Proceedings of the First National Symposium on Physics Graduate Research (1^{st} NSPG)
29 June – 1 July 2006, Chulabhorn Dam, Khon San, Chaiyaphum, Thailand

Cosmological Dynamics from Modified $f(R)$ Gravity in Einstein Frame*

พลวัตรเชิงจักรวาลวิทยาจากการปรับแต่งความโน้มถ่วงแบบ $f(R)$ ในกรอบไอน์สไตน์

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In this paper, we investigate and analyze the cosmological dynamics of the universe which is an effect of modified $f(R)$ gravity emerging at cosmological scale. We choose Einstein frame as a physical frame. We consider phase portraits at present and late time of the universe from modified $f(R)$ model gravity. This result yields an acceleration phase of the universe without existence of dark energy.

งานวิจัยชิ้นนี้ เราได้สำรวจและวิเคราะห์พลวัตรเชิงจักรวาลวิทยาของเอกภพ โดยที่ผลของการปรับแต่งความโน้มถ่วงแบบ $f(R)$ จะปรากฏขึ้นที่ระดับระยะทางเชิงจักรวาลวิทยา เราเลือกกรอบของ ไอน์สไตน์ เป็นกรอบทางกายภาพ และพิจารณาเฟสของแผนภาพที่ช่วงปัจจุบัน และช่วงปลายของเอกภพจากแบบจำลองการปรับแต่งความโน้มถ่วงแบบ $f(R)$ ผลที่ได้คือเอกภพมีช่วงการขยายตัวออกแบบเร่งโดยไม่จำเป็นต้องอาศัยพลังงานมืด

PACS numbers: 04.50.+h, 98.80.-k
Keywords: alternative theory of gravity, accelerating universe, cosmological dynamics, conformal transformation, Einstein frame.

INTRODUCTION

The alternative explanation to solve Dark Energy problem is modification of Einstein gravity theory to be source of acceleration universe. Carroll *et al.* [1] purposed of modified model of gravity by adding inverse power of Ricci scalar into Einstein–Hilbert action giving deviation from general relativity at small curvature and causing acceleration at large scale. This happens either with de Sitter and anti de Sitter solutions in vacuum case which provides purely gravitation driven accelerating universe and good agreement for testing in current cosmological observation data. The modified $f(R)$ gravity model is very interesting because it was shown it can be derived from string/M-theory [3]. Many works on various aspects and extends this model. We will consider cosmological dynamics of modified gravity like [1, 2], but we use metric variation formalisms in Einstein frame phase portraits diagrams. We consider this scenario in Einstein frame as the physical frame and this frame gives us to self gravity of scalar field effective potential $V(\phi)$ [4, 5, 6].

METHODOLOGY

In this section we review the conformal transformation method follow [4, 5, 6]. The action for $f(R)$ gravity of Jordan frame without matter field can be written in form,

$$S_{\rm J} = \frac{1}{2\kappa^2}\int f(R)\sqrt{-g}\,{\rm d}^4x \qquad (1)$$

where $\kappa^2 = 8\pi G$. The field equation from (1) in Jordan frame gives us the fourth order and very complicate to solve [1, 2]. We choose Einstein frame as physical frame for it is simplicity. We define the conformal transformation between Jordan and Einstein frame as [13]

$$^{\rm E}g_{ab} = {\rm e}^{\phi}g_{ab}. \qquad (2)$$

where $\phi \equiv \ln f'(R)$. We obtain action in Einstein frame by use conformal transformation in (2) i.e.

*cf. arXiv: astro-ph/0606612.

$g_{ab} \rightarrow e^{\phi} g_{ab}$ [3, 4, 5, 6]

$$S_{\rm E} = \frac{1}{2\kappa^2} \int \left[{}^{\rm E}R - \frac{3}{2} \nabla_a \phi \nabla^a \phi - V(\phi) \right] \sqrt{-{}^{\rm E}g}\, {\rm d}^4 x \qquad (3)$$

where $V(\phi)$ is effective potential which depends on choice of $f(R)$ models determined by $V = \left[Rf'(R) - f(R) \right] / [f'(R)]^2$ where $f'(R) \equiv {\rm d}f(R)/{\rm d}R$. We vary action (3) with respect to ${}^{\rm E}g^{ab}$, we get

$${}^{\rm E}R_{ab} - \frac{1}{2}\, {}^{\rm E}g_{ab}\, {}^{\rm E}R = 3 \nabla_a \phi \nabla_b \phi - {}^{\rm E}g_{ab} \left[\frac{3}{2} g^{cd} \nabla_c \phi \nabla_d \phi + V(\phi) \right] \qquad (4)$$

and when varying respect to ϕ, we get the Klein–Gordon equation

$$\nabla_a \nabla^a \phi + \frac{V_\phi}{3} = 0 \qquad (5)$$

where $V_\phi \equiv {\rm d}V/{\rm d}\phi$. Now we consider present universe based on FRW metric which is spatially flat,

$${\rm d}s^2 = -{\rm d}t^2 + a^2(t) \left[{\rm d}x^2 + {\rm d}y^2 + {\rm d}z^2 \right]. \qquad (6)$$

where $a(t)$ is scale factor of universe.

Now we calculate the modified Friedman equation in Einstein frame from line element (6), the $(0, 0)$ component of field equation (4) the Friedman equation and the Klein–Gordon equation become

$$H^2 = \frac{1}{4} \dot{\phi}^2 + \frac{V(\phi)}{6} \qquad (7)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{V_\phi}{3} = 0. \qquad (8)$$

where $H \equiv \dot{a}/a$ is Hubble parameter and $\dot{a} \equiv {\rm d}a/{\rm d}t$. We also obtain the time derivative of Hubble parameter by using (7) and (8)

$$\dot{H} = -\frac{3}{4} \dot{\phi}^2. \qquad (9)$$

Next section we will consider cosmological dynamics in phase portrait in some particular types of $f(R)$ gravity model.

We consider scalar field dominate the late universe. The acceleration can happen when the field at late time moves very slow. Therefore we can use the slow-roll approximation in this phase [8] i.e. $\ddot{\phi} \approx 0$ and $\dot{\phi}^2 \ll V$. Using the slow-roll approximation condition directly to the Friedman equation in equation of motion of scalar field (8), we get the late time trajectory of scalar field evolving to approach an asymptotic attractor solution in phase space as [14]

$$\dot{\phi} \simeq -\frac{V_\phi}{9H}. \qquad (10)$$

We define a new variables as

$$X = \phi\,, \quad Y = \dot{\phi}\,, \quad Z = V(\phi). \qquad (11)$$

Using new variables in (11) substitute into (8) and (9) we obtain autonomous system,

$$Y = \dot{\phi} \qquad (12)$$

$$\dot{H} = -\frac{3}{4} Y \qquad (13)$$

$$\dot{Y} = -3HY - \frac{1}{3} \frac{{\rm d}Z}{{\rm d}X}. \qquad (14)$$

Next we will investigate and analyze the phase space portraits for some types of the effective potential $V(\phi)$ which depends on choices of $f(R)$ models of gravity. By using autonomous system of equations in (12), (13) and (14).

RESULTS, DISCUSSION AND CONCLUSION

1. The $f(R) = R - \mu^{2(n+1)}/R^n$ model was proposed by [1, 2] (for $n = 1$). In this model the effective potential (at the late time, R is very small) in Einstein frame is [7]

$$V(\phi) \simeq \frac{2}{\mu^2} e^{-(3/2)\phi}. \qquad (15)$$

This potential in our approximation look like the power law inflation, which gives scale factor increase proportion to $a \sim t^{4/3}$. We note that potential increases when curvature is minimum and

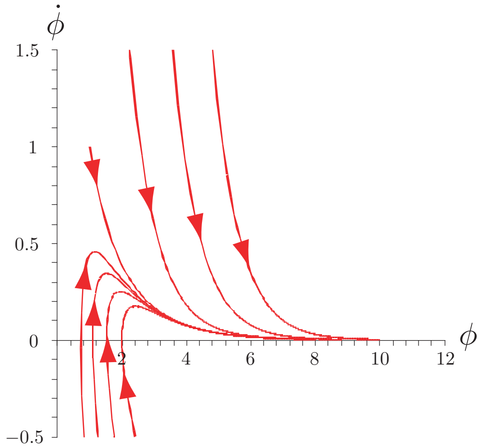

Figure 1: The phase portrait of $\dot{\phi}$ vs ϕ in the modified $1/R$ gravity model, when $\mu = 1$.

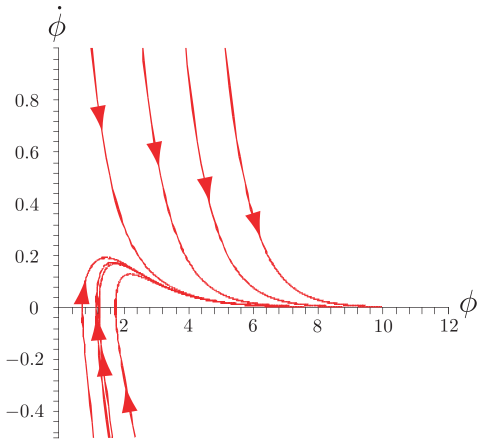

Figure 2: The phase portrait of $\dot{\phi}$ vs ϕ in the modified $\ln R$ gravity model, when $\alpha = 1$.

therefore the universe begins to have the power law expansion. We also consider phase space portrait of scalar fields of modified $1/R$ gravity model, which presents attractor in phase plane as Figure 1.

2. The $f(R) = R + \alpha \ln\left(R/\gamma^2\right)$ **model** was purposed by [9]. The effective potential in (at the late time, R is very small) Einstein frame is [9]

$$V(\phi) \simeq \alpha\, \mathrm{e}^{-2\phi}\phi. \tag{16}$$

We note that when the curvature rolls slow down until reaching local minimum ($R \neq 0$) the curvature stay here. This may correspond to the present accelerating universe [9]. This potential can be shown to have $w \to -1$ (where w is the equation of state) by fine-tuning parameters. We also consider phase space portrait of scalar fields of modified $\ln R$ gravity model, which has attractor in phase plane as Figure 2.

Let we discuss from above phase space portraits both $1/R$ and $\ln R$ gravity model, both models give late time attractor. These results mean our universe is in accelerating expansion. The evolution when scalar field is at some given point on the potential has to be independent of any initial conditions. It is surprised our analysis in Einstein frame gives similar result as in [11, 12] which chooses Jordan frame as physical frame. They was shown that $1/R$ model gravity has late time attractor on phase space portrait. Both Jordan and Einstein frame are analyzed similarly $1/R$ gravity model. We still can not tell which one is physical frame. This work is still ongoing.

Acknowledgement

I would like to thanks Itzadah Thongkool for helpful discussion and critical comments, Kiattisak Thepsuriya for proofreading and calculation, Sarayut Pantian for very helpful and support for numerical results, Chakkrit Kaeonikhom for a nice graphics and finally thanks Burin Gumjudpai for a useful discussion, critical comments and financial support.

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[13] For convenient to find conformal factor e^{ϕ}, we use Palatini formalisms varying action with respect ${\Gamma^c}_{ab}$.

[14] For detail of slow-roll approximation and relate topics see [10].