Phantom Acceleration of the Universe from Loop Quantum Cosmology

Daris Samart

Submitted in partial fulfilment of
the requirements for the award of the degree of

Master of Science in Applied Physics
M.S.(Applied Physics)

Fundamental Physics & Cosmology Research Unit
The Tah Poe Academia Institute for Theoretical Physics & Cosmology
Department of Physics, Faculty of Science
Naresuan University

February 13, 2008

To My Mother and My Dreams

Phantom Acceleration of the Universe from Loop Quantum Cosmology

Daris Samart

The candidate has passed oral examination by members of examination panel. This report has been accepted by the panel as partially fulfilment of the course 271599 Thesis.

. . . . Chair & Supervisor
Dr. Burin Gumjudpai, BS MSc PhD AMInstP FRAS

. . . . Member
Dr. Chanun Sricheewin, BSc MPhil PhD

. . . . Member
Assoc. Prof. Dr. Chayan Boonyarak, BEd (Hons) MEd PhD

. . . . Member
Assoc. Prof. Dr. Yupeng Yan, BS MS Dr rer nat.

. . . . Secretary
Assoc. Prof. Charun Promsuwan, BEd MSc

Acknowledgements

I would like to thanks Burin Gumjudpai, my suporvisor, who gave motivation and opportunity to me to open up my eyes in the real physics world that I have never known. Thank for his support in every way and motivating me to be a good physicist and nice guys, thank for his good will to me all the time, thank for his explanation of difficult concepts and thank for his training of LaTeX program. Anyway I feel indebtedness for everything that he gave me so much whereas I don't know how to give back to him. I also thank Itzadah Thongkool for willingness to spend his time discussing to me, thank for help on some usually gave useful attitudes to me. I also thank Shinji Tsujikawa and M. Sami for their kindness, good will and gave illustration on many things in cosmology and physics researches. Thank Narit Pidojkrat for his supports in many good papers and his usual encouragement. Thanks Kiattisak Tepsuriya for his usually support like many good text books to me and a great friendship. I also thank Sarayut Pantian my classmate for his great friendship and his support on computational techniques. Thank Warintorn Sreethawong for her support in many good papers. Thank Chankit Kanchong for some discussions and way out of some difficult problems. Thank Chanun Sricheewin for helpful discussion in Quantum Mechanics. Thank Alongkorn Khudwilat for discussion in interesting problems. Thank Chakkrit Keaownikom for editing nice figures. All other members in the new founded Tah Poe Academia Institute for Theoretical Physics *&* Cosmology (formerly the Tah Poe Group of Theoretical Physics: TPTP). Thank for their encouragement that push me to do my works. Finally, thank for great encouragement, support and kindness of my mother who has been giving my life, my heart, my blood, my soul and love to me all the time.

Title: `Acceleration of the Universe from Loop Quantum Cosmology`

Candidate: `Mr.Daris Samart`

Supervisor: `Dr.Burin Gumjudpai`

Degree: `Master of Science Programme in Applied Physics`

Academic Year: `2006`

Abstract

This thesis present the features of loop quantum gravity in the cosmological setting so called loop quantum cosmology. We have shown the quantization of the flat, homogeneous and isotropic FRW spacetime via the loop representation using the classical approximation of the Hamiltonian constraint. This procedure gives the effective Friedmann equation. The loop quantum-geometrical correction in the effective Friedmann equation can avoid the Big Rip singularity from the phantom field dark energy background in this scenario.

Contents

1 Introduction **1**

1.1 Background 1
1.2 Objectives 3
1.3 Frameworks 3
1.4 Expected Use 3
1.5 Tools 3
1.6 Procedure 4
1.7 Outcome 4

2 Related Theory :Standard and Loop Quantum Cosmology **5**

2.1 Standard Cosmology and Dark Energy 5
2.1.1 The Standard Model of Cosmology and the Evidence of the Accelerating Universe
2.1.2 The Scalar Field Models of Dark Energy 9
2.2 Loop Quantum Cosmology 15
2.2.1 The Ashtekar Variables in General Relativity 15
2.2.2 The Isotropic Hamiltonian of the flat FRW spacetime 19
2.2.3 Isotropic loop quantum cosmology 24

3 Methodology:Dynamical System in Cosmology **32**

3.1 Introduction to the Dynamical System 32
3.2 Dynamical System in Standard Cosmology 34
3.2.1 Dynamics of Scalar Field in the FRW Cosmology 35

4 Results:The Phantom Field Dynamics in Loop Quantum Cosmology **40**

4.1 Dynamical Analysis 40
4.1.1 Autonomous System 41

4.1.2 Fixed points 43
4.2 Stability Analysis 43
4.3 Numerical Results 46
4.3.1 Class I solutions 46
4.3.2 Class II solutions 47

5 Conclusion **51**

A Conventions **52**

B $3+1$ ADM and Hamiltonian Formalisms in GR **55**
B.1 The $3+1$ ADM Formulation in General Relativity 55
B.2 Hamiltonian Formalisms in General Relativity 59

C Detailed calculation **63**
C.1 Derivation of the Spin Connection 63
C.2 Derivation of the Holst Action 64
C.3 Determination of the Perturbation Matrixand their Eigenvalues in Section 4.1.1 70

Chapter 1

Introduction

1.1 Background

Recently, present accelerating expansion of the universe has been confirmed with observations via cosmic microwave background anisotropies [1, 2], large scale galaxy surveys [3] and type IA supernovae [4, 5]. However, the problem is that the acceleration can not be understood in the regime of standard cosmology. This motivates many groups of cosmologists to find out the answers. Proposals to explain this acceleration made till today could be, in general, categorized into three ways of approach [6]. In the first approach, in order to achieve acceleration, we need some form of scalar fluid so called dark energy with equation of state $P = w\rho$ where $w < -1/3$. Various types of models in this category have been proposed and classified (for a recent review see Ref. [7, 8]). The other two ways are that accelerating expansion is an effect of backreaction of cosmological perturbations [10] or late acceleration is an effect of modification in action of general relativity. This modified gravity approach includes braneworld models (for review, see [11]). Till today there has not yet been truly satisfied explanation of the present acceleration expansion.

Considering dark energy models, precise observational data analysis (combining CMB, Hubble Space Telescope, type Ia Supernovae and 2dF datasets) allows equation of state $P = w\rho$ with constant w value between -1.38 and -0.82 at the 95 % of confidence level [12]. The interpretation of various data bring about a suggestion that dark energy should be in the form of phantom field-a fluid with $w < -1$ which violates dominant energy condition, $\rho \geq |P|$, rather than quintessence field [13]. The phantom equation of state $P < -\rho$ can be attained by negative kinetic energy term of the phantom field. However there are some types of braneworld model [14] as well as Brans-Dicke scalar-tensor theory that can also yield phantom energy [17]. There has been investigation on dynamical properties of the phantom field in the standard FRW background with exponential and inverse-power law potentials by [20, 21] and with other forms of potentials by [22, 23]. These studies describes fate of the phantom dominated universe with different steepness of the potentials.

A problem for phantom field dark energy in standard Friedmann-Robertson-Walker (FRW) cosmology is that it leads to singularity. Fluid with w less than -1 can end up with future singularity so called the Big Rip [24] which is of type I singularity according to classification by [26, 27]. The Big Rip singularity corresponds to $a \to \infty, \rho \to \infty$ and $|p| \to \infty$ at finite time $t \to t_s$ in future. Choosing particular class of potential for phantom field enables us to avoid future singularity. However, the avoidance does not cover general classes of potential [22]. In addition, alternative model in which two scalar fields appear with inverse power-law and exponential potentials can also avoid the Big Rip singularity [28]. Nevertheless, nature of the scalar field is still an open question.

In this thesis, a fundamental background theory in which we are interested is Loop Quantum Gravity (LQG). This theory is a non-perturbative type of quantization of gravity and is background-independent [29, 30]. It has been applied in cosmological context as seen in various literatures where it is known as Loop Quantum Cosmology (LQC) (for review, see Ref. [32]). Effective loop quantum modifies standard Friedmann equation by adding a correction term $-\rho^2/\rho_{\rm lc}$ into the equation [33, 34, 35, 31, 36]. When this term becomes dominant, the universe begins to bounce and then expands backwards. A merit of LQG is the resolution of singularity problem in various situations [37, 30, 33, 38]. Nice feature of LQC is avoidance of the future singularity from the correction quadratic term $-\rho^2/\rho_{\rm lc}$ in the modified Friedmann equation of LQC [39] as well as the singularity avoidance at semi-classical regime [40]. The early-universe inflation has been also studied in the context of LQC at semi-classical limit [41, 42, 36, 43, 44, 45, 47]. Investigation of phantom field dynamics and its late time behavior in the loop quantum cosmological context could reveal some interesting features of the model.

1.2 Objectives

We wish to study the cosmological dynamics of the phantom field dark in the LQC energy background. Such model of dark energy in standard cosmology can produce future singularity so-called the Big Rip singularity at the late time. Our hypothesis, is the Big Rip singularity can be avoided by the quantum-geometrical effect from LQC.

1.3 Frameworks

- To study acceleration of the universe via the scalar field model of dark energy.
- To explain the Big Rip singularity in the standard model of the universe.
- To obtain the effective Friedmann equation from LQC.
- To use the dynamical system for doing analysis dynamics of the phantom field dark energy in LQC background.
- To understand the nature of singularity and dark energy in context of LQC.

1.4 Expected Use

- To obtain the effective friedmann equation from LQC.
- To avoid the Big Rip singularity from the LQC effect.
- A derived-in-detailed report for those who interest with thoroughly calculation from the $3 + 1$ ADM formulation, the Ashtekar variables and the LQC.

1.5 Tools

- Text books in physics and mathematics.
- A high efficiency personal computer.
- Software e.g. LaTeX, WinEdit , Mathematica and Photoshop.

1.6 Procedure

- Studying basic theory of the LQG.
- Studying quantization of the Friedmann-Robertson-Walker (FRW) spacetime.
- Extracting the effective Friedmann equation from the LQC.
- Studying dynamical system in cosmology.
- Setting the autonomous system from the effective Friedmann equation in the phantom dark energy background.
- Finding the critical points and demonstrating their stabilities.
- Concluding cosmological consequence from the dynamical system.

1.7 Outcome

- Understanding of basic ideas of LQG, LQC and nature of dark energy.
- Understanding behavior of phantom dynamics under LQC background.

Chapter 2

Related Theory : Standard and Loop Quantum Cosmology

2.1 Standard Cosmology and Dark Energy

In the first section of this chapter we will briefly review on the standard model of cosmology, evidence of the accelerating universe and the scalar field model of dark energy.

2.1.1 The Standard Model of Cosmology and the Evidence of the Accelerating Universe

The hot big bang universe

The standard model of cosmology is the hot big bang theory which based on general relativity [48, 49, 50] (for readable review see [52]). The hot big bang universe is confirmed by the existence of cosmic microwave background (CMB) radiation [53]. General relativity has strongly suggested the universe must be created from the explosion of primordial singularity. The cosmological solutions from general relativity also give the non-static or expanding universe [54]. The cosmological model is based

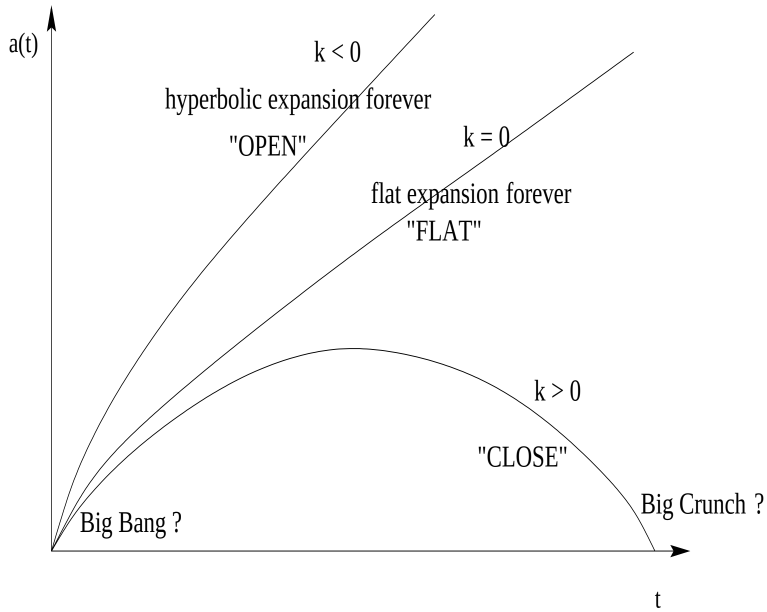

Figure 2.1: Curvature and fate of the universe (expand forever or re-collapse) from [52] .

on the cosmological principle i.e. the universe is homogeneous and isotropic (more detail see [52]) at the very large scale. From such principle it implies the geometrical line element of the universe,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\theta^2 + r^2 \sin\theta \mathrm{d}\phi^2 \right] \tag{2.1}$$

where $a(t)$ is the scale factor which implies size of the universe and k is the curvature parameter, such parameter has values as -1 for open universe, 0 for flat universe, 1 for close universe (see figure 2.1).

Let us start by considering the dynamical equation of the universe. The Einstein field equation can be written as [49]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{2.2}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci or curvature scalar, G is the Newton's gravitational constant and $T_{\mu\nu}$ is the energy-momentum tensor. The energy-momentum tensor for the perfect fluid is given by

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu} \tag{2.3}$$

where ρ is energy density, P is pressure and u_μ is four-velocity.

The FRW line element gives the Ricci tensor and scalar as

$$R_{00} = -3\,\frac{\ddot{a}}{a} \tag{2.4}$$
$$R_{ij} = (a\ddot{a} + 2\dot{a}^2 + 2k)\delta_{ij} \tag{2.5}$$
$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). \tag{2.6}$$

The $(0,0)$ component of the Einstein field equation, using above three equations and equation in (2.3) also, we obtain

$$H^2 = \frac{8\pi G}{3}\,\rho + \frac{k}{a^2} \tag{2.7}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, this parameter mean the rate of expansion of the universe. For the $(i\,,j)$ component, we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3P)\,. \tag{2.8}$$

The equation (2.7) is known as the Friedmann equation describing the dynamics of the universe. The equation (2.8) is known as the acceleration equation or the second Friedmann equation. From the covariant conservation of the energy-momentum tensor i.e. $(\nabla^\mu T_{\mu\nu} = 0)$, we obtain

$$\dot{\rho} + 3H(\rho + P) = 0\,. \tag{2.9}$$

The Friedmann equation (2.7) can be written as

$$\Omega - 1 = \frac{k}{a^2H^2} \tag{2.10}$$

where $\Omega \equiv \rho/\rho_c$ is the density parameter, $\rho_c \equiv 3H^2/8\pi G$ is the critical density. The density parameter Ω can be determined the geometry of the universe as

$$\begin{aligned} k &= \ \ 0 \quad \Rightarrow \quad \Omega = 1 \\ k &= -1 \quad \Rightarrow \quad \Omega < 1 \\ k &= \ \ 1 \quad \Rightarrow \quad \Omega > 1\,. \end{aligned}$$

The precise cosmological observation of CMB strongly suggested that our universe is nearly flat [55]. Using equations (2.7), (2.8) and (2.9) to solve the flat-Friedmann equation $k = 0$, we obtain

$$
\begin{aligned}
H &= \frac{2}{3(1+w)t} && (2.11) \\
a &= a_0\, t^{\frac{2}{3(1+w)}} && (2.12) \\
\rho &= \rho_0\, a^{-3(1+w)} && (2.13)
\end{aligned}
$$

where a_0, ρ_0 are the arbitrary constants and $w \equiv P/\rho$ is the equation of state of the perfect fluid. For the dust matter has $w = 0$ and the radiation matter has $w = 1/3$.

Let us consider the acceleration equation in (2.8). We can use this equation regardless of the geometry of the universe because this equation does not contain factor k. When we neglect the small value of the cosmological constant, the equation of state has value $w < -1/3$ fluid (the dark energy hypothesis) with $P < -\rho/3$. Thus $\ddot{a}$ is positive value and the universe is in acceleration.

The Evidence of the Accelerating Universe

In 1998 there were project to study supernovae type Ia, by two groups of physicists, Supernova Cosmology Project and the High-z Supernova Team. These groups discovered important results that make physicists alert. These are,

- The redshift spectrum measurement from supernovae type Ia has higher values than the redshift predicted in the open universe model ($k = -1$). This fact tells us that the expansion of universe is speeding up rather than slowing down.

- The component of the universe measurement found that the CMB results tell us the universe is flat. Such model must have the total density equal to the critical density. However the realistic CMB measurement found the total component of the universe equal to one-third of the critical density. It rise to gives critical question, why the 2/3 times of the critical density component are missing in the CMB measurement?

The acceleration expansion of the universe is most directly provided by the supernovae Ia observation and also strongly supported by many astronomical and cosmological phenomena e.g. CMB measurement, gravitational lensing, redshift galaxy survey and the large scale structure formation. After discovery these puzzles give doubt to physicists, and cosmologists. To solve the situation, they need to introduce the existence of some mystery fluid with negative pressure. The mystery fluid is known

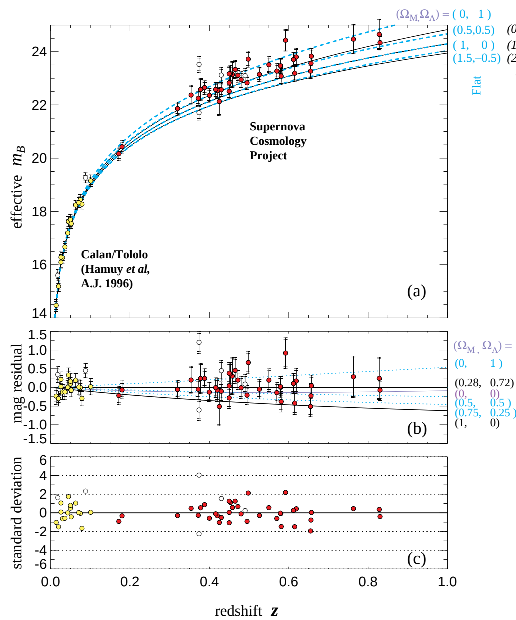

Figure 2.2: Hubble diagram from the Supernova Cosmology Project [56]. The bottom plot shows the number of standard deviations of each point from the best-fit curve.

as **dark energy**. The dark energy also give new questions, why the universe is accelerating, expansion? what dark energy actually is, and etc. Next section we will discuss about the resolutions of these problems.

2.1.2 The Scalar Field Models of Dark Energy

This section we introduce the resolution of the accelerating expansion of the universe. The most interesting and quite successful explanation of these problems is the dark energy hypothesis. Here we discuss the cosmological constant and some simple the scalar field models (the quintessence and the phantom dark energy).

The Cosmological constant

After the cosmological solutions are obtained by many physicist from general relativity and the cosmological principle. Such the solutions told us that the universe model is dynamics. Einstein who is the father of general relativity was not happy with the dynamic universe models. He added some constant by hand into the Einstein field

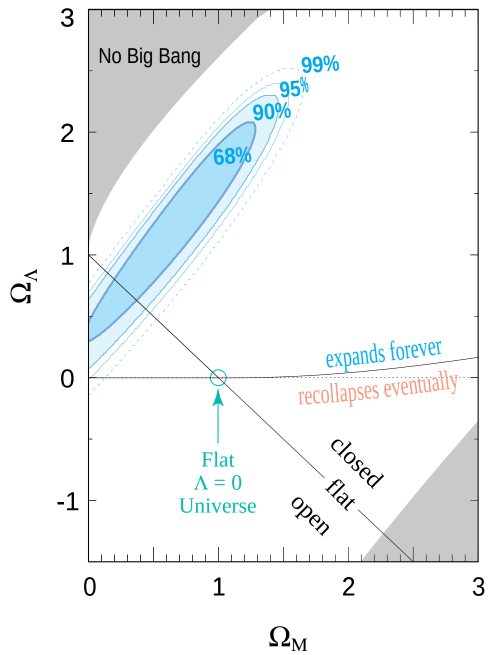

Figure 2.3: Constraints in the Ω_M-Ω_Λ plane from the Supernova Cosmology Project [56].

equation as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.14}$$

where the constant Λ is known as the cosmological constant. The cosmological solution of modified Einstein field equation with adding Λ gives static universe model. We can rewrite the above equation as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} + T^{\Lambda}_{\mu\nu}) \tag{2.15}$$

where $T^{\Lambda}_{\mu\nu}$ is given by

$$T^{\Lambda}_{\mu\nu} \equiv \rho_\Lambda \, g_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} \, . \tag{2.16}$$

We re-interpret the cosmological constant Λ as the energy density ρ_Λ .

Until Edwin Hubbles discovered the the redshift of the galaxy in 1929 , this phenomenon obviously indicates that universe is expanding. Then Einstein must drop his cosmological constant and said deathless phrase “It's my greatest blunder!” . The cosmological constant was ignored by physicist for almost 70 years.

The reborn of the cosmological constant become the first candidate of the dark

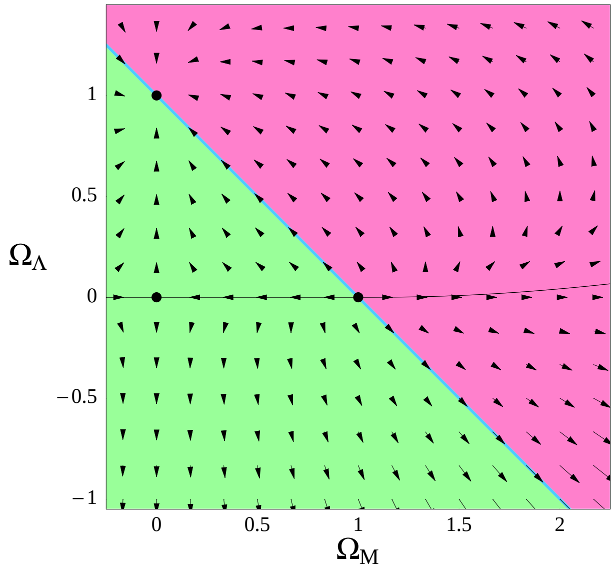

Figure 2.4: Dynamics for $\Omega = \Omega_{\rm M} + \Omega_\Lambda$. The arrows indicate the direction of evolution of the parameters in an expanding universe from [9].

energy model (vacuum energy) for solving the accelerating universe problem. The cosmological constant is the simplest model of dark energy, it also provides a best agreement with astronomical and cosmological observations.

Let us consider the Friedmann equation with $k = 0$ of the field equation in (2.15), we obtain

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_\Lambda)\,. \tag{2.17}$$

We can reduce the energy density and the pressure of both matter and the cosmological constant with $\rho \to \rho + \rho_\Lambda$ and $P \to P + P_\Lambda$. We also use the conservation of the matter-(perfect fluid) in equation (2.9), it is easy to show that

$$\dot{\rho_\Lambda} + 3H(\rho_\Lambda + P_\Lambda) = 0\,. \tag{2.18}$$

Since ρ_Λ is constant, that implies

$$\rho_\Lambda = -P_\Lambda \quad \Rightarrow \quad w_\Lambda = -1\,. \tag{2.19}$$

As we know, if the equation of state of any perfect fluid is less than $1/3$, such perfect fluid (here our perfect fluid is the cosmological constant) can drive the acceleration expansion. The cosmological constant also passes this condition, then the cosmological constant can be candidate of dark energy model.

Although the cosmological constant is best fitted with the observational data so-called lambda cold dark matter (ΛCDM) model, but unfortunately the cosmological constant causes the theoretical inconsistency problem i.e. the prediction values of the cosmological constant from quantum field theory are very larger than the observational values allowed (in level 120 orders of magnitude $\rho_{\Lambda \text{ theory}}/\rho_{\Lambda \text{ obs}} \sim 10^{120}$). This unsolved problem in particle physics is known as the cosmological constant problem.

The Quintessential Dark Energy

The scalar field model becomes the candidate of dark energy model instead the cosmological constant because the the cosmological constant has the fine tuning problem and it also gives the constant equation of state $w = -1$. While the scalar field model has the time variation equation of state i.e. $w = w(t)$. The scalar field has a good motivation on particle physics including string theory. Then the scalar field can behave like dark energy in various ways. We will follow Ref. [7] .

The quintessence is the ordinary scalar field ϕ , such field is non minimally coupled to gravity. We begin with the action of the quintessence given by

$$S = \int \sqrt{-g} \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right) \mathrm{d}^4 x \,, \tag{2.20}$$

where $V(\phi)$ is the potential of the quintessence. In the flat FRW metric, the Klein-Gorgon equation can be written as[1]

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \,. \tag{2.21}$$

The energy-momentum tensor of the quintessence is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \,, \tag{2.22}$$

we obtain

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + V(\phi) \right) \,. \tag{2.23}$$

Comparing the above equation with the standard form of energy-momentum tensor of the perfect fluid in (2.3), we obtain the energy density and the pressure density of quintessence as

$$\rho \;=\; T^0_0 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \,, \tag{2.24}$$

$$P \;=\; T^a_a = \frac{1}{2}\dot{\phi}^2 + V(\phi) \,. \tag{2.25}$$

[1]For detail calculation see [15].

Using the definition of the energy density and the pressure density of quintessence substitute into equations (2.7) and (2.8), we obtain

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{2.26}$$

and

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left(\dot{\phi}^2 - V(\phi)\right) . \tag{2.27}$$

We take time derivative in (2.26) and using (2.21) also, we get

$$\dot{H} = -4\pi G\dot{\phi}^2 . \tag{2.28}$$

We can express the the potential $V(\phi)$ and field ϕ in terms of H and $\dot{H}$ as

$$V = \frac{3H^2}{8\pi G}\left(1 + \frac{\dot{H}}{3H^2}\right) \tag{2.29}$$

$$\phi = \int \left(-\frac{\dot{H}}{4\pi G}\right)^{1/2} \mathrm{d}t . \tag{2.30}$$

The equation of state of quintessence is defined by

$$w_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} . \tag{2.31}$$

Let us consider the potential of the quintessence, which gives the exact solution (power law inflation's potential), i.e. the exponential potential take form

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{16\pi G}{p}}\,\phi\right) . \tag{2.32}$$

Substituting this potential in equation (2.21), (2.26) and (2.28). The solution from the exponential potential can be written as

$$a(t) \propto t^p . \tag{2.33}$$

The acceleration expansion occurs when $p > 1$. Thus the quintessence with exponential potential may provide dark energy. For more details on the several types of the quintessence potentials see [7].

The Phantom Field Dark Energy

The kinetic energy scalar field dark energy is known as the phantom field, was proposed by Ref. [13, 24]. Such fields were motivated from observational constraint that may be allows equation of state $w < -1$ with constant w value between -1.38 and -0.82 at the 95 % of confidence level [12], from S-brane construction in string/M theory [16] and from the scalar-tensor gravity [17]. This model also violated the null dominated energy condition in classical general relativity. Actually, the phantom fields were first proposed by Fred Hoyle, he introduced the creation (C)-field to reconcile the observational expanding universe with the steady state universe model. The C-field will create new matter and making universe to be homogeneity [18]. Later the C-field is extended to the Hoye-Narlikar theory of gravity [19].

The phantom fields are non-minimally coupling with gravity, the action can be written as

$$S = \int \sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right)\mathrm{d}^4x \tag{2.34}$$

In the same way as the case of quintessence, we obtain

$$\rho = -\frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2.35}$$

$$P = -\frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{2.36}$$

The equation of state of the phantom fields is given by

$$w \equiv \frac{P}{\rho} = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)} \tag{2.37}$$

from this fact we have $w < -1$ when $\dot{\phi}^2 \ll V(\phi)$. Recalling the solution of the energy density in term of scale factor in (2.13) gives

$$\rho \propto a^{-3(1+w)}\,.$$

When the phantom fields dominated universe at late time the equation of state will be $w < -1$. This obviously indicates that the energy density will approach infinite. The fields of the phantom fields will roll up the potential instead of rolling down like ordinary scalar field, then the energy of field will diverge (i.e. for the exponential potential). Such phenomena is known as the Big Rip singularity, that implies that at the late time everything in the universe must be ripped apart. But if the shape of potentials of the phantom field have peak of the potential (finite maximum values of the potential) the field will oscillate around the peak, so the energy of phantom field can not diverge. This is a good strategy was propose by Ref. [25] to avoid the Big Rip singularity. Unfortunately we do not know what is the actual shape of the dark energy potential. This mechanism can not generally avoid the Big Rip singularity in the phantom field dark energy.

2.2 Loop Quantum Cosmology

2.2.1 The Ashtekar Variables in General Relativity

This section we review in the Ashtekar variables. We have shown the $3+1$ ADM formulation in general relativity at appendix B. The Hamiltonian in general relativity is based on phase space with the dynamical variables the spatial 3-dimensional metric q_{ab} and canonical conjugate momentum π^{ab}. But the Ashtekar variables is formulated by the densititized triad E^a_i and the $su(2)$ connection A^i_a [57] .[2] The $su(2)$ connection A^i_a and the densititized triad E^a_i is obtained by canonical transformation of the $3+1$ ADM formulation. The features of the Ashtekar variables are the standard model of particle physics cannot written in term of metric tensor [58] and bringing general relativity closer contact to the gauge theory where the gauge theory is the available theory to quantization [60]. The Ashtekar variables is understood in the Einstein-Cartan geometry without matter which equivalence to general relativity (for more detail see [58]) . Such variables also provide the polynomial of the constraints in the canonical variables.

This section we will follow [60] . Considering the 3-dimensional metric relate to the triad e^a_i as follow

$$q^{ab} = e^a_i e^b_j \delta^{ij} \tag{2.38}$$

and for the co-triad e^i_a

$$q_{ab} = e^i_a e^j_b \delta_{ij} \,. \tag{2.39}$$

The densitized triad E^a_i can be written in the form of co-triad e^i_a as

$$E^a_i = \frac{1}{2}\, \epsilon_{ijk} \epsilon^{abc} e^j_b e^k_c \,. \tag{2.40}$$

[2] For conventions in this section and thesis see at appendix.

Using equations (2.39) and (2.40), we obtain the relation between the densitized triad and the 3-dimensional tensor as

$$\begin{aligned}
E_i^a E^{di} &= \left(\frac{1}{2}\,\epsilon_{ijk}\epsilon^{abc}e_b^j e_c^k\right)\left(\frac{1}{2}\,\epsilon^{ilm}\epsilon^{dfh}e_{fl}e_{hm}\right) \\
&= \frac{1}{4}\delta_{jk}^{lm}\epsilon^{abc}\epsilon^{dfh}e_b^j e_c^k e_{fl}e_{hm} = \frac{1}{4}(\delta_j^l\delta_k^m - \delta_k^l\delta_j^m)\epsilon^{abc}\epsilon^{dfh}e_b^j e_c^k e_{fl}e_{hm} \\
&= \frac{1}{4}\left(\epsilon^{abc}\epsilon^{dfh}e_b^j e_c^k e_{fj}e_{hk} - \epsilon^{abc}\epsilon^{dfh}e_b^j e_c^k e_{fk}e_{hj}\right) \\
&= \frac{1}{4}\left(\epsilon^{abc}\epsilon^{dfh}q_{bf}q_{ch} - \epsilon^{abc}\epsilon^{dfh}q_{bh}q_{cf}\right) = \frac{1}{2}\,\epsilon^{abc}\epsilon^{dfh}q_{bf}q_{ch} \\
&= \frac{1}{2}\,\epsilon^{abc}\epsilon^{dfh}q_{bf}q_{ch}\frac{\delta_a^a}{3} = \frac{1}{6}\,\epsilon^{abc}\epsilon^{dfh}q_{bf}q_{ch}q_{ab}q^{ab} \\
&= |\,q|\,q^{ab}
\end{aligned} \tag{2.41}$$

we used identities $\epsilon_{ijk}\epsilon^{ilm} = \delta_{jk}^{lm}$ where $\delta_{jk}^{lm} = \delta_j^l\delta_k^m - \delta_k^l\delta_j^m$ is generalized Kronecker delta, $e_i^a e_a^j = \delta_i^j$, $e_{ai}e_b^i = q_{ab}$ and $|\,q| = (1/3!)\epsilon^{abc}\epsilon^{dfh}q_{bf}q_{ch}q_{ab}$ is determinant of metric q_{ab} . We have the relation between the jacobian of q_{ab} and e_i^a, using equation (2.38) or (2.39) , we obtain

$$\begin{aligned}
q_{ab} &= e_a^i e_{bi} \\
\det(q_{ab}) &= \det(e_a^i e_{bi}) \\
q &= \det(e_a^i)\det(e_{bi}) \\
q &= e^2\,.
\end{aligned} \tag{2.42}$$

Using equations (2.41) and (2.42) , then we obtain

$$E_i^a = \sqrt{e}\;e_i^a\,. \tag{2.43}$$

The densitized triad E_i^a have carried information in the 3-dimensional hypersurface Σ_t from the 3-dimensional metric q_{ab} .

Next we consider the spin connection ω_a^i . The compatible of the spin connection with the co-triad e_a^i is satisfied by condition

$$\partial_{[a}e_{b]}^i + \omega_{[aj}^i e_{b]}^j = 0\,. \tag{2.44}$$

We can solve above equation and giving the explicit form of the spin connection ω_a^i as (see detail derivation at appendix C.1)

$$\omega_{aj}^i = \frac{1}{2}e_a^k\left(e_k^b e_j^c\partial_{[b}e_{c]}^i + e^{bi}e_k^c\partial_{[b}e_{c]j} - e_j^b e^{ci}\partial_{[b}e_{c]k}\right) \tag{2.45}$$

for any antisymmetric quantity of two 3d indices $i\,,j$ can be represent in one index via

$$v^i = \frac{1}{2}\,\epsilon^i{}_{jk} v^{jk}\,, \qquad v^{ij} = \epsilon^{ij}{}_k v^k\,. \tag{2.46}$$

From this fact, we can represent the spin connection in term of one index form as

$$\omega^i_a = -\epsilon^{ijk} e^b_j (\partial_{[a} e_{b]k} + \frac{1}{2} e^c_k e^l_a \partial_{[c} e_{b]l}) \tag{2.47}$$

Certainly the spin connection also preserves information in the 3-dimensional hypersurface Σ_t via the relation between triad, co-triad and 3-dimensional metric i.e. $e^a_i e^{bi} = q^{ab}$ and $e^i_a e_{bi} = q_{ab}$. The extrinsic curvature one-form K^i_a is defined by

$$K^i_a = K_{ab} e^{bi}\,. \tag{2.48}$$

The $su(2)$ connection A^a_i is defined by a sum of the spin connection and the extrinsic curvature one-form i.e.

$$A^i_a = \omega^i_a + \chi K^i_a \tag{2.49}$$

where χ is the Barbero-Immirzi parameter. Such parameter is play important role in the level spacing geometric eigenvalues and set by the black hole thermodynamics in LQG, as $\chi \sim 0.2375$. As we seen like the densitized triad E^a_i, the $su(2)$ connection also preserve the information of the 3-dimensional hypersurface Σ_t via the spin connection ω^i_a and the extrinsic curvature K^i_a in explicitly way.

After we have shown how the Ashtekar variables come from, we can be written the densitized triad and the $su(2)$ connection in the Poisson bracket like the canonical dynamical variables (B.33) in $3+1$ ADM formulation as

$$\{A^j_b\,,\ E^a_i\} = 8\pi G\chi \delta^a_b \delta^j_i \delta^3(x\,,\ y) \tag{2.50}$$

where G is the Newton's gravitational constant.
Using the Ashtekar variables is the canonical dynamics variables. The gravitational action of the $3+1$ ADM formulation under the Legend transformation, we obtain the Holst action as [61, 62] (see detail calculation at appendix C.2)

$$S_{GR}[E, A, \lambda, N^a, N] = \int\int \left(\frac{-1}{8\pi G\chi} E^a_i \pounds_t A^i_a - (\lambda^i G_i + N^a \mathcal{C}_a + N\mathcal{C}_{GR}) \right) \mathrm{d}^3x\ \mathrm{d}t \tag{2.51}$$

where $\lambda^i \equiv A^i_t$, the polynomial set of the constraints are given by

$$G_i = D_a E^a_i = \partial_a E^a_i + \epsilon_{ij}{}^k A^j_a E^a_k \tag{2.52}$$

where D_a is the covariant derivative operator compatible with co-triad i.e. $D_{[a}e_{b]i} = 0$, the constraint G_i is known as the Gauss constraint. Such constraint additional arise from the Ashtekar variables.

$$\mathcal{C}_a = \frac{1}{8\pi G\chi} E^b_i F^i_{ab} - \left(\frac{1+\chi^2}{\chi}\right) K^i_a G_i \tag{2.53}$$

the constraint $\mathcal{C}_a$ is the diffeomorphisms constraint like the $3+1$ ADM formulation.

$$\mathcal{C}_{GR} = \frac{1}{16\pi G\sqrt{q}} E^a_i E^b_j \left(\epsilon^{ij}{}_k F^k_{ab} - (1+\chi^2) K^i_{[a} K^j_{b]}\right) \tag{2.54}$$

the constraint $\mathcal{C}_{GR}$ is the Hamiltonian constraint like the $3+1$ ADM formulation also. The quantity F^i_{ab} is the curvature of the $su(2)$ connection is defined by

$$F^i_{ab} = \partial_{[a} A_{b]} + \epsilon^i{}_{jk} A^j_a A^k_b \,. \tag{2.55}$$

Thus we can be obtained the total gravitational Hamiltonian in the Ashtekar variables as

$$H_{GR} = \int (\lambda^i G_i + N^a \mathcal{C}_a + N \mathcal{C}_{GR}) \mathrm{d}^3 x \,. \tag{2.56}$$

As in the $3+1$ ADM formulation, the total Hamiltonian is a sum of the G_i, $\mathcal{C}_a$ and $\mathcal{C}_{GR}$, but the total Hamiltonian in the Ashtekar variables have raising the additional constraint i.e. the Gauss constraint G_i. This is feature of the Ashtekar variables when it is used in the canonical theory. The Gauss constraint G_i can be generated the $su(2)$ rotational property of the densitized triad E^a_i and the connection A^i_a. The diffeomorphisns constraintt $\mathcal{C}_a$ can be generated diffeomorphisms along the 3-dimensional hypersurface Σ_t. The Hamiltonian constraintt $\mathcal{C}_{GR}$ can be determined dynamics and generated the time evolution of the 3-dimensional hypersurface Σ_t. The Hamiltonian $\mathcal{C}_{GR}$ also separately consider as

$$\mathcal{C}_{GR} = -\mathcal{C}_E + \mathcal{C}_L \tag{2.57}$$

where $\mathcal{C}_E$ is known as the Euclidean part, given by

$$-\mathcal{C}_E = \frac{1}{16\pi G\sqrt{q}} E^a_i E^b_j \epsilon^{ij}{}_k F^k_{ab} \tag{2.58}$$

and $\mathcal{C}_L$ is known as the Lorentz part, given by

$$\mathcal{C}_L = -\frac{1}{16\pi G\sqrt{q}} (1+\chi^2) E^a_i E^b_j K^i_{[a} K^j_{b]} \,. \tag{2.59}$$

When we quantize the Hamiltonian constraint. The Euclidean part is the first quantized separately to be the operator. The Lorentz part is quantized later by using the operator of the Euclidean constraint.

The equation of motions of general relativity in the (canonical) Ashtekar variables are obtained by analogy the $3+1$ ADM formulation's equation of motion as

$$\dot{A}_a^i = \{A_a^i\,,\ H_{GR}\} = 8\pi G\chi \frac{\delta H_{GR}}{\delta E_i^a} \tag{2.60}$$

and

$$\dot{E}_i^a = \{E_i^a\,,\ H_{GR}\} = -8\pi G\chi \frac{\delta H_{GR}}{\delta A_a^i}\,. \tag{2.61}$$

2.2.2 The Isotropic Hamiltonian of the flat FRW spacetime

This section we will construct the form of isotropic the densitized triads and the $su(2)$ connection for the flat FRW spacetime $(k=0)$. This type of solution can be obtained in the Bianchi I model, we obtain the FRW line element following

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{2.62}$$

where $N(t)$ is lapse function and $a(t)$ is a scale factor. The homogeneous connection and triads can be decomposed by using the basis one-form and vector fields as [63]

$$\begin{aligned} A_a^i &= c_j^i(t)\,\theta_a^j \\ E_i^a &= p_i^j(t)\,X_j^a \end{aligned} \tag{2.63}$$

where θ_a^i, X_i^a are the one-form and vector fields basis respectively and $c_j^i(t)$, $p_i^j(t)$ are relating dynamical quantities with the metric. For an isotropic case, the connection take the form [63]

$$A_a^i = \widetilde{c}\,(t)\,\theta_a^i \tag{2.64}$$

and the densitized triad take the form

$$E_i^a = \sqrt{^0q}\,\widetilde{p}\,(t)\,X_i^a \tag{2.65}$$

where $\widetilde{c}(t)$ is the dynamical component of the connection A_a^i $\widetilde{p}(t)$ is the dynamical component of the densitized triad after symmetry reduction [63], ^0q is determinant

of the fiducial background metric [3] ${}^0q_{ab} = \theta^i_a\theta_{bi}$
The 3-dimensional metric q_{ab} is given by

$$q_{ab} = a^2(t)\theta_{ai}\theta^i_b = a^2(t)\,{}^0q_{ab}\,. \tag{2.66}$$

We will use equation (2.41) to determine the relation between $\widetilde{p}(t)$ and $a(t)$, we obtain

$$\begin{aligned} E^a_i E^{bi} &= \sqrt{q}q^{ab} \\ \sqrt{{}^0q}\,\widetilde{p}\,(t)\,X^a_i\sqrt{{}^0q}\,\widetilde{p}\,(t)\,X^{bi} &= (a^6(t)\,{}^0q)\,\frac{\delta^{ab}}{a^2(t)} \\ \widetilde{p}^{\,2}(t)\delta^a_i\delta^{bi} &= a^4(t)\delta^{ab} \\ |\,\widetilde{p}\,(t)| &= a^2(t)\,. \end{aligned} \tag{2.67}$$

The triad oppose to the scale factor. An orientation of the triad can be determined by the sign of $\widetilde{p}$ i.e. $\widetilde{p}$ allow both positive and negative values. Here we denote "the sign of $\widetilde{p} \equiv \mathrm{sgn}(\widetilde{p}\,)$" . The triad in term of both $\widetilde{p}$ and a using equations (2.43) and (2.67), we obtain

$$e^a_i = \mathrm{sgn}(\widetilde{p}\,)a^{-1}X^a_i\,. \tag{2.68}$$

Let us consider the spin connection ω^i_a following the definition in (2.47) . The spin connection is identically vanish in the flat FRW line element due to the basis vector fields X^a_i is orthogonal coordinate (or in flat space). Using the definition of extrinsic curvature K_{ab} in equation (B.24) with FRW metric, we obtain

$$\begin{aligned} K_{ab} &= \frac{1}{2N}\,(\dot{q_{ab}} - D_aN_b - D_bN_a) \\ &= \frac{1}{2N}\left(\frac{\mathrm{d}}{\mathrm{d}t}(a^2(t)\theta_{ai}\theta^i_b) - D_a(0) - D_b(0)\right) \\ &= N^{-1}a\dot{a}\,\theta_{ai}\theta^i_b\,. \end{aligned} \tag{2.69}$$

We use above equation and the definition of triad in equation (2.68), the extrinsic curvature one-form is given by

$$\begin{aligned} K^i_a &= K_{ab}e^{bi} \\ &= N^{-1}a\dot{a}\,\theta^j_a\theta_{bj}\,\mathrm{sgn}(\widetilde{p}\,)a^{-1}X^{bi} \\ &= \mathrm{sgn}(\widetilde{p}\,)N^{-1}\dot{a}\,\theta^i_a\,. \end{aligned} \tag{2.70}$$

[3] The fiducial metric for the flat FRW line element is given by $\theta_{ai}\theta^i_b\mathrm{d}x^a\mathrm{d}x^b = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.

we also used identity $\theta_{ai}X^{aj} = \delta_i^j$. We directly obtained the $su(2)$ connection A_a^i by using the facts from above as

$$\begin{aligned} A_a^i &= \omega_a^i + \chi K_a^i \\ &= \chi\,\mathrm{sgn}(\widetilde{p})N^{-1}\dot{a}\,\theta_a^i\,. \end{aligned} \tag{2.71}$$

We use the homogeneous and isotropic connection in (2.64) and densitized triad in (2.65) substituting to the gravitational action in (2.51) , we obtain

$$\begin{aligned} E_i^a \pounds_t A_a^i &= E_i^a(t^b\partial_b A_a^i + A_b^i\partial_a t^b) \\ &= \sqrt{{}^0q}\,\widetilde{p}\,X_i^a\,\theta_a^i\left(\frac{\mathrm{d}x^b}{\mathrm{d}t}\,\partial_b\widetilde{c}\right) \\ &= 3\sqrt{{}^0q}\,\widetilde{p}\,\dot{\widetilde{c}}. \end{aligned} \tag{2.72}$$

For the Gauss constraint will identically vanish due to the basis vector fields X_i^a is orthogonal basis in the Bianchi I model i.e.

$$\begin{aligned} \partial_a E_i^a + \epsilon_{ij}{}^k A_a^j E_k^a &= \partial_a(\widetilde{p}\,X_i^a) + \epsilon_{ij}{}^k\,\widetilde{c}\,\widetilde{p}\,\theta_a^j X_k^a \\ &= \epsilon_{ij}{}^k\,\widetilde{c}\,\widetilde{p}\,\delta_k^j \\ &= \epsilon_{ij}{}^j\,\widetilde{c}\,\widetilde{p} = 0\,. \end{aligned} \tag{2.73}$$

The curvature F_{ab}^i can easy calculation, we obtain

$$\begin{aligned} F_{ab}^i &= \partial_{[a}(\widetilde{c}\,\theta_{b]}^i) + \epsilon^i{}_{jk}A_a^j A_b^k \\ &= \epsilon^i{}_{jk}\widetilde{c}\,\theta_a^j\widetilde{c}\,\theta_b^k \\ &= \epsilon^i{}_{jk}\widetilde{c}^{\,2}\theta_a^j\,\theta_b^k\,. \end{aligned} \tag{2.74}$$

For the diffeomorphisms constraint, we use the curvature F_{ab}^i in above equation and the fact $G_i = 0$ from Gauss constraint, substituting in equation (2.53) we obtain

$$\begin{aligned} \frac{1}{8\pi G\chi}E_i^b F_{ab}^i - \left(\frac{1+\chi^2}{\chi}\right)K_a^i G_i &= \frac{1}{8\pi G\chi}\sqrt{{}^0q}\,\widetilde{p}\,\theta_i^b\epsilon^i{}_{jk}\widetilde{c}^{\,2}\theta_a^j\,\theta_b^k \\ &= \frac{\sqrt{{}^0q}}{8\pi G\chi}\,\widetilde{p}\,\widetilde{c}^{\,2}\epsilon^i{}_{jk}X_i^b\theta_a^j\,\theta_b^k = \frac{\sqrt{{}^0q}}{8\pi G\chi}\,\widetilde{p}\,\widetilde{c}^{\,2}\epsilon^i{}_{jk}\delta_i^k\,\theta_a^j \\ &= \frac{\sqrt{{}^0q}}{8\pi G\chi}\,\widetilde{p}\,\widetilde{c}^{\,2}\epsilon^i{}_{ji}\,\theta_a^j = 0\,. \end{aligned} \tag{2.75}$$

For the Hamiltonian constraint $\mathcal{C}_{GR}$ The Euclidean part is given by

$$\begin{aligned}
-\mathcal{C}_E &= \frac{1}{16\pi G\chi\sqrt{q}}\,\epsilon^{ij}{}_k E^a_i E^b_j F^k_{ab} \\
&= \frac{1}{16\pi G\chi\sqrt{{}^0q}\,\sqrt{\widetilde{p}^{\,3}}}\,\epsilon^{ij}{}_k\epsilon^k{}_{lm}\sqrt{{}^0q}\,\widetilde{p}\,X^a_i\sqrt{{}^0q}\,\widetilde{p}\,X^b_j\,\widetilde{c}^{\,2}\,\theta^l_a\,\theta^m_b \\
&= \frac{1}{16\pi G}\sqrt{{}^0q}\,\sqrt{\widetilde{p}}\,\widetilde{c}^{\,2}\,\delta^{ij}_{lm}\,X^a_i\,X^b_j\,\theta^l_a\,\theta^m_b \\
&= \frac{1}{16\pi G}\sqrt{{}^0q}\,\sqrt{\widetilde{p}}\,\widetilde{c}^{\,2}\,(\delta^i_l\delta^j_m-\delta^i_m\delta^j_l)\,X^a_i\,X^b_j\,\theta^l_a\,\theta^m_b \\
&= \frac{1}{16\pi G}\sqrt{{}^0q}\,\sqrt{\widetilde{p}}\,\widetilde{c}^{\,2}\,(\delta^i_l\delta^j_m\,X^a_i\,X^b_j\,\theta^l_a\,\theta^m_b-\delta^i_m\delta^j_l\,X^a_i\,X^b_j\,\theta^l_a\,\theta^m_b) \\
&= \frac{1}{16\pi G}\sqrt{{}^0q}\,\sqrt{\widetilde{p}}\,\widetilde{c}^{\,2}\,(\delta^i_l\delta^j_m\delta^l_i\delta^m_j-\delta^i_m\delta^j_l\delta^l_i\delta^m_j)=\frac{1}{16\pi G}\sqrt{{}^0q}\,\sqrt{\widetilde{p}}\,\widetilde{c}^{\,2}\,(\delta^i_i\delta^j_j-\delta^i_i) \\
&= \frac{3}{8\pi G}\sqrt{{}^0q}\,\sqrt{\widetilde{p}}\,\widetilde{c}^{\,2}
\end{aligned} \tag{2.76}$$

and the Lorentzian term $\mathcal{C}_L$, for the fact $\omega^i_a=0$ and using equation (2.70), we get

$$\begin{aligned}
\mathcal{C}_L &= -\frac{1}{8\pi G\sqrt{q}}(1+\chi^2)E^a_iE^b_jK^i_{[a}K^j_{b]}=-\frac{(1+\chi^2)}{8\pi G\sqrt{q}}E^a_iE^b_j\left(\frac{1}{2}\right)(K^i_aK^j_b-K^i_bK^j_a) \\
&= -\frac{(1+\chi^2)}{16\pi G\sqrt{{}^0q}\,\sqrt{\widetilde{p}^{\,3}}}\sqrt{{}^0q}\,\widetilde{p}\,X^a_i\sqrt{{}^0q}\,\widetilde{p}\,X^b_j \\
&\quad\times\left(\mathrm{sgn}(\widetilde{p})N^{-1}\dot{a}\,\theta^i_a\mathrm{sgn}(\widetilde{p})N^{-1}\dot{a}\,\theta^j_b-\mathrm{sgn}(\widetilde{p})N^{-1}\dot{a}\,\theta^i_b\mathrm{sgn}(\widetilde{p})N^{-1}\dot{a}\,\theta^j_a\right) \\
&= -\frac{(1+\chi^2)}{16\pi G}\sqrt{{}^0q}\,\sqrt{\widetilde{p}}(\mathrm{sgn}(\widetilde{p})^2N^{-2}\dot{a}^2)(\,\theta^i_a\theta^j_bX^a_i\,X^b_j-\,\theta^i_b\theta^j_aX^a_i\,X^b_j) \\
&= -\frac{(1+\chi^2)}{16\pi G}\sqrt{{}^0q}\,\sqrt{\widetilde{p}}\left(\frac{\widetilde{c}^{\,2}}{\chi^2}\right)(\delta^a_a\delta^b_b-\delta^a_a) \\
&= -\frac{3}{8\pi G}(1+\chi^{-2})\sqrt{{}^0q}\,\sqrt{\widetilde{p}}\,\widetilde{c}^{\,2}\,.
\end{aligned} \tag{2.77}$$

Using above two equation, the total Hamiltonian constraint is given by

$$\mathcal{C}_{GR}=-\mathcal{C}_E+\mathcal{C}_L=\frac{3\Omega}{8\pi G\chi^2}\sqrt{\widetilde{p}}\,\widetilde{c}^{\,2}\,. \tag{2.78}$$

where $\Omega=\int \mathrm{d}^3x\sqrt{{}^0q}$ is the fiducial volume.

Using all the constraint values insert to the gravitational action of the homogeneous and isotopic include the action of matter field S_M, we get

$$S_{GR}[N,\widetilde{p}\,,\widetilde{c}\,,\mathrm{matter}]=\int\mathrm{d}t\left[\frac{3\Omega}{8\pi G\chi}\,\widetilde{p}\,\dot{\widetilde{c}}+N\left(\frac{3\Omega}{8\pi G\chi^2}\sqrt{\widetilde{p}}\,\widetilde{c}^{\,2}\right)\right]+S_M \tag{2.79}$$

Comparing this action with action in classical mechanics, we obtain the total Hamiltonian as

$$H_{GR} = N\left(-\frac{3\Omega}{8\pi G\chi^2}\sqrt{\widetilde{p}}\,\widetilde{c}^{\,2}\right) + H_M \tag{2.80}$$

and we immediately write the poisson bracket relation between $\widetilde{c}$ and $\widetilde{p}$ as

$$\{\widetilde{c}\,,\widetilde{p}\} = \frac{8\pi G\chi}{3\Omega} \tag{2.81}$$

The equations (2.80) and (2.81) are known as the symmetry reduction. The hamiltonian in (2.80) and the Poisson bracket in (2.81) are the preparation of the homogeneous and isotopic for flat-FRW spacetime to be quantization process.
For simplifying, we can be dropped the fiducial volume factor Ω by changing the new variables to absorb this factor following

$$\begin{aligned} p &\equiv \Omega^{2/3}\,\widetilde{p} \\ c &\equiv \Omega^{1/3}\,\widetilde{c}\,. \end{aligned} \tag{2.82}$$

Using these variables, we obtain the action as

$$S_{GR} = \int\left(\frac{3}{8\pi G\chi}\,p\,\dot{c} + c^2 N\,\frac{3}{8\pi G\chi^2}\right)\mathrm{d}t\,. \tag{2.83}$$

The Poisson brackets of the untilde c and p becomes

$$\{c\,,p\} = \frac{8\pi G\chi}{3}\,. \tag{2.84}$$

The Hamiltonian constraint can be written as

$$\mathcal{C}_{GR} = -\frac{3}{8\pi G\chi^2}\sqrt{|p|}\;c^2\,. \tag{2.85}$$

The relation between the untilde c and p and the FRW metric variables as

$$\begin{aligned} |p| &= \Omega^{2/3}a^2 \\ c &= \frac{\chi}{N}\,\mathrm{sgn}(p)\Omega^{1/2}\dot{a}\,. \end{aligned} \tag{2.86}$$

We will use c and p variables at the beginning to quantization. The features of the untilde c and p variables does not for simply formalisms only, these variables also invariant under the coordinate gauge freedom when we quantize gravity. That means, our quantum theory of gravity will be invariant under gauge freedom.

To obtain the Hamiltonian constraint of the scalar field (here we follow above

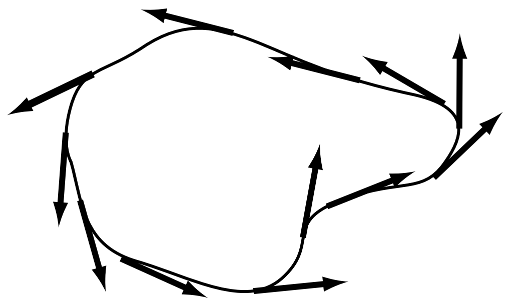

Figure 2.5: The notion of holonomy, figure from [58].

procedure step by step), the action of the scalar field in the flat FRW can be written as

$$S_\phi = \int \sqrt{-g} a^3 N \Omega \left(\frac{\dot{\phi}^2}{2N^2} - V(\phi) \right) \mathrm{d}t \tag{2.87}$$

where $V(\phi)$ is the potential energy of the scalar field. Let us define the canonical conjugate momentum Π_ϕ of the scalar field as

$$\Pi_\phi = \frac{a^3 \Omega}{N} \dot{\phi} \, . \tag{2.88}$$

Using the Legendre transformation of action in (2.87), we obtain

$$S_\phi = \int \left[\Pi_\phi \dot{\phi} - N \left(\frac{\Pi_\phi^2}{2a^3\Omega} - a^3 \Omega V(\phi) \right) \right] \mathrm{d}t \, . \tag{2.89}$$

Thus the Hamiltonian of the scalar field and it constraint can be written as

$$H_\phi = N \left(\frac{\Pi_\phi^2}{2a^3\Omega} + a^3 \Omega V(\phi) \right) \tag{2.90}$$

$$\mathcal{C}_\phi = \frac{\Pi_\phi^2}{2|\widetilde{p}|^{3/2}\Omega} + |\widetilde{p}|^{3/2} \Omega V(\phi) \, . \tag{2.91}$$

2.2.3 Isotropic loop quantum cosmology

The Loop Quantized Representation

This section we present the Dirac's quantization formalisms via Wilson's loop approach, known as **loop quantization** and we will see below "why loop ?". Dirac quantization procedures as following [64]

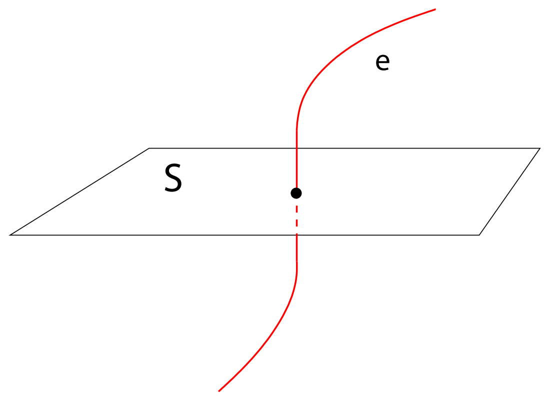

Figure 2.6: A notion of holonomy and flux as elementary conjugate variables, figure from [65].

- Quantize the Poisson bracket of phase space variables as the commutator in the kinematical Hilbert space i.e. $\{\ ,\ \}_{\mathrm{PB}} \longrightarrow -i/\hbar\,[\ ,\ ]\,.$
- Promote the gravitation constraints in (2.52) , (2.53) , (2.54) to the self-adjoint operator.
- Find the physical Hilbert space i.e characterize the space of the constraint solutions and define the inner product that gives notion of physical probability.
- Find a complete set of gauge invariant observables (i.e. loop variables operators) with commute to the constraints.

This strategy is very successful and powerful in the LQG and LQC as well. Such approach also gives the background independent in our theory. In our quantization procedure, the Hamiltonian will be ignored i.e. $H_{\mathrm{GR}} \approx 0\,.$ This thesis, we especially treat the quantization of the flat FRW spacetime ($k = 0$) only and using a classical approximation to obtain the effective Friedmann equation.

Let us start from the basic configuration variable holonomies (or Wilson's loop) of the connection along a given edge

$$h_e(A) = \mathcal{P}\exp\left(\int (\dot{\gamma}^{\mu}(s)A^i_{\mu}(\gamma(s))\tau_i)\mathrm{d}s\right) \tag{2.92}$$

where $\mathcal{P}$ denotes a path order of the exponential, $\dot{\gamma}^\mu$ is tangent vector at the edge, and $\tau_i = -\frac{i}{2}\sigma_i$ is a basis of the $SU(2)$ of the Lie algebra, with σ_i is Pauli spin matrices. The holonomy of the connection $A^i_a = \widetilde{c}\,\theta^i_a$ is given by

$$
\begin{aligned}
h_i(A) &= \exp\left(\int (\widetilde{c}\,\theta^j_a X^a_i \tau_j)\mathrm{d}s\right) \\
&= \exp({}^0l\,\widetilde{c}\,\tau_i) \\
&= \cos\left(\frac{{}^0l\,\widetilde{c}}{2}\right) + 2\sin\left(\frac{{}^0l\,\widetilde{c}}{2}\right)\tau_i
\end{aligned}
\tag{2.93}
$$

where ${}^0l$ is the oriented edge length. We can re-write it in the untilde variables and define new parameter $\mu' = {}^0l/\Omega^{1/3}$, obtaining

$$
h_i(A) = \cos\left(\frac{\mu' c}{2}\right)\mathbb{I} + 2\sin\left(\frac{\mu' c}{2}\right)\tau_i \tag{2.94}
$$

where $\mathbb{I}$ is identity matrix in 2×2 dimension and μ' is the kinematical length of the square loop as area of the loop is given by minimum eigenvalue of LQG area operator [35] . The basis momentum variables are fluxes of the triad through a two-surface S

$$
F_S(E) = \int_S \epsilon_{abc} E^c_i \tau^i f'_i \mathrm{d}^a x \mathrm{d}^b x \tag{2.95}
$$

where f'_i is a test function. Using the fact at above, the flux of the triad is proportional to the triad itself

$$
F_S(E) = A_{S,f}\Omega^{-\frac{2}{3}}\,p \,\propto p \tag{2.96}
$$

where $A_{S,f}$ is area of S . We note that both the holonomy and flux variables do not need a background metric to defined and the flux variable are conjugate variable to holonomy as we see in figure 2.6 . The Poisson bracket of these variable are not non-zero (we will see below) , if the edge of the holonomy intersect to the surface of the flux. These variables are well defined to promote quantum operator (i.e. a gauge invariant observables) that created a loop state. From this reason, that is why we call **loop quantization** . In LQG (and LQC also) , spacetime is formed by loop state i.e. loop state associate with respect to other loop only without refering to the background metric. Therefore LQG is background independent (quantum gravity) theory.

In the homogeneous and isotropy cosmological setting of Dirac's quantization, we can construct the algebra function to be represented on the kinematical Hilbert space which is defined by holonomy and flux also. The algebra almost periodic function is constructed from a finite holonomies [30]

$$f(c) = \sum_{j} f_j \, \exp\left(i \frac{\mu'_j \, c}{2}\right) \tag{2.97}$$

where j is the finite integer labeling number of edges, $\mu'_j \in \mathbb{R}$ (real) and $f_j \in \mathbb{C}$ (complex). We note that the almost periodic algebra function depends on c only but it is not c directly. As we see the almost periodic algebra function, we can represent all continuous functions from the almost periodic algebra functions, like view point of a fourier series. We have shown at above the flux of triad is proportional to itself, therefore we can use p to be algebra function directly. We will construct the holonomy-flux algebra function via the Poisson bracket as

$$\begin{aligned}
\{f(c)_j \, , p\} &= \frac{8\pi G \, \chi}{3} \left(\frac{\partial}{\partial c} \left(\sum_{j} f_j \, \exp\left(i \frac{\mu'_j \, c}{2}\right)\right) \frac{\partial p}{\partial p} \right. \\
&\qquad \left. - \frac{\partial}{\partial p} \left(\sum_{j} f_j \, \exp\left(i \frac{\mu'_j \, c}{2}\right)\right) \frac{\partial p}{\partial c} \right) \\
&= \frac{8\pi G \, \chi}{3} \sum_{j} f_j \, \mu'_j \, \exp\left(i \frac{\mu'_j \, c}{2}\right)
\end{aligned} \tag{2.98}$$

the Poisson bracket is almost periodic also, that means this algebra is closed. After, we prepared the classical phase space the holonomy-flux algebra to be represented on the Hilbert space already which satisfies and passes requirement from Dirac quantization. The almost periodic algebra function can represent the constitution of an orthonomal basis in kinematical Hilbert space, by setting

$$f_\mu = \exp\left(i \, \frac{\mu \, c}{2}\right) . \tag{2.99}$$

Analogous conventional quantum mechanics, we can write the orthonormal basis state in bra-ket notation as

$$\exp\left(i \, \frac{\mu \, c}{2}\right) = < c | \, \mu > \; . \tag{2.100}$$

Orthonormality of basis states is given by

$$< \mu \, | \, \mu' >= \, \delta_{\mu \mu'} \, . \tag{2.101}$$

A general state $| \, \Psi >$ in kinetic Hilbert space can write as

$$| \, \Psi >= \sum_{\mu} \, \Psi_\mu \, | \, \mu > \; . \tag{2.102}$$

Inner product of a general state becomes

$$< \Psi \,|\, \Psi' >= \sum_{\mu} \Psi^*_{\mu}\, \Psi'_{\mu} \tag{2.103}$$

where Ψ^*_{μ} is the complex conjugate of Ψ_{μ}. From requirement of state kinematical Hilbert space must have finite norm that gives,

$$\sum_{\mu} \Psi^*_{\mu}\, \Psi'_{\mu} \, < \, \infty \, . \tag{2.104}$$

Next, we will promote $f(c)$ and p to be the operators which satisfy

$$\{f(c)_j \, , p\} = -\frac{i}{\hbar}[\widehat{f}(c) \, , \widehat{p}\,] \, . \tag{2.105}$$

The configuration variables (holonomy) $f(c)$, is promoted to be operator as

$$\hat{f}(c) \equiv \widehat{e^{i\mu\, c/2}} \, . \tag{2.106}$$

The eigenvalue equation for holonomies operators is given by

$$\hat{f}(c)|\, \mu >= \, |\, \mu + \mu' > \; . \tag{2.107}$$

The conjugate momentum variables (flux) p is also promoted to be operator as

$$\hat{p} \equiv -\frac{i\, 8\pi G\, \chi\hbar}{3} \frac{\mathrm{d}}{\mathrm{d}c} \, . \tag{2.108}$$

The eigenvalue equation for holonomies operator is given by

$$\widehat{p}\,|\, \mu >= \, \frac{4\pi G\, \chi\hbar\, \mu}{3} |\, \mu > \; . \tag{2.109}$$

where $|\mu >$ is orthonormal eigenstate in the kinetic Hilbert space and we will demonstrate its representation the commutator relation by using (2.107) and (2.109), we get

$$[\hat{f}(c) \, , \widehat{p}\,] \;\; = \;\; -\frac{4\pi G\, \chi\hbar\, \mu}{3} \, \widehat{e^{i\mu\, c/2}} \, . \tag{2.110}$$

Comparing this result with respect to (2.98), we found that it satisfies quantization procedure in (2.105).

This section we have briefly introduced the (first-look) loop quantization in cosmological setting (i.e. LQC) and have shown how to construct LQC from classical theory at the beginning of this theory. We will stop here because LQC is still on progress to go on and not explicitly complete. For comprehensive reviews, more details, applications and literatures of LQC see [30, 32, 60, 61, 66].

The Effective Friedmann Equation in LQC

LQC naturally gives rise to inflationary phase of the early universe with graceful exit, however the same mechanism leads to a prediction that present-day acceleration must be very small [41]. At late time when universe and at large scale, the semi-classical approximation in LQC formalisms can be used [46]. This section we will derive the effective Friedmann equation from LQC following [34] which is main material of this thesis. The classical Hamiltonian constraint of the flat FRW in (2.111) with matter part is given by

$$\mathcal{C}_{GR} = -\frac{3}{8\pi G\chi^2}\sqrt{|p|}\,c^2 + \mathcal{C}_{\rm m}\,. \tag{2.111}$$

The effective Friedman equation can be obtained by using an effective Hamiltonian with loop quantum modifications [34, 39, 47]:

$$\mathcal{C}_{\rm eff} \;=\; -\frac{3}{\kappa^2\chi^2\mu'^{\,2}}\,a\sin^2(\mu' c) + \mathcal{C}_{\rm m} \tag{2.112}$$

where $\kappa^2 = 8\pi G$. The matter part[4] of effective Hamiltonian is given by substituting the equation (2.82) into equation (2.91). we obtain

$$\mathcal{C}_{\rm m} = \mathcal{C}_\phi = \frac{1}{2}\frac{\Pi_\phi^2}{p^{3/2}} + p^{3/2}V(\phi) \tag{2.113}$$

where $\Pi_\phi/p^{3/2} \equiv \dot{\phi}$ is obtained by using the the Hamilton's equation of motion. We can rewrite above equation in function of energy density of the scalar field as

$$\mathcal{C}_{\rm m} = p^{3/2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = p^{3/2}\rho\,. \tag{2.114}$$

The Hamilton's equation of motion is

$$\dot{p} \;=\; \{p, \mathcal{C}_{\rm eff}\} = -\frac{\kappa^2\chi}{3}\frac{\partial \mathcal{C}_{\rm eff}}{\partial c}\,. \tag{2.115}$$

where c and p are respectively conjugate connection and triad satisfying $\{c,p\} = \kappa^2\chi/3$ as we have discussed in chapter 2. These are two variables in the simplified phase space structure under FRW symmetries [32] and relates the two variables to scale factor as $p = a^2$ and $c = \chi\dot{a}$ that we have demonstrated in above. Substituting the effective Hamiltonian constraint in (2.112) into the Hamilton's equation of motion

[4] Considering the matter part constitute the scalar fields only.

in (2.115) we get

$$\begin{aligned}
\dot{p} &= -\frac{\kappa^2\chi}{3}\frac{\partial}{\partial c}\left(-\frac{3}{\kappa^2\chi^2\mu'^{\,2}}\, a\sin^2(\mu' c) + \frac{1}{2}\frac{\Pi_\phi^2}{p^{3/2}} + p^{3/2}V(\phi)\right) \\
&= \frac{2a}{\chi\mu'}\,\sin(\mu' c)\cos(\mu' c) \qquad (2.116)
\end{aligned}$$

we use the relation $\dot{p} = 2a\dot{a}$ with the above equation, therefore the time derivative of the scale factor is given by

$$\dot{a} = \frac{1}{\chi\mu'}\,\sin(\mu' c)\cos(\mu' c) \qquad (2.117)$$

Using the Equations (2.114) and (2.115) with constraint from realization that loop quantum correction of effective Hamiltonian $\mathcal{C}_{\rm eff}$ is small at large scale, $\mathcal{C}_{\rm eff} \approx 0$ [32, 35, 34, 39], i.e.

$$\begin{aligned}
\mathcal{C}_{\rm eff} &= -\frac{3}{\kappa^2\chi^2\mu'^{\,2}}\, a\sin^2(\mu' c) + \mathcal{C}_{\rm m} = 0 \\
\sin^2(\mu' c) &= \frac{\kappa^2\chi^2\mu'^{\,2}}{3a}\,\mathcal{C}_{\rm m} = \frac{\kappa^2\chi^2\mu'^{\,2}p}{3}\,\rho\,. \qquad (2.118)
\end{aligned}$$

Let us consider the minimum eigenvalue of the area operator in LQG is $\alpha'\ell_{\rm p}^2$ where $\ell_{\rm p}^2 \equiv G\hbar$ and α' are the Planck length and the order unity respectively. Comparing with the area of the loop in the flat FRW geometry i.e. $\mathcal{A} = \mu'^{\,2}|p|$, then we get equality of the minimum eigenvalue of the area operator between LQG and LQC [34] as

$$\mu'^{\,2}a^2 = \alpha'\ell_{\rm p}^2\,. \qquad (2.119)$$

We can obtain (effective) modified Friedmann equation by using (2.117), (2.118) and (2.119) as

$$\begin{aligned}
H^2 &= \left(\frac{\dot{a}}{a}\right)^2 = \frac{(1/\chi\mu'\,\sin(\mu' c)\cos(\mu' c))^2}{a^{\,2}} = \frac{(1/\chi\mu')^2\,\sin(\mu' c)^2(1-\sin(\mu' c)^2)}{p} \\
&= \frac{\kappa^2\chi^2\mu'^{\,2}p\,\rho\,(1-\kappa^2\chi^2\mu'^{\,2}p\,\rho/3)}{3\alpha^2\mu'^{\,2}p} = \frac{\kappa^2}{3}\,\rho\left(1-\frac{\kappa^2\chi^2\mu'^2\alpha'\ell_{\rm m}^2}{3\mu'^2}\,\rho\right) \\
&= \frac{\kappa^2}{3}\,\rho\left(1-\frac{\rho}{\rho_{\rm lc}}\right) \qquad (2.120)
\end{aligned}$$

where $\rho_{\rm lc} = 3/(\alpha'\kappa^2\chi^2\ell_{\rm m}^2)$.

The effective Friedmann equation of LQC rises the correction term ρ^2 from the discrete quantum geometric effect in classical regime. This equation was first derived in [34] recently, such equation is more similar to the effective Friedmann equation in the Randall-Sumdrum (RS II) [67, 68, 69] brane-world cosmology model which is motivated by M/String theory but in the RS II model has + sign in ρ^2 term. In the high energy regime i.e. $\rho \gg \rho_{\rm lc}$ the ρ^2 term will dominate in this equation and in the low energy regime i.e. $\rho \ll \rho_{\rm lc}$ the effective Friedmann equation will reduce to the Friedmann equation in standard GR. We note that the effective Friedmann equation in (2.120) gives the bouncing universe in this frame work due to the $-$ sign of the ρ^2 term in the left hand side of this equation, it also identical to the bouncing in brane-world [14]. The Hubble parameter H^2 will be equal to 0 when $\rho = \rho_{\rm lc}$. From this situation the universe will stop expanding when the energy density ρ grow equal to the critical energy density $\rho_{\rm lc}$ and then the universe will turn from the expansion phase to the collapsing phase and vice versa. Featuring of the bouncing universe from both LQC and brane-world avoid many singularities in cosmology [33, 37, 38, 39, 14]. The dualities of the effective Friedmann equation and their cosmological consequences between LQC and brane-world give impressive signal of the quantum gravity theory.

We will use the effective Friedmann equation inspired by LQC to analyze the future fate of the universe in the Phantom field DE via the the standard dynamical system in chapter 4 in this thesis.

Chapter 3

Methodology: Dynamical System in Cosmology

This chapter, we will begin brief introduction to the dynamical system in view point of the first order non-linear differential equation. Later we enter the dynamical system to the cosmological analysis of the scalar field in the standard cosmology.

3.1 Introduction to the Dynamical System

At the end of the nineteenth century the French mathematician Henri Poincare introduced a new approach to the study of differential equation. Realizing that analytic solution were unattainable for most nonlinear equation, he focused his efforts on finding descriptive properties of solutions of differential equation. This approach is known as the qualitative theory of differential equation. For example, in the qualitative theory, one would like to know the limiting behavior of all solution of the equation as $t \to \pm\infty$. Does the solution approach a constant, a periodic cycle, infinity, or something else? There are very important questions, since many phenomena live in their limiting behavior. In many systems transient solution approach zero so rapidly that only limiting behavior is observed.

The cornerstone of the Poincare qualitative approach is the phase plane, which often reveal many important properties of solutions of the differential equation, even when solution is unknown. The use of the phase plane gives the qualitative theory of differential equation a more geometric flavor rather than analytic flavor.

Let us consider the system of two first-order differential equations follow as

$$\begin{aligned} \dot{x} &= f(x,\ y) \\ \dot{y} &= g(x,\ y) \end{aligned} \tag{3.1}$$

where $\dot{} \equiv \mathrm{d}/\mathrm{d}t$, f and g are continuous function of x and y with continuous first partial derivative. If function f and g in system (3.1) do not depend explicitly on t, this system is known as the autonomous system. The values of x and y define a point $(x,\ y)$ in the phase space, so-called the state vector.
The simplest trajectories are those that settle down to a steady equilibrium. A points $(x_0,\ y_0)$ is known as the critical points or fixed points i.e.

$$\begin{aligned} \dot{x} &= f(x_0,\ y_0) = 0 \\ \dot{y} &= g(x_0,\ y_0) = 0\,. \end{aligned} \tag{3.2}$$

The critical points are points where the motion of the state vector is at rest.

Next we will study the shapes of the trajectories of the two-dimensional nonlinear autonomous that we mentioned in (3.1). It is possible to approximate the trajectory of nonlinear system. Nearly the critical points $(x_0,\ y_0)$ with the trajectories of the linear system. To find the linear system that approximates the nonlinear in (3.1), make the substitution

$$x = x_0 + \delta x\,, \quad y = y_0 + \delta y\,. \tag{3.3}$$

From the original variables $(x,\ y)$ to new variables $(\delta x,\ \delta y)$, which gives rise to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \mathcal{M}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \tag{3.4}$$

where $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x=x_0, y=y_0)}\,. \tag{3.5}$$

This possesses two eigenvalues μ_1 and μ_2. The general solution for the evolution of linear perturbations can be written as

$$\delta x = C_1 e^{\mu_1 N} + C_2 e^{\mu_2 N}\,, \tag{3.6}$$

$$\delta y = C_3 e^{\mu_1 N} + C_4 e^{\mu_2 N}\,, \tag{3.7}$$

where C_1, C_2, C_3, C_4 are integration constants. Thus the stability around the fixed points depends upon the nature of the eigenvalues. One generally uses the following classification [7] :

- (i) Stable node: $\mu_1 < 0$ and $\mu_2 < 0$.
- (ii) Unstable node: $\mu_1 > 0$ and $\mu_2 > 0$.
- (iii) Saddle point: $\mu_1 < 0$ and $\mu_2 > 0$ (or $\mu_1 > 0$ and $\mu_2 < 0$).
- (iv) Stable spiral: The determinant of the matrix $\mathcal{M}$ is negative and the real parts of μ_1 and μ_2 are negative.

A fixed point is an attractor in the cases (i) and (iv), but it is not so in the cases (ii) and (iii).

3.2 Dynamical System in Standard Cosmology

The dynamical system has played important role in cosmology for study the dynamics of inflaton field (scalar field) and inflation attractor properties at the early time for the first proposed [70, 71] . Such approach has many features in cosmology such as it does not emphasize the initial value of the universe at the very early time, having well physically interpret asymptotical behavior of the universe at both early and late time, giving a suitable and viable to explain the evolution of expansion history in our universe and also free from the fine-tuning problem etc.

From the many features of the dynamical system in cosmology as we mention in above leading many authors use this approach to study the scalar field model of DE (for the quintessence model at first and see [7] for review and reference in there) . The dynamical system may solve the **coincidences problem**[1] via the scaling solution. This section we will briefly review on the dynamics of the scalar field models of DE with the exponential potential follow [7, 71] in the flat FRW metric background.

[1]This problem was posed by the question from observational data, why the cosmological constant (or DE) and the (dark)-matter fluid are emerge at the same time scale? We will discuss on the resolution of this problem below.

3.2.1 Dynamics of Scalar Field in the FRW Cosmology

The Autonomous system and Scaling Solution in Cosmology

Let us recall the Friedmann equation (with the matter), the Klein-Gordon equation and the acceleration equation in the previous chapter follow as,

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_m\right) \tag{3.8}$$

$$\ddot{\phi} = -3H\dot{\phi} - V' \tag{3.9}$$

$$\dot{H} = -\frac{\kappa^2}{2}\left(\dot{\phi}^2 + (1+w_m)\rho_m\right) \tag{3.10}$$

where ρ_m and w_m are the energy density and the equation of state of matter respectively. We will define the new dimensionless parameter as [71]

$$\begin{aligned} x &\equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H} \\ \lambda &\equiv -\frac{V'}{\kappa V}, \quad \Gamma \equiv \frac{VV''}{V'^{\,2}} \end{aligned} \tag{3.11}$$

we will discuss the physical meaning of the variables λ and Γ below. Using the system of equation of the universe that we shown in above (3.8) , (3.9) and (3.10) , then we get the autonomous system of equation as

$$\frac{\mathrm{d}x}{\mathrm{d}N} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x\left((1-w_m)x^2 + (1+w_m)(1-y^2)\right) \tag{3.12}$$

$$\frac{\mathrm{d}y}{\mathrm{d}N} = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y\left((1-w_m)x^2 + (1+w_m)(1-y^2)\right) \tag{3.13}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}N} = -\sqrt{6}\lambda^2(\Gamma-1)x \tag{3.14}$$

where $N \equiv \mathrm{d}a/a$ is the e-folding number. We also get the constraint equation of the new variables from the Friedmann equation is

$$x^2 + y^2 + \frac{\kappa^2\rho_m}{3H^2} = 1\,. \tag{3.15}$$

We can rewrite the equation of state and the energy of the scalar filed in the new dimensionless variables following

$$w_\phi = \frac{x^2 - y^2}{x^2 + y^2} \tag{3.16}$$

$$\Omega_\phi = x^2 + y^2\,. \tag{3.17}$$

The acceleration equation of the universe in the present of the matter can be written as

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{\kappa^2}{6}(\rho_\phi + \rho_m + 3(P_\phi + P_m)) \\ &= -\frac{\kappa^2}{6}(1 + 3w_{\text{eff}})\rho_{\text{eff}}\end{aligned} \tag{3.18}$$

where $w_{\text{eff}} \equiv P_{\text{eff}}/\rho_{\text{eff}} = (P_\phi + P_m)/(\rho_\phi + \rho_m)$ is equation of state of the effective fluid (the total energy density and the total pressure of the scalar field and the matter). From the acceleration equation in (3.18), it easily see that the universe has to the accelerating expansion ($\ddot{a} > 0$) when the effective equation of state is $w_{\text{eff}} < -1/3$.

Here we will discuss the viable resolution for the coincidence problem via the scaling solution as we mention at above. From many observational data have strongly indicated our universe have component of dark matter $\sim 30\,\%$ and DE $\sim 70\,\%$ approximately. From observations also tell us the current value of the density of the cosmological constant (or DE) and the density of the matter are the same order ($\rho_\phi \sim \rho_{\text{matter}}$). Now there is a problem so called the **coincidence problem**. The density of the DE is subdominant during the radiation and matter dominated eras. The possible resolution of this problem is giving the energy density of scalar field DE (quintessence model) mimics the background matter (and radiation) fluid energy density [77] characterized by

$$\frac{\rho_\phi}{\rho_{\text{matter}}} = \text{constant}\,. \tag{3.19}$$

The energy density of scalar field DE also decrease proportion to the energy density of the background matter fluid dominated eras or otherwise the scalar field DE and matter density rapidly evolve as

$$\frac{\Omega_\phi}{\Omega_{\text{matter}}} \propto a^3\,. \tag{3.20}$$

Such cosmological solutions are called **scaling solution** i.e. the two fluids (both scalar field DE and ordinary matter) have same scaling with time. Therefore a viable DE models must be existed the scaling solution. The scaling solution is the attractor solution, in the sense of the dynamical system.

In this section, we will consider the exponential potential[2] of the scalar field i.e.

$$V(\phi) = V_0 e^{-\kappa\lambda\phi}\,, \tag{3.21}$$

[2] For comprehensive analysis in other type of potential see [7, 78].

Name	x	y	Existence	Stability	Ω_ϕ	γ_ϕ
(a)	0	0	All λ and γ	Saddle point for $0 < \gamma < 2$	0	-
(b1)	1	0	All λ and γ	Unstable node for $\lambda < \sqrt{6}$ Saddle point for $\lambda < \sqrt{6}$	1	0
(b2)	-1	0	All λ and γ	Unstable node for $\lambda > -\sqrt{6}$ Saddle point for $\lambda < -\sqrt{6}$	1	2
(c)	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1-\frac{\lambda^2}{6}}$	$\lambda^2 < 6$	Stable node for $\lambda^2 < 3\gamma$ Saddle point for $3\gamma < \lambda^2 < 6$	1	$\frac{\lambda^2}{3}$
(d)	$\sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}$	$\sqrt{\frac{3\gamma(2-\gamma)}{2\lambda^2}}$	0	Stable node for $3\gamma < \lambda^2 < \frac{24\gamma^2}{(9\gamma-2)}$ Stable spiral for $\lambda^2 > \frac{24\gamma^2}{(9\gamma-2)}$	$\frac{3\gamma}{\lambda^2}$	γ

Table 3.1: The properties of the critical points for the autonomous system in Eq. (3.12), (3.13) and (3.14).

such potential type has well motivation from many theories e.g. particle physics, supersymmetry, supergravity, string theory, scalar-tensor gravity and etc. The exponential potential provide a accelerating expansion of the universe with the scale factor evolve proportion to the power of time (power law inflation) [72]. This potential also give the exact solution of the Friedmann, the Klein-Gordon and the acceleration equation (see in more detail in [76]).

The critical (or fixed) points of the autonomous system of equation in (3.12), (3.13) and (3.14) can be obtained following the definition in (3.2) i.e. $\mathrm{d}x/\mathrm{d}N = \mathrm{d}y/\mathrm{d}N = \mathrm{d}\lambda/\mathrm{d}N = 0$. The lists of critical points and its properties were shown in table (3.1). Here the $\lambda \equiv -V'/\kappa V$ variable play role of the slope of the potential. For the exponential potential give the λ is the constant $(\mathrm{d}\lambda/\mathrm{d}N = 0)$, this potential will simply for analysis in our autonomous system i.e. our autonomous system reduce to 2 variables.

The eigenvalues of linear perturbation each the critical points can be obtained by using equations (3.4) and (4.23) as

- Point (a):

$$\mu_1 = -\frac{3}{2}(2-\gamma)\,, \quad \mu_2 = \frac{3}{2}\gamma\,. \tag{3.22}$$

- Point (b1):

$$\mu_1 = 3 - \frac{\sqrt{6}}{2}\lambda\,, \quad \mu_2 = 3(2-\gamma)\,. \tag{3.23}$$

- Point (b2):

$$\mu_1 = 3 + \frac{\sqrt{6}}{2}\lambda\,, \quad \mu_2 = 3(2-\gamma)\,. \tag{3.24}$$

- Point (c):

$$\mu_1 = \frac{1}{2}(\lambda^2 - 6)\,, \quad \mu_2 = \lambda^2 - 3\gamma\,. \tag{3.25}$$

- Point (d):

$$\mu_{1,2} = -\frac{3(2-\gamma)}{4}\left[1 \pm \sqrt{1 - \frac{8\gamma(\lambda^2 - 3\gamma)}{\lambda^2(2-\gamma)}}\,\right]. \tag{3.26}$$

Next we will briefly analyze and clarify the properties of the five critical points[3] was shown in table 3.1 .

From the classification of the eigenvalues that we have discussed at last in previous section, this shows that the point (a) , (b1) and (b2) are not the attractor points because their eigenvalues are once more than zero and another once more than zero which shown at the first three equations in (3.26) . The points (b1) and (b2) are called the kinetic dominated solution, this type of solution can be interpreted as the universe will collapse in this phase. The point (c) is stable node for $\lambda^2 < 3\gamma$ and it is a saddle point for $3\gamma < \lambda^2 < 6$ it also give the scalar field dominated universe $(\Omega_\phi = 1)$. The effective equation of state of this point is given by $w_{\rm eff} = -1 + \lambda^2/3$. This point also satisfied the acceleration condition i.e. $\lambda^2 < 2$, then the universe will accelerately expand at the late time and the value of the effective equation of state may well pass from observations. Thus this point is suitable for the physical attractor point of this system i.e. all the solutions will approach to the point (c). Although the point (d) also has a stable node in this case but it is not suitable for explain the accelerating expansion at the late time because this point is not satisfied the condition $\Omega_\phi \le 1$, when point (d) is saddle point for $\lambda^2 < 3\gamma$. However the point (d) have provided the scaling regime before the universe pass to the scalar field (DE) dominated and accelerated expansion at the late time. To obtain the scaling solution via the new effective equation of state γ_ϕ defined by

$$\gamma_\phi \equiv 1 + w_\phi = \frac{\rho_\phi + P_\phi}{\rho_\phi} = \frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2V(\phi)} = \frac{2x^2}{x^2 + y^2}\,. \tag{3.27}$$

[3] For detail analysis and calculation see [79] .

For any critical points on phase plane give the solutions of scale factor evolve with power of time as

$$a \sim t^p \,, \quad p = \frac{2}{3\gamma_\phi} \,. \tag{3.28}$$

Next chapter we will perform following this dynamical systematic analysis for the phantom scalar field DE in the effective LQC background.

Chapter 4

Results: The Phantom Field Dynamics in Loop Quantum Cosmology

4.1 Dynamical Analysis

This section we will derive the cosmological equations in LQC for using in the autonomous system that we will see below. We also define the new dimensionless variables for analysis the fate of the universe at the late time of the phantom field DE background in LQC via the dynamical system approach.

Differentiating the equation (2.120) and using the fluid equation (2.9), we obtain

$$\dot{H} = -\frac{(\rho + p)}{2M_{\rm P}^2}\left(1 - \frac{2\rho}{\rho_{\rm c}}\right) \tag{4.1}$$

where $M_{\rm P}^2 \equiv \kappa^{-2} \equiv 8\pi G$ is the Planck mass. We will use such convenient throughout chapters 4 and 5 . The equation (2.120) , (2.9) and (4.1), in domination of the phantom field, the evolution equations of the flat FRW universe in LQC become

$$H^2 = \frac{1}{3M_{\rm P}^2}\left(-\frac{\dot{\phi}^2}{2}+V\right)\left(1-\frac{\rho}{\rho_{\rm c}}\right) \tag{4.2}$$

$$\dot{\rho} = -3H\rho\left(1+\frac{\dot{\phi}^2+2V}{\dot{\phi}^2-2V}\right) \tag{4.3}$$

$$\dot{H} = \frac{\dot{\phi}^2}{2M_{\rm P}^2}\left(1-\frac{2\rho}{\rho_{\rm c}}\right). \tag{4.4}$$

Now we have the key equation for study the dynamics of the phantom field and analyze the future fate of the universe in context of LQC. We define dimensionless variables following the style of [71]

$$X \equiv \frac{\dot{\phi}}{\sqrt{6}M_{\rm P}H} \;\;,\;\; Y \equiv \frac{\sqrt{V}}{\sqrt{3}M_{\rm P}H} \;\;,\;\; Z \equiv \frac{\rho}{\rho_{\rm c}} \tag{4.5}$$

$$\lambda \equiv -\frac{M_{\rm P}V'}{V} \;\;,\;\; \Gamma \equiv \frac{V\,V''}{(V')^2} \;\;,\;\; \frac{\rm d}{{\rm d}N} \equiv \frac{1}{H}\frac{\rm d}{{\rm d}t} \tag{4.6}$$

where $N \equiv \ln a$. Using new variables in equation (4.2), we obtain a constraint equation as

$$1 = (-X^2+Y^2)(1-Z)\,. \tag{4.7}$$

Using the equation (4.4) and the new defined variables above, the acceleration condition,

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0, \tag{4.8}$$

becomes

$$\frac{Y^2}{X^2} > 1 - 3\frac{(1-Z)}{(1-2Z)}\,. \tag{4.9}$$

We will use above condition for check each a critical points that giving the universe to accelerating expansion.

4.1.1 Autonomous System

We will use the cosmological equation in LQC that we have obtained in previous section for set up the autonomous system of equation in LQC. We use the new dimensionless variables which play important role for analyze the cosmological consequence in context of LQC that we have defined at above. Differential equations of new variables with respect to the e-folding number ($N \equiv \ln a$) in our autonomous system are

$$\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}N} &= -3X - \sqrt{\frac{3}{2}}\,\lambda Y^2 - 3X^3\,(1-2Z) && (4.10)\\
\frac{\mathrm{d}Y}{\mathrm{d}N} &= -\sqrt{\frac{3}{2}}\,\lambda XY - 3X^2Y\,(1-2Z) && (4.11)\\
\frac{\mathrm{d}Z}{\mathrm{d}N} &= -3Z\left(1+\frac{X^2+Y^2}{X^2-Y^2}\right) && (4.12)\\
\frac{\mathrm{d}\lambda}{\mathrm{d}N} &= -\sqrt{6}(\Gamma-1)\lambda^2 X\,. && (4.13)
\end{aligned}$$

We use exponential potential in the form

$$V = V_0 \exp\left(-\frac{\lambda}{M_{\mathrm{P}}}\,\phi\right). \tag{4.14}$$

The potential is known to yield power-law inflation in standard cosmology with canonical scalar field with slow-roll parameter $\epsilon = \eta/2 = 1/p$ where $\lambda = \sqrt{2/p}$ and $p > 1$ [72, 73]. Although the potential has been applied to the quintessence scalar field with tracking behavior [74], the quintessence field can not dominate the universe due to constancy of the ratio between densities of matter and quintessence field (see discussion in pages 37-38 of [7]). In case of phantom field in standard cosmology, the stable node is a scalar-field dominated solution with the equation of state, $w = -1 - \lambda^2/3$ [23, 21, 75]. In our LQC phantom domination context, we begin our analysis from equation (4.13) where we can see that for the exponential potential, $\Gamma = 1$. This yields trivial value of $\mathrm{d}\lambda/\mathrm{d}N$ and therefore λ is a constant which is non-zero otherwise the potential is flat. Let $f \equiv \mathrm{d}X/\mathrm{d}N, g \equiv \mathrm{d}Y/\mathrm{d}N$ and $h \equiv \mathrm{d}Z/\mathrm{d}N$. We can find critical points of the autonomous system under condition:

$$(\,f\,,\ g\,,\ h\,)\,|_{(X_{\mathrm{c}}\,,\,Y_{\mathrm{c}}\,,\,Z_{\mathrm{c}})} = 0\,. \tag{4.15}$$

The critical points of this system are

- Point (a) : $\left(\dfrac{-\lambda}{\sqrt{6}},\ \ \sqrt{1+\dfrac{\lambda^2}{6}}\,,\ 0\right)$ (4.16)
- Point (b) : $\left(\dfrac{-\lambda}{\sqrt{6}}, -\sqrt{1+\dfrac{\lambda^2}{6}}\,,\ 0\right)$. (4.17)

Name	X	Y	Z	Existence	Stability	w	Acceleration
(a)	$-\frac{\lambda}{\sqrt{6}}$	$\sqrt{1+\frac{\lambda^2}{6}}$	0	All λ	Saddle point for all λ	$-1-\frac{\lambda^2}{3}$	For all λ (i.e. $\lambda^2 > -2$)
(b)	$-\frac{\lambda}{\sqrt{6}}$	$-\sqrt{1+\frac{\lambda^2}{6}}$	0	All λ	Saddle point for all λ	$-1-\frac{\lambda^2}{3}$	For all λ (i.e. $\lambda^2 > -2$)

Table 4.1: Properties of fixed points of phantom field dynamics in LQC background under the exponential potential Eq.(4.14).

4.1.2 Fixed points

Let $f \equiv \mathrm{d}X/\mathrm{d}N, g \equiv \mathrm{d}Y/\mathrm{d}N$ and $h \equiv \mathrm{d}Z/\mathrm{d}N$. We can find fixed points of the autonomous system under condition:

$$(f \, , \, g \, , \, h \,) \, |_{(X_\mathrm{c} \, , \, Y_\mathrm{c} \, , \, Z_\mathrm{c})} = 0 \, . \tag{4.18}$$

The are two real fixed points of this system: [1]

- Point (a) : $$\left(\frac{-\lambda}{\sqrt{6}}, \quad \sqrt{1+\frac{\lambda^2}{6}} \, , \, 0 \right) , \tag{4.19}$$
- Point (b) : $$\left(\frac{-\lambda}{\sqrt{6}}, -\sqrt{1+\frac{\lambda^2}{6}} \, , \, 0 \right) . \tag{4.20}$$

4.2 Stability Analysis

Suppose that there is a small perturbation δX, δY and δZ about the fixed point $(X_\mathrm{c}, Y_\mathrm{c}, Z_\mathrm{c})$, i.e.,

$$X = X_\mathrm{c} + \delta X \, , \quad Y = Y_\mathrm{c} + \delta Y \, , \quad Z = Z_\mathrm{c} + \delta Z \, . \tag{4.21}$$

[1] The other two imaginary fixed points $(i, 0, 0)$ and $(-i, 0, 0)$ also exist. However they are not interesting here since we do not consider model that includes complex scalar field.

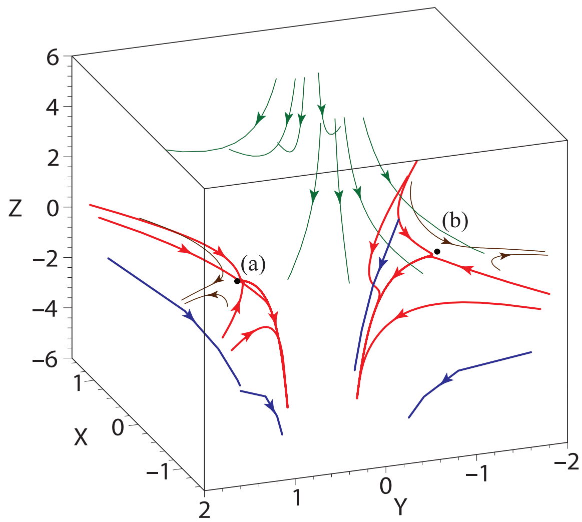

Figure 4.1: Three dimensional phase space of X, Y and Z. The saddle points (a) (-0.40825, 1.0801, 0) and (b) (-0.40825, -1.0801, 0) appear in the figure. λ is set to 1. If the initial condition points are under $Z = 0$ plane (i.e. $Z < 0$ which is non physical), Z approaches $-\infty$ when X and Y approach 0. The solutions in this region are marked with red and blue colours. The green lines (class I solution) are in region $Z \geq 0$ but also non physical since they yields imaginary value of H. The only physical solutions are the black lines (class II solution) above (a) and (b) of which H is real (see section 4.3.2).

From Eqs. (4.10), (4.11) and (4.12), neglecting higher order term in the perturbations, we obtain first-order differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}N}\begin{pmatrix}\delta X\\ \delta Y\\ \delta Z\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta X\\ \delta Y\\ \delta Z\end{pmatrix}. \tag{4.22}$$

The matrix $\mathcal{M}$ defined at a fixed point $(X_\mathrm{c}, Y_\mathrm{c}, Z_\mathrm{c})$ is given by

$$\mathcal{M} = \begin{pmatrix}\frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} & \frac{\partial f}{\partial Z}\\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} & \frac{\partial g}{\partial Z}\\ \frac{\partial h}{\partial X} & \frac{\partial h}{\partial Y} & \frac{\partial h}{\partial Z}\end{pmatrix}_{(X=X_\mathrm{c}, Y=Y_\mathrm{c}, Z=Z_\mathrm{c})}. \tag{4.23}$$

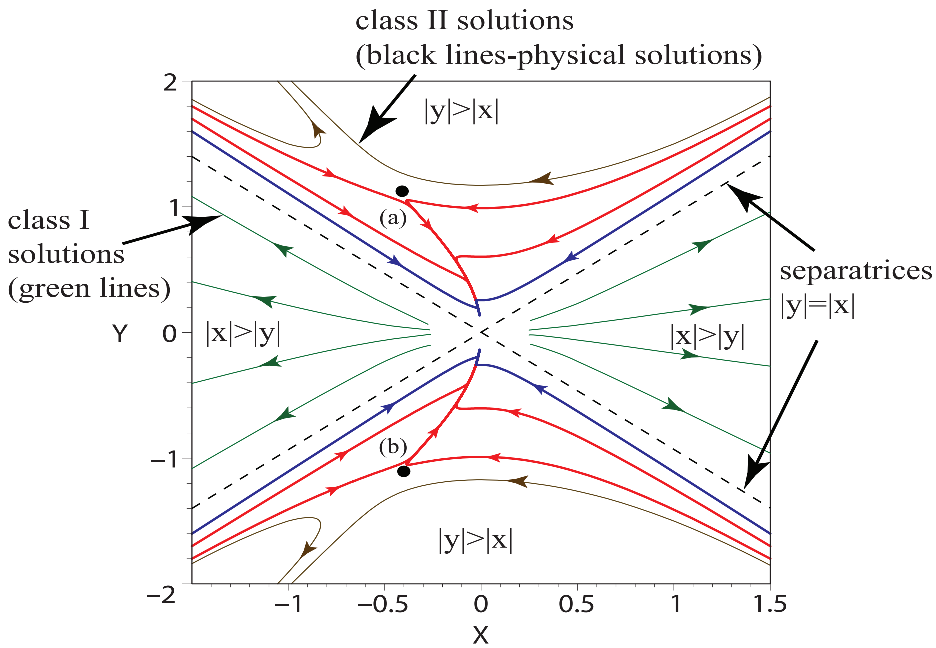

Figure 4.2: Phase space of the kinetic part X and potential part Y (top view). The saddle points (a) (-0.40825, 1.0801) and (b) (-0.40825, -1.0801) are shown here. The blue lines and red lines are in the region $Z < 0$ which is not physical. Green lines are of class I solutions which yields imaginary H. Only class II solutions shown as black lines are physical.

We find eigenvalues of the matrix $\mathcal{M}$ for each fixed point:

- At point (a):

$$\mu_1 = \lambda^2\,, \quad \mu_2 = -\lambda^2\,, \quad \mu_3 = -3 - \frac{\lambda^2}{2}\,. \tag{4.24}$$

- At point (b):

$$\mu_1 = \lambda^2\,, \quad \mu_2 = -\lambda^2\,, \quad \mu_3 = -3 - \frac{\lambda^2}{2}\,. \tag{4.25}$$

From the two fixed points in TABLE 4.1, we found only two physical fixed points which are both saddle points. There is no any stable node in the system. Location of the points depends on only λ and exists for all values of λ. Both points correspond to the equation of state $-1-\lambda^2/3$, that is to say, it has phantom equation of state for all values of $\lambda \neq 0$. Since there is no any attractor in the system, the phase trajectory is very sensitive to initial conditions given to the system. At late time, therefore, there is no singularity.

4.3 Numerical Results

Numerical results from the autonomous set (4.10), (4.11) and (4.12) are presented in Figs. 4.1 and 4.2 where we set $\lambda = 1$. Locations of the two saddle points are: point (a) $(X_{\rm c} = -0.40825, Y_{\rm c} = 1.0801, Z_{\rm c} = 0)$ and point (b) $(X_{\rm c} = -0.40825, Y_{\rm c} = -1.0801, Z_{\rm c} = 0)$ which match our analytical results. Since there is no any stable node and the solutions are sensitive to initial conditions, we need to make classification of solutions and analyze them separately. Note that the condition, $Z \geq 0$ must hold for physical solutions since the density can not be negative, i.e. $\rho \geq 0$, therefore the physical solutions must be in the region $Z \geq 0$. The blue lines and red lines are solutions in the region $Z < 0$ hence not physical and will no longer be of our interest. From now on we consider only the region $Z \geq 0$. There are two separatrices satisfying equation $|X| = |Y|$. The separatices determine borders of solutions into four regions of XY plane. In regions with $|X| > |Y|$, the solutions therein are green lines (hereafter classified as class I). The other regions with $|Y| > |X|$ contain solutions seen as black line (classified as class II).

4.3.1 Class I solutions

Consider the Friedmann equation (4.2), the Hubble parameter, H takes real value only if

$$\frac{1}{3M_{\rm P}^2}\left(-\frac{\dot{\phi}^2}{2}+V\right)\left(1-\frac{\rho}{\rho_{\rm lc}}\right) \;\; \geq \;\; 0\,. \tag{4.26}$$

Divided by $3M_{\rm P}^2H^2$ on both sides, the expression above becomes

$$(-X^2+Y^2)(1-Z) \;\; \geq \;\; 0\,. \tag{4.27}$$

It is clear from (4.27) that, in order to obtain real value of H, class I solutions (green line) must obey both conditions $|X| > |Y|$ and $Z \geq 1$ together. As we consider

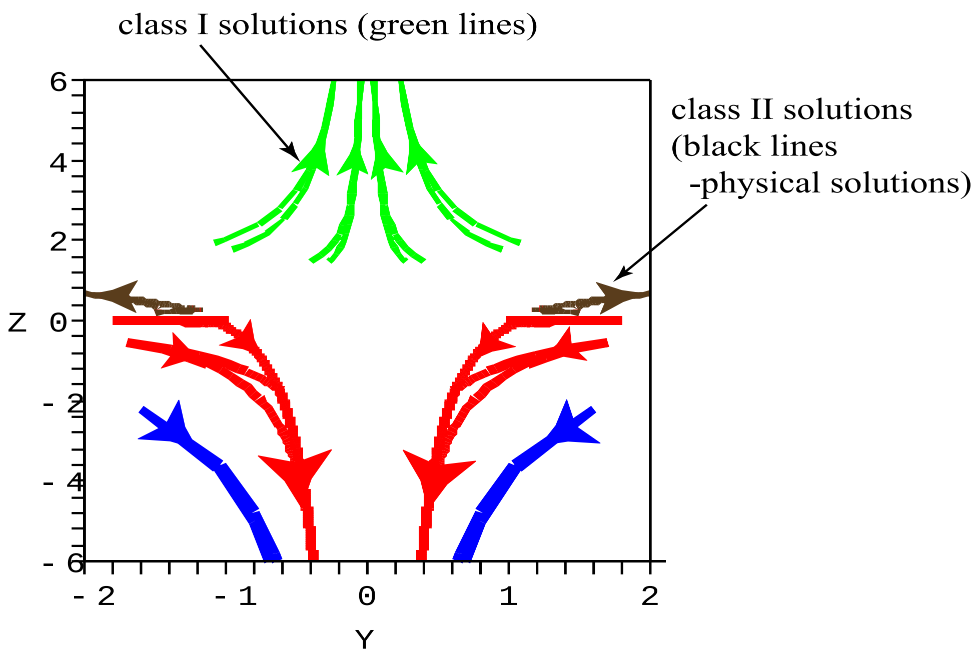

Figure 4.3: Phase space of Y and Z. Figure shown explicit classification between class I and class II solution. All solutions of the blue lines and red lines will converge to the region $Z < 0$ which is not physical. Green lines are not physical solutions when $Z > 1$. Only class II solutions shown as black lines are physical and physical solutions are bounded at range $0 < Z < 1$.

$Z = \rho/\rho_{\rm lc}$ with $\rho = -(\dot{\phi}^2/2) + V$, we can rewritten it as

$$\frac{\rho_{\rm lc} Z}{3M_{\rm P}^2 H^2} = -X^2 + Y^2 \,. \tag{4.28}$$

Imposing $|X| > |Y|$ to Eq. (4.28), we obtain $Z < 0$ instead, hence contradict to $Z \geq 1$. Therefore this class of solutions does not possess any real values of H and hence not physical solutions.

4.3.2 Class II solutions

As we proceed the same analysis as done for class I, we found that in order for H to be real, class II solutions must obey both $|Y| > |X|$ and $0 \leq Z \leq 1$ together. In

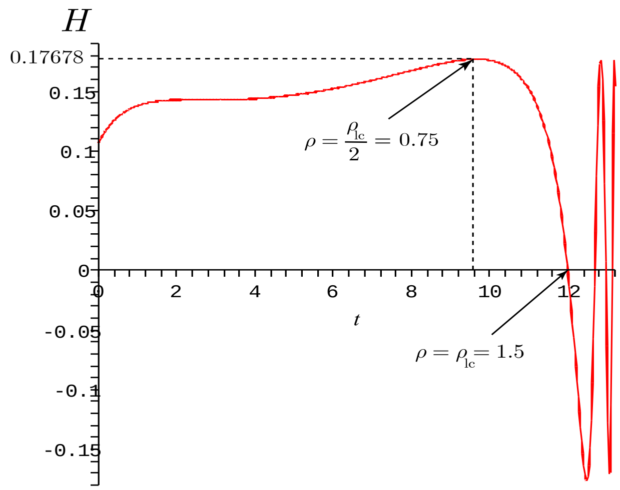

Figure 4.4: Evolution of H with time of a class II solution. Set values are $\lambda = 1, \rho_{\rm lc} = 1.5, V_0 = 1$ and $M_{\rm P} = 2$. The universe undergoes acceleration from the beginning until reaching turning point at $\rho = \rho_{\rm lc}/2 = 0.75$ where $H = H_{\rm max} = 0.17678$. Beyond this point, the universe expands with deceleration until halting $(H = 0)$ at $\rho = \rho_{\rm lc} = 1.5$. After halting, it undergoes contraction until H bounces. The oscillating in H goes on forever.

Eq. (4.28), when imposing $|Y| > |X|$, we obtain $Z > 0$. Therefore as we combine both results, it is concluded that class II solutions can possess real H in the region $|Y| > |X|$ and $0 < Z \leq 1$. This implies $0 < \rho \leq \rho_{\rm lc}$ as in the case of standard non-phantom field in LQC [81]. The class II is therefore only class of physical solutions. We consider another set of autonomous equations from which the evolution of cosmological variables are conveniently obtained by using numerical approach. In the new autonomous set, instead of using N [80] , which could decrease after bouncing from LQC effect, we use time as independent variable. We define new variable as

$$\dot{\phi} \;=\; S\,. \tag{4.29}$$

The Eqs. (2.21) and (4.4) are therefore rewritten as

$$\dot{H} \;=\; \frac{S^2}{2M_{\rm P}^2}\left[1 - \frac{2}{\rho_{\rm lc}}\left(-\frac{S^2}{2} + V(\phi)\right)\right], \tag{4.30}$$

$$\dot{S} \;=\; -3HS + V'\,. \tag{4.31}$$

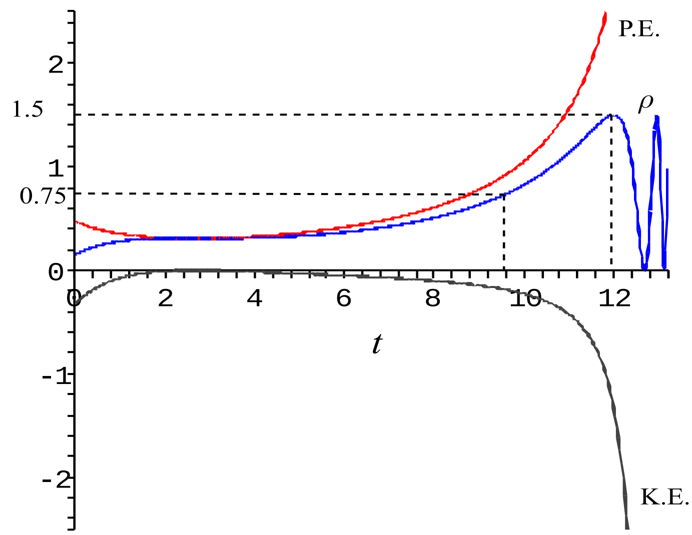

Figure 4.5: Time evolution of potential energy density (P.E.), kinetic potential energy (K.E.) and $\rho =$ K.E. + P.E. of the field for a class II solution. K.E. is always negative and, at late time, it goes to $-\infty$ while P.E. is always positive. ρ is maximum when $\rho = \rho_{\rm lc} = 1.5$. Other features are discussed as in Fig. 4.4.

The Eqs. (4.29), (4.30) and (4.31) form another closed autonomous system. Numerical integrations from the new system yields result plotted in Figs. 4.4 and 4.5 in which set values are $\lambda = 1, \rho_{\rm lc} = 1.5, V_0 = 1$ and $M_{\rm P} = 2$. From Eq. (2.120) the slope of H with respect to ρ, $\mathrm{d}H/\mathrm{d}\rho$, is flat when $\rho = \rho_{\rm lc}/2$ [81]. Another fact is

$$\left(\frac{\mathrm{d}^2 H}{\mathrm{d}\rho^2}\right)_{\rho=\rho_{\rm lc}/2} = \frac{-2}{M_{\rm P}\sqrt{3\rho_{\rm lc}^3}} \ < \ 0\,, \tag{4.32}$$

hence, as $\rho = \rho_{\rm lc}/2$, H takes maximum value $H_{\rm max} = \sqrt{\rho_{\rm lc}/12M_{\rm P}^2}$. This result is valid in LQC scenario regardless of types of fluid. We use set parameters given above in Figs. 4.4 and 4.5. As $\rho = \rho_{\rm lc}/2 = 0.75$, H is maximum and $H_{\rm max} = 0.17678$. When $H = 0$ i.e. $\rho = \rho_{\rm lc} = 1.5$, the expansion halts. After halting, H turns negative (contracting of scale factor) after that it bounces forward and backward faster and faster in time. The fast bounce in H is an effect from the bounce in ρ as illustrated in Fig. 4.5 where the red line represents potential energy density $V(\phi)$, the green line represents kinetic energy density $-\dot{\phi}^2/2$ and the blue line is total energy density

ρ. The negative magnitude of kinetic energy density becomes very large as the field rolling faster and faster up the potential. The exponential potential energy density increases more and more in positive axis. However magnitude of potential part is always greater than kinetic part, therefore sum of them, ρ, is positive. Oscillation of ρ affects in H oscillation.

Chapter 5

Conclusion

A dynamical system of phantom canonical scalar field evolving in background of loop quantum cosmology is considered and analyzed in this work. Exponential potential is used in the system. The analytical dynamical analysis of autonomous system renders two real fixed points $(-\lambda/\sqrt{6},\ \sqrt{1+\lambda^2/6}\,,\ 0\,)\,, (-\lambda/\sqrt{6}, -\sqrt{1+\lambda^2/6}\,,\ 0\,)$ which are saddle points corresponding to equation of state, $w = -1 - \lambda^2/3$. (Note that in case of standard cosmology, this fixed point is Big Rip attractor at $(-\lambda/\sqrt{6},\ \sqrt{1+\lambda^2/6})$ with the same equation of state, $w = -1-\lambda^2/3$ [21].) Due to absence of stable nodes, the late time behavior depends very much on initial conditions given. Therefore we do numerical plots to investigate solutions of the system and classify the solutions. A physical solution must locate in region $Z \geq 0$ i.e. $\rho \geq 0$. Within this $Z \geq 0$ region, we call solutions satisfying $|X| > |Y|$ and $Z \geq 1$ as class I. In order to obtain real value of H in class I, Z must be negative which, however, contradicts to $Z \geq 1$. Then the class I are non physical. Class II solutions identified by $|Y| > |X|$ and $0 < Z \leq 1$ are the only physical solutions. The universe with class II solution undergoes acceleration from the beginning until $\rho = \rho_{\rm lc}/2$ where $H = H_{\rm max} = \sqrt{\rho_{\rm lc}/12 M_{\rm P}^2}$. After that the universe decelerates until it stops expansion $(H = 0, \rho = \rho_{\rm lc})$. Accelerating contraction happens right after halting, however the accelerating contraction does not go on forever but turn to decelerating contraction. The universe bounces again and turn to expand accelerately. This oscillation goes on and on forever.

Appendix A

Conventions

Throughout of this thesis except appendix B. We used the convention following [60] .

Indices notations

The middle of Greek alphabet i.e. $\mu\,,\nu\,,\sigma\,,... = 0,\ 1,\ 2,\ 3$ are 4 dimensional spacetime coordinate indices.

The middle of the upper Latin alphabet i.e. $I\,,J\,,K\,,... = 0,\ 1,\ 2,\ 3$ are 4 dimensional local Lorentz frame indices.

The beginning of the lower Latin alphabet i.e. $a\,,b\,,c\,,... = 1,\ 2,\ 3$ are 3 dimensional spatial coordinate indices.

The middle of the lower Latin alphabet i.e. $i\,,j\,,k\,,... = 1,\ 2,\ 3$ are 3 dimensional spatial local Lorentz frame indices.

We also used the signature $(-,+,+,+)$ for $g_{\mu\nu}$ and η_{ij} spacetime and local Lorentz frame metric respectively and setting the velocity of light equal to 1 .

Symmetrization and Antisymmetrization of Tensors

For symmetry tensor is defined as

$$T_{(ab)} = T_{ab} + T_{ba} \tag{A.1}$$

and for antisymmetry tensor is defined as

$$T_{[ab]} = T_{ab} - T_{ba}\,. \tag{A.2}$$

Definition of the curvature tensor [1]

[1] We use convention following [85] .

The curvature (Riemann) tensor of the affine connection $\Gamma^{\rho}_{\mu\nu}$ (or Christoffel symbol) with torsion free i.e. $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$, is defined by

$$\nabla_{[\mu}\nabla_{\nu]}t_{\rho} = R_{\mu\nu\rho}{}^{\sigma}t_{\sigma} \tag{A.3}$$

and given by

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_{[\mu}\Gamma^{\sigma}_{\nu]\rho} + \Gamma^{\sigma}_{\alpha[\mu}\Gamma^{\alpha}_{\nu]\rho} \tag{A.4}$$

where t_{μ} is vector field on 4-dimensional spacetime coordinate. We also manifest the curvature tensor of the spin connection as [49]

$$R_{\mu\nu\,I}{}^{J} = \partial_{[\mu}\omega_{\nu]I}^{\;\;J} + \omega_{[\mu\,K}^{\;\;J}\omega_{\nu]I}^{\;\;K} \,. \tag{A.5}$$

Multiplying tetrad e^I_{μ} into (A.3) and using identity $t_{\mu} = e^I_{\mu}t_I$, we get

$$e^{\rho}_I R_{\mu\nu\rho}{}^{\sigma}t_{\sigma} = R_{\mu\nu\rho}{}^{\sigma}e^{\rho}_I e^J_{\sigma}\,t_J = R_{\mu\nu\,I}^{\;\;J}t_J \,. \tag{A.6}$$

Therefore we can express The curvature tensor in other form as

$$R_{\mu\nu\,I}^{\;\;J} = R_{\mu\nu\rho}{}^{\sigma}e^{\rho}_I e^J_{\sigma} \,. \tag{A.7}$$

The curvature $F_{\mu\nu\,I}{}^{J}$ of $SU(2)$ spin connection A^{IJ}_{μ} is defined by

$$F_{\mu\nu\,I}{}^{J} = \partial_{[\mu}A_{\nu]I}^{\;\;J} + A_{[\mu\,K}^{\;\;J}A_{\nu]I}^{\;\;K} \,. \tag{A.8}$$

The Levi-civita symbol

The Levi-civita symbol $\epsilon^{\,\mu_1\cdots\mu_n}$ and $\epsilon_{\,\mu_1\cdots\mu_n}$ are total asymmetric density tensor weight 1 and -1 respectively of n dimensional spacetime. Properties of The Levi-civita are as follows

- $$\epsilon^{\,\mu_1\cdots\mu_n} = \begin{cases} 1\,, & \text{for even permutation arranging indices} \\ -1\,, & \text{for odd permutation arranging indices} \\ 0\,, & \text{for otherwise}\,. \end{cases} \tag{A.9}$$

- $$\epsilon_{\,\mu_1\cdots\mu_n} = \frac{1}{g}\,g_{\mu_1\nu_1}\cdots g_{\mu_n\nu_n}\,\epsilon^{\,\nu_1\cdots\nu_n} \tag{A.10}$$

for any non-degeneracy metric tensor $g_{\mu\nu}$, where g is determinant of metric tensor $g_{\mu\nu}$, and g is defined by

$$g \equiv \det\left(g_{\mu\nu}\right) = \frac{1}{n!}\,\epsilon^{\,\mu_1\cdots\mu_n}\,\epsilon^{\,\nu_1\cdots\nu_n}\,g_{\mu_1\nu_1}\cdots g_{\mu_n\nu_n} \,.$$

- $$\epsilon^{\,\mu_1 \dots \mu_n}\,\epsilon_{\,\nu_1 \dots \nu_n} = \delta^{[\mu_1}_{\nu_1}\,\delta^{\mu_2}_{\nu_2} \dots \delta^{\mu_n]}_{\nu_n} \tag{A.11}$$

Let us examine the properties of the Levi-civita symbol from above in 4 dimensional spacetime as

$$\begin{aligned}
-\,\epsilon_{\,0123} &= \epsilon^{\,0123} = \ \ 1 \\
-\epsilon_{\,1023} &= \epsilon^{\,1023} = -1 \\
\epsilon_{\,1123} &= \epsilon^{\,1123} = \ \ 0 \\
\epsilon_{\,\mu\nu\sigma\rho} &= \frac{1}{g}\, g_{\mu\alpha} g_{\nu\beta} g_{\sigma\gamma} g_{\rho\lambda}\, \epsilon^{\alpha\beta\gamma\lambda} \\
g \equiv \det\left(g_{\mu\nu}\right) &= \frac{1}{4!}\; g_{\mu\alpha} g_{\nu\beta} g_{\sigma\gamma} g_{\rho\lambda}\, \epsilon^{\,\mu\nu\sigma\rho}\, \epsilon^{\alpha\beta\gamma\lambda} \\
\epsilon_{\,\mu\nu\sigma\rho}\, \epsilon^{\mu\beta\gamma\lambda} &= \delta^{[\beta}_{\nu} \delta^{\gamma}_{\sigma} \delta^{\lambda]}_{\rho} \\
\epsilon_{\,\mu\nu\sigma\rho}\, \epsilon^{\mu\nu\gamma\lambda} &= 2!\, \delta^{[\gamma}_{\sigma} \delta^{\lambda]}_{\rho} \\
\epsilon_{\,\mu\nu\sigma\rho}\, \epsilon^{\mu\nu\sigma\lambda} &= 3!\, \delta^{\lambda}_{\rho} \\
\epsilon_{\,\mu\nu\sigma\rho}\, \epsilon^{\mu\nu\sigma\rho} &= 4!
\end{aligned}$$

Appendix B

$3+1$ ADM and Hamiltonian Formalisms in GR

[1]

B.1 The $3+1$ ADM Formulation in General Relativity

The Hamiltonian formulation of GR is very powerful tool applications in frontiers theoretical physics. For beautiful discussions in this topic see [49] and more easier [84]. Certainly LQC also require the Hamiltonian of FRW for quantization. This section we will brief review The $3+1$ ADM formulation following [49] . This formalisms was formulated by Arnowitt, Deser and Misner [83]. Considering the 4-dimensional global spacetime (M, g) where M is 4-dimension manifold and g is spacetime metric on M. We can split 4-dimensional spacetime into the 3-dimensional spatial hepersurfaces Σ_t and foliated by 1 dimensional timelike curve vector field. The Σ_t hypersurface parameterized by a global time function t. Let n^a is the normal unit vector field[2] on the hypersurface Σ_t. We introduce the metric q_{ab} on each the hypersurface Σ_t. This

[1] We use notations and conventions following [49] in appendix B only.

[2] We impose the dot produce of the normal unit vector field is $n_a n^a = -1$. That means n^a perpendicular to every points of spacelike hypersurface and propagate in timelike curve.

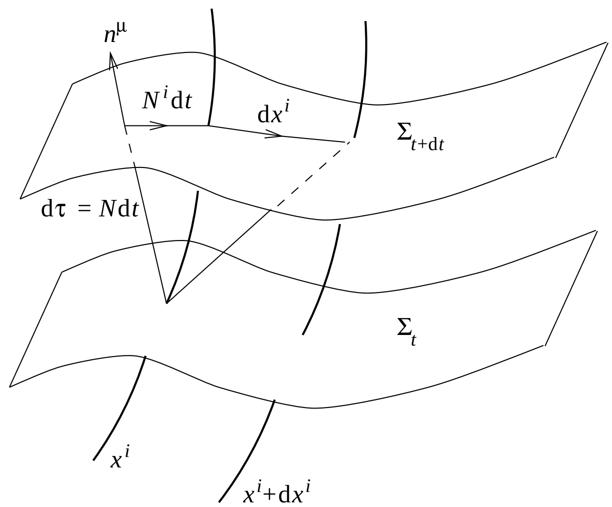

Figure B.1: The evolution of the 3-dimensional hypersurface evolve with time (3 + 1 formalisms). The dynamics of spacetime is illustrated by the lapse function N and the shift vector N_i, figure from [51].

variable is induce by the spacetime metric g_{ab}, the q_{ab} is given by

$$q_{ab} = g_{ab} + n_a n_b \tag{B.1}$$

and the inverse of q_{ab} is given by

$$q^{ab} = g^{ab} + n^a n^b \,. \tag{B.2}$$

Since the 3-dimensional metric q_{ab} are perpendicular to the normal unit vectior field n^a i.e.

$$\begin{aligned} q_{ab} n^a &= (g_{ab} + n_a n_b) n^a \\ &= n_b + (-1) n_b = 0 \end{aligned} \tag{B.3}$$

we also use relations $g_{ab} n^a = n_b$ and $n_a n^a = -1$ in the first line. The lapse function N of Σ_t. That meaning of N is the normal part of the Σ_t, given by

$$N = -t^a n_a = (n^a \nabla_a t)^{-1} \tag{B.4}$$

where t^a is vector field[3] on M and satisfied by $t^a\nabla_a t = 1$. We also gives ∇_a is the covariant derivative on 4-dimensional spacetime which compatible with the g_{ab} metric i.e. $\nabla_c g_{ab} = 0$. Next we will introduce the shift vector N^a, given by

$$N_a = h_{ab}t^b\,. \tag{B.5}$$

The shift vector is tangential part of the Σ_t. The lapse function N and shift vector N^a represent how to move forward on time of hypersurface Σ_t in the manifold.

After we split 4-dimensional spacetime (or manifold) to $3+1$ (3-space + 1-time) spacetime. The view of spacetime in ADM formalisms is the time evolution of g_{ab} metric on a fixed 3-dimensional manifold. Thus we can respect the metric q_{ab} on hypersurface Σ_t. We will introduce the notion of extrinsic curvature as representing the bending of hypersursurface Σ_t which embeding in spacetime manifold M. The extrinsic curvature is defined by

$$K_{ab} = \nabla_a n_b \tag{B.6}$$

where the extrinsic curvature K_{ab} is symmetric tensor i.e. $K_{[ab]} = 0$. The extrinsic curvature can rising and lower indices via contraction of q_{ab} metric (e.g. $K_a^b = q_{ac}K^{bc}$). We also express the extrinsic curvature via the time coordinate derivative from the covariant derivative of any vector field which is normal to hypersurface Σ_t.

$$\begin{aligned} K_{ab} &= q_a^c \nabla_c n_b \\ &= \frac{1}{2}\pounds_n q_{ab}\,. \end{aligned} \tag{B.7}$$

From these relations we can reconstruct the $3+1$ spacetime manifold in GR. Now we have new manifold which consist three dynamical variables (Σ, q_{ab}, K_{ab}) where Σ is the 3-dimensional manifold, q_{ab} is the metric on Σ and K_{ab} is the symmetric tensor field on Σ. We will establish the relation between the 3-dimensional metric q_{ab} and it's covariant derivative compatible D_a on hypersurface i.e.

$$D_c q_{ab} = q_c^d q_a^e q_b^f \nabla_d (g_{ef} + n_e n_f) = 0 \tag{B.8}$$

since we also used equation (B.3) and $\nabla_c g_{ab} = 0$. let $T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$ is the 4-dimensional tensor, we can project 4-dimensional tensor on manifold M to the hypersurface Σ. By using q_{ab} metric follow as

$$T^{a_1 \ldots\, a_k}{}_{b_1 \ldots\, b_l} = q^{a_1}_{c_1} \ldots q^{a_k}_{c_k} q^{d_1}_{b_1} \ldots q^{d_l}_{b_l} T^{c_1 \ldots\, c_k}{}_{d_1 \ldots\, d_l} \tag{B.9}$$

[3] We can represent the vector field t^a is the flow of time throughout the 4-dimension spacetime.

Let us continually construct the 4-dimensional curvature ${R^d}_{abc}$ by expression of the 3-dimensional curvature ${\mathcal{R}^d}_{abc}$ and the extrinsic curvature K_{ab}. We analogue definition of the 4-dimensional curvature ${R^d}_{abc}$ then the 3-dimensional curvature ${\mathcal{R}^d}_{abc}$ given by

$$ {\mathcal{R}^d}_{abc} = D_{[a}D_{b]}\omega_c \, . \tag{B.10} $$

After use equations (B.3), (B.6), (B.8), (B.9) and (B.10), after simple calculation but quite lengthly manipulation, we obtain

$$ {\mathcal{R}^d}_{abc} = q^e_a q^f_b q^k_c q^d_j {R^j}_{efk} - K_{ac}{K_b}^d + K_{bc}{K_a}^d \, . \tag{B.11} $$

and

$$ D_a {K_b}^a - D_b {K_a}^a = R_{ac} n^c q^a_b \, . \tag{B.12} $$

We so-called equations (B.11) and (B.12) as the Gauss-Codacci relations.

We consider the 4-dimensional Einstein field equation in vacuum case reads

$$ G_{ab} = 0 \tag{B.13} $$

where $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$ is the Einstein tensor. Contracting the Einstein tensor by q_{ab} and n^a, we obtain the initial value constraint in vacuum case as

$$ 0 = G_{bc} q^b_a n^c = R_{bc} q^b_a n^c = D_b K^a_a - D_a K^a_b \tag{B.14} $$

In addition, we have relation between contraction of the 4-dimensional curvature by 3-dimensional metric and contraction of the Einstein tensor by normal unit vector as

$$ \begin{aligned} R_{abcd} q^{ac} q^{bd} &= R_{abcd}(g^{ac} + n^a n^c)(g^{bd} + n^b n^d) \\ &= R + 2R_{ac} n^a n^c \\ &= 2G_{ac} n^a n^c \end{aligned} \tag{B.15} $$

we use the identity $R_{abcd} n^a n^b n^c n^d = 0$ from page 78 of [84] at the second line. We also use the first Gauss-Codacci relation in (B.11) into above equation, we obtain

$$ \begin{aligned} 0 &= G_{ab} n^a n^b \\ &= \frac{1}{2}\left(\mathcal{R} + (K^a_a)^2 - K_{ab}K^{ab}\right) \, . \end{aligned} \tag{B.16} $$

The equations (B.14) and (B.16) are the initial value constraint equations of general relativity in the vacuum case.

B.2 Hamiltonian Formalisms in General Relativity

As we saw in previous section. We have shown the splitting the 4-dimensional spacetime manifold to the 3-dimensional hypersurface and timelike curve. We have new view point of spacetime is 3-dimensional surface slicing each a time evolution. This view point is suitable in the classical mechanics formulation i.e. Hamiltonian formalisms. This section we will formulate the Hamiltonian in general relativity.

Before we reconstruct of the Hamiltonian formalisms. We will discuss on the relation between 4-dimensional world volume and 3-dimensional world volume. The 4-dimensional world volume element $\mathbf{e}_{abcd}$ at a fixed time on each hypersurface gives the relation ${}^{(3)}\mathbf{e}_{abc} = \mathbf{e}_{abcd}t^d$ on Σ_t where ${}^{(3)}\mathbf{e}_{abc}$ is 3-dimensional world volume. Using this fact we obtain the weight of Jacobian of q_{ab} from g_{ab} is

$$\sqrt{-g} = N\sqrt{q} \tag{B.17}$$

where q is the spatial (space) component of q_{ab} and N is lapse function that we have defined. We can express the normal unit vector in term of the vector field t^a , lapse function N and shift vector N^a as

$$n^a = \frac{1}{N}(t^a - N^a)\,. \tag{B.18}$$

Using above equation the inverse metric g^{ab} has form

$$g^{ab} = q^{ab} - N^{-2}(t^a - N^a)(t^b - N^b)\,. \tag{B.19}$$

In the langauge of $3+1$ spacetime, we can write the gravitational Lagrangian in the expression of function (q_{ab}, N, N^a) and their time and space derivative. Let us start from the contraction of the Einstein field equation with normal unit vector n^a , we obtain

$$R = 2(G_{ab}n^a n^b - R_{ab}n^a n^b)\,. \tag{B.20}$$

We recall second constraint equation in (B.16) , we have

$$G_{ab}n^a n^b = \frac{1}{2}\left(\mathcal{R} + (K^a_a)^2 - K_{ab}K^{ab}\right)\,. \tag{B.21}$$

Considering the second term in equation (B.20) . We use definition of the Riemanian curvature tensor, we get

$$\begin{aligned} R_{ab}n^a n^b &= R^c{}_{acb}n^a n^b \\ &= -n^a(\nabla_a\nabla_c - \nabla_c\nabla_a)n^c \\ &= (\nabla_a n^a)(\nabla_c n^c) - (\nabla_a n^c)(\nabla_c n^a) - \nabla_a(n^a\nabla_c n^c) + \nabla_c(n^a\nabla_a n^c) \\ &= (K^a_a)^2 - K_{ac}K^{ac}\,. \end{aligned} \tag{B.22}$$

The last two terms in the third line are divergence term and these terms vanish at the boundary. Using equations (B.17) , (B.20) , (B.21) and (B.22) . The lagrangian density of the gravitational action is

$$\mathcal{L}_{GR} = N\sqrt{q}(\mathcal{R} + K_{ab}K^{ab} - K^2) \tag{B.23}$$

The relation between the extrinsic curvature K_{ab} and the time derivative of q_{ab} or $\dot{q_{ab}} \equiv \pounds_t q_{ab}$ is given by

$$\begin{aligned} K_{ab} &= \frac{1}{2}\pounds_n q_{ab} = \frac{1}{2}\left(n^c\nabla_c q_{ab} + q_{ac}\nabla_b n^c + q_{bc}\nabla_a n^c\right) \\ &= \frac{1}{2N}\left(Nn^c\nabla_c q_{ab} + q_{ac}\nabla_b(Nn^c) + q_{bc}\nabla_a(Nn^c) - q_{ac}n^c\nabla_b N - q_{bc}n^c\nabla_a N\right) \\ &= \frac{1}{2N}\left(Nn^c\nabla_c q_{ab} + q_{ac}\nabla_b(Nn^c) + q_{bc}\nabla_a(Nn^c)\right) \\ &= \frac{1}{2N}\left(\pounds_t q_{ab} - \pounds_n q_{ab}\right) \\ &= \frac{1}{2N}\left(\dot{q_{ab}} - D_a N_b - D_b N_a\right)\,. \end{aligned} \tag{B.24}$$

The last two term in the second line will vanish from $q_{ab}n^b = 0$ and using equations (B.18) and (B.19) in the third line.

Let us consider the canonical conjugate momentum π^{ab} of q_{ab} is given by

$$\begin{aligned} \pi^{ab} &\equiv \frac{\partial\mathcal{L}_{GR}}{\partial\dot{q_{ab}}} = \frac{\partial K_{ab}}{\partial\dot{q_{ab}}}\frac{\partial\mathcal{L}_{GR}}{\partial K_{ab}} \\ &= \left(\frac{\partial}{\partial\dot{q_{ab}}}(2N)^{-1}\left(\dot{q_{ab}} - D_a N_b - D_b N_a\right)\right)\left(\frac{\partial}{\partial K_{ab}}N\sqrt{q}(\mathcal{R} + K_{ab}K^{ab} - K^2)\right) \\ &= \sqrt{q}(K^{ab} - q^{ab}K) \end{aligned} \tag{B.25}$$

we also use equations (B.23) and (B.24) to obtain the canonical conjugate momentum. Conversely we can write the extrinsic curvature K_{ab} in term of the canonical conjugate momentum π^{ab} follow as[4]

$$\begin{aligned}\pi^{ab} &= \sqrt{q}(K^{ab} - q^{ab}K) \\ q_{ab}\pi^{ab} &= \sqrt{q}q_{ab}(K^{ab} - q^{ab}K) \\ \pi &= \sqrt{q}(K - 3K) \\ K &= -\frac{\pi}{2\sqrt{q}}\,. \end{aligned} \tag{B.26}$$

Using the above equation substitute to the definition of the canonical conjugate momentum in equation (B.25), we get

$$\begin{aligned}\pi^{ab} &= \sqrt{q}K^{ab} + \frac{1}{2}q^{ab}\pi \\ K^{ab} &= \frac{1}{\sqrt{q}}\left(\pi^{ab} - \frac{1}{2}q^{ab}\pi\right)\,. \end{aligned} \tag{B.27}$$

Following the definition of the Hamiltonian in the classical mechanics. The Hamiltonian density in general relativity is given by

$$\begin{aligned}\mathcal{H}_{GR} &= \pi^{ab}\dot{q_{ab}} - \mathcal{L}_{GR} \\ &= \pi^{ab}(2NK_{ab} + D_aN_b + D_bN_a) - N\sqrt{q}(\mathcal{R} + K_{ab}K^{ab} - K^2) \\ &= \frac{2N\pi^{ab}}{\sqrt{q}}\left(\pi_{ab} - \frac{1}{2}q_{ab}\pi\right) + \pi^{ab}(D_aN_b + D_bN_a) \\ &\quad -N\sqrt{q}\left(\mathcal{R} + \frac{1}{q}\left(\pi_{ab} - \frac{1}{2}q_{ab}\pi\right)\left(\pi^{ab} - \frac{1}{2}q^{ab}\pi\right) - \left(-\frac{\pi}{2\sqrt{q}}\right)^2\right) \\ &= -N\sqrt{q}\mathcal{R} + \frac{N}{\sqrt{q}}\left(\pi_{ab}\pi^{ab} - \frac{1}{2}\pi^2\right) + 2\pi^{ab}D_aN_b \\ &= \sqrt{q}\left(N\left(-\mathcal{R} + \frac{1}{q}\pi_{ab}\pi^{ab} - \frac{1}{2q}\pi^2\right) - 2N_bD_a\left(\frac{\pi^{ab}}{\sqrt{q}}\right) + D_a\left(\frac{2N_b\pi^{ab}}{\sqrt{q}}\right)\right)\,. \end{aligned} \tag{B.28}$$

The last term in the above equation is also a divergent term which vanishes at the boundary where $H_{GR} = \int \mathcal{H}_{GR}\ {}^{(3)}\mathbf{e}$. Varying the Hamiltonian H_{GR} with respect to N and N_a gives what follows

$$\mathcal{C} \equiv -\sqrt{q}\left(\mathcal{R} - \frac{1}{q}\pi_{ab}\pi^{ab} + \frac{1}{2q}\pi^2\right) = 0 \tag{B.29}$$

[4]The canonical conjugate momentum π_{ab} is the symmetric tensor like the extrinsic curvature K_{ab} i.e. $\pi_{ab} = \pi_{ba}$. The contraction of it is $\pi_{ab}q^{ab} = \pi^a_a = \pi$.

and

$$\mathcal{C}^a \equiv -\sqrt{q} D_b \left(\frac{\pi^{ab}}{\sqrt{q}} \right) = 0 \,. \tag{B.30}$$

The first constraint $\mathcal{C}$ is known as the Hamiltonian constraint and the second constraint $\mathcal{C}^a$ is known as the diffeomorphisms constraint. We can re-write the Hamiltonian formulation of general relativity in the sum of two above constraints as

$$H_{GR} = \int (N\mathcal{C} + N_a\mathcal{C}^a) \ {}^{(3)}\mathbf{e} \,. \tag{B.31}$$

Following classical mechanics, the Hamilton's equation of motion is given by

$$\dot{x} = \{x \,, H_{GR}\} \tag{B.32}$$

for any phase space variable x. The dynamical variables q_{ab} and π^{ab} are canonically related by the Poisson's bracket i.e.

$$\{q_{ab}(x) \,, \pi^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\delta(x \,, y) \,. \tag{B.33}$$

The Hamilton's equation of motion of general relativity is given by

$$\begin{aligned} \dot{q_{ab}} &= \{q_{ab} \,, H_{GR}\} = \frac{\delta H_{GR}}{\delta \pi^{ab}} \\ &= \frac{2N}{\sqrt{q}} \left(\pi_{ab} - \frac{1}{2} q_{ab}\pi \right) + 2D_{(a}N_{b)} \end{aligned} \tag{B.34}$$

and

$$\begin{aligned} \dot{\pi^{ab}} &= \{\pi^{ab} \,, H_{GR}\} = -\frac{\delta H_{GR}}{\delta q_{ab}} \\ &= -N\sqrt{q} \left(\mathcal{R}^{ab} - \frac{1}{2} q^{ab}\mathcal{R} \right) + \frac{1}{2\sqrt{q}} N q^{ab} \left(\pi_{ab}\pi^{ab} - \frac{1}{2}\pi^2 \right) \\ &\quad -\frac{2N}{\sqrt{q}} \left(\pi^{ac}\pi_c^b - \frac{1}{2}\pi\pi^{ab} \right) + \sqrt{q}(D^aD^bN - q^{ab}D^cD_cN) \\ &\quad +\sqrt{q}D_c \left(\frac{N^c\pi^{ab}}{\sqrt{q}} \right) - 2\pi^{c(a}D_cN^{b)} \end{aligned} \tag{B.35}$$

The equation of motions in (B.34) and (B.35) is known as the $3+1$ ADM formulation [83]. Such formalism is based on dynamical variables q_{ab} and π^{ab}. The $3+1$ ADM formulation is very powerful to study quantum cosmology and semi-classical theory of general relativity. The quantization of this formalisms is known as the Wheeler-DeWitt quantization. Although the Wheeler-DeWitt quantization success to study quantum cosmology. But non-perturbative of such theory is not realized and difficult due to complicated form of constraint in equations (B.29) and (B.30).

Appendix C

Detailed calculation

C.1 Derivation of the Spin Connection

We begin with co-triad ($e_{b\,i}$) compatible with respect to the covariant derivative (D_a) on spatial coordinate as

$$D_{[a}e_{b]i} = \partial_{[a}e_{b]i} + \omega^{l}_{[a\,i}e_{b]l} = 0\,. \tag{C.1}$$

Multiplying the the cyclic permutation indices i.e. i , j , k of twice co-triads into equation (C.1) as follow

$$\begin{aligned} e^a_j e^b_k(\partial_{[a}e_{b]i} + \omega^l_{[a\,i}e_{b]l}) &= 0 && \text{(C.2)}\\ e^a_i e^b_j(\partial_{[a}e_{b]k} + \omega^l_{[a\,k}e_{b]l}) &= 0 && \text{(C.3)}\\ e^a_k e^b_i(\partial_{[a}e_{b]j} + \omega^l_{[a\,j}e_{b]l}) &= 0\,. && \text{(C.4)}\end{aligned}$$

Adding equation (C.2), (C.3) and subtracting (C.4) together, we get

$$\begin{aligned} &e^a_j e^b_k\partial_{[a}e_{b]i} + e^a_j e^b_k\omega^l_{[a\,i}e_{b]l} + e^a_i e^b_j\partial_{[a}e_{b]k} + e^a_i e^b_j\omega^l_{[a\,k}e_{b]l} - e^a_k e^b_i\partial_{[a}e_{b]j} - e^a_k e^b_i\omega^l_{[a\,j}e_{b]l}\\ &= e^a_j e^b_k\partial_{[a}e_{b]i} + e^a_j e^b_k e_{b\,l}\omega^l_{a\,i} - e^a_j e^b_k e_{a\,l}\omega^l_{b\,i} + e^a_i e^b_j\partial_{[a}e_{b]k} + e^a_i e^b_j e_{b\,l}\omega^l_{a\,k} - e^a_i e^b_j e_{a\,l}\omega^l_{b\,k}\\ &\quad - e^a_k e^b_i\partial_{[a}e_{b]j} - e^a_k e^b_i e_{b\,l}\omega^l_{a\,j} - e^a_k e^b_i e_{a\,l}\omega^l_{b\,j} =\\ &e^a_j e^b_k\partial_{[a}e_{b]i} + e^a_i e^b_j\partial_{[a}e_{b]k} - e^a_k e^b_i\partial_{[a}e_{b]j} - 2e^a_j\omega_{a\,ik} = 0 \qquad \text{(C.5)}\end{aligned}$$

therefore solution of the spin connection can write in form

$$\begin{aligned} 2e^a_j\omega_{a\,ik} &= e^a_j e^b_k\partial_{[a}e_{b]i} + e^a_i e^b_j\partial_{[a}e_{b]k} - e^a_k e^b_i\partial_{[a}e_{b]j}\\ \omega_{a\,ik} &= \frac{1}{2}\,e^j_a\left(e^a_j e^b_k\partial_{[a}e_{b]i} + e^a_i e^b_j\partial_{[a}e_{b]k} - e^a_k e^b_i\partial_{[a}e_{b]j}\right) && \text{(C.6)}\end{aligned}$$

or equivalently to

$$\omega_{a\,i}^{k} = \frac{1}{2}\, e_a^j \left(e_i^a e_j^b \partial_{[a} e_{b]}^k + e_j^a e^{b\,k} \partial_{[a} e_{b]i} - e^{a\,k} e_i^b \partial_{[a} e_{b]j} \right) . \tag{C.7}$$

C.2 Derivation of the Holst Action

We perform a detail derivation following the original paper [62]. We begin with the Holst's modification of Palatini action as

$$\begin{aligned} S &= \frac{1}{2}\int e e_\mu^I e_\nu^J (R_{\mu\nu}{}^{IJ} - \alpha\, {}^*R_{\mu\nu}{}^{IJ})\, \mathrm{d}^4x \\ &= \frac{1}{2}\int e e_\mu^I e_\nu^J (R_{\mu\nu}{}^{IJ} - \frac{\alpha}{2}\, \epsilon^{IJ}{}_{KL} R_{\mu\nu}{}^{KL})\, \mathrm{d}^4x \end{aligned} \tag{C.8}$$

where $e \equiv \det e_\mu^I = \sqrt{-g}$ is determinate of tetrad, we used the self-dual identity i.e. ${}^*R_{\mu\nu}{}^{IJ} = \frac{1}{2}\,\epsilon^{IJ}{}_{KL} R_{\mu\nu}{}^{KL}$ and ${}^{**}R_{\mu\nu}{}^{IJ} = -R_{\mu\nu}{}^{IJ}$. For simply we set $8\pi G = 1$. Let us define new quantity following as

$$\widetilde{F}_{\mu\nu}{}^{IJ} \equiv R_{\mu\nu}{}^{IJ} - \frac{\alpha}{2}\, \epsilon^{IJ}{}_{KL} R_{\mu\nu}{}^{KL} \tag{C.9}$$

Before we perform the $3+1$ ADM decomposition, we will decompose the co-tetrad e_μ^I as

$$e_\mu^I = \begin{pmatrix} N & N^a e_a^i \\ 0 & e_a^i \end{pmatrix} \tag{C.10}$$

and for the tetrad e_I^μ as

$$e_I^\mu = \begin{pmatrix} 1/N & 0 \\ -N^a/N & e_i^a \end{pmatrix} . \tag{C.11}$$

where N and N^a are lapse funtion and shift vector as we discussed in appendix B. For convenient, we will use indices $\mu = t$ and $I = 0$ as the time component of the spacetime coordinate and the local Lorentz frame respectively. From above two equations, we get

$$e_t^0 = N\,, \quad e_t^i = N^a e_a^i\,, \quad e_a^0 = 0\,, \quad e_a^i = e_a^i \tag{C.12}$$

and

$$e_0^t = \frac{1}{N}\,, \quad e_0^a = -\frac{N^a}{N}\,, \quad e_i^t = 0\,, \quad e_i^a = e_i^a\,. \tag{C.13}$$

The $3+1$ ADM decomposition of the Holst action can be written as

$$
\begin{aligned}
S &= \frac{1}{2}\int e e^I_\mu e^J_\nu (R_{\mu\nu}{}^{IJ} - \frac{\alpha}{2}\,\epsilon^{IJ}{}_{KL} R_{\mu\nu}{}^{KL})\,\mathrm{d}^4x \\
&= \frac{1}{2}\int e e^I_\mu e^J_\nu \widetilde{F}_{\mu\nu}{}^{IJ}\,\mathrm{d}^4x \\
&= \frac{1}{2}\int e(e^t_0 e^t_0 \widetilde{F}_{tt}{}^{00} + e^t_0 e^t_i \widetilde{F}_{tt}{}^{0i} + e^t_i e^t_0 \widetilde{F}_{tt}{}^{i0} + e^t_i e^t_j \widetilde{F}_{tt}{}^{ij} + e^t_0 e^a_0 \widetilde{F}_{ta}{}^{00} \\
&\qquad + e^t_0 e^a_i \widetilde{F}_{ta}{}^{0i} + e^t_i e^a_0 \widetilde{F}_{ta}{}^{i0} + e^t_i e^a_j \widetilde{F}_{ta}{}^{ij} + e^a_0 e^t_0 \widetilde{F}_{at}{}^{00} \\
&\qquad + e^a_0 e^t_i \widetilde{F}_{at}{}^{0i} + e^a_i e^t_0 \widetilde{F}_{at}{}^{i0} + e^a_i e^t_j \widetilde{F}_{at}{}^{ij} + e^a_0 e^b_0 \widetilde{F}_{ab}{}^{00} \\
&\qquad + e^a_0 e^b_i \widetilde{F}_{ab}{}^{0i} + e^a_i e^b_0 \widetilde{F}_{ab}{}^{i0} + e^a_i e^b_j \widetilde{F}_{ab}{}^{ij})\,\mathrm{d}^4x \\
&= \int N\sqrt{q}\,(e^t_0 e^a_i \widetilde{F}_{at}{}^{i0} + e^a_i e^b_0 \widetilde{F}_{ab}{}^{i0} + \frac{1}{2}\,e^a_i e^b_j \widetilde{F}_{ij}{}^{ij})\,\mathrm{d}t\mathrm{d}^3x \\
&= \int \sqrt{q}(e^a_i \widetilde{F}_{at}{}^{i0} + N^b e^a_i \widetilde{F}_{ab}{}^{i0} + \frac{1}{2} N e^a_i e^b_j \widetilde{F}_{ab}{}^{ij})\,\mathrm{d}t\mathrm{d}^3x\,, \qquad \text{(C.14)}
\end{aligned}
$$

we used antisymmetry tensor of $\widetilde{F}_{\mu\nu}{}^{IJ}$ i.e. $\widetilde{F}_{\mu\nu}{}^{IJ} = -\widetilde{F}_{\mu\nu}{}^{JI}$ and $\widetilde{F}_{\mu\nu}{}^{IJ} = -\widetilde{F}_{\nu\mu}{}^{IJ}$. Therefore any components of $\widetilde{F}_{\mu\mu}{}^{IJ} = \widetilde{F}_{\mu\nu}{}^{II} = 0$. We also used $e = N\sqrt{q}$ as we discuss in appendix B.

The non vanish component of $\widetilde{F}_{\mu\nu}{}^{IJ}$ in (C.14) are given by

$$
\begin{aligned}
\widetilde{F}_{at}{}^{i0} &= R_{at}{}^{i0} - \frac{\alpha}{2}\,\epsilon_{JK}{}^{i0} R_{at}{}^{JK} \\
&= \partial_{[a}\omega_{t]}{}^{i0} + \omega_{[aJ}{}^{i}\omega_{t]}{}^{J0} - \frac{\alpha}{2}\,\epsilon_{JK}{}^{i0}\left(\partial_{[a}\omega_{t]}{}^{JK} + \omega_{[a}{}^{JL}\omega_{t]L}{}^{K}\right) \\
&= \partial_{[a}\omega_{t]}{}^{i0} + \omega_{[a0}{}^{i}\omega_{t]}{}^{00} + \omega_{[aj}{}^{i}\omega_{t]}{}^{j0} - \frac{\alpha}{2}\,\epsilon_{00}{}^{i0}\left(\partial_{[a}\omega_{t]}{}^{00} + \omega_{[a}{}^{0L}\omega_{t]L}{}^{0}\right) \\
&\quad - \frac{\alpha}{2}\,\epsilon_{j0}{}^{i0}\left(\partial_{[a}\omega_{t]}{}^{j0} + \omega_{[a}{}^{jL}\omega_{t]L}{}^{0}\right) - \frac{\alpha}{2}\,\epsilon_{0k}{}^{i0}\left(\partial_{[a}\omega_{t]}{}^{0k} + \omega_{[a}{}^{0L}\omega_{t]L}{}^{k}\right) \\
&\quad - \frac{\alpha}{2}\,\epsilon_{jk}{}^{i0}\left(\partial_{[a}\omega_{t]}{}^{jk} + \omega_{[a}{}^{jL}\omega_{t]L}{}^{k}\right) \\
&= \partial_a\omega_t{}^{i0} - \partial_t\omega_a{}^{i0} + \omega_{aj}{}^{i}\omega_t{}^{j0} - \omega_{tj}{}^{i}\omega_a{}^{j0} - \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\partial_a\omega_t{}^{jk} + \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\partial_t\omega_a{}^{jk} \\
&\quad - \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\omega_a{}^{j0}\omega_{t0}{}^{k} - \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\omega_a{}^{jl}\omega_{tl}{}^{k} + \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\omega_t{}^{j0}\omega_{a0}{}^{k} + \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\omega_t{}^{jl}\omega_{al}{}^{k}
\end{aligned}
$$

(C.15)

$$
\begin{aligned}
\widetilde{F}_{ab}{}^{i0} &= R_{ab}{}^{i0} - \frac{\alpha}{2}\,\epsilon_{JK}{}^{i0} R_{ab}{}^{JK} \\
&= \partial_{[a}\omega_{b]}{}^{i0} + \omega_{[aJ}{}^{i}\omega_{b]}{}^{J0} - \frac{\alpha}{2}\,\epsilon_{JK}{}^{i0}\left(\partial_{[a}\omega_{b]}{}^{JK} + \omega_{[a}{}^{JL}\omega_{b]L}{}^{K}\right) \\
&= \partial_{[a}\omega_{b]}{}^{i0} + \omega_{[a0}{}^{i}\omega_{b]}{}^{00} + \omega_{[aj}{}^{i}\omega_{b]}{}^{j0} - \frac{\alpha}{2}\,\epsilon_{00}{}^{i0}\left(\partial_{[a}\omega_{b]}{}^{00} + \omega_{[a}{}^{0L}\omega_{b]L}{}^{0}\right) \\
&\quad - \frac{\alpha}{2}\,\epsilon_{j0}{}^{i0}\left(\partial_{[a}\omega_{b]}{}^{j0} + \omega_{[a}{}^{jL}\omega_{b]L}{}^{0}\right) - \frac{\alpha}{2}\,\epsilon_{0k}{}^{i0}\left(\partial_{[a}\omega_{b]}{}^{0k} + \omega_{[a}{}^{0L}\omega_{b]L}{}^{k}\right) \\
&\quad - \frac{\alpha}{2}\,\epsilon_{jk}{}^{i0}\left(\partial_{[a}\omega_{b]}{}^{jk} + \omega_{[a}{}^{jL}\omega_{b]L}{}^{k}\right) \\
&= \partial_a\omega_b{}^{i0} - \partial_b\omega_b{}^{i0} + \omega_{aj}{}^{i}\omega_b{}^{j0} - \omega_{bj}{}^{i}\omega_a{}^{j0} - \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\partial_a\omega_b{}^{jk} + \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\partial_b\omega_a{}^{jk} \\
&\quad - \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\omega_a{}^{j0}\omega_{b0}{}^{k} - \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\omega_a{}^{jl}\omega_{bl}{}^{k} + \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\omega_b{}^{j0}\omega_{a0}{}^{k} + \frac{\alpha}{2}\,\epsilon_{jk}{}^{i}\omega_b{}^{jl}\omega_{al}{}^{k}
\end{aligned}
\tag{C.16}
$$

$$
\begin{aligned}
\widetilde{F}_{ab}{}^{ij} &= R_{ab}{}^{ij} - \frac{\alpha}{2}\,\epsilon_{KL}{}^{ij} R_{ab}{}^{KL} \\
&= \partial_{[a}\omega_{b]}{}^{ij} + \omega_{[aK}{}^{i}\omega_{b]}{}^{Kj} - \frac{\alpha}{2}\,\epsilon_{KL}{}^{ij}\left(\partial_{[a}\omega_{b]}{}^{KL} + \omega_{[a}{}^{KM}\omega_{b]M}{}^{L}\right) \\
&= \partial_{[a}\omega_{b]}{}^{ij} + \omega_{[a0}{}^{i}\omega_{b]}{}^{0j} + \omega_{[ak}{}^{i}\omega_{b]}{}^{kj} - \frac{\alpha}{2}\,\epsilon_{00}{}^{ij}\left(\partial_{[a}\omega_{b]}{}^{00} + \omega_{[a}{}^{0M}\omega_{b]M}{}^{0}\right) \\
&\quad - \frac{\alpha}{2}\,\epsilon_{k0}{}^{ij}\left(\partial_{[a}\omega_{b]}{}^{k0} + \omega_{[a}{}^{kM}\omega_{b]M}{}^{0}\right) - \frac{\alpha}{2}\,\epsilon_{0l}{}^{ij}\left(\partial_{[a}\omega_{b]}{}^{0l} + \omega_{[a}{}^{0M}\omega_{b]M}{}^{l}\right) \\
&\quad - \frac{\alpha}{2}\,\epsilon_{kl}{}^{ij}\left(\partial_{[a}\omega_{b]}{}^{kl} + \omega_{[a}{}^{jM}\omega_{b]M}{}^{l}\right)
\end{aligned}
\tag{C.17}
$$

$$
\begin{aligned}
&= \partial_a\omega_b{}^{ij} - \partial_b\omega_b{}^{ij} + \omega_{ak}{}^{i}\omega_b{}^{kj} - \omega_{bk}{}^{i}\omega_a{}^{kj} - \frac{\alpha}{2}\,\epsilon_k{}^{ij}\partial_a\omega_b{}^{k0} + \frac{\alpha}{2}\,\epsilon_k{}^{ij}\partial_b\omega_a{}^{k0} \\
&\quad - \frac{\alpha}{2}\,\epsilon_k{}^{ij}\omega_a{}^{k0}\omega_{b0}{}^{0} - \frac{\alpha}{2}\,\epsilon_k{}^{ij}\omega_a{}^{km}\omega_{bm}{}^{0} + \frac{\alpha}{2}\,\epsilon_k{}^{ij}\omega_b{}^{k0}\omega_{a0}{}^{0} + \frac{\alpha}{2}\,\epsilon_k{}^{ij}\omega_b{}^{km}\omega_{am}{}^{0} \\
&\quad - \frac{\alpha}{2}\,\epsilon_l{}^{ij}\partial_a\omega_b{}^{0l} - \frac{\alpha}{2}\,\epsilon_l{}^{ij}\partial_b\omega_a{}^{0l} - \frac{\alpha}{2}\,\epsilon_l{}^{ij}\omega_a{}^{00}\omega_{b0}{}^{l} - \frac{\alpha}{2}\,\epsilon_l{}^{ij}\omega_a{}^{0m}\omega_{bm}{}^{l} \\
&\quad + \frac{\alpha}{2}\,\epsilon_l{}^{ij}\omega_b{}^{00}\omega_{a0}{}^{l} + \frac{\alpha}{2}\,\epsilon_l{}^{ij}\omega_b{}^{0m}\omega_{am}{}^{l} \\
&= \partial_a\omega_b{}^{ij} - \partial_b\omega_b{}^{ij} + \omega_{ak}{}^{i}\omega_b{}^{kj} - \omega_{bk}{}^{i}\omega_a{}^{kj} - \frac{\alpha}{2}\,\epsilon_k{}^{ij}\partial_a\omega_b{}^{k0} + \frac{\alpha}{2}\,\epsilon_k{}^{ij}\partial_b\omega_a{}^{k0} \\
&\quad - \frac{\alpha}{2}\,\epsilon_k{}^{ij}\omega_a{}^{km}\omega_{bm}{}^{0} + \frac{\alpha}{2}\,\epsilon_k{}^{ij}\omega_b{}^{km}\omega_{am}{}^{0} - \frac{\alpha}{2}\,\epsilon_l{}^{ij}\partial_a\omega_b{}^{0l} - \frac{\alpha}{2}\,\epsilon_l{}^{ij}\partial_b\omega_a{}^{0l} \\
&\quad - \frac{\alpha}{2}\,\epsilon_l{}^{ij}\omega_a{}^{0m}\omega_{bm}{}^{l} + \frac{\alpha}{2}\,\epsilon_l{}^{ij}\omega_b{}^{0m}\omega_{am}{}^{l}
\end{aligned}
\tag{C.18}
$$

where we have used the time gauge [62] i.e. $\epsilon_{jk}{}^{i0} = \epsilon_{jk}{}^{i}$ (other terms of Levi-civita vanish due to their properties that is shown in appendix A) and using antisymmetry identity of spin connection 1-form i.e. $\omega_\mu{}^{IJ} = -\omega_\mu{}^{JI}$, then any $\omega_\mu{}^{II} = 0$.

We will introduce new variables following definition in (C.9) as

$$ {}^{(+)}A_a^i = \omega_a^{0i} + \frac{1}{2}\,\epsilon^i{}_{jk}\omega_a^{jk} \tag{C.19} $$

$$ {}^{(-)}A_a^i = \omega_a^{0i} - \frac{1}{2}\,\epsilon^i{}_{jk}\omega_a^{jk}\,. \tag{C.20} $$

and their inverse as

$$ \omega_a^{0i} = \frac{1}{2}({}^{(+)}A_a^i +{}^{(-)} A_a^i) \tag{C.21} $$

$$ \omega_a^{jk} = \frac{1}{2\alpha}\,\epsilon_i{}^{jk}({}^{(+)}A_a^i -{}^{(-)} A_a^i) \tag{C.22} $$

Using (C.21) and substitute (C.22) into (C.15), (C.16) and (C.18), we obtain

$$ \begin{aligned} \widetilde{F}_{at}{}^{i0} &= -\partial_t\,{}^{(-)}A_a^i + \partial_a\left(\omega_t{}^{i0} - \frac{\alpha}{2}\,\epsilon^i{}_{jk}\omega_t{}^{jk}\right) + \frac{(\alpha^2+1)}{\alpha}\,\epsilon^i{}_{jk}\omega_t{}^{0j}\,{}^{(+)}A_a^k \\ &\quad + \frac{(\alpha^2-1)}{\alpha}\,\epsilon^i{}_{jk}\omega_t{}^{0j}\,{}^{(-)}A_a^k - \omega_{tj}{}^{i}\,{}^{(-)}A_a^j \end{aligned} \tag{C.23} $$

$$ \begin{aligned} \widetilde{F}_{ab}{}^{i0} &= \partial_{[a}\,{}^{(-)}A_{b]}^i - \frac{(\alpha^2+1)}{2\alpha}\,\epsilon^i{}_{jk}\,{}^{(+)}A_{[a}^j\,{}^{(+)}A_{b]}^k - \frac{(\alpha^2-3)}{2\alpha}\,\epsilon^i{}_{jk}\,{}^{(-)}A_{[a}^j\,{}^{(-)}A_{b]}^k \\ &\quad - \frac{(\alpha^2+1)}{\alpha}\,\epsilon^i{}_{jk}\,{}^{(+)}A_{[a}^j\,{}^{(-)}A_{b]}^k \end{aligned} \tag{C.24} $$

$$ \begin{aligned} \widetilde{F}_{ab}{}^{ij} &= 2\frac{(\alpha^2+1)}{\alpha}\,\epsilon^{ij}{}_k\,\partial_{[a}\,{}^{(+)}A_{b]}^k + 2\frac{(\alpha^2-1)}{\alpha}\,\epsilon^{ij}{}_k\,\partial_{[a}\,{}^{(-)}A_{b]}^k - \frac{(\alpha^2+1)}{\alpha^2}\,{}^{(+)}A_{[a}^i\,{}^{(+)}A_{b]}^j \\ &\quad \frac{(3\alpha^2-1)}{\alpha^2}\,{}^{(-)}A_{[a}^i\,{}^{(-)}A_{b]}^j + 4\frac{(\alpha^2+1)}{\alpha^2}\,{}^{(+)}A_{[a}^{[i}\,{}^{(-)}A_{b]}^{j]}\,. \end{aligned} \tag{C.25} $$

Using some identity of densitized triad i.e.

$$ \frac{1}{2}\,\epsilon^{abc}\epsilon_{ijk}e_b^je_c^k = \frac{1}{(2)(3)}\,\epsilon^{abc}\epsilon_{ijk}e_i^ae_a^ie_b^je_c^k = \sqrt{q}\,e_i^a \equiv E_i^a \tag{C.26} $$

and a important identity was shown by [62] i.e.

$$ {}^{(+)}A_a^i = {}^{(-)}A_a^i - 2\alpha\,\omega_a^i \tag{C.27} $$

where $\omega_a^i = \frac{1}{2}\, e^i{}_{jk}\omega_a{}^{jk}$. Substituting (C.27) into (C.23) (C.24) and (C.25) and plugging these results in the Holst action at (C.14), we obtain

$$
\begin{aligned}
S \;=\; & \int \sqrt{q}\, \Big[e_i^a \Big(-\partial_t\, {}^{(-)}A_a^i + \partial_a \Big(\omega_t{}^{i0} - \frac{\alpha}{2}\, \epsilon^i{}_{jk}\omega_t{}^{jk} \Big) \\
& + \frac{(\alpha^2+1)}{\alpha}\, \epsilon^i{}_{jk}\omega_t{}^{0j} ({}^{(-)}A_a^i - 2\alpha\, \omega_a^i) + \frac{(\alpha^2-1)}{\alpha}\, \epsilon^i{}_{jk}\omega_t{}^{0j}\, {}^{(-)}A_a^k - \omega_{tj}{}^{i}\, {}^{(-)}A_a^j \Big) \\
& + N^b e_i^a \Big(\partial_{[a}\, {}^{(-)}A_{b]}^i - \frac{(\alpha^2+1)}{2\alpha}\, \epsilon^i{}_{jk} ({}^{(-)}A_{[a}^j\, {}^{(-)}A_{b]}^k - 2\alpha\, {}^{(-)}A_{[a}^j\omega_{b]}^k - 2\alpha\, \omega_{[a}^j\, {}^{(-)}A_{b]}^k \\
& + 4\alpha^2 \omega_{[a}^j\omega_{b]}^k) - \frac{(\alpha^2-3)}{2\alpha}\, \epsilon^i{}_{jk}\, {}^{(-)}A_{[a}^j\, {}^{(-)}A_{b]}^k \\
& \qquad\qquad - \frac{(\alpha^2+1)}{\alpha}\, \epsilon^i{}_{jk} ({}^{(-)}A_{[a}^j\, {}^{(-)}A_{b]}^k - 2\alpha\, \omega_{[a}^j\, {}^{(-)}A_{b]}^k) \Big) \\
& + \frac{1}{2}\, N e_i^a e_j^b \Big(2\frac{(\alpha^2+1)}{\alpha}\, \epsilon^{ij}{}_k\, (\partial_{[a}\, {}^{(-)}A_{b]}^k - 2\alpha\partial_{[a}\omega_{b]}^k) + 2\frac{(\alpha^2-1)}{\alpha}\, \epsilon^{ij}{}_k\, \partial_{[a}\, {}^{(-)}A_{b]}^k \\
& - \frac{(\alpha^2+1)}{\alpha^2} \Big({}^{(-)}A_{[a}^i\, {}^{(-)}A_{b]}^j - 2\alpha\, {}^{(-)}A_{[a}^i\omega_{b]}^j - 2\alpha\, \omega_{[a}^i\, {}^{(-)}A_{b]}^j + 4\alpha^2\omega_{[a}^i\omega_{b]}^j \Big) \\
& + \frac{(3\alpha^2-1)}{\alpha^2}\, {}^{(-)}A_{[a}^i\, {}^{(-)}A_{b]}^j + 4\frac{(\alpha^2+1)}{\alpha^2} \big({}^{(-)}A_{[a}^{[i}\, {}^{(-)}A_{b]}^{j]} - 2\alpha\, \omega_{[a}^{[i}\, {}^{(-)}A_{b]}^{j]} \big) \Big) \Big]\, \mathrm{d}t\mathrm{d}^3x \\
=\; & \int \Big[- E_i^a(\partial_t\, {}^{(-)}A_a^i) - \alpha(\alpha\, \omega_t{}^{i0} - \frac{1}{2}\, \epsilon^i{}_{jk}\omega_t{}^{jk})(\partial_a E_i^a + \frac{1}{\alpha}\, \epsilon_{ij}{}^k\, {}^{(-)}A_a^j E_k^a) \\
& + N^a E_i^b \Big(\partial_{[a}\, {}^{(-)}A_{b]}^i + \frac{1}{\alpha}\, \epsilon^i{}_{jk}\, {}^{(-)}A_{[a}^j\, {}^{(-)}A_{b]}^k \Big) + \frac{1}{2}\, N E_i^a E_j^b \Big(\alpha\, \epsilon^{ij}{}_k\, (\partial_{[a}\, {}^{(-)}A_{b]}^k \\
& + \epsilon_{ij}{}^{k}\, {}^{(-)}A_{[a}^i\, {}^{(-)}A_{b]}^j) - (1+\alpha^2)\epsilon^{ij}{}_k(\partial_{[a}\omega_{b]}^k + \epsilon_{ij}{}^{k}\omega_{[a}^i\omega_{b]}^j) \Big) \Big]\, \mathrm{d}t\mathrm{d}^3x \\
=\; & \int \Big[-\alpha\, E_i^a(\pounds_t\, {}^{(-)}A_a^i) - \alpha\, {}^{(-)}A_t^i D_a E_i^a + N^a E_i^b\, {}^{(-)}\widetilde{F}_{ab}{}^i \\
& \qquad\qquad + \frac{1}{2\sqrt{q}}\, \epsilon^{ij}{}_k\, N E_i^a E_j^b \Big(\alpha\, {}^{(-)}\widetilde{F}_{ab}{}^k - (1+\alpha^2) R_{ab}{}^k \Big) \Big]\, \mathrm{d}t\mathrm{d}^3x
\end{aligned}
\tag{C.28}
$$

where ${}^{(-)}\widetilde{F}_{ab}^i = \partial_{[a}^{(-)}A_{b]}^i + \alpha^{-1}\epsilon^i{}_{jk}\, {}^{(-)}A_{[a}^j\, {}^{(-)}A_{b]}^k$ and $R_{ab}{}^i = \partial_{[a}\omega_{b]}^i + \epsilon^i{}_{jk}\omega_{[a}^j\omega_{b]}^k$. We neglect any surface integral terms e.g. $\int D_a[(\omega_t{}^{i0} - \frac{\alpha}{2}\, \epsilon^i{}_{jk}\omega_t{}^{jk})E_i^a]\mathrm{d}^3x \approx 0$. The Lie derivative $\pounds_t$ of the local Lorentz frame indices will be ignored. Using definition of Ashtekar variable ${}^{(-)}A_a^i \equiv \omega_a^i + \alpha K_a^i$ substitute at $R_{ab}{}^i$ term in (C.28) and Barbero-Immirzi parameter given by $\chi = 1/\alpha$ [62]. After simple manipulation but quite tediously, we obtain

$$
\begin{aligned}
S &= \int \Big[-\frac{1}{\chi}\, E^a_i\, \pounds_t\, {}^{(-)}A^i_a - \frac{1}{\chi}\; {}^{(-)}A^i_t D_a E^a_i + N^a E^b_i\, {}^{(-)}\widetilde{F}_{ab}{}^i \\
&\quad + \frac{1}{2\sqrt{q}}\, \epsilon^{ij}{}_k\, N E^a_i E^b_j \Big[\frac{1}{\chi}\, {}^{(-)}\widetilde{F}_{ab}{}^k - (1+\chi^{-2})\big(\partial_{[a}\, {}^{(-)}A^k_{b]} - \frac{1}{\chi}\, \partial_{[a} K^k_{b]} \\
&\qquad\qquad + \epsilon_{ij}{}^k \big({}^{(-)}A^i_{[a}\, {}^{(-)}A^j_{b]} - \frac{1}{\chi}\, K^i_{[a} K^j_{b]} \big) \big) \Big] \Big] \mathrm{d}t\mathrm{d}^3x \\
&= \int \left[-\frac{1}{\chi}\, E^a_i\, \pounds_t\, {}^{(-)}A^i_a - \frac{1}{\chi}\; {}^{(-)}A^i_t D_a E^a_i + N^a E^b_i\, {}^{(-)}\widetilde{F}_{ab}{}^i \right. \\
&\quad + \frac{1}{2\sqrt{q}}\, \epsilon^{ij}{}_k\, N E^a_i E^b_j \left[\frac{2}{\chi}\, {}^{(-)}\widetilde{F}_{ab}{}^k - \frac{2}{\chi}(1+\chi^{-2}) \left(\chi\, \partial_{[a}\, {}^{(-)}A^k_{b]} - \partial_{[a} K^k_{b]} \right.\right. \\
&\qquad\qquad \left.\left.\left. + \epsilon_{ij}{}^k \left(\chi\, {}^{(-)}A^i_{[a}\, {}^{(-)}A^j_{b]} - K^i_{[a} K^j_{b]} \right) \right) \right] \right] \mathrm{d}t\mathrm{d}^3x \\
&= \int \left[-\frac{1}{\chi}\, E^a_i\, \pounds_t\, {}^{(-)}A^i_a - \frac{1}{\chi}\; {}^{(-)}A^i_t D_a E^a_i + N^a \left(\frac{1}{\chi}\, E^b_i\, {}^{(-)}\widetilde{F}_{ab}{}^i - \chi(1+\chi^{-2}) K^i_a\, D_a E^a_i \right) \right. \\
&\quad \left. + \frac{1}{2\sqrt{q}}\, N E^a_i E^b_j \left(\epsilon^{ij}{}_k \frac{1}{\chi}\, {}^{(-)}\widetilde{F}_{ab}{}^k - \chi(1+\chi^{-2}) K^i_{[a} K^j_{b]} \right) \right] \mathrm{d}t\mathrm{d}^3x \\
&= \int \left[-\frac{1}{\chi}\, E^a_i\, \pounds_t A^i_a - \frac{1}{\chi}\, A^i_t D_a E^a_i + N^a \left(\frac{1}{\chi}\, E^b_i F_{ab}{}^i - \frac{(1+\chi^2)}{\chi}\, K^i_a\, D_a E^a_i \right) \right. \\
&\quad \left. + \frac{1}{2\sqrt{q}\,\chi}\, N E^a_i E^b_j \left(\epsilon^{ij}{}_k\, F_{ab}{}^k - (1+\chi^2) K^i_{[a} K^j_{b]} \right) \right] \mathrm{d}t\mathrm{d}^3x \\
&= \int \left[-\frac{1}{\chi}\, E^a_i\, \pounds_t A^i_a - A^i_t D_a E^a_i + N^a \mathcal{C}_a + N \mathcal{C}_{GR} \right] \mathrm{d}t\mathrm{d}^3x
\end{aligned}
\tag{C.29}
$$

we used the by part technique in the $\partial_{[a} K^i_{b]}$ term and neglected any surface terms like above. These results gives equivalent to the constraints in (2.52), (2.53) and (2.54) as we seen at chapter 2, if we set $8\pi G = 1$, ${}^{(-)}\widetilde{F}_{ab}{}^i = F_{ab}{}^i$ and ${}^{(-)}A^i_a = A^i_a$.

C.3 Determination of the Perturbation Matrix and their Eigenvalues in Section 4.1.1

In the $\lambda =$ constant case. Equations for our autonomous system are reduced to

$$\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}N} \equiv f &= -3X - \sqrt{\frac{3}{2}}\,\lambda Y^2 - 3X^3\,(1-2Z) \\
\frac{\mathrm{d}Y}{\mathrm{d}N} \equiv g &= -\sqrt{\frac{3}{2}}\,\lambda XY - 3X^2Y\,(1-2Z) \\
\frac{\mathrm{d}Z}{\mathrm{d}N} \equiv h &= -3Z\left(1+\frac{X^2+Y^2}{X^2-Y^2}\right).
\end{aligned}$$

Following the definition of the perturbation matrix in (4.23) , we obtain

$$\mathcal{M} = \begin{pmatrix} -3-9X_c^2(1-2Z_c) & -\sqrt{6}Y_c & 6X_c^3 \\ -\sqrt{\frac{3}{2}}\lambda Y_c - 6X_cY_c(1-2Z_c) & -\sqrt{\frac{3}{2}}\lambda X_c - 3X_c^2(1-2Z_c) & 6X_c^2Y_c \\ -3Z_c\left(\frac{2X_c}{X_c^2-Y_c^2} - \frac{2X_c(X_c^2+Y_c^2)}{(X_c^2-Y_c^2)^2}\right) & -3Z_c\left(\frac{2Y_c}{X_c^2-Y_c^2} + \frac{2Y_c(X_c^2+Y_c^2)}{(X_c^2-Y_c^2)^2}\right) & -3\left(1+\frac{X_c^2+Y_c^2}{X_c^2-Y_c^2}\right) \end{pmatrix}. \tag{C.30}$$

- The perturbation matrix at point (a) : $\left(-\frac{\lambda}{\sqrt{6}}\,, \sqrt{1+\frac{\lambda^2}{6}}\,,\ 0\right)$ is given by

$$\mathcal{M} = \begin{pmatrix} -3-\frac{3}{2}\lambda^2 & -\lambda\sqrt{6+\lambda^2} & -\frac{\lambda^3}{\sqrt{6}} \\ \left(-\sqrt{\frac{3}{2}}+\sqrt{6}\right)\lambda\sqrt{1+\frac{\lambda^2}{6}} & 0 & \lambda^2\sqrt{1+\frac{\lambda^2}{6}} \\ 0 & 0 & \lambda^2 \end{pmatrix}. \tag{C.31}$$

The cubic equation of the eigenvalues μ for point (a) as

$$(\mu+3+\frac{3}{2}\lambda^2)(\mu)(\mu-\lambda^2) - (\mu-\lambda^2)(-3\lambda^2)\left(1+\frac{\lambda^2}{6}\right) = 0\,.$$

The solutions for eigenvalues μ are given by

$$\mu_1 = \frac{\lambda^2}{2}\,,\quad \mu_2 = -\lambda^2\,,\quad \mu_3 = -3-\frac{\lambda^2}{2}\,. \tag{C.32}$$

- The perturbation matrix at point (b) : $\left(-\frac{\lambda}{\sqrt{6}}\,, -\sqrt{1+\frac{\lambda^2}{6}}\,,\ 0\right)$ is given by

$$\mathcal{M} = \begin{pmatrix} -3-\frac{3}{2}\lambda^2 & \lambda\sqrt{6+\lambda^2} & -\frac{\lambda^3}{\sqrt{6}} \\ \left(\sqrt{\frac{3}{2}}-\sqrt{6}\right)\lambda\sqrt{1+\frac{\lambda^2}{6}} & 0 & -\lambda^2\sqrt{1+\frac{\lambda^2}{6}} \\ 0 & 0 & \lambda^2 \end{pmatrix}. \tag{C.33}$$

The cubic equation of the eigenvalues μ for point (b) as

$$(\mu + 3 + \frac{3}{2}\lambda^2)(\mu)(\mu - \lambda^2) - (\mu - \lambda^2)(-3\lambda^2)\left(1 + \frac{\lambda^2}{6}\right) = 0\,.$$

The solutions for eigenvalues μ are given by

$$\mu_1 = \frac{\lambda^2}{2}\,, \quad \mu_2 = -\lambda^2\,, \quad \mu_3 = -3 - \frac{\lambda^2}{2}\,. \tag{C.34}$$

Bibliography

[1] C. L. Bennett *et al.*, Observations: Astrophys. J. Suppl. **148**, 1 (2003) [arXiv:astro-ph/0302207]; D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148** (2003) 175 [arXiv:astro-ph/0302209].

[2] S. Masi *et al.*, Prog. Part. Nucl. Phys. **48**, 243 (2002) [arXiv:astro-ph/0201137].

[3] R. Scranton *et al.* [SDSS Collaboration], [arXiv:astro-ph/0307335].

[4] A. G. Riess *et al.* [Supernova Search Team Collaboration], Universe and a Astron. J. **116**, 1009 (1998) [arXiv:astro-ph/9805201]; A. G. Riess, arXiv:astro-ph/9908237; J. L. Tonry *et al.* [Supernova Search Team Collaboration], Astrophys. J. **594**, 1 (2003) [arXiv:astro-ph/0305008].

[5] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], Supernovae," Astrophys. J. **517**, 565 (1999) [arXiv:astro-ph/9812133]; G. Goldhaber *et al.* [The Supernova Cosmology Project Collaboration], arXiv:astro-ph/0104382.

[6] N. Straumann, Mod. Phys. Lett. A **21**, 1083 (2006) [arXiv:hep-ph/0604231].

[7] E. J. Copeland, M. Sami and S. Tsujikawa, arXiv:hep-th/0603057.

[8] T. Padmanabhan, Curr. Sci. **88**, 1057 (2005) [arXiv:astro-ph/0411044]; T. Padmanabhan, arXiv:astro-ph/0603114.

[9] S. M. Carroll, Living Rev. Rel. **4**, 1 (2001) [arXiv:astro-ph/0004075].

[10] E. W. Kolb, S. Matarrese, A. Notari and A. Riotto, arXiv:hep-th/0503117; E. W. Kolb, S. Matarrese and A. Riotto, arXiv:astro-ph/0506534.

[11] S. Nojiri and S. D. Odintsov, arXiv:hep-th/0601213.

[12] A. Melchiorri, L. Mersini-Houghton, C. J. Odman and M. Trodden, Phys. Rev. D **68**, 043509 (2003) [arXiv:astro-ph/0211522].

[13] R. R. Caldwell, Phys. Lett. B **545**, 23 (2002) [arXiv:astro-ph/9908168].

[14] V. Sahni and Y. Shtanov, JCAP **0311**, 014 (2003) [arXiv:astro-ph/0202346]; V. Sahni and Y. Shtanov, Int. J. Mod. Phys. D **11**, 1515 (2000) [arXiv:gr-qc/0205111].

[15] S. Pantian, *Tachyonic Cosmological Dynamics*, M.S. thesis Naresuan University, (2007).

[16] C. M. Chen, D. V. Gal'tsov and M. Gutperle, Phys. Rev. D **66**, 024043 (2002); P. K. Townsend and M. N. R. Wohfarth, Phys. Rev. Lett. **91**, 061302 (2003); N. Ohta, Prog. Theor. Phys. **110**, 269 (2003).

[17] E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D **70**, 043539 (2004) [arXiv:hep-th/0405034].

[18] F. Hoyle, Mon. Not. Roy. Astron. Soc. **108**, 372 (1948); Mon. Not. Roy. Astron. Soc. **109**, 365 (1949).

[19] F. Hoyle and J. V. Narlikar, Proc. Roy. Soc. Lond. **A282**, 191 ; Mon. Not. Roy. Astron. Soc. **115**, 305 (1972).

[20] X. z. Li and J. g. Hao, Phys. Rev. D **69**, 107303 (2004) [arXiv:hep-th/0303093]; J. g. Hao and X. z. Li, Phys. Rev. D **67**, 107303 (2003) [arXiv:gr-qc/0302100]; L. A. Urena-Lopez, JCAP **0509**, 013 (2005) [arXiv:astro-ph/0507350].

[21] J. G. Hao and X. z. Li, Phys. Rev. D **70**, 043529 (2004) [arXiv:astro-ph/0309746].

[22] P. Singh, M. Sami and N. Dadhich, Phys. Rev. D **68**, 023522 (2003) [arXiv:hep-th/0305110].

[23] M. Sami and A. Toporensky, Mod. Phys. Lett. A **19**, 1509 (2004) [arXiv:gr-qc/0312009].

[24] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003) [arXiv:astro-ph/0302506]; S. Nesseris and L. Perivolaropoulos, Phys. Rev. D **70**, 123529 (2004) [arXiv:astro-ph/0410309].

[25] S. M. Carroll, M. Hoffman and M. Trodden, Phys. Rev. D **68**, 023509 (2003) [arXiv:astro-ph/0301273].

[26] J. D. Barrow, Class. Quant. Grav. **21**, L79 (2004) [arXiv:gr-qc/0403084].

[27] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005) [arXiv:hep-th/0501025].

[28] H. Wei and R. G. Cai, Phys. Rev. D **72**, 123507 (2005) [arXiv:astro-ph/0509328].

[29] T. Thiemann, Lect. Notes Phys. **631**, 41 (2003) [arXiv:gr-qc/0210094];

[30] A. Ashtekar, M. Bojowald and J. Lewandowski, Adv. Theor. Math. Phys. **7**, 233 (2003) [arXiv:gr-qc/0304074].

[31] G. Date and G. M. Hossain, Class. Quant. Grav. **21**, 4941 (2004) [arXiv:gr-qc/0407073].

[32] M. Bojowald, Living Rev. Rel. **8**, 11 (2005) [arXiv:gr-qc/0601085].

[33] A. Ashtekar, T. Pawlowski and P. Singh, arXiv:gr-qc/0604013.

[34] P. Singh, Phys. Rev. D **73**, 063508 (2006) [arXiv:gr-qc/0603043].

[35] A. Ashtekar, arXiv:gr-qc/0605011.

[36] G. M. Hossain, Class. Quant. Grav. **21**, 179 (2004) [arXiv:gr-qc/0308014]; K. Banerjee and G. Date, Class. Quant. Grav. **22**, 2017 (2005) [arXiv:gr-qc/0501102].

[37] M. Bojowald, Phys. Rev. Lett. **86**, 5227 (2001) [arXiv:gr-qc/0102069]; M. Bojowald, G. Date and K. Vandersloot, Class. Quant. Grav. **21**, 1253 (2004) [arXiv:gr-qc/0311004]; G. Date, Phys. Rev. D **71**, 127502 (2005) [arXiv:gr-qc/0505002].

[38] A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. Lett. **96**, 141301 (2006) [arXiv:gr-qc/0602086]

[39] M. Sami, P. Singh and S. Tsujikawa, arXiv:gr-qc/0605113.

[40] P. Singh and A. Toporensky, Phys. Rev. D **69**, 104008 (2004) [arXiv:gr-qc/0312110].

[41] M. Bojowald, Phys. Rev. Lett. **89**, 261301 (2002) [arXiv:gr-qc/0206054].

[42] M. Bojowald and K. Vandersloot, Phys. Rev. D **67**, 124023 (2003) [arXiv:gr-qc/0303072]; G. Calcagni and M. Cortes, arXiv:gr-qc/0607059.

[43] S. Tsujikawa, P. Singh and R. Maartens, Class. Quant. Grav. **21**, 5767 (2004) [arXiv:astro-ph/0311015].

[44] E. J. Copeland, J. E. Lidsey and S. Mizuno, Phys. Rev. D **73**, 043503 (2006) [arXiv:gr-qc/0510022].

[45] M. Bojowald and M. Kagan, arXiv:gr-qc/0606082.

[46] M. Bojowald, Class. Quant. Grav. **18**, L109 (2001) [arXiv:gr-qc/0105113].

[47] P. Singh and K. Vandersloot, Phys. Rev. D **72**, 084004 (2005) [arXiv:gr-qc/0507029].

[48] G. Gamow, Phys. Rev. **70**, 572 (1946); R. A. Alpher, H. Bethe and G. Gamow, Phys. Rev. **73**, 803 (1948).

[49] R. M. Wald, *General Relativity*, The University of Chicago Press (1984).

[50] S. M. Carroll, *Spacetime and Geometry: An introduction to General Relativity*, Addison Wesley (2004).

[51] D. L. Wiltshire, arXiv:gr-qc/0101003.

[52] B. Gumjudpai, arXiv:astro-ph/0305063.

[53] A. A. Penzias and R. W. Wilson, Astrophys. J. **142**, 419 (1965).

[54] A. Friedman, Z. Phys. **10**, 377 (1922) [Gen. Rel. Grav. **31**, 1991 (1999)].

[55] P. de Bernardis *et al.* [Boomerang Collaboration], Nature **404**, 955 (2000) [arXiv:astro-ph/0004404]; S. Hanany *et al.*, Astrophys. J. **545**, L5 (2000) [arXiv:astro-ph/0005123]; A. Balbi *et al.*, Astrophys. J. **545**, L1 (2000) [Erratum-ibid. **558**, L145 (2001)] [arXiv:astro-ph/0005124]; P. de Bernardis *et al.*, Astrophys. J. **564**, 559 (2002) [arXiv:astro-ph/0105296]; C. L. Bennett *et al.*, Astrophys. J. Suppl. **148**, 1 (2003) [arXiv:astro-ph/0302207]; D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148**, 175 (2003) [arXiv:astro-ph/0302209].

[56] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], Astrophys. Journ. **517**, 565 (1999) [astro-ph/9812133].

[57] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); A. Ashtekar, Phys. Rev. D **36**, 1587 (1987)

[58] C. Rovelli, *Quantum Gravity*, Cambridge university press (2004).

[59] C. Rovelli, Class. Quant. Grav. **8**, 1613 (1991).

[60] K. Vandersloot, *Loop Quantum Cosmology*, PhD thesis, on-line avaliable at http://igpg.gravity.psu.edu/archives/thesis/2006/kevin.pdf , cited on 1 December 2006.

[61] A. Ashtekar and J. Lewandowski, Class. Quant. Grav. **21**, R53 (2004) [arXiv:gr-qc/0404018].

[62] S. Holst, Phys. Rev. D **53**, 5966 (1996) [arXiv:gr-qc/9511026].

[63] M. Bojowald and H. A. Kastrup, Class. Quant. Grav. **17**, 3009 (2000) [arXiv:hep-th/9907042].

[64] A. Perez, arXiv:gr-qc/0409061.

[65] H. Nicolai and K. Peeters, arXiv:hep-th/0601129.

[66] A. Ashtekar, arXiv:gr-qc/0702030.

[67] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [arXiv:hep-th/9906064].

[68] T. Shiromizu, K. i. Maeda and M. Sasaki, Phys. Rev. D **62**, 024012 (2000) [arXiv:gr-qc/9910076].

[69] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B **477**, 285 (2000) [arXiv:hep-th/9910219]; E. E. Flanagan, S. H. H. Tye and I. Wasserman, Phys. Rev. D **62**, 044039 (2000) [arXiv:hep-ph/9910498].

[70] P. G. Ferreira and M. Joyce, Phys. Rev. Lett. **79**, 4740 (1997) [arXiv:astro-ph/9707286]; P. G. Ferreira and M. Joyce, Phys. Rev. D **58**, 023503 (1998) [arXiv:astro-ph/9711102].

[71] E. J. Copeland, A. R. Liddle and D. Wands, Phys. Rev. D **57**, 4686 (1998) [arXiv:gr-qc/9711068].

[72] F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985).

[73] A. R. Liddle, Phys. Lett. B **220**, 502 (1989).

[74] T. Barreiro, E. J. Copeland and N. J. Nunes, Phys. Rev. D **61**, 127301 (2000) [arXiv:astro-ph/9910214]; S. C. C. Ng, N. J. Nunes and F. Rosati, Phys. Rev. D **64**, 083510 (2001) [arXiv:astro-ph/0107321].

[75] J. Kujat, R. J. Scherrer and A. A. Sen, arXiv:astro-ph/0606735.

[76] For detail of slow-roll approximation and relate topics see: Andrew R. Liddle and David H. Lyth, *Cosmological Inflation and Large-Scale Structure*, Cambridge University Press (2000) .

[77] I. Zlatev, L. M. Wang and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999) [arXiv:astro-ph/9807002].

[78] S. C. C. Ng, N. J. Nunes and F. Rosati, Phys. Rev. D **64**, 083510 (2001) [arXiv:astro-ph/0107321]; A. R. Liddle and R. J. Scherrer, Phys. Rev. D **59**, 023509 (1999) [arXiv:astro-ph/9809272].

[79] E. Aonkaew, *Dynamics of Scalar Field with Cosmological Barotropic Fluid under Exponential Potential*, B.S. Independent Study Report, Naresuan University, (2007). Avilable on http://www.sci.nu.ac.th/tptp/centralOutstandinglists.htm

[80] G. N. Felder, A. V. Frolov, L. Kofman and A. V. Linde, Phys. Rev. D **66**, 023507 (2002) [arXiv:hep-th/0202017].

[81] P. Singh, K. Vandersloot and G. V. Vereshchagin, Phys. Rev. D **74**, 043510 (2006) [arXiv:gr-qc/0606032].

[82] J. Brunnemann and T. Thiemann, Class. Quant. Grav. **23**, 1395 (2006) [arXiv:gr-qc/0505032].

[83] R. Arnowitt, S. Deser and C. W. Misner, "The dynamics of general relativity," Gravitation: an introduction to current research, Louis Witten ed. (Wilew 1962), arXiv:gr-qc/0405109.

[84] E. Poisson, *A Relativist's Toolkit: The Mathematic of Black-Hole Mechanics*, Cambridge University Press (2004).

[85] P. Peldan, Class. Quant. Grav. **11**, 1087 (1994) [arXiv:gr-qc/9305011].