this result for E_n , we have

$$\alpha = \frac{me^2}{4\pi\epsilon_0\hbar^2}\cdot\frac{1}{n} = \frac{1}{na_0} \tag{1.9}$$

called "Bohr formula" with the Bohr radius.

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529 \times 10^{-10}\text{m}$$

Thus, the final normalized hydrogen wave function is written as

$$\psi_{n\ell m}(r,\theta,\phi) = \left[\left(\frac{2}{na_0}\right)^3\frac{(n-\ell-1)!}{2n(n+\ell)!}\right]^{1/2}e^{-\alpha r/2}(\alpha r)^\ell L_{n-\ell-1}^{2\ell+1}(\alpha r)Y_\ell^m(\theta,\phi) \tag{1.10}$$

Regular solutions exist for $n \geq \ell + 1$, so the lowest state with $\ell = 1$ (called a $2P$ state) occurs only with $n = 2$. The hydrogen atom ground state $(\ell = 0, m = 0)$ may be described by the spatial wave function

$$\psi(r) = \left(\frac{1}{\pi a_0^3}\right)^{1/2}e^{-r/a_0} \tag{1.11}$$

3
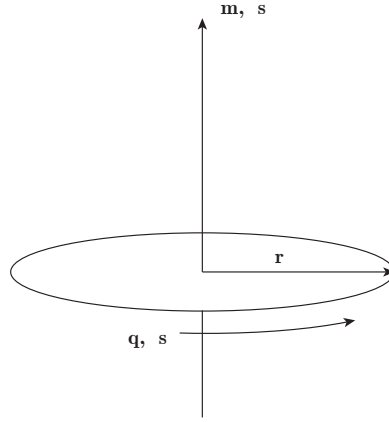
1.1.2   The magnetic dipole moment of the electron

Figure 1.2: A charge q smeared out around a ring of radius r

The magnetic dipole moment of a spining charge is related to its (spin) angular momentum. Consider first a charge q smeard out around a ring of radius r (see Fig. 1.2), which rotates about the axis with period T.

The magnetic dipole moment($m_{\rm d}$) of the ring is defined as the current $\left(\dfrac{q}{T}\right)$ times the area (πr^2).

$$m_{\rm d} = \left(\frac{q}{T}\right) \cdot \pi r^2 \qquad (1.12)$$

If the mass of the ring is m, its angular momentum is the moment of inertia (mr^2) times the angular velocity $\left(\dfrac{2\pi}{T}\right)$:

$$S = \left(mr^2\right) \cdot \left(\frac{2\pi}{T}\right) = \frac{2\pi mr^2}{T} \qquad (1.13)$$

from $\gamma = \dfrac{m_{\rm d}}{S} = \dfrac{q}{2m}$, then Eq.(1.12) becomes

$$\mathbf{m_{\rm d}} = \left(\frac{q}{2m}\right) \mathbf{S} \qquad (1.14)$$

The electron's magnetic moment is twice the classical value :

$$\mathbf{m}_e = -\frac{e}{m} \cdot \mathbf{S} \qquad (1.15)$$

1.1.3 Hyperfine structure

As we know, the hydrogen atom of an electron setting in the neighborhood of the proton, where it can exist in any one of a number of discrete energy states in each one of which the pattern of motion of the electron is different. The first excited state, for example, lie 3/4 of the Rydberg, or about 10 eV, above the ground state. But even the so-called ground state of hydrogen is not really a single, definite - energy state, because of the spins of the electron and proton. These spins are responsible for the "hyperfine structure" in the energy levels, which splits all the the energy levels into several nearly *equal levels*.

The electron can have its spin either "up" or "down" and, the proton can have its spin either "up" or "down". There are, therefore, *four* possible spin states for energy dynamical condition of the atom. That is, when people say "the ground state" of hydrogen, the really mean the "four ground state", and not just the very lowest state. The shifts are, however, much, much smaller then the 10 volts or 50 volts from the ground state to the next state above. As a consequence, each dynamical state has its energy split into a set of very close energy levels - the so-called *hyperfine splitting*.

The hyperfine splitting is due to the interaction of the magnetic moments of the electron and proton, which gives a slightly different magnetic energy for each spin state. These energy shifts are only about ten - millionths of an electron volt - really very small compared with 10 volts. It is because of this large gap that we can thing about the fact that there really many more states at higher energies. We are going to limit ourselves here to a study of the hyperfine structure of the ground state of the hydrogen atom. The nucleus has been assumed to interact with the electron only through its electric field. However, like the electron, the proton has spin angular momentum with $s = 1/2$, and associated with this angular momentum is an intrinsic dipole moment.

$$\mathbf{m}_p = \gamma_p \frac{e}{Mc} \mathbf{S}_p \qquad (1.16)$$

where M is the proton mass and γ_p is a numerical factor known experimentally to be $\gamma_p = 2.7928$. Note that the proton dipole moment is weaker than the electron dipole moment by roughly a factor of $M/m \sim 2000$, and hence one expects the associated effects to be small, even in comparison to fine structure, proton dipole moment will interact with both the spin dipole moment of the electron and the orbital dipole moment of the electron, and so there are two new contributions to the Hamiltonian, the nuclear spin-orbit interaction and the spin-spin interaction. The derivation for the nuclear spin-orbit Hamiltonian is the same as for the electron spin-orbit Hamiltonian, except that the calculation is done in the frame of the proton and hence there is no factor of 1/2 from the Thomas precession. The nuclear spin-orbit Hamiltonian is

$$\Delta H_{pso} = \frac{\gamma_p e^2}{mMc^2 r^3} \mathbf{L} \cdot \mathbf{S}_p \qquad (1.17)$$

The spin-spin Hamiltonian can be de derived by considering the field produced by

the proton spin dipole, which can be written

$$\mathbf{B}(\mathbf{r}) = \frac{1}{r^3}\left[3\frac{(\mathbf{m}_p \cdot \mathbf{r})\mathbf{r}}{r^2} - \mathbf{m}_p\right] + \frac{8\pi}{3}\mathbf{m}_p\delta^3(\mathbf{r}) \tag{1.18}$$

The first term is just the usual field associated with a magnetic dipole, but the second term requires special explanation. Normally, when are considers a dipole field, it is implicit that one is interested in the field for from the dipole-that is, at distances for from the source compared to the size of the current loop producing the dipole. However, every field line outside the loop must return inside the loop, as show in Fig. 1.3. If the size of the current loop goes to zero, then the field will be infinite at the origin, and this contribution is what is what is reflected by the second term in Eq.(1.18)
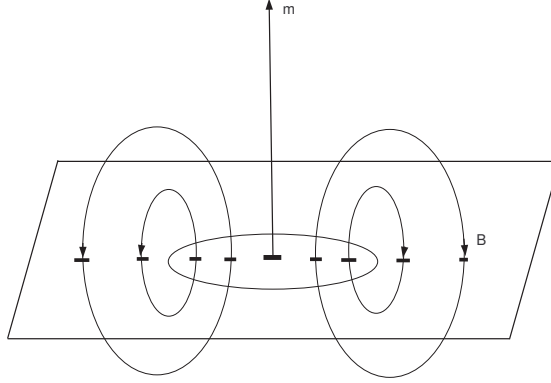

Figure 1.3: The field of a magnetic dipole. All **B** field lines cross the plane of the dipole going up inside the loop and down outside the loop

The electron has additional energy

$$\Delta E_{ss} = -\boldsymbol{\mu}_e \cdot \mathbf{B} \tag{1.19}$$

due to the interaction of its spin dipole with this field, and hence the spin-spin Hamiltonian is

$$\Delta H_{ss} = \frac{\gamma_p e^2}{mMc^2}\left\{\frac{1}{r^3}[3(\mathbf{S}_p \cdot \hat{\mathbf{r}})(\mathbf{S}_e \cdot \hat{\mathbf{r}}) - (\mathbf{S}_p \cdot \mathbf{S}_e)] + \frac{8\pi}{3}(\mathbf{S}_p \cdot \mathbf{S}_e)\delta^3(\mathbf{r})\right\} \tag{1.20}$$

Consider first the case $\ell = 0$, since the hyperfine splitting of the hydrogen atom ground state is of the most interest. Since the electron has no orbital angular momentum, there is no nuclear spin-orbit effect. It can be shown that become the wave function has spherical symmetry, only the delta function term contribution from the spin-spin Hamiltonian.

First order perturbation theory yields

$$\Delta E_{hf} = \frac{8\pi\gamma_p e^2}{3mMe^2}(\mathbf{S}_p \cdot \mathbf{S}_e)|\psi(0)|^2 \qquad (1.21)$$

Fig. 1.4. Shows a revised version of the structure of the hydrogen atom, including the Lamp shift and hyperfine structure.
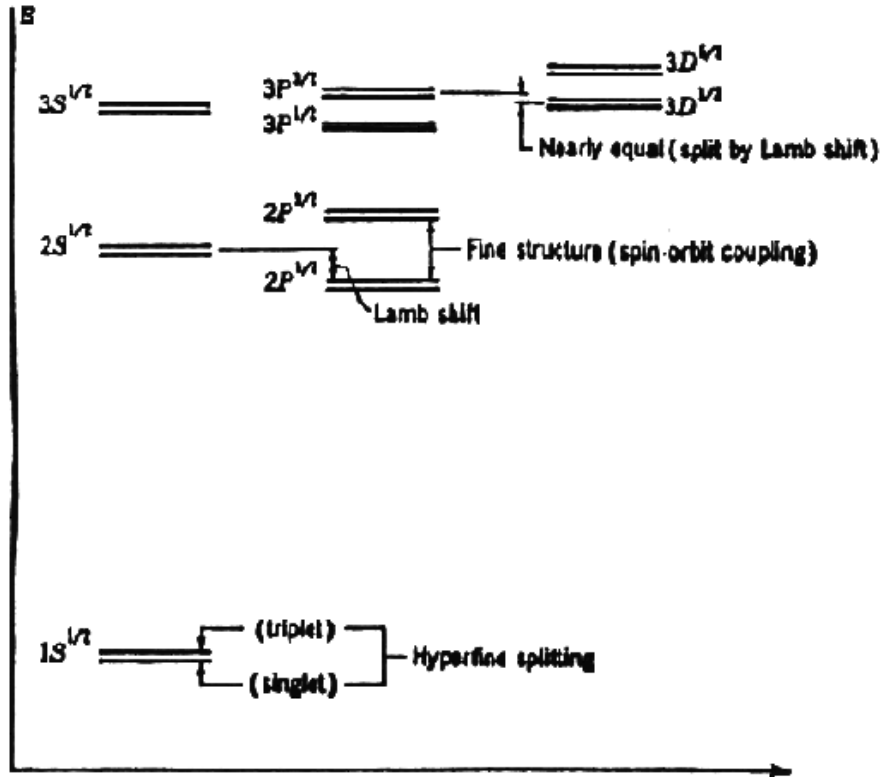

Figure 1.4: Some low - energy states of the hydrogen atom, including fine structure, hyperfine structure, and the lamb shift.

1.2 Objectives

To study the hyperfine structure of atomic hydrogen, By using electrodynamics and quantum theory to explain this system.

1.3   Frameworks

- Atomic hydrogen single system.

- By using to the electric field and the magnetic field action to system.

- By using to the electrodynamics and quantum theory to find the level of energy.

1.4   Expected use

- Can find the level of energy.

- Analyse graphs of energy.

- The explain effect in frame of electrodynamic.

- The explain effect in frame of quantum theory.

1.5   Tools

- Text books in physics and mathematics.

- A high efficiency personal computer.

- Software e.g. LaTeX, WinEdit, Maple, Mayura Draw and Photoshop.

1.6   Procedure

- Research data and reference to related with the hyperfine structure.

- Find electric field of electric dipole, by using the gradient of electric potential.

- Find magnetic field of magnetic dipole, by using the curl of magnetic potential.

- Find a value of interaction energy between spin of nucleus and spin of electron that related the orientation of dipole moment, by applying both of electrodynamics and of quantum theory.

- Analyzing level - energy and conclusion.

1.7 Outcome

- Obtaining the energy gap between the singlet state and the triplet state, in which the spins are antiparallel, carries a somewhat lower energy than the triplet combination.

- Show that the interaction energy is maximum value when angles occurs at $\theta = 0$ and $\theta = \pi$; if they free to rotate then, the compass needles will tend to line up parallel to one another, along the common axis.

- Show that the interaction energy between spin of nucleus and spin of electron relate to the orientation of dipole moment.

Chapter 2

Field of an electric dipole

The electron and the proton in atomic hydrogen constitute tiny magnetic dipole. Unfortunately, we can not directly calculate magnetic dipole. In general, we calculate electric dipole easier than the magnetic dipole. This chapter, we begin with a calculation of the electric field of an electric dipole. With considering system what consist of two charges of equal magnitude but of opposite sign, so called electric dipole. We then calculate the electric field of an electric dipole.

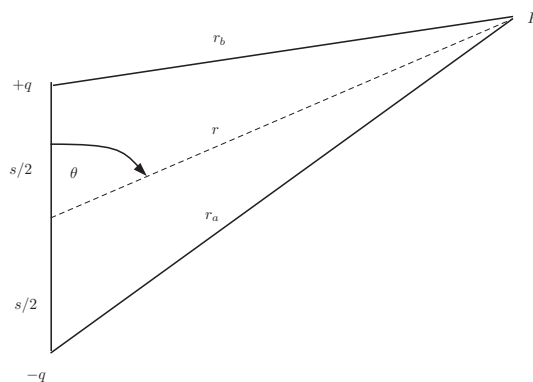## 2.1   Field of an electric dipole

Figure 2.1: An elementary charge-pair dipole that have opposite sign each situated a distant s from the origin, which is taken to lie on the line connecting.

Let us first consider an elementary example of a static system of charges. Our

system will consist (see Fig. 2.1) of two charges of equal magnitude, but of opposite sign, each situated a distant s from the origin O, which is taken to lie on the line connecting the charges. Such a system of charges is the simplest example of an *electric dipole*. The potential at the point $P(r, \theta, \phi)$ is given by

$$V = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_b} - \frac{1}{r_a} \right) \tag{2.1}$$

where r_a and r_b are the distance from $+q$ $(-q)$ to the point of the potential. we wish,however,to express the potential in terms of the magnitude of the vector $\mathbf{r}(|\mathbf{r}| = r)$ and the angle θ (Because the charge distribution is axially symmetric, clearly the potential must be independent of the azimuthal angle ϕ.). In order to do this, we first express r_a and r_b as functions of r and θ. Using the cosine law, we may write

$$r_a^2 = r^2 + \left(\frac{s}{2}\right)^2 + 2r\left(\frac{s}{2}\right)\cos\theta$$

$$\left(\frac{r}{r_a}\right)^2 = \frac{1}{1 + \left(\frac{s}{2r}\right)^2 + \left(\frac{s}{r}\right)\cos\theta}$$

$$\frac{r}{r_a} = \frac{1}{\left[1 + \left(\frac{s}{2r}\right)^2 + \left(\frac{s}{r}\right)\cos\theta\right]^{\frac{1}{2}}}$$

$$\frac{r}{r_a} = \left[1 + \left(\frac{s}{2r}\right)^2 + \left(\frac{s}{r}\right)\cos\theta\right]^{-\frac{1}{2}} \tag{2.2}$$

By applying a power series, is given by

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \dots \tag{2.3}$$

and giving in term of a distant s, is written as

$$x = \left(\frac{s}{2}\right)^2 + \left(\frac{s}{2}\right)\cos\theta \tag{2.4}$$

giving $m = 1/2$ and Eq.(2.3)

$$(1+x)^{1/2} = 1 + \left(-\frac{1}{2}\right)(x) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)x^2}{2!} + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)x^3}{3!} + \dots$$

$$= 1 - \frac{1}{2}\left[\left(\frac{s}{2r}\right)^2 + \left(\frac{s}{r}\right)\cos\theta\right] + \frac{3}{8}\left[\left(\frac{s}{2r}\right)^2 + \left(\frac{s}{r}\right)\cos\theta\right]^2 - \frac{5}{16}\left[\left(\frac{s}{2r}\right)^2 + \left(\frac{s}{r}\right)\cos\theta\right]^3$$

$$= 1 - \frac{1}{2}\left(\frac{s^2}{4r^2} + \frac{s}{r}\cos\theta\right) + \frac{3}{8}\left(\frac{s^2}{4r^2} + \frac{s}{r}\cos\theta\right)^2 - \frac{5}{6}\left(\frac{s^2}{4r^2} + \frac{s}{r}\cos\theta\right)^3 \tag{2.5}$$

11

If we neglect terms of order higher than s^2/r^2, Eq.(2.2) becomes

$$\frac{r}{r_a} = 1 - \frac{s}{2r}\cos\theta + \frac{s^2}{4r^2}\frac{3\cos^2\theta - 1}{2} \tag{2.6}$$

Finally, we obtain

$$\frac{1}{r_a} = \frac{1}{r} - \frac{s}{2r^2}\cos\theta + \frac{s^2}{4r^3}\frac{3\cos^2\theta - 1}{2} \tag{2.7}$$

Where we have assumed $s << r$ in order to expand the radial. We shall restrict our often to field point P that are at distances large compared with the dimension of the dipole. Therefore, we are an approximately

$$\frac{1}{r_a} = \frac{1}{r} - \frac{s}{2r^2}\cos\theta + \frac{s^2}{4r^3}\frac{3\cos^2\theta - 1}{2} \tag{2.8}$$

Similarly,

$$\frac{1}{r_b} = \frac{1}{r} + \frac{s}{2r^2}\cos\theta + \frac{s^2}{4r^3}\frac{3\cos^2\theta - 1}{2} \tag{2.9}$$

where the minus sign in expression for $\dfrac{1}{r_a}$ arises from $\cos(\pi - \theta) = -\cos\theta$. Thus the potential becomes approximately

$$V = q\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left[\left(\frac{1}{r} - \frac{s}{2r^2}\cos\theta + \frac{s^2}{4r^3}\frac{3\cos^2\theta - 1}{2}\right) - \left(\frac{1}{r} + \frac{s}{2r^2}\cos\theta + \frac{s^2}{4r^3}\frac{3\cos^2\theta - 1}{2}\right)\right]$$

$$= \frac{q}{4\pi\epsilon_0}\frac{s}{r^2}\cos\theta \tag{2.10}$$

The potential due to a dipole therefore decreases with distance as $\dfrac{1}{r^2}$, where as the potential due to a single decreases as $\dfrac{1}{r}$. It is reasonable that the potential due to a dipole should decrease with distance more rapidly than the potential due to a single charge because,as the observation point P is moved father and farther away, the dipole charge distribution appears more and to be simply a small unit with zero charge.

We define the *electric dipole moment* of the pair of equal charges as the product of q and separation $2s$:

$$\mathbf{p} = q\mathbf{s} \tag{2.11}$$

The dipole moment is a vector whose direction is defined as the direction from negative to the positive charge (see Fig. 2.1). The dipole potential may be expressed as

$$V = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} \tag{2.12}$$

The electric field vector $\mathbf{E}$ for the dipole is give by the negative of the gradient of scalar potential (V)

$$\mathbf{E} = -\boldsymbol{\nabla}V \tag{2.13}$$

The spherical component of $\mathbf{E}$ may be calculated most easily by referring to Eq.(2.13). Writing $\mathbf{p} = q\mathbf{s}$, we have

$$E_r = \frac{q}{4\pi\epsilon_0} \frac{2s}{r^3} \cos\theta \tag{2.14}$$

$$E_\theta = \frac{q}{4\pi\epsilon_0} \frac{s}{r^3} sin\theta \tag{2.15}$$

$$E_\phi = 0 \tag{2.16}$$

(see Fig. 2.2) show some lines of equal potential and some electric field - lines. Both sets of curves are symmetric about the polar axis so that the equipotential surfaces may be obtained by rotating the curves of (see Fig. 2.2) about the symmetry axis. The potential of an ideal electric dipole is given by[5]
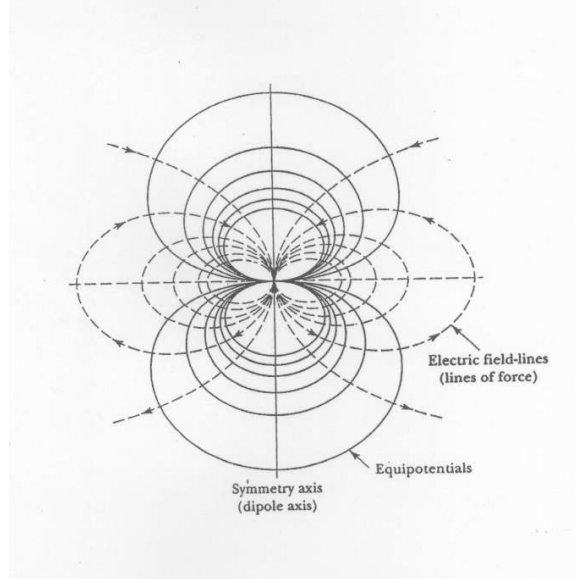

Figure 2.2: Dipole equipotential and field lines

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} \tag{2.17}$$

13

where $\mathbf{p}$ is the dipole moment and $\mathbf{r}$ is the vector from the dipole to the point of observation. From Eqs.(2.10), (2.12) and (2.13), we have

$$\mathbf{E}(r) = -\boldsymbol{\nabla}V$$

$$= -\frac{\partial V(r)}{\partial r}$$

$$= -\frac{\partial}{\partial r}\frac{q}{4\pi\epsilon_0}\frac{s}{r^2}\cos\theta\hat{\mathbf{r}}$$

$$= -\frac{1}{4\pi\epsilon_0}\cos\theta\frac{\partial}{\partial r}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$= -\frac{1}{4\pi\epsilon_0}\cos\theta\left[\frac{r^2\frac{\partial}{\partial r}(\mathbf{p}\cdot\hat{\mathbf{r}})-2r(\mathbf{p}\cdot\hat{\mathbf{r}})}{r^4}\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\cos\theta\frac{1}{r^3}\left[\frac{r^2\frac{\partial}{\partial r}(\mathbf{p}\cdot\hat{\mathbf{r}})-2r(\mathbf{p}\cdot\hat{\mathbf{r}})}{r}\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\cos\theta\frac{1}{r^3}\left[r\frac{\partial}{\partial r}(\mathbf{p}\cdot\frac{\mathbf{r}}{r})-2r(\mathbf{p}\cdot\hat{\mathbf{r}})\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\cos\theta\frac{1}{r^3}\left[r\mathbf{p}\frac{\partial}{\partial r}(r^{-1}\cdot\mathbf{r})-2r(\mathbf{p}\cdot\hat{\mathbf{r}})\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\cos\theta\frac{1}{r^3}[(\mathbf{p}\cdot\hat{\mathbf{r}})-(\mathbf{p}\cdot\hat{\mathbf{r}})-2(\mathbf{p}\cdot\hat{\mathbf{r}})]$$

$$= -\frac{1}{4\pi\epsilon_0}\cos\theta\frac{1}{r^3}[(\mathbf{p}\cdot\hat{\mathbf{r}}-3(\mathbf{p}\cdot\hat{\mathbf{r}})]$$

$$= \frac{1}{4\pi\epsilon_0}\cos\theta\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})-(\mathbf{p}\cdot\hat{\mathbf{r}})]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}] \tag{2.18}$$

Taking the gradient of V, we obtain the dipole field.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}] \tag{2.19}$$

But this familiar result[6] cannot be correct, for it is incompatible with the following general theorem[7], which applies to all static charge configurations.

14

Theorem 1. The average electric field over a spherical volume of radius R, due to an arbitrary distribution of stationary charges within the sphere, is

$$\mathbf{E}_{\text{av}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3} \tag{2.20}$$

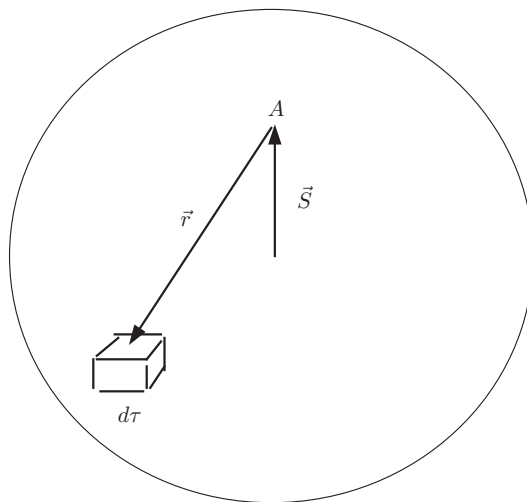where $\mathbf{p}$ is the total dipole moment with respect to the center of the sphere.

Figure 2.3: Average field, over a sphere, due to a point charge at A.

$$\mathbf{E}_{\text{av}} = \frac{1}{\tau}\int \mathbf{E}_z d\tau$$

$$= \frac{1}{\tau}\frac{1}{4\pi\epsilon_0}\int \frac{q}{r^2}\hat{\mathbf{r}}d\tau \tag{2.21}$$

The average field due to a *single* charge q located at point A within the sphere (Fig. 2.3) is given by

$$\mathbf{E}_{\text{av}} = \frac{1}{\tau}\int \mathbf{E}_z d\tau$$

$$= \frac{1}{\tau}\frac{1}{4\pi\epsilon_0}\int \frac{q}{r^2}\hat{\mathbf{r}}d\tau$$

where $\tau = \frac{4}{3}\pi R^3$ is the volume of sphere, we have $d\tau = r^2\sin\theta dr d\theta d\phi$ and $\mathbf{E}_z = \frac{q}{4\pi\epsilon_0 r}\cos\theta\hat{\mathbf{e}}_z$, Eq.(2.21) is written as

$$\mathbf{E}_{\text{av}} = \frac{1}{\tau}\frac{q}{\epsilon_0}\int_{\theta=\frac{\pi}{2}}^{\theta=\pi}\int_{r=0}^{r=-2s\cos\theta}\frac{1}{2}\frac{1}{r^2}\cos\theta r^2\sin\theta dr d\theta\hat{\mathbf{e}}_z$$

15

$$= \frac{1}{\tau} \frac{q}{\epsilon_0} \int_{\theta=\frac{\pi}{2}}^{\theta=\pi} \left(-\frac{s\cos\theta}{3} \right) \cos\theta \sin\theta d\theta \hat{\mathbf{e}}_z$$

$$= \frac{1}{\tau} \left(-\frac{qs}{3\epsilon_0} \right) \left[-\int_{\theta=\frac{\pi}{2}}^{\theta=\pi} \cos^2\theta d\cos\theta \right] \hat{\mathbf{e}}_z$$

$$= \frac{1}{\tau} \left(\frac{qs}{3\epsilon_0} \right) \left[\frac{\cos^3\theta}{3} \right]_{\frac{\pi}{2}}^{\pi} \hat{\mathbf{e}}_z$$

$$= \frac{1}{\tau} \left(\frac{qs}{3\epsilon_0} \right) (-1)\hat{\mathbf{e}}_z$$

$$= \frac{3}{4\pi R^3} \left(-\frac{qs}{3\epsilon_0} \right) \hat{\mathbf{e}}_z$$

$$= -\frac{q\mathbf{s}}{4\pi\epsilon_0 R^3} \qquad ; \mathbf{s} = s\hat{\mathbf{e}}_z \tag{2.22}$$

For an arbitrary distribution of charges within the sphere, qs is replaced by

$$\Sigma q_i \mathbf{s}_i = \mathbf{p} \tag{2.23}$$

(the total dipole moment of sphere) and the theorem is proved and written as

$$\mathbf{E}_{\text{av}} = -\frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}}{R^3} \tag{2.24}$$

Let us apply this theorem to the simplest possible case: an ideal dipole at the origin, pointing in the z direction (see Fig. 2.4) if we take the dipole field in Eq.(2.24) as it stands, we have

$$\mathbf{E}_{\text{av}} = \frac{1}{\tau} \frac{1}{4\pi\epsilon_0} p \int \int \frac{1}{r^3} (3\cos^2\theta - 1)r^2 \sin\theta dr d\theta d\phi \tag{2.25}$$

But the integral give zero , while the r infinite so the result is indeterminate. Evidently in Eq.(2.19) is incorrect or at best ambiguous. Because the source of the problem is the point $r = 0$, where the potential of the dipole is singular. Our formula for the field is unobjectionable everywhere *else*, but at that one point we must be more careful.

An ideal dipole is, after all, the point limit of a real (extended), dipole, so let us approach it from that perspective. The usual model - equal and opposite charges $\pm q$ separated by displacement $\mathbf{s}$, with $\mathbf{p} = q\mathbf{s}$ - is Cumbersome, for our present purposed. Most tractable is a uniformly polarized sphere, of radius a, polarization $\mathbf{P}$, and dipole moment.

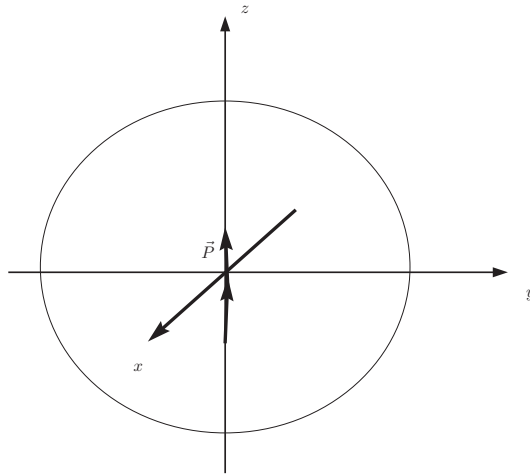$$\mathbf{p} = \frac{4}{3}\pi a^3 \mathbf{P} \tag{2.26}$$

16

Figure 2.4: Average field, over a sphere, due to a point charge at A.

It is well know[9] that the field *outside* such a sphere is given precisely by Eq.(2.19):

$$\mathbf{E}_{\text{out}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] \quad \text{for} \quad r > a \tag{2.27}$$

while (surprisingly) the field *inside* the sphere is uniform (Fig. 2.5)

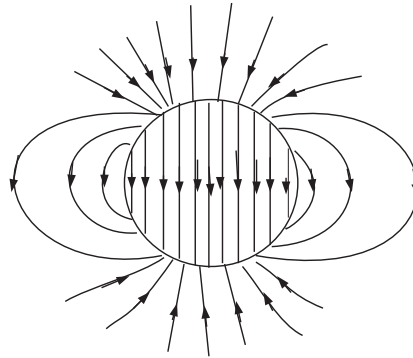$$\mathbf{E}_{\text{in}}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{a^3} \quad \text{for} \quad r < a \tag{2.28}$$

Figure 2.5: Field of a uniformly polarized sphere.

In the ideal dipole limit $(a \to 0)$ the interior region shrinks to zero, and one might suppose that this contribution disappears altogether. However, E_{in} itself blows up,

17

in the same limit, and in just such a way that its integral over the sphere.

$$\int \mathbf{E}_{\text{in}} d\tau = \left(-\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{a^3} \right) \left(\frac{4}{3}\pi a^3 \right) = -\frac{\mathbf{p}}{3\epsilon_0} \tag{2.29}$$

remains constant, no matter how small the sphere become. We recognize here the defining conditions for a Dirac delta function; evidently, as a the field inside the sphere goes to

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] - \frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r}) \tag{2.30}$$

on the understanding that the first term applies only to the region *outside* an infinitesimal sphere about the point $r = 0$. With the radial integral thus truncated, Eq.(2.19) now yields zero unambiguously but there is an extra contribution to E_{av}, coming from the delta function :

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r}) \tag{2.31}$$

which is exactly what **Theorem** 1 requires. Although the delta function only affects the field at the point $r = 0$, it is crucial in establishing the consistency of the theory[10].

18

Chapter 3

Field of a magnetic dipole

In this chapter, we shall deal with the magnetic field of a magnetic dipole. This familiar result is identical in form to the electric field of an electric dipole seen (Chapter 2). We start with the calculation of the vector potential $\mathbf{A}$. We then calculate the magnetic field of a magnetic dipole.
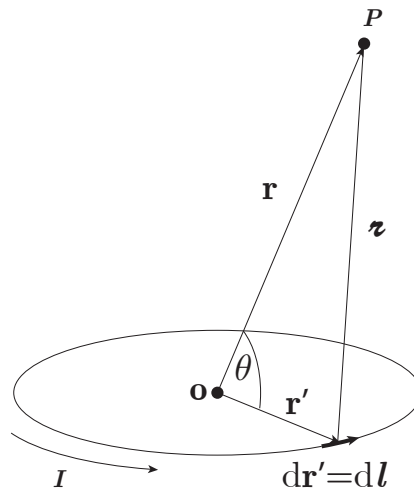
3.1   Field of a magnetic dipole

Figure 3.1: The vector potential of a localized current distribution

We shall now show that a small loop of wire of area S, situated at the origin

in a plane perpendicular to the z - axis and carrying a current I. The vector $\mathbf{A}$ at the point $P(r, \theta, \phi)$ is directed in the azimuthal direction. is given by

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint \frac{1}{r'} dl \hat{\mathbf{e}}_\phi \tag{3.1}$$

Since the dl vectors have no z component, A can only have x and y components. A little thought will show that, for any given value of r'. If a is the radius of the loop, then

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a \cos \phi}{r'} d\phi \hat{\mathbf{e}}_\phi \tag{3.2}$$

We must now express r' in terms of r and of ϕ, in the form of a power series in $\frac{1}{r'}$

$$r'^2 = r^2 + a^2 - 2ar \cos \phi \tag{3.3}$$

by rewriting Eq.(3.3), we have

$$\left(\frac{r}{r'}\right)^2 = \frac{1}{1 - \left(\frac{a}{r'}\right)^2 + \left(\frac{2ar}{r'^2}\right) \cos \phi}$$

$$\frac{r}{r'} = \frac{1}{\left[1 - \left(\frac{a}{r'}\right)^2 + \left(\frac{2ar}{r'^2}\right) \cos \phi\right]^{\frac{1}{2}}}$$

$$= \left[1 - \left(\frac{a}{r'}\right)^2 + \left(\frac{2ar}{r'^2}\right) \cos \phi\right]^{-\frac{1}{2}}$$

$$= 1 - \frac{a^2}{2r^2} + \frac{a}{r} \cos \phi$$

thus

$$\frac{1}{r'} = \frac{1}{r} - \frac{a^2}{2r^3} + \frac{a}{r^2} \cos \phi \tag{3.4}$$

We make the assumption that $a << r$. Since $\frac{r}{r'} \approx 1$, we have substituted for r' in the two correction terms on the right - hand side. Thus the potential becomes approximately

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a \cos \phi}{r'} d\phi \hat{\mathbf{e}}_\phi$$

$$= \frac{\mu_0 I}{4\pi} \frac{a^2 \pi}{r^2} \sin \theta \hat{\mathbf{e}}_\phi \tag{3.5}$$

we define the magnetic dipole moment of circular loop of wire carrying a current I, except that πa^2 must be replaced by the area S of the loop.

$$\mathbf{m} = I\mathbf{S} \tag{3.6}$$

20

The magnetic dipole potential may be expressed as[11]

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \tag{3.7}$$

Where $\mathbf{m}$ is the dipole moment. The magnetic field vector $\mathbf{B}$ for dipole is give by the curl of the potential $\mathbf{A}$,

$$\mathbf{B} = \mathbf{\nabla} \times \mathbf{A} \tag{3.8}$$

By using formula :

$$\mathbf{\nabla} \times [\mathbf{C} \times \mathbf{D}] = (\mathbf{D} \cdot \mathbf{\nabla})\mathbf{C} - (\mathbf{C} \cdot \mathbf{\nabla})\mathbf{D} + \mathbf{C}(\mathbf{\nabla} \cdot \mathbf{D}) - \mathbf{D}(\mathbf{\nabla} \cdot \mathbf{C}) \tag{3.9}$$

$$\mathbf{\nabla} \times [\mathbf{\nabla} \times \mathbf{A}] = \frac{\mu_0}{4\pi} \mathbf{\nabla} \times \left[\mathbf{m} \times \frac{\hat{\mathbf{r}}}{r^3} \right]$$

$$= \frac{\mu_0}{4\pi} \left[\left(\frac{\hat{\mathbf{r}}}{r^3} \cdot \mathbf{\nabla} \right) \mathbf{m} - (\mathbf{m} \cdot \mathbf{\nabla}) \frac{\hat{\mathbf{r}}}{r^3} + \mathbf{m} \left(\mathbf{\nabla} \cdot \frac{\hat{\mathbf{r}}}{r^3} \right) - \frac{\hat{\mathbf{r}}}{r^3}(\mathbf{\nabla} \cdot \mathbf{m}) \right] \tag{3.10}$$

But the term $\mathbf{m} \left(\mathbf{\nabla} \cdot \dfrac{\hat{\mathbf{r}}}{r^3} \right)$ equal zero, therefore, Eq.(3.10) becomes

$$\mathbf{\nabla} \times [\mathbf{\nabla} \times \mathbf{A}] = \frac{\mu_0}{4\pi} \left[(\frac{\hat{\mathbf{r}}}{r^3} \cdot \mathbf{\nabla})\mathbf{m} - (\mathbf{m} \cdot \mathbf{\nabla}) \frac{\hat{\mathbf{r}}}{r^3} - \frac{\hat{\mathbf{r}}}{r^3}(\mathbf{\nabla} \cdot \mathbf{m}) \right]$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[(\hat{\mathbf{r}} \cdot \mathbf{\nabla})\mathbf{m} - (\mathbf{m} \cdot \mathbf{\nabla})\hat{\mathbf{r}} - \hat{\mathbf{r}}(\mathbf{\nabla} \cdot \mathbf{m}) \right] \tag{3.11}$$

by using formula

$$(\nabla)_i \frac{x_j}{r^3} = \frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \tag{3.12}$$

Hence

$$(\mathbf{\nabla} \times \mathbf{A})_j = \frac{r^2 m_j - 3 x_j (\mathbf{m} \cdot \hat{\mathbf{r}})}{r^5}$$

$$= \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - r^2 \mathbf{m}}{r^5}$$

$$= \frac{1}{r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] \tag{3.13}$$

thus

$$\mathbf{B}(\mathbf{r}) = \mathbf{\nabla} \times \mathbf{A} = \frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] \tag{3.14}$$

21

Finally, the magnetic field by the curl yields

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] \tag{3.15}$$

This familiar result[12] is identical form to the *electric* field of an *electric* dipole Eq.(2.19), and once again it cannot be correct, for it is incompatible with the following general theorem[13].

Let us first consider the vector potential of an ideal magnetic dipole is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^3} \tag{3.16}$$

The spherical component of $\mathbf{B}$ may be calculated most easily by referring to Eq.(3.8), we do find that

$$B_r = \frac{\mu_0}{4\pi}\frac{2m}{r^3}\cos\theta \tag{3.17}$$

$$B_\theta = \frac{\mu_0}{4\pi}\frac{m}{r^3}\sin\theta \tag{3.18}$$

$$B_\phi = 0 \tag{3.19}$$

Theorem 2 the average magnetic field over a spherical volume of radius R, due to an arbitrary configuration of steady currents within the sphere, is

$$\mathbf{B}_{\text{av}} = \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3} \tag{3.20}$$

where $\mathbf{m}$ is the total dipole moment of the sphere.
By definition

$$\mathbf{B}_{\text{av}} = \frac{1}{\tau}\int\mathbf{B}d\tau \tag{3.21}$$

where $\tau = \frac{4}{3}\pi R^3$, as before. Writing $\mathbf{B}$ as the curl of $\mathbf{A}$, and invoking the vector indentity[14]

$$\int_{\text{volume}}(\boldsymbol{\nabla}\times\mathbf{A})d\tau = -\int_{\text{surface}}\mathbf{A}\times d\mathbf{a} \tag{3.22}$$

Therefore, Eq.(3.21) becomes

$$\mathbf{B}_{\text{av}} = -\frac{1}{\tau}\int\mathbf{A}\times d\mathbf{a} \tag{3.23}$$

where $d\mathbf{a}$ is an infinitesimal element of area at the surface of the sphere, pointing in the radial direction. Now, the vector potential is itself an integral one the current distribution[15].

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}}{r}d\tau \tag{3.24}$$

22

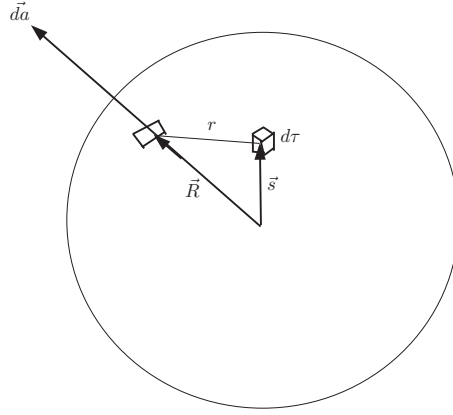Figure 3.2: The vector potential of a localized current distribution

and hence

$$\mathbf{B}_{av} = -\frac{1}{\tau}\frac{\mu_0}{4\pi}\int\int\frac{1}{r}(\mathbf{J}\times\mathbf{da})d\tau \qquad (3.25)$$

We propose to do the surface integral first, setting the polar axis along the vector ($\mathbf{s}$) from the center to $d\tau$ (Fig. 3.2), so that

$$r = (R^2 + S^2 - 2Rs\cos\theta)^{1/2}$$

$$\mathbf{da} = R^2\sin\theta d\theta d\phi\hat{\mathbf{R}}$$

and therefore

$$\int\frac{1}{r}\mathbf{da} = \int\left(R^2 + s^2 - 2Rs\cos\theta\right)^{-\frac{1}{2}}\times R^2\sin\theta\cos\theta d\theta d\phi\hat{\mathbf{s}}$$

$$= \frac{4}{3}\pi s \qquad (3.26)$$

Finally, the volume integral yields

$$\mathbf{B}_{av} = -\frac{1}{\tau}\frac{\mu_0}{4\pi}\int\frac{4}{3}\pi(\mathbf{J}\times\hat{\mathbf{s}})d\tau$$

$$= -\frac{\mu_0}{4\pi R^3}\int(\mathbf{J}\times\hat{\mathbf{s}})d\tau \qquad (3.27)$$

given by

$$\mathbf{m} = \frac{1}{2}\int(\hat{\mathbf{s}}\times\mathbf{J})d\tau \qquad (3.28)$$

23

where $\mathbf{m}$ is the total dipole moment of the sphere[16]. we have

$$\mathbf{B}_{\text{av}} = -\frac{\mu_0}{4\pi R^3}\left(-2\int(\mathbf{J}\times\hat{\mathbf{s}})d\tau\right)$$

$$= \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3} \tag{3.29}$$

Suppose we wish **Theorem** 2 for the simplest possible case : an ideal magnetic dipole $\mathbf{m}$ at origin. If we attempt to calculate the average magnetic field, using Eq.(3.28), while the r integral is infinite so the result is indeterminate. This familiar result in form to the electric field, Eq.(2.19). Once again, the source of the problem is the point $r = 0$; there is an extra delta - function contribution to the field, which Eq.(3.28) ignores. In order to obtain this extra term, we treat the ideal dipole as the point limit of a uniformly magnetized sphere, of radius a, magnetization $\mathbf{M}$, and dipole moment

$$\mathbf{m} = \frac{4}{3}\pi a^3\mathbf{M} \tag{3.30}$$

outside such a sphere is given precisely by Eq.(3.15)

$$\mathbf{B}_{\text{out}} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] \qquad \text{for} \quad r > a \tag{3.31}$$

while field inside the sphere is *uniform*[17]

$$\mathbf{B}_{\text{in}} = \frac{\mu_0}{2\pi}\frac{\mathbf{m}}{a^3} \qquad \text{for} \quad r < a \tag{3.32}$$

In the ideal dipole limit $(a \to 0)$ the interior region shrinks to zero, but the field goes to infinity : their product remains constant :

$$\int\mathbf{B}_{\text{in}}d\tau = \left(\frac{\mu_0}{2\pi}\frac{\mathbf{m}}{a^3}\right)\left(\frac{4}{3}\pi a^3\right)$$

$$= \frac{2}{3}\mu_0\mathbf{m} \tag{3.33}$$

As $a \to 0$, therefore, the field inside the sphere goes to a delta function

$$\mathbf{B}_{\text{in}}(\mathbf{r}) = \frac{2}{3}\mu_0\mathbf{m}\delta^3(\mathbf{r}) \tag{3.34}$$

The magnetic field of an ideal dipole can thus be written

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] + \frac{2}{3}\mu_0\mathbf{m}\delta^3(\mathbf{r}) \tag{3.35}$$

24

with the understanding that the first term applies only to the region outside an infinitesimal sphere at the origin. The *average* field (over sphere of radius R) comes exclusively from the delta - function term :

$$\mathbf{B}_{\text{av}} = \frac{1}{\tau} \int \mathbf{B}_{\text{in}} d\tau$$

$$= \frac{1}{\tau} \int \left[\frac{2}{3} \mu_0 \mathbf{m} \delta^3(\mathbf{r}) \right] d\tau$$

$$= \frac{\mu_0}{2\pi} \frac{\mathbf{m}}{R^3} \tag{3.36}$$

which is exactly what **Theorem** 2 requires. Once again, although it only affect the one point $r = 0$, the delta - function contribution is essential for the consistency of the theory[18].

Chapter 4

Hyperfine structure in the ground state of hydrogen

In this chapter, are work out the formula for the interaction energy of two magnetic dipole. Up to this point the calculation lies entirely within the real in of classical electrodynamics; The quantum mechanics enters only in the final step, where the classical interaction energy is interpreted as the hyperfine structure hamiltonian, and the energies of the singlet and triplet spin states are evaluated in first - order perturbation theory.

4.1   The Hamiltonian

The electron in orbit around the nucleus. This orbiting positive charge sets up a magnetic field $\mathbf{B}$ in the electron frame, which exerts a torque on the spinning electron, tending to align its magnetic moment ($\mathbf{m}$) along the direction of the field. The Hamiltonian is[19]

$$H = -\mathbf{m} \cdot \mathbf{B} \tag{4.1}$$

In particular, the energy of one magnetic dipole ($\mathbf{m}_1$) in the field of another magnetic dipole ($\mathbf{m}_2$) is

$$H = -\mathbf{m} \cdot \mathbf{B} \tag{4.2}$$

$$= -\mathbf{m}_1 \cdot \left(\frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_2] - \frac{2}{3}\mu_0 \mathbf{m}_2 \delta^3(\mathbf{r}) \right)$$

$$= -\frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - \mathbf{m}_1 \cdot \mathbf{m}_2] - \frac{2}{3}\mu_0 (\mathbf{m}_1 \cdot \mathbf{m}_2) \delta^3(\mathbf{r})$$

26

$$= -\frac{\mu_0}{4\pi}\frac{1}{r^3}[3(m_1 \cos\theta)(m_2 \cos\theta) - \mathbf{m}_1 \cdot \mathbf{m}_2] - \frac{2}{3}\mu_0(\mathbf{m}_1 \cdot \mathbf{m}_2)\delta^3(\mathbf{r})$$

$$= -\frac{\mu_0}{4\pi}\frac{1}{r^3}(\mathbf{m}_1 \cdot \mathbf{m}_2)(3\cos^2\theta - 1) - \frac{2}{3}\mu_0(\mathbf{m}_1 \cdot \mathbf{m}_2)\delta^3(\mathbf{r}) \qquad (4.3)$$

where r is their separation. The formula is symmetric in its treatment of $\mathbf{m}_1$ and $\mathbf{m}_2$, as of course it *should* be it represents the energy of interaction of the two dipole. In most applications $\mathbf{m}_1$ and $\mathbf{m}_2$ are physically separated, and the delta-function term can be ignored ; however, it is precisely this part which accounts for hyperfine splitting in the ground state of hydrogen.

In first-order perturbation theory, the change in energy of a quantum state is given by the expectation value of the perturbing Hamiltonian[20]. The ground-state wave function for atomic hydrogen is[21]

$$\psi_0 = (\pi a^3)^{-\frac{1}{2}}e^{-\frac{r}{a}}|s> \qquad (4.4)$$

where $a = 0.5291770$ Å is the Bohr radius[22] and $|s>$ denotes the spin of the electron.

Treating the dipole-dipole interaction Eq.(4.3) as a perturbation, the energy of the ground state is shifted by an amount

$$E = \int \psi_0^* H \psi_0 d\tau$$

$$= \int \psi_0^*[-\frac{\mu_0}{4\pi}\frac{1}{r^3}(\mathbf{m}_1 \cdot \mathbf{m}_2)(3\cos^2\theta - 1) - \frac{2}{3}\mu_0(\mathbf{m}_1 \cdot \mathbf{m}_2)\delta^3(\mathbf{r})\psi_0 d\tau$$

$$= -\frac{\mu_0}{4\pi}\mathbf{m}_1 \cdot \mathbf{m}_2 \int |\psi_0|^2\frac{1}{r^3}(3\cos^2\theta - 1)d\tau - \frac{2}{3}\mu_0(\mathbf{m}_1 \cdot \mathbf{m}_2)\int |\psi_0|^2\delta^3(\mathbf{r})d\tau \qquad (4.5)$$

Because ψ_0 (and indeed any $\ell = 0$ state) is spherically symmetrical[23], the θ integral gives zero, just as it did in Eq.(2.25). Accordingly

$$E = -\frac{\mu_0}{4\pi}\mathbf{m}_1 \cdot \mathbf{m}_2 \int |\psi_0|^2\frac{1}{r^3}(3\cos^2\theta - 1)d\tau$$

$$= 0 \qquad (4.6)$$

Thus

$$E = -\frac{2}{3}\mu_0(\mathbf{m}_1 \cdot \mathbf{m}_2)\int |\psi_0|^2\delta^3(\mathbf{r})d\tau$$

$$= -\frac{2}{3}\mu_0(\mathbf{m}_1 \cdot \mathbf{m}_2)\int \frac{1}{\pi a^3}\delta^3(\mathbf{r})d\tau$$

$$= -\frac{2}{3}\mu_0(\mathbf{m}_1 \cdot \mathbf{m}_2)\frac{1}{\pi a^3}\int \delta^3(\mathbf{r})d\tau$$

27

$$E = -\frac{2}{3}\frac{\mu_0}{\pi a^3}(\mathbf{m}_1 \cdot \mathbf{m}_2) \tag{4.7}$$

Here $\mathbf{m}_1$ is the magnetic dipole moment of the proton and $\mathbf{m}_2$ is that of the electron; they are proportional to the respective spins:

$$\mathbf{m_1} = \gamma_p \mathbf{S}_p \quad , \quad \mathbf{m_2} = -\gamma_e \mathbf{S}_e \tag{4.8}$$

where γ is are the two gyromagnetic ratios[24].Thus

$$E = \left(\frac{2}{3}\frac{\mu_0}{\pi a^3}\right)\gamma_e\gamma_p (\mathbf{S}_e \cdot \mathbf{S}_p) \tag{4.9}$$

In the presence of such "spin - spin coupling" the z components of $\mathbf{S}_e$ and $\mathbf{S}_p$ are no longer separately conserved; the quantum numbers for the system are rather the eigenvalues of the *total* angular momentum.

$$\mathbf{J} = \mathbf{S}_e + \mathbf{S}_p \tag{4.10}$$

$$\mathbf{J}^2 = (\mathbf{S}_e + \mathbf{S}_p)^2 \tag{4.11}$$

$$\mathbf{J}^2 = \mathbf{S}_e^2 + S_p^2 + 2\mathbf{S}_e \cdot \mathbf{S}_p \tag{4.12}$$

so that

$$\mathbf{S}_e \cdot \mathbf{S}_p = \frac{1}{2}(J^2 - S_e^2 - S_p^2) \tag{4.13}$$

The electron and proton carry spin$-\frac{1}{2}$, so the eigenvalues of S_e^2 and S_p^2 are $\frac{3}{4}\hbar^2$. The two spins combine to form a spin–1 , so called a triplet state ($J^2 = 2\hbar^2$) and a spin–0, so called a singlet state ($J^2 = 0$)[25].Thus

$$(\mathbf{S}_e \cdot \mathbf{S}_p) = \frac{1}{4}\hbar^2 \quad \text{(triplet)} \tag{4.14}$$

$$(\mathbf{S}_e \cdot \mathbf{S}_p) = -\frac{3}{4}\hbar^2 \quad \text{(singlet)} \tag{4.15}$$

and hence

$$E = \left(\frac{2}{3}\frac{\mu_0}{\pi a^3}\right)\gamma_e\gamma_p \left(\frac{1}{4}\hbar^2\right) \quad \text{(triplet)} \tag{4.16}$$

$$E = \left(\frac{2}{3}\frac{\mu_0}{\pi a^3}\right)\gamma_e\gamma_p \left(-\frac{3}{4}\hbar^2\right) \quad \text{(singlet)} \tag{4.17}$$

Evidently the singlet state, in which the spins are antiparallel, carries a some what lower energy than the triplet combination (Fig. 4.1). The energy gap is

$$\Delta E = E_{\text{trip.}} - E_{\text{sing.}} \tag{4.18}$$

$$= \left(\frac{2}{3}\frac{\mu_0}{\pi a^3}\right)\gamma_e\gamma_p\left(\frac{1}{4} - \left(-\frac{3}{4}\right)\right) \tag{4.19}$$

thus

$$\Delta E = \left(\frac{2}{3}\frac{\mu_0}{\pi a^3}\right)\gamma_e\gamma_p \tag{4.20}$$

Now, the gyromagnetic ratios are given by

$$\gamma = \left(\frac{e}{2m}\right)g \tag{4.21}$$

where e is the proton charge, m is the mass of the particle, and g is its "g factor" (2.0023 for the electron, 5.5857 for the proton). So, finally[26]

$$\Delta E_{\text{hyd}} = \left(\frac{\mu_0\hbar^2 e^2}{6\pi a^3}\right)\frac{g_e g_p}{m_e m_p} \tag{4.22}$$

$$= \left[\frac{(14.680812 \times 10^{50}\text{NA}^{-2}\text{kg}^{-2})(1.68960995 \times 10^{-53}\text{J}\cdot\text{s}\cdot\text{C}^2)}{4.4445541342 \times 10^{-31}\text{m}^3}\right]$$

$$= \left[\frac{(14.680812 \times 10^{50}(\frac{\text{N}}{\text{A}^2\cdot\text{kg}^2}))(2.854781783 \times 10^{-106}\text{J}^2\cdot\text{s}^2\cdot\text{C}^2)}{4.4445541342 \times 10^{-31}\text{m}^3}\right]$$

$$= 9.427539063 \times 10^{-25}\frac{\text{m}^5\cdot\text{kg}}{\text{m}^3\cdot\text{s}^2}$$

$$= 9.427539561 \times 10^{-25}\text{N}\cdot\text{m}$$

$$= 9.427539561 \times 10^{-25}\text{J}$$

$$= 5.88420775 \times 10^{-6}\text{eV} \tag{4.23}$$

The frequency of the photon emitted in a transition from the triplet to the singlet state is then

$$\nu = \frac{\Delta E}{h}$$

$$= 1422.8 \quad \text{MHz} \tag{4.24}$$

and its wavelength is then

$$\lambda = \frac{c}{\nu}$$

$$= 21.07 \quad \text{cm} \tag{4.25}$$

The experimental value is[27]

$$\nu = 1420.4057517667 \quad \text{MHz} \tag{4.26}$$

the 0.2% discrepancy is attributable to quantum electrodynamical corrections[28]. It is instructive to express the hyperfine splitting Eq.(4.23) in terns of the binding energy $(R = 13.6058 \text{ eV})$ of the ground state :

$$\Delta E_{\text{hyd}} = \frac{8}{3} \left(\frac{R^2}{m_p c^2} \right) g_e g_p \tag{4.27}$$

By contrast, the fine structure goes like $(R^2/m_e c^2)$, and is therefore typically greater by a factor on the order of $(m_p/m_e) = 1836$. In the case of positronium, where the proton is replaced by a positron, the fine and hyperfine splittings are roughly equal in size. If we apply Eq.(4.23) to positronium cussing the reduced mass, of course, in calculating the "Bohr radius", we obtain

$$\Delta E_{\text{pos}} = \frac{1}{8} \left(1 + \frac{m_e}{m_p} \right)^3 \left(\frac{g_e}{g_p} \right) \left(\frac{m_p}{m_e} \right) \times \Delta E_{\text{hyd}}$$

$$= 4.849 \times 10^{-4} \quad \text{eV} \tag{4.28}$$

as compared with an experimental value of 8.411×10^{-4} eV[29]. The large discrepancy is due primarily to pair annihilation, which splits the levels by an additional amount, $\frac{3}{4}\Delta E_{pos}$[30], and does not occur, of course, in hydrogen. Muonium (in which a *muon* substitutes for the proton) offers a cleaner application of Eq.(4.23). The g factor of the muon is 2.0023[22], (identical, up to corrections of very high order, with that of the electron), so

$$\Delta E_{\text{muon}} = \left(\frac{1 + m_e/m_p}{1 + m_e/m_p} \right)^3 \frac{g_\mu}{g_p} \frac{m_p}{m_\mu} \Delta E_{\text{hyd}}$$

$$= 1.8493 \times 10^{-5} \quad \text{eV} \tag{4.29}$$

which compares very well with the experimental value[31] 1.845885×10^{-5} eV the 0.2% discrepancy in, a quantum electrodynamical correction[32]. Incidentally, the hyperfine splitting in muonic hydrogen (muon substitutions for electron) would be "gigantic"

$$\Delta E_{\text{muon}} = \left(\frac{1 + m_p/m_e}{1 + m_p/m_\mu} \right)^3 \frac{g_\mu}{g_e} \frac{m_e}{m_\mu} \Delta E_{\text{hyd}}$$

$$= 1.8493 \times 10^{-5} \quad \text{eV} \tag{4.30}$$

which corresponds to a wavelength of 67800 Å, in the infrared region. however, as for as we know this quantity has not yet been measured directly in thee laboratory[33].
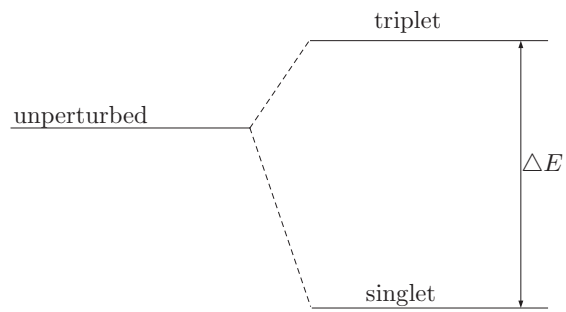
Figure 4.1: Hyperfine splitting in the ground state of hydrogen.

Chapter 5

Analysis

5.1 Why is the singlet level lower?

In the singlet state, the proton and electron spins are antiparallel, which is to say that their magnetic moments are parallel. Why should this be the configuration of lowest energy ? On a formal level, it is a consequence of that sign of the delta-function term in the interaction energy in Eq.(5.2); electric dipole, by contrast, would line up antiparallel compare Eqs.(2.28) and (3.34). But we would like to understand this on a more intuitive basis.

Imagine two compass needles, held a distance r is substantially greater than the length of each needle, they interact essentially as ideal magnetic dipoles, and the energy of the system is given by the first term in Eq.(5.2):
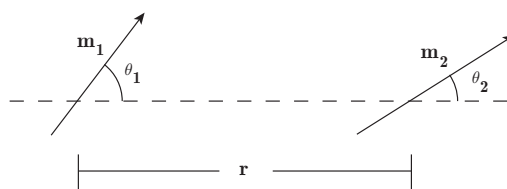

Figure 5.1: Interaction of two magnetic dipole

$$H = -\mathbf{m} \cdot \mathbf{B} \qquad (5.1)$$

$$= -\mathbf{m}_1 \cdot \left(\frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - \mathbf{m}_2] - \frac{2}{3} \mu_0 \mathbf{m}_2 \delta^3(\mathbf{r}) \right)$$

$$= -\frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - \mathbf{m}_1 \cdot \mathbf{m}_2] - \frac{2}{3} \mu_0 (\mathbf{m}_1 \cdot \mathbf{m}_2) \delta^3(\mathbf{r}) \qquad (5.2)$$

From Fig. 5.1, yields

$$\mathbf{m}_1 \cdot \hat{\mathbf{r}} = m_1 \cos \theta_1 \qquad (5.3)$$

$$\mathbf{m}_2 \cdot \hat{\mathbf{r}} = m_2 \cos \theta_2 \qquad (5.4)$$

$$\mathbf{m}_1 \cdot \mathbf{m}_2 = m_1 m_2 \cos(\theta_1 - \theta_2) \qquad (5.5)$$

Finally, the energy of the system is given by the first term in Eq.(5.2)

$$H = -\frac{\mu_0}{4\pi} [3m_1 \cos \theta_1 m_2 \cos \theta_2 - m_1 m_2 \cos(\theta_1 - \theta_2)] \qquad (5.6)$$

thus

$$W = \frac{\mu_0}{4\pi} \frac{1}{r^3} m_1 m_2 [cos(\theta_1 - \theta_2) - 3cos\theta_1 cos\theta_2] \qquad (5.7)$$

By using Eq.(5.7), we simulate the interaction energy that varies according to the relative orientation of their dipole moment. when we fix $\mathbf{m}_1$, but $\mathbf{m}_2$ is free to rotate (see Fig. 5.2), and we consider two case of the magnetic dipole setting. The fist case, the spins are parallel (or the magnetic dipoles are antiparallel), so called the triplet state. We found that the interaction energy varies according to the relative orientation of their dipole moment (see Fig. 5.2) the second case, the spins are antiparallel (or the
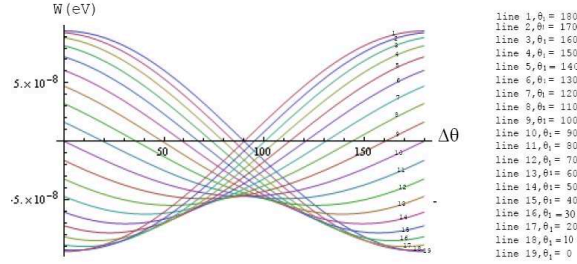

Figure 5.2: Interaction energy of two magnetic dipole

magnetic dipoles are parallel), so called "the singlet state". Similarly, the interaction energy varies according to the relative orientation of their dipole moment (see Fig. 5.3) It is easy to show that the minimum occurs at $\theta_1 - \theta_2 = 0$ (or, equivalently, at $\theta_1 = \theta_2 = \pi$); if they are free to rotate,then, the compass needles will tend to line
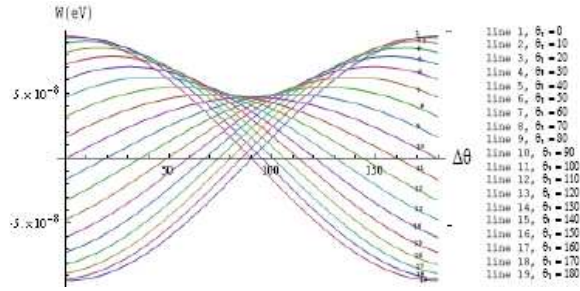
Figure 5.3: Interaction energy of two magnetic dipole

up parallel to one another, along the common axis. And the same goes for electric dipoles.

Consider first the case of electric dipoles. Plus and minus charges separated by a distance s. As long as r is greater than s, they line up along the axis (Fig. 5.4). But when the positive end of one meets the negative end of the other, these ends stick together and move off the line of centers (Fig. 5.5). Until finally, as $r \to 0$, the two dipole are oriented *antiparallel* to one another, and *perpendicular* to the line joining them (Fig. 5.6).
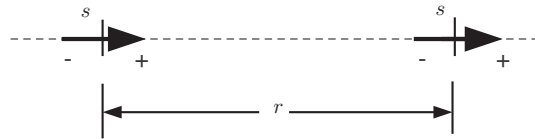

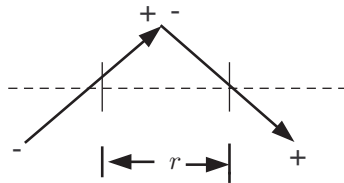Figure 5.4: Stable configuration for electric dipoles : $r > s$

Figure 5.5: Stable configuration for electric dipoles : $r < s$

If we now repeat the process with *magnetic dipoles* represent by circular current loops of diameter d no such reversal occurs. Since parallel currents attract, there will occur a time when the circles tilt over to touch one another (Fig. 5.7). But as $r \to 0$
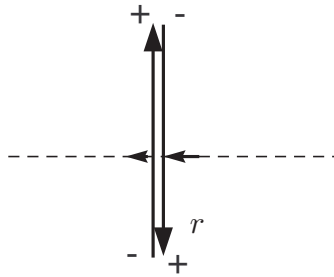
Figure 5.6: Stable configuration for electric dipoles : $r \to 0$

the loops come together with their currents in the same direction (Fig. 5.8). The stable configuration for superimposed *magnetic* dipole, then, is one in which they lie *parallel* to each other, and to the line joining them.
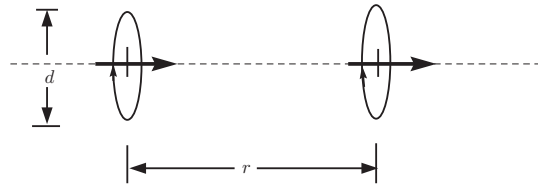

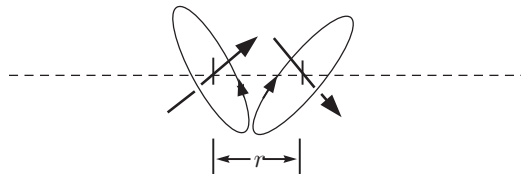Figure 5.7: Stable configuration for magnetic dipoles : $r > d$

Figure 5.8: Stable configuration for magnetic dipoles : $r < d$
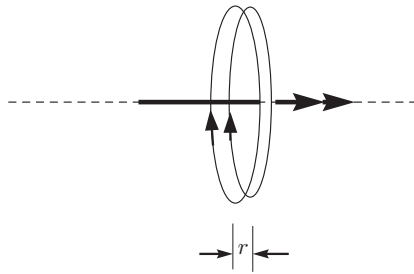
Figure 5.9: Stable configuration for magnetic dipoles : $r \to 0$

Chapter 6

Conclusion

A study of the hyperfine structure of atomic hydrogen. That there is the electron and the proton. They constitute tiny magnetic dipole, whose interaction energy varies according to the relative orientation of their dipole moments. We found the spins are parallel (or the magnetic dipoles are antiparallel), so called the triplet state, the energy is high that it is when the spins are antiparallel (or the magnetic dipoles are parallel), so called the singlet state. The difference of the energy, or the energy gap, is 5.884×10^{-6} eV.

We approach a study of the hyperfine structure with the advantage of the classical electrodynamics and the quantum theory. With the classical electrodynamics, we obtain the electric field of an ideal dipole as follows.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] - \frac{1}{3\epsilon_0}\mathbf{p}\cdot\delta^3(\mathbf{r}) \qquad (6.1)$$

where $\mathbf{p}$ is the dipole moment, $\mathbf{r}$ is the vector from the dipole to the point of observation (r is its magnitude and $\hat{\mathbf{r}} = \dfrac{\mathbf{r}}{r}$), $\epsilon_0 = 8.854187817 \times 10^{-12}$ $\frac{\text{F}}{\text{m}}$ is *permittivity of free space*. On the understanding that the first term applies only to the region *outside* an infinitesimal sphere about the point $r = 0$. With the radial integral thus the first term yields zero unambiguously - but there is an extra contribution to the average electric field, $\mathbf{E}_{\text{av}}$, coming from only the second term :

$$\mathbf{E}_{\text{av}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3} \qquad (6.2)$$

where R is the radial of a spherical volume.

We then are the advantage of electric field of an electric dipole to calculate. The magnetic field of an ideal dipole can be written :

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] + \frac{2}{3}\mu_0\mathbf{m}\delta^3(\mathbf{r}) \qquad (6.3)$$

where $\mathbf{m}$ is the dipole moment and $\mu_0 = 4\pi \times 10^{-7}$ N $\cdot$ A^{-2} is *permeability of free space*. With the understanding that the first term applies only to the region *outside* an infinitesimal sphere at the origin. The *average* field (over a sphere of radius R) come from the second term :

$$\mathbf{B}_{\text{av}} = \frac{\mu_0}{2\pi} \frac{\mathbf{m}}{R^3} \tag{6.4}$$

The quantum mechanics enter in the final step, where the energy of a magnetic dipole $\mathbf{m}$, in the presence of a magnetic field $\mathbf{B}$, is given by the familiar formula

$$H = -\mathbf{m} \cdot \mathbf{B} \tag{6.5}$$

In particular, the energy of one magnetic dipole $(\mathbf{m}_1)$ in the field of another magnetic dipole $(\mathbf{m}_2)$ is

$$H = -\frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - \mathbf{m}_1 \cdot \mathbf{m}_2] - \frac{2}{3}\mu_0 \mathbf{m}_1 \cdot \mathbf{m}_2 \delta^3(\mathbf{r}) \tag{6.6}$$

where r is their separation, and the classical interaction energy is interpreted as the hyperfine structure Hamiltonian, and the energies of the singlet and triplet spin one evaluated in first - order perturbation theory Hamiltonian. The ground - state wave function for atomic hydrogen.is

$$\psi_0 = (\pi a^3)^{-1/2} e^{-r} |s> \tag{6.7}$$

where $a = 0.52917706$ Å is the bohr radius and $|s>$ denotes the spin of the electron. Treating the dipole - dipole interaction (see in Eq.(6.6)) as a perturbation, the energy of the general state is shifted :

$$E = \frac{2}{3} \frac{\mu_0 \hbar^2}{\pi a^2} \gamma_e \gamma_p \tag{6.8}$$

where the γ are the two gyromagnetic ratios, $\hbar = 1.05457168 \times 10^{-34}$ J $\cdot$ s is Planck's constant. The energy gap is

$$\Delta E = \frac{2}{3} \left(\frac{\mu_0 \hbar^2}{\pi a^3} \right) \gamma_e \gamma_p$$

$$= 5.884 \times 10^{-6} \text{ eV} \tag{6.9}$$

From this result, the frequency(ν) of the photon emitted in a transition from the triplet to the singlet state equals 1422.8 MHz, which differ from the experimental value about 0.2% that is attributable to quantum electrodynamical corrections, and its wavelength(λ) approximates 21 cm.

The observation of the 21-cm wavelength line (or called the 21 cm line) of hydrogen marked the birth of spectral-line radio astronomy. It was first observed in 1951 by Harold Ewen and Edward M. Purcell at Harvard, followed soon afterward by observers in Holland and Australia. The prediction that he 21 cm line should be observable in emission was made in 1944 by Dutch astronomer H. C. van de Hulst.

References

[1] D. Park, Introduction to the Quantum Theory, 2nd ed. (McGraw-Hill, New York, 1974), Sec. 14.3.

[2] A.A. Penzuas and R.W. Wilson, Astrophys. J. 142, 419 (1965).

[3] H.I.Ewen and E.M.Purcell, Nature168, 356 (1951); G.Cocconi and P.Morrison, Nature 184, 844 (1959).

[4] Some of the material presented here will be found (on a more sophisticated level) in J.D. Jackson, Classical Electrodynamics, 2nd ed. (Wiley, New York, 1975) Secs. 4.1, 5.6, and 5.7.

[5] P. Lorrain and D. Corson, Electromagnetic Fields and Waves, 2nd ed. (Freeman, San Francisco, 1970) , p.62.

[6] J.R. Reitz, F.J.Milford, and R.w. Christy, Foundation of Electromagnetic Theory, 3rd ed. (Addision-Wesley, Reading, MA, 1979), p.39.

[7] Reference 5. Sec.2.13. A somewhat different proof is given here.

[8] Reference 5, p.61.

[9] E.M.Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1965), p.327.

[10] If the ideal dipole is represented by infinitesimally displaced plus and minus charges, the delta-function term pertains to the region between them, where (unlike all other points on the axis) the field is opposite to p.

[11] Reference 5, p. 320.

[12] Reference 6, p. 177.

[13] Reference 5, p. 389-391. A somewhat different proof is given here.

[14] This follows from the divergence theorem, $\int \nabla \cdot v d\tau = \int V \cdot da$, if you let $V = A \times c$, where **c** is an arbitrary constant vector.

[15] Reference 6, p. 175.

[16] Reference 5, p. 321.The entire proof of Theorem 2 can be cast into the (perhaps more familiar) language of line currents by the standard transcription $\int()\mathbf{J}d\tau \rightarrow I\int()d\mathbf{I}$.

[17] Reference 6, p. 208.

[18] If the ideal dipole is represented by infinitesimal current loop, the delta-function term pertains to the region inside the loop, where (unlike all other points on the plane) the field is parallel to m.

[19] Reference 4,p. 186.

[20] Reference 1,p. 264.

[21] Reference 1,p. 229.

[22] Particle Data Group, Rev. Mod.Phys. 52. (2), Part II (1980).

[23] By contrast, when $l > 0, |\psi(0)|^2 = 0$, and the hyperfine structure is due exclusively to the *first* term in H. See L. D. Landau and E.M. Lifschitz, *Quantum Mechanics*, 3rd ed. (Pergamon, New York, 1977), p. 499.

[24] Reference 6, p. 233.

[25] Reference 1, Sec 6.4.

[26] Reference 22.

[27] P. Kusch and A. G. Prodell,Phys.Rev. 79, 1009(1950); S. Crampton, D Kleppner, and N.F. Ramsey, Phys. Rev. Lett. 11, 338 (1963) ; L. Essen, R.W. Donaldson, M.J Bangham, and E.G.Hope, Nature 229, 110 (1971).

[28] S. J. Brodsky and S. D. Drell, Ann. Rev.Nucl. Sci.20, 147(1970). The lowest-order hyperfine splitting [Eq.(37)] was first obtained by E. Fermi, Z.Phys. 60, 320(1930).

[29] A. P. Mills, Jr. and G.H. Bearman, Phys.Rev. Lett. 34, 246(1975); P.O. Egan, W.E. Frieze, V.W. Hughes, and M.H. Yam, Phys. Lett. 54A, 412(1975).

[30] H. A. Bethe and E. E. Salpeter, *Quantum Mechanic of One-and Two electron Atoms* (Plenum, New York, 1977), p.116.

[31] D. E. Casperson, T. W. Crane, A. B. Denison, P. O. Egan, V. W. Hughes, F. G. Mariam, H. Orth, H. W. Reist, P. A. Souder, R. D. Stambaugh, P.A. Thomson, and G.zu Putlitz, Phys.Rev. Lett.38, 956(1977).

[32] G.T. Bodwin and D. R. Yennie, Phys. Rep. (Phys. Lett. C) 43, 267(1978); W.E. Caswell and G.P. Lepage, Phys Rev. Lett.41, 1092(1978).

[33] See V. W. Hughes, in *High Energy Physics and Nuclear Structure-1975* (AIP Conference Proceedings, Number 26) edited by D.E. Nagle (AIP, New York, 1975), p. 533.

[34] D. J. Griffiths, Am. J. Phys. 50(8), 698-703(1982).