Israel Junction Conditions on Hypersurface from Variational Principle Approach

เงื่อนไขการเชื่อมต่ออิสราเอลบนพื้นผิวมิติเกินจากการใช้หลักการแปรผัน

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Einstein's field equation can be derived using variational with boundary term that vanishes at infinity. In case of bounded spacetime, we can include boundary terms into gravitational action. The result is Israel junction conditions on a hypersurface. The method has been widely applied to braneworld gravity and braneworld cosmology.

สมการสนามของไอน์สไตน์เป็นกฎที่สำคัญข้อหนึ่งในทฤษฎีสัมพัทธภาพทั่วไป สามารถถูกหาได้โดยใช้วิธีการแปรผัน โดยเลือกให้พจน์ที่ขอบเขตเป็นศูนย์ที่อนันต์ ในรายงานนี้ กาลอวกาศของเรามีขอบเขต เราสามารถเพิ่มพจน์ที่ขอบเขตให้กับแอคชันความโน้มถ่วงได้ อันนำมาซึ่งเงื่อนไขการเชื่อมต่ออิสราเอลบนพื้นผิวมิติเกิน วิธีการนี้ถูกประยุกต์ใช้อย่างแพร่หลายในทฤษฎีความโน้มถ่วงแบบภพแผ่นและจักรวาลวิทยาแบบภพแผ่น

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INTRODUCTION

Among several variational principles used in general relativity, the most common one is to use the Einstein–Hilbert action which an integral over the invariant four-volume to the Ricci scalar (curvature scalar). The structure of the variation can be split into two parts: one yields Einstein's field equation in vacuum (excluding matter term), and another, with a four-divergence form resulting a surface term known as the Gibbons–Hawking boundary term. In general, surface terms are neglected in derivation of field equation from an action. But in this case, our manifold (spacetime) is bounded. An action considered here is the Einstein–Hilbert action with surface term. When our manifold is bounded, it automatically splits into two parts. They are called a bulk and a hypersurface. Variation of surface terms is done with respect to an induced metric on hypersurface, not the metric on the manifold. The result leads to the Israel junction conditions, if matter action is included on hypersurface.

METHODOLOGY

We start by reviewing the standard derivation of Einstein's field equation from the Einstein–Hilbert action.

$$S_{\rm EH} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\, R\, {\rm d}^4x, \qquad (1)$$

where $\mathcal{M}$ denotes the manifold, g is a determinant of the metric $g_{\mu\nu}$ which has a signature of $(-,+,+,+)$ in our convention, R is the Ricci scalar and $\kappa = 8\pi G$. Varying this action with respect to the metric, we get

$$\delta S_{\rm EH} = \frac{1}{2\kappa}\int_{\mathcal{M}} \sqrt{-g}\, G_{\mu\nu}\, \delta g^{\mu\nu}\, {\rm d}^4x + \delta S_{\rm S}, \qquad (2)$$

where $G_{\mu\nu}$ is the Einstein tensor and $\delta S_{\rm S}$ is a surface term. It has the following form

$$S_{\rm S} = -\frac{1}{\kappa}\int_{\Sigma} \sqrt{-h}\, K\, {\rm d}^3x. \qquad (3)$$

Σ denotes the hypersurface or our boundary of the manifold, h is the determinant of the induced metric on Σ and K is the trace of the extrinsic curvature of that boundary. We can define the energy–momentum tensor in the usual way,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm M}}{\delta g^{\mu\nu}},$$

where $S_{\rm M}$ denotes the matter action. The full action is $S = S_{\rm EH} + S_{\rm M}$, which does not lead to the Einstein's field equation since our manifold is assumed to be a restrictive spacetime. The surface term therefore cannot be set to zero. In the derivation of gravitational action, the metric and induced metric do not vary on the boundary such that $\delta g_{\mu\nu} = 0$ and $\delta h_{ab} = 0$. (Here we use Greek indices $(\mu, \nu, \ldots)$ which run from 0 to 3 to denote coordinates on the manifold, and Latin indices $(a, b, \ldots)$ which run from 1 to 3 to denote coordinates on Σ.) We generalize the equation (2) by naming the first term as the variation of gravitational action, and get the result

$$S_{\rm Gravity} = \frac{1}{2\kappa}\left[\int_{\mathcal{M}} \sqrt{-g}\, R\, {\rm d}^4x + 2\int_{\Sigma} \sqrt{-h}\, K\, {\rm d}^3x\right]. \quad (4)$$

This action is used to derive of Israel condition. Hereafter, we will not fix the metric and the induced metric on Σ for this derivation. The variation then gives three terms as follows,

$$\delta S_{\rm Gravity} = \frac{1}{2\kappa}\left[\int_{\mathcal{M}} \sqrt{-g}\, G_{\mu\nu}\, \delta g^{\mu\nu}\, {\rm d}^4x + 2\int_{\Sigma} \sqrt{-h}\, \big(K_{ab} - h_{ab}K\big)\, \delta h^{ab}\, {\rm d}^3x + \int_{\Sigma} \sqrt{-h}\, {\rm D}_b\big(h^b{}_a\, \delta n^a\big)\, {\rm d}^3x\right]. \quad (5)$$

${\rm D}_b$ in the last term denotes a covariant derivative associated with the hypersurface Σ, and n^a is the normal unit vector to Σ, whose normalization condition is $n_a n^a = -1$ for Σ is spacelike [3]. The last term in the equation (5) is a divergence term. It vanishes by using Stoke's theorem with a boundary on Σ at infinity. Finally, the variation of gravitational action is

$$\delta S_{\rm Gravity} = \frac{1}{2\kappa}\left[\int_{\mathcal{M}} \sqrt{-g}\, G_{\mu\nu}\, \delta g^{\mu\nu}\, {\rm d}^4x + \int_{\Sigma} \sqrt{-h}\, \big(K_{ab} - h_{ab}K\big)\, \delta h^{ab}\, {\rm d}^3x\right]. \quad (6)$$

The energy–momentum tensor,

$$t_{ab} = \frac{2}{\sqrt{-h}}\, \frac{\delta S_{\rm Mat}}{\delta h^{ab}},$$

is included on Σ by adding an action for matter, $S_{\rm Mat}$, in the gravitational action. The Einstein tensor, $G_{\mu\nu}$ in the bulk vanishes because t_{ab} is only included on Σ. Therefore, the total action, $S_{\rm Gravity} = S_{\rm EH} + S_{\rm S} + S_{\rm Mat}$ gives the *Israel junction conditions*

$$K_{ab} - h_{ab}K = -8\pi G\, t_{ab}. \quad (7)$$

The energy–momentum tensor on a hypersurface does not necessarily conserve because energy can flow from the hypersurface to the bulk. This can be seen by taking the divergence of the Israel junction conditions:

$${\rm D}_a t^{ab} = -\frac{1}{8\pi G}\Big[{\rm D}_a K^{ab} - {\rm D}_a\big(h^{ab}K\big)\Big]. \quad (8)$$

The right side can be evaluated using Gauss–Codacci relations [2], giving

$$\begin{aligned} {\rm D}_a t^{ab} &= -\frac{1}{8\pi G}\, R_{cd}\, n^c h^{db} \\ &= -T_{cd}\, n^c h^{db}, \end{aligned} \quad (9)$$

where T_{cd} is the bulk energy–momentum tensor. This equation describes conservation of energy when T_{cd} moves from the bulk to the boundary.

CONCLUSION

Usually, in deriving the Einstein's field equation using the variational principle, we use a boundary term that vanishes at infinity. But because of an attempt to explain bounded manifolds or space-times, e.g. black holes, a boundary term cannot be set zero. By varying this term, the Israel junction condition on a hypersurface is obtained. It acts as a connection between the events inside and on the boundary. There is wide application of the Israel junction condition to cosmology, and to the idea of Braneworld gravity and Braneworld cosmology.

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