Cosmological Field Equations: Metric Formalism

Amornthep Tita
54411277

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Amornthep Tita

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.... Advisor
(Asst. Prof. Pornrad Srisawad, Ph.D.)

.... Co - advisor
(Assoc. Prof. Burin Gumjudpai, Ph.D.)

.... Member
(Attapon Amthong, Ph.D.)

.... Director, The Institute for Fundamental Study
(Seckson Sukhasena, Ph.D.)

.... Head of the Department of Physics
(Sarawut Thountom, Ph.D.)

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Candidate :	Amornthep Tita
Adviser:	Asst. Prof. Dr. Pornrad Srisawad
Co-Adviser:	Assoc. Prof. Dr. Burin Gumjudpai
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Abstract

General relativity – GR is a theory for gravity which Newton theory of gravity fails to explain. Postulates in GR lead to the Einstein's field equation. We applying to cosmological and local spherical bodies. Postulates in GR and cosmology are the cosmological principle. These are symmetries of isotropy and homogeneity and the existence of cosmic time. The postulates lead to the Friedmann–Lemaître–Robertson–Walker metric (FLRW metric) which we show its derivation in detail here. We apply FLRW metric to the field equation to obtain the Freidmann equation. We as well show derivation of the static spherical case of Schwarzschild metric.

Contents

Chapter 1

Introduction

1.1 Background

Classical dynamics is able to explain motions at low velocity, small scale distances or weak gravitational force. When the velocity reaches relativistic regime, i.e. $v \to c$, large scale distances where spatial curvature comes to play its role or strong gravitation intensity, classical dynamics fails to be responsible for being the description but general relativity - GR instead takes the role. Major consequences of postulates in GR lead to the Einstein's field equation, when applying to cosmology and local spherical bodies, it results in explicit equation of motion - the Friedmann equation which governs dynamics of the large scale universe.

1.2 Objectives

- To derive in detailed of the FLRW metric and the Friedmann equation.
- To derive in detailed of the Schwarzschild metric.

1.3 Frameworks

- Standard general relativity
- Based on cosmological principles and Weyl's postulate and spherically symmetric space

1.4 Expected Use

- Obtaining detailed derivation of FLRW metric, the Friedmann equation and the Schwarzschild metric

1.5 Procedures

- Studying tensor analysis and calculational skills
- Studying concepts of general relativity.
- Applying variational principle method to Einstein - Hilbert action
- Deriving FLRW metric and Friedmann equation
- Deriving Schwarzschild metric
- Conclusion

1.6 Outcome

- Detailed derivation of the FLRW metric and the Friedmann equation
- Detailed derivation of the Schwarzschild metric

Chapter 2

Failure of classical mechanics

2.1 Inertial reference frames

Newton introduced his three laws of motion as axioms of classical mechanics. These laws introduce a frame of reference called inertial frame.
To measure velocity of a an object, we need a frame of reference. The Earth and is not really inertial frame due it motion.
In the absence of gravity, if S and S' are two inertial frames then S' can differ from S only by (i) a translation, and/or (ii) a rotation and/or (iii) a motion of one frame with respect to the other at a constant velocity.

2.2 Special relativity

Consider inertial frames of reference moving with constant velocity to each other

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{2.1}$$

that is so called Galilean transformation. Newton's law are invariant under Galilean transformation

$$F_i = m\ddot{x}_i = m\ddot{x}'_i = F'_i. \tag{2.2}$$

However electromagnetic wave equation is not invariant under Galilean transformation. Consider electromagnetic wave equation:

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

or

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\partial^2\phi}{\partial t^2} = 0 \tag{2.3}$$

Using chain rule and equation (2.1) to transform coordinate, the wave equation become

$$\frac{c^2 - v^2}{c^2}\frac{\partial^2\phi}{\partial x'^2} + \frac{2v}{c^2}\frac{\partial^2\phi}{\partial t'\partial x'} + \frac{\partial^2\phi}{\partial y'^2} + \frac{\partial^2\phi}{\partial z'} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} = 0 \tag{2.4}$$

This equation contradicts to Einstein's postulates in special relativity that physical laws should be the same in all inertial frames. Therefore we require new transformation law, **Lorentz transformation**.

Einstein's principle of special relativity states that

- the laws of physical phenomena are the same in all inertial reference frames.
- the velocity of light is the same in all inertial reference frames.

Newtonian mechanics considers only three-dimensional space while special relativity considers space and time as one single entity called spacetieme.

The spacetime interval in four dimensional spacetime is

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{2.5}$$

The Lorentz transformation between two frames is written as

$$\begin{aligned} c\mathrm{d}t' &= \gamma(c\mathrm{d}t - v\mathrm{d}x/c) \\ \mathrm{d}x' &= \gamma(\mathrm{d}x - v\mathrm{d}t) \\ \mathrm{d}y' &= \mathrm{d}y \\ \mathrm{d}z' &= \mathrm{d}z. \end{aligned} \tag{2.6}$$

Using equation (2.5) and equation (2.6), we obtain

$$\begin{aligned} -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 &= -c^2\mathrm{d}t'^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2 \\ \mathrm{d}s^2 &= \mathrm{d}s'^2 \end{aligned} \tag{2.7}$$

since speed of light is the same in all inertial frame.

Problem of Newton's theory of gravity is that the theory permits action at a distance. A point mass at one place may then act instantaneously on a point mass at another remote position. According to the special theory of relativity, instantaneous action at a distance is impossible, because the limitations on speeds faster than the speed of light c is not possible at arbitrary distances.

Considering at strong gravitational force(i.e. closer to the Sun), Newton's theory of gravity is not acceptable, the theory needs to be modified to general relativistic case where curvature effect comes to play the role. Example, As it orbits the Sun, this planet follows an ellipse but Mercury to the sun does not always occur at the same place but that it slowly moves around the sun.This rotation of the orbit is called a precession. Newton's theory does not fully explain the precession of Mercury's orbit but General Relativity provides full explanation for the observed precession of Mercury's orbit. As Mercury moves toward closer to the Sun, it moves deeper into the Sun's gravity well. Its motion into this region of greater curvature of space-time.

Chapter 3

Introduction to general relativity

3.1 Tensor and curvature

A knowledge of tensor is needed for understanding general relativity. Vectors and scalars are subsets of tensors indicated by rank of tensor. Tensors are defined on **manifold** μ which is n-dimensional generalized object that locally looks like Euclidian space.

3.1.1 Vectors

Vectors are expressed in general form $\vec{A} = A^{\alpha} e_{\alpha}$ where A^{α} is components of vector and e_{α} is a basisvector.

We can write vectors in frames S and S' as

$$\vec{A} = A^{\alpha} e_{\alpha} = A^{\alpha'} e_{\alpha'}. \tag{3.1}$$

These express the frame-independent nature of any four-vectors. We can transform vectors from frames S to S' by

$$A^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} A^{\beta} = \Lambda^{\alpha'}_{\beta} A^{\beta} \tag{3.2}$$

where $\Lambda^{\alpha'}_{\beta}$ is a general transformation metric.
We call vectors which transforms in this way, **"Contravariant vector"** or **"tangent vector A^{α}"**
Consider a scalar differentiate ϕ with respect to $x^{\alpha'}$
we obtain

$$\frac{\partial \phi}{\partial x^{\alpha'}} = \frac{\partial \phi}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x^{\alpha'}} = \Lambda^{\beta}_{\alpha'} \frac{\partial \phi}{\partial x^{\beta}}. \tag{3.3}$$

These quantities are called **"Covariant vectors"** or **"one-forms"**

3.1.2 The metric tensor

Metrices are used to define distance and length of vector.

Consider a scalar product,

$$\vec{A} \cdot \vec{B} = A_\mu B^\mu \tag{3.4}$$

$$\vec{A} \cdot \vec{B} = g_{\mu\nu} A^\nu B^\mu \tag{3.5}$$

$g_{\mu\nu}$ are components of a tensor called **" metric tensor "**. We define the inverse of $g_{\mu\nu}$ as $g^{\mu\nu}$ whereas

$$g_{\mu\nu} g^{\nu\alpha} = \delta^\alpha_\mu. \tag{3.6}$$

The metric tensor and inverse metric tensor, $g_{\mu\nu}$ and $g^{\mu\nu}$ can be used to lower and raise any given index of tensor,

$$A_\mu = g_{\mu\nu} A^\nu \tag{3.7}$$

$$A^\mu = g^{\mu\nu} A_\nu. \tag{3.8}$$

The square of the infinitesimal distance distance or interval in special relativity.

Consider scalar product

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu \tag{3.9}$$

$$\mathrm{d}s^2 = \eta_{\mu\nu} \mathrm{d}x^\nu \mathrm{d}x^\mu \tag{3.10}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

$\eta_{\mu\nu}$ is a flat Minkowski metric.
In general relativity we are interested in curved space. We write $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$ to obtain

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\nu \mathrm{d}x^\mu \tag{3.12}$$

3.1.3 Covariant derivative

Suppose the vector field $\vec{V}(x)$ is defined over some region of a manifold, We will consider derivative of vector field.

Consider of contravariant components of $\vec{V} = V^\mu e_\mu$ we thus obtain

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} e_\alpha + V^\alpha \frac{\partial e_\alpha}{\partial x^\beta}. \tag{3.13}$$

In the second term, the coordinate basis vector varies with the position in the manifold, and hence can be expanded over the basis

$$\frac{\partial e_\alpha}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta} e_\gamma \tag{3.14}$$

where the $\Gamma^\gamma_{\alpha\beta}$ are a set of coefficients depending on position.
They are call **"connection coefficients"** or **"Christoffel symbols"**. In flat space $\Gamma^\gamma_{\alpha\beta} = 0$. But in curve space it is impossibleto make all the $\Gamma^\gamma_{\alpha\beta}$ vanish over all space.
From (3.13),

$$\frac{\partial V}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} e_\alpha + V^\alpha \Gamma^\gamma_{\alpha\beta} e_\gamma \tag{3.15}$$

Interchanging indices α and γ, we obtain

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} e_\alpha + V^\gamma \Gamma^\alpha_{\gamma\beta} e_\alpha = \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta} V^\gamma \right) e_\alpha \tag{3.16}$$

so

$$\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta} V^\gamma \tag{3.17}$$

are the components of a tensor, call the covariant derivative of a tensor. The component is expressed as

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \Gamma^\alpha_{\gamma\beta} V^\gamma \tag{3.18}$$

The notation ∂_β is introduced by $\partial_\beta \equiv \partial / \partial x^\beta$

Next, we considering the derivative of $\vec{V} = V_\alpha e^\alpha$

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V_\alpha}{\partial x^\beta} e^\alpha + V_\alpha \frac{\partial e^\alpha}{\partial x^\beta} \tag{3.19}$$

The derivatives of the dual basis vectors with respect to the coordinates are given by

$$\frac{\partial e^{\alpha}}{\partial x^{\beta}} = \partial_{\beta} e^{\alpha} = -\Gamma^{\alpha}_{\gamma\beta} e^{\gamma}. \tag{3.20}$$

We obtain

$$\frac{\partial \vec{V}}{\partial x^{\beta}} = \frac{\partial V_{\alpha}}{\partial x^{\beta}} e^{\alpha} + V_{\alpha} \left(-\Gamma^{\alpha}_{\gamma\beta} e^{\gamma} \right). \tag{3.21}$$

Interchanging indices γ and α, we obtain

$$\frac{\partial \vec{V}}{\partial x^{\beta}} = \frac{\partial V_{\alpha}}{\partial x^{\beta}} e^{\alpha} + V_{\gamma} \left(-\Gamma^{\gamma}_{\alpha\beta} e^{\alpha} \right) = \left(\partial_{\beta} V_{\alpha} - \Gamma^{\gamma}_{\alpha\beta} V_{\gamma} \right) e^{\alpha}. \tag{3.22}$$

Covariant derivative for covariant vector is hence

$$\nabla_{\beta} V_{\alpha} = \partial_{\beta} V_{\alpha} - \Gamma^{\gamma}_{\alpha\beta} V_{\gamma}. \tag{3.23}$$

Covariant derivative of the metric tensor vanishes,

$$\nabla_{\beta} g_{\alpha\gamma} = 0 \tag{3.24}$$

and

$$\nabla_{\beta} g^{\alpha\gamma} = 0. \tag{3.25}$$

Considering the **Cristoffel symbols** we can shows that it can be expressed with the metric.

$$\Gamma^{\alpha}_{\gamma\beta} = \frac{1}{2} g^{\alpha\delta} \left(\partial_{\gamma} g_{\delta\beta} + \partial_{\beta} g_{\gamma\delta} - \partial_{\delta} g_{\gamma\beta} \right). \tag{3.26}$$

The Cristoffel symbols are necessarily symmetric under interchanging of lower indices

$$\Gamma^{\alpha}_{\gamma\beta} = \Gamma^{\alpha}_{\beta\gamma} \tag{3.27}$$

3.1.4 Parallel transport

This is extended to the curved spacetime of GR by the notion of parallel transport in which a vector is moved along a curve staying parallel to itself and of constant magnitude.

Consider the change of a vector $\vec{V}$ along a line parameter by λ

$$\frac{\mathrm{d}\vec{V}}{\mathrm{d}\lambda} = \frac{\mathrm{d}V^\alpha}{\mathrm{d}\lambda} e_\alpha + V^\alpha \frac{\mathrm{d}e_\alpha}{\mathrm{d}\lambda}. \tag{3.28}$$

We can write

$$\frac{\mathrm{d}e_\alpha}{\mathrm{d}\lambda} = \frac{\partial e_\alpha}{\partial x^\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}. \tag{3.29}$$

Using definition of the connection

$$\frac{\partial e_\alpha}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta} e_\gamma, \tag{3.30}$$

hence

$$\frac{\mathrm{d}\vec{V}}{\mathrm{d}\lambda} = \frac{\mathrm{d}V^\alpha}{\mathrm{d}\lambda} e_\alpha + V^\alpha \Gamma^\gamma_{\alpha\beta} e_\gamma \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} \tag{3.31}$$

swapping indices α and γ in second term

$$\frac{\mathrm{d}\vec{V}}{\mathrm{d}\lambda} = \frac{\mathrm{d}V^\alpha}{\mathrm{d}\lambda} e_\alpha + V^\gamma \Gamma^\alpha_{\gamma\beta} e_\alpha \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}V^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha_{\gamma\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} V^\gamma \right) e_\alpha \tag{3.32}$$

vector components is

$$\frac{\mathrm{D}V^\alpha}{\mathrm{D}\lambda} = \frac{\mathrm{d}V^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha_{\gamma\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} V^\gamma \tag{3.33}$$

where $U^\alpha = dx^\alpha/d\lambda$ is the " tangent vector " pointing along the line.

Considering covariant derivative

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \Gamma^\alpha_{\gamma\beta} V^\gamma, \tag{3.34}$$

the component $\mathrm{D}V^\alpha/\mathrm{D}\lambda$ is similar to the covariant derivative,

$$\frac{\mathrm{D}V^\alpha}{\mathrm{D}\lambda} = \frac{\partial V^\alpha}{\partial x^\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} + \Gamma^\alpha_{\gamma\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} V^\gamma = \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\gamma\beta} V^\gamma \right) U^\beta \tag{3.35}$$

where

$$U^{\beta} = \frac{dx^{\beta}}{d\lambda}. \tag{3.36}$$

Therefore

$$\frac{\mathrm{D}V^{\alpha}}{\mathrm{D}\lambda} = \nabla_{\beta} V^{\alpha} U^{\beta}. \tag{3.37}$$

If a vector $\vec{V}$ is " parallel transported" along a line then

$$\frac{\mathrm{d}\vec{V}}{\mathrm{d}\lambda} = 0, \tag{3.38}$$

or in component form

$$\frac{\mathrm{D}V^{\alpha}}{\mathrm{D}\lambda} = \nabla_{\beta} V^{\alpha} U^{\beta} = 0 \tag{3.39}$$

or

$$\frac{\mathrm{d}V^{\alpha}}{\mathrm{d}\lambda} + \Gamma^{\alpha}_{\gamma\beta} \frac{\mathrm{d}x^{\beta}}{d\lambda} V^{\gamma} = 0. \tag{3.40}$$

3.1.5 Straight line or geodesics

A line is " straight " if it parallel transports its own tangent vector.

$$V^{\alpha} = U^{\alpha} \tag{3.41}$$

where V^{α} is parallel transports and U^{α} is a tangent vector

$$\frac{\mathrm{d}^2 x^{\alpha}}{\mathrm{d}\lambda^2} + \Gamma^{\alpha}_{\gamma\beta} \frac{\mathrm{d}x^{\beta}}{\mathrm{d}\lambda} \frac{\mathrm{d}x^{\gamma}}{\mathrm{d}\lambda} = 0. \tag{3.42}$$

This equation is known as the geodesic equation. These are force-free equation of motion.

3.1.6 Curvature tensor

An important concept of general relativity, is Riemannian geometry which is described in tensorial form. The Riemann curvature tensor $R^{\rho}_{\alpha\beta\gamma}$ is defined by the commutator of covariant derivatives

$$[\nabla_\gamma, \nabla_\beta] V_\alpha = R^{\rho}_{\alpha\beta\gamma} V_\rho. \tag{3.43}$$

Consider commutator of covariant derivatives

$$[\nabla_\gamma, \nabla_\beta] V_\alpha = \nabla_\gamma \nabla_\beta V_\alpha - \nabla_\beta \nabla_\gamma V_\alpha. \tag{3.44}$$

The first term in right hand equation

$$\nabla_\gamma (\nabla_\beta V_\alpha) = \partial_\gamma (\nabla_\beta V_\alpha) - \Gamma^{\sigma}_{\alpha\gamma} \nabla_\beta V_\sigma - \Gamma^{\sigma}_{\beta\gamma} \nabla_\sigma V_\alpha \tag{3.45}$$

$$= \partial_\gamma \left(\partial_\beta V_\alpha - \Gamma^{\sigma}_{\alpha\beta} V_\sigma\right) - \Gamma^{\sigma}_{\alpha\gamma} \left(\partial_\beta V_\sigma - \Gamma^{\rho}_{\sigma\beta} V_\rho\right) - \Gamma^{\sigma}_{\beta\gamma} \left(\partial_\sigma V_\alpha - \Gamma^{\rho}_{\alpha\sigma} V_\rho\right). \tag{3.46}$$

Interchanging the indices β and γ

$$\nabla_\beta (\nabla_\gamma V_\alpha) = \partial_\beta \left(\partial_\gamma V_\alpha - \Gamma^{\sigma}_{\alpha\gamma} V_\sigma\right) - \Gamma^{\sigma}_{\alpha\beta} \left(\partial_\gamma V_\sigma - \Gamma^{\rho}_{\sigma\gamma} V_\rho\right) - \Gamma^{\sigma}_{\gamma\beta} \left(\partial_\sigma V_\alpha - \Gamma^{\rho}_{\alpha\sigma} V_\rho\right), \tag{3.47}$$

We obtain

$$\begin{aligned} [\nabla_\gamma, \nabla_\beta] V_\alpha = {} & \partial_\gamma \left(\partial_\beta V_\alpha - \Gamma^{\sigma}_{\alpha\beta} V_\sigma\right) - \Gamma^{\sigma}_{\alpha\gamma} \left(\partial_\beta V_\sigma - \Gamma^{\rho}_{\sigma\beta} V_\rho\right) \\ & - \Gamma^{\sigma}_{\beta\gamma} \left(\partial_\sigma V_\alpha - \Gamma^{\rho}_{\alpha\sigma} V_\rho\right) \\ & - \partial_\beta \left(\partial_\gamma V_\alpha - \Gamma^{\sigma}_{\alpha\gamma} V_\sigma\right) + \Gamma^{\sigma}_{\alpha\beta} \left(\partial_\gamma V_\sigma - \Gamma^{\rho}_{\sigma\gamma} V_\rho\right) \\ & + \Gamma^{\sigma}_{\gamma\beta} \left(\partial_\sigma V_\alpha - \Gamma^{\rho}_{\alpha\sigma} V_\rho\right). \end{aligned} \tag{3.48}$$

The Cristoffel symbols are symmetric

$$\Gamma^{\alpha}_{\gamma\beta} = \Gamma^{\alpha}_{\beta\gamma}, \tag{3.49}$$

thus

$$[\nabla_\gamma, \nabla_\beta] V_\alpha = \partial_\gamma \left(\partial_\beta V_\alpha - \Gamma^{\sigma}_{\alpha\beta} V_\sigma\right) - \Gamma^{\sigma}_{\alpha\beta} \left(\partial_\beta V_\sigma - \Gamma^{\rho}_{\sigma\beta} V_\rho\right) \tag{3.50}$$

$$- \partial_\beta \left(\partial_\gamma V_\alpha - \Gamma^{\sigma}_{\alpha\gamma} V_\sigma\right) + \Gamma^{\sigma}_{\alpha\beta} \left(\partial_\gamma V_\sigma - \Gamma^{\rho}_{\sigma\gamma}\right) V_\rho$$

$$= \partial_\gamma(\partial_\beta V_\alpha) - \partial_\gamma(\Gamma^{\sigma}_{\alpha\beta} V_\sigma) - \Gamma^{\sigma}_{\alpha\gamma} \partial_\beta V_\sigma + \Gamma^{\sigma}_{\alpha\gamma} \Gamma^{\rho}_{\sigma\beta} V_\rho \tag{3.51}$$

$$- \partial_\beta(\partial_\gamma V_\alpha) + \partial_\beta(\Gamma^{\sigma}_{\alpha\gamma} V_\sigma) + \Gamma^{\sigma}_{\alpha\beta} \partial_\gamma V_\sigma - \Gamma^{\sigma}_{\alpha\beta} \Gamma^{\rho}_{\sigma\gamma} V_\rho.$$

Use the commutativity of partial differentiation, all the terms with derivatives of the $\vec{V}$ cancel out,

$$[\nabla_\gamma, \nabla_\beta] V_\alpha = -\partial_\gamma \Gamma^\sigma_{\alpha\beta} V_\sigma + \Gamma^\sigma_{\alpha\gamma}\Gamma^\rho_{\sigma\beta} V_\rho + \partial_\beta \Gamma^\sigma_{\alpha\gamma} V_\sigma - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\gamma} V_\rho. \quad (3.52)$$

Rename indices ρ to σ in $\partial_\beta \Gamma^\sigma_{\alpha\gamma} V_\sigma$ and $\partial_\gamma \Gamma^\sigma_{\alpha\beta} V_\sigma$

$$\begin{aligned}
[\nabla_\gamma, \nabla_\beta] V_\alpha &= \partial_\beta \Gamma^\rho_{\alpha\gamma} V_\rho - \partial_\gamma \Gamma^\rho_{\alpha\beta} V_\rho + \Gamma^\sigma_{\alpha\gamma}\Gamma^\rho_{\sigma\beta} V_\rho - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\gamma} V_\rho \\
&= \left[\partial_\beta \Gamma^\rho_{\alpha\gamma} - \partial_\gamma \Gamma^\rho_{\alpha\beta} + \Gamma^\sigma_{\alpha\gamma}\Gamma^\rho_{\sigma\beta} - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\gamma}\right] V_\rho \\
&= R^\rho_{\alpha\beta\gamma} V_\rho. \quad (3.53)
\end{aligned}$$

We have Riemann tensor expressed in term of Cristoffel symbols

$$R^\rho_{\alpha\beta\gamma} = \partial_\beta \Gamma^\rho_{\alpha\gamma} - \partial_\gamma \Gamma^\rho_{\alpha\beta} + \Gamma^\sigma_{\alpha\gamma}\Gamma^\rho_{\sigma\beta} - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\sigma\gamma}. \quad (3.54)$$

$R^\rho_{\alpha\beta\gamma}$ it is anti-symmetric on its last pair of indices

$$R^\rho_{\alpha\beta\gamma} = -R^\rho_{\alpha\gamma\beta}. \quad (3.55)$$

Lowering the first index with the metric, the lowered tensor is symmetric under interchanging of the first and last pair of indices

$$R_{\rho\alpha\beta\gamma} = R_{\beta\gamma\rho\alpha}. \quad (3.56)$$

The tensor is anti-symmetric within its last pair of indices as

$$R_{\rho\alpha\beta\gamma} = -R_{\rho\alpha\gamma\beta}. \quad (3.57)$$

We can use the curvature tensor to define **Ricci tensor**

$$g^{\rho\gamma} R_{\rho\alpha\gamma\beta} = R^\gamma_{\alpha\gamma\beta} = R_{\alpha\beta}, \quad (3.58)$$

$$R_{\alpha\beta} = \partial_\gamma \Gamma^\gamma_{\alpha\beta} - \partial_\beta \Gamma^\gamma_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta}\Gamma^\gamma_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\gamma_{\sigma\beta}. \quad (3.59)$$

Contraction of Ricci tensor then also defines **Ricci scalar**

$$g^{\alpha\beta} R_{\alpha\beta} = R. \quad (3.60)$$

These two tensors can be used to define **Einstein tensor**

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R. \quad (3.61)$$

3.2 The equivalence principle

According to Newtonian gravity, when gravity acts on a body, it acts on the gravitational mass m_{G}.The result of the force is acceleration of the inertial mass m_{I}

In a small laboratory falling freely in gravitational field, mechanical phenomena are the same as those observed in an inertial frame in the absence of gravitational field, This implies $m_{\mathrm{I}} = m_{\mathrm{G}}$ .That is to say $\vec{a} = \vec{g}$ and hence there is no distinct between inertial frames and freely falling frames.

3.3 Einstein's law of gravitational

3.3.1 The energy - momentum tensor for perfect fluids

A perfect fluid is a fluid with (i) no heat conduction (ii) no viscosity (iii) no an isotropic stress. We write the energy - momentum tensor for perfect fluids in a rest frame as

$$T_{\alpha\beta} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{3.62}$$

where ρ is the energy density and p is the pressure.
This can be written as well as

$$T_{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) u_\alpha u_\beta + p g_{\alpha\beta}. \tag{3.63}$$

The four-velocity u_α defined as $u_\alpha = (c^2, 0, 0, 0)$ for rest frame.

In the limit $p \to 0$, perfect fluid reduces as :

$$T_{\alpha\beta} = \rho u_\alpha u_\beta. \tag{3.64}$$

This equation is simplest kind of matter field, that is **non-relativistic matter** or **dust** .

3.3.2 Einstein's field equation

Einstein's field equation told us that the metric is correspondent to geometry and geometry is the effect of an amount of matter which expressed in energy-momentum tensor.

This section we introduce the Einstein's field equation derived by variational principle method in order to get the field equation.

The least action principle is

$$\delta S = 0 \tag{3.65}$$

Consider action

$$S = \int \mathcal{L} \mathrm{d}^4 x \tag{3.66}$$

where $\mathcal{L}$ is Lagrangian density.
The well definition of Lagrangian density is $\mathcal{L} = \sqrt{-g}R$, therefore

$$S_{\text{EH}} = \int \sqrt{-g} R \mathrm{d}^4 x \tag{3.67}$$

is known as the **Einstein - Hilbert action**.
We derive field equation by variation of action

$$\begin{aligned}
\delta S_{\text{EH}} &= \delta \int \sqrt{-g} R \mathrm{d}^4 x \\
\delta S_{\text{EH}} &= \delta \int \sqrt{-g} g^{\alpha\beta} R_{\alpha\beta} \mathrm{d}^4 x \\
\delta S_{\text{EH}} &= \int \sqrt{-g} g^{\alpha\beta} \delta R_{\alpha\beta} \mathrm{d}^4 x + \int \sqrt{-g} \delta g^{\alpha\beta} R_{\alpha\beta} \mathrm{d}^4 x \\
& \int \delta \sqrt{-g} g^{\alpha\beta} R_{\alpha\beta} \mathrm{d}^4 x.
\end{aligned} \tag{3.68}$$

The first term is

$$\delta S_{\text{EH}(1)} = \int \sqrt{-g} g^{\alpha\beta} \delta R_{\alpha\beta} \mathrm{d}^4 x. \tag{3.69}$$

Considering variation of Ricci tensor

$$\begin{aligned}
R_{\alpha\beta} &= R^{\gamma}_{\alpha\gamma\beta} = \partial_\gamma \Gamma^{\gamma}_{\alpha\beta} - \partial_\beta \Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\gamma\rho} \Gamma^{\rho}_{\beta\alpha} - \Gamma^{\gamma}_{\beta\rho} \Gamma^{\rho}_{\alpha\gamma} \\
\delta R_{\alpha\beta} &= \partial_\gamma \delta \Gamma^{\gamma}_{\alpha\beta} - \partial_\beta \delta \Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\gamma\rho} \delta \Gamma^{\rho}_{\beta\alpha} + \delta \Gamma^{\gamma}_{\gamma\rho} \Gamma^{\rho}_{\beta\alpha} - \delta \Gamma^{\gamma}_{\beta\rho} \Gamma^{\rho}_{\alpha\gamma} - \Gamma^{\gamma}_{\beta\rho} \delta \Gamma^{\rho}_{\alpha\gamma} \\
\delta R_{\alpha\beta} &= (\partial_\gamma \delta \Gamma^{\gamma}_{\alpha\beta} + \Gamma^{\gamma}_{\gamma\rho} \delta \Gamma^{\rho}_{\beta\alpha} - \Gamma^{\rho}_{\alpha\gamma} \delta \Gamma^{\gamma}_{\beta\rho} - \Gamma^{\rho}_{\beta\gamma} \delta \Gamma^{\gamma}_{\alpha\rho}) - \\
&\quad (\partial_\beta \delta \Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\beta\rho} \delta \Gamma^{\rho}_{\alpha\gamma} - \Gamma^{\rho}_{\beta\alpha} \delta \Gamma^{\gamma}_{\gamma\rho} - \Gamma^{\rho}_{\beta\gamma} \delta \Gamma^{\gamma}_{\alpha\rho})
\end{aligned} \tag{3.70}$$

Consider covariant derivative formula

$$\nabla_\gamma \delta \Gamma^{\gamma}_{\alpha\beta} = \partial_\gamma \delta \Gamma^{\gamma}_{\alpha\beta} + \Gamma^{\gamma}_{\gamma\rho} \delta \Gamma^{\rho}_{\beta\alpha} - \Gamma^{\rho}_{\alpha\gamma} \delta \Gamma^{\gamma}_{\beta\rho} - \Gamma^{\rho}_{\beta\gamma} \delta \Gamma^{\gamma}_{\alpha\rho} \tag{3.71}$$

and

$$\nabla_\beta \delta\Gamma^\gamma_{\alpha\gamma} = \partial_\beta \delta\Gamma^\gamma_{\alpha\gamma} + \Gamma^\gamma_{\beta\rho}\delta\Gamma^\rho_{\alpha\gamma} - \Gamma^\rho_{\beta\alpha}\delta\Gamma^\gamma_{\gamma\rho} - \Gamma^\rho_{\beta\gamma}\delta\Gamma^\gamma_{\alpha\rho} \tag{3.72}$$

Substituting equation (3.70) and (3.71) to (3.69) to get

$$\delta R_{\alpha\beta} = \nabla_\gamma \delta\Gamma^\gamma_{\alpha\beta} - \nabla_\beta \delta\Gamma^\gamma_{\alpha\gamma}. \tag{3.73}$$

Substituting equation (3.72) to (3.68)

$$S_{\text{EH}(1)} = \int \sqrt{-g} g^{\alpha\beta}(\nabla_\gamma \delta\Gamma^\gamma_{\alpha\beta} - \nabla_\beta \delta\Gamma^\gamma_{\alpha\gamma})\mathrm{d}^4x \tag{3.74}$$

$$= \int \sqrt{-g}[\nabla_\gamma(g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta}) - \delta\Gamma^\gamma_{\alpha\beta}\nabla_\gamma g^{\alpha\beta} - \nabla_\beta(g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\gamma}) \tag{3.75}$$

$$+\delta\Gamma^\gamma_{\alpha\gamma}\nabla_\beta g^{\alpha\beta}]\mathrm{d}^4x. \tag{3.76}$$

That the covariant derivative of metric is zero

$$\nabla_\gamma g^{\alpha\beta} = 0. \tag{3.77}$$

Therefore we get

$$\begin{aligned}\delta S_{\text{EH}(1)} &= \int_\mu \sqrt{-g}[\nabla_\gamma(g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta}) - \nabla_\beta(g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\gamma})]\mathrm{d}^4x \\ &= \int_\mu \sqrt{-g}[\nabla_\gamma(g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta}) - \nabla_\gamma(g^{\alpha\gamma}\delta\Gamma^\beta_{\alpha\beta})]\mathrm{d}^4x \\ &= \int_\mu \sqrt{-g}\nabla_\gamma[(g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta}) - (g^{\alpha\gamma}\delta\Gamma^\beta_{\alpha\beta})]\mathrm{d}^4x \\ &= \int_\mu \sqrt{-g}\nabla_\gamma J^\gamma \mathrm{d}^4x. \end{aligned} \tag{3.78}$$

Where we introduce

$$J^\gamma \equiv g^{\alpha\beta}\delta\Gamma^\gamma_{\alpha\beta} - g^{\alpha\gamma}\delta\Gamma^\beta_{\alpha\beta} \tag{3.79}$$

If J^γ ia a vector field over a region μ with boundary Σ
Using Gauss-Stoke theorem.

$$\int_V \nabla_\alpha A^\alpha \sqrt{-g}\mathrm{d}^4x = \oint_{\partial V} A^\alpha \mathrm{d}\Sigma_\alpha \tag{3.80}$$

Consider equation (3.77) we obtain.

$$\int_\mu \nabla_\gamma J^\gamma \sqrt{-g}\mathrm{d}^4x = \oint_\Sigma J^\gamma \mathrm{d}\Sigma_\gamma \tag{3.81}$$

where $d\Sigma_\gamma = n_\gamma\sqrt{|h|}d^3x$, n_γ is a unit area vector and $d^3x\sqrt{|h|}$ are the size of the area.
therefore we get

$$\int_\mu \sqrt{-g}\nabla_\gamma J^\gamma d^4x = \oint_\Sigma J^\gamma n_\gamma \sqrt{|h|}d^3x. \tag{3.82}$$

Where $h_{\alpha\beta}$ is induced metric on hypersurface defined by

$$h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta. \tag{3.83}$$

Therefore the first term of the action becomes

$$\delta S_{\text{SH(1)}} = \oint_\Sigma J^\gamma n_\gamma \sqrt{|h|}d^3x. \tag{3.84}$$

This equation is an integral with respect to the volume element. Using Gauss-Stoke theorem, this is equal to integral over all boundary.
From equation (3.78) $\delta\Gamma = 0$ at all boundaries, we have the first term as

$$\delta S_{\text{SH(1)}} = 0. \tag{3.85}$$

Variation of the metric

Consider metric $g_{\alpha\beta}$ since the contravariant and covariant metrics are symmetric metrics then

$$g_{\gamma\alpha}g^{\beta\alpha} = \delta^\beta_\gamma. \tag{3.86}$$

Inverse of the metric

$$g^{\alpha\beta} = \frac{1}{g}(A^{\alpha\beta})^T, \tag{3.87}$$

where g is determinant and $A^{\beta\alpha}$ is the adjoint of the metric $g_{\alpha\beta}$,
inverse metric becomes

$$g^{\alpha\beta} = \frac{1}{g}(A^{\alpha\beta})^T = \frac{1}{g}A^{\beta\alpha}. \tag{3.88}$$

Contracting with metric

$$\begin{aligned} gg^{\alpha\beta} &= A^{\beta\alpha} \quad (3.89)\\ g(g_{\gamma\alpha}g^{\alpha\beta}) &= g_{\gamma\alpha}A^{\beta\alpha}\\ g\delta^\beta_\gamma &= g_{\gamma\alpha}A^{\beta\alpha}\\ g &= g_{\beta\alpha}A^{\beta\alpha} \quad (3.90)\end{aligned}$$

Using property in symmetric of $g_{\alpha\beta} = g_{\beta\alpha}$,
we have

$$g = g_{\alpha\beta} A^{\alpha\beta} \tag{3.91}$$

If we perform partial differentiation on both side with respect to $g_{\alpha\beta}$,

$$\frac{\partial g}{\partial g_{\alpha\beta}} = \frac{\partial g_{\alpha\beta}}{\partial g_{\alpha\beta}} A^{\alpha\beta} = A^{\alpha\beta}. \tag{3.92}$$

Consider variation of determinant g using chainrule

$$\begin{aligned} \delta g &= \frac{\partial g}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} \\ \delta g &= A^{\alpha\beta} \delta g_{\alpha\beta} \\ \delta g &= g g^{\alpha\beta} \delta g_{\alpha\beta} \end{aligned} \tag{3.93}$$

consider

$$\begin{aligned} \delta\sqrt{-g} &= \delta(-g)^{\frac{1}{2}} = \frac{1}{2}(-g)^{-\frac{1}{2}}(-1)\delta g \\ \delta\sqrt{-g} &= -\frac{\delta g}{2\sqrt{-g}} \\ \delta\sqrt{-g} &= -\frac{1}{2}\frac{g}{\sqrt{-g}} g^{\alpha\beta} \delta g_{\alpha\beta}. \end{aligned} \tag{3.94}$$

We shall convert $\delta g_{\alpha\beta}$ to $\delta g^{\alpha\beta}$,
consider

$$\begin{aligned} \delta\delta^d_\alpha = \delta(g_{\alpha\gamma} g^{\gamma\rho}) &= 0 \\ g^{\gamma\rho}\delta g_{\alpha\gamma} + g_{\alpha\gamma}\delta g^{\gamma\rho} &= 0 \\ g^{\gamma\rho}\delta g_{\alpha\gamma} &= -g_{\alpha\gamma}\delta g^{\alpha\gamma} \\ g_{\beta\rho} g^{\gamma\rho}\delta g_{\alpha\gamma} &= -g_{\beta\rho} g_{\alpha\gamma}\delta g^{\gamma\rho} \\ \delta^\gamma_\beta \delta g_{\alpha\gamma} &= -g_{\beta\rho} g_{\alpha\gamma}\delta g^{\gamma\rho} \\ \delta g_{\alpha\beta} &= -g_{\beta\rho} g_{\alpha\gamma}\delta g^{\gamma\rho}. \end{aligned} \tag{3.95}$$

Substituting (3.94) to (3.93)

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2}\frac{g}{\sqrt{-g}} g^{\alpha\beta}(-g_{\beta\rho} g_{\alpha\gamma}\delta g^{\gamma\rho}) \\ &= \frac{1}{2}\frac{g}{\sqrt{-g}}\frac{\sqrt{-g}}{\sqrt{-g}}\delta^\alpha_\rho g_{\alpha\gamma}\delta g^{\gamma\rho} \\ &= -\frac{1}{2}\sqrt{-g} g_{\rho\gamma}\delta g^{\gamma\rho} \\ &= -\frac{1}{2}\sqrt{-g} g_{\gamma\rho}\delta g^{\gamma\rho}. \end{aligned} \tag{3.96}$$

Renaming indices γ to α and ρ to β

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}. \tag{3.97}$$

Hence variation of the Einstein - Hilbert action becomes

$$\begin{aligned} \delta S_{\text{EH}} &= \delta S_{\text{EH(2)}} + \delta S_{\text{EH(3)}} \\ &= \int \sqrt{-g}R_{\alpha\beta}\delta g^{\alpha\beta}\text{d}^4x + \int g^{\alpha\beta}R_{\alpha\beta}\delta\sqrt{-g}\text{d}^4x \\ &= \int \sqrt{-g}R_{\alpha\beta}\delta g^{\alpha\beta}\text{d}^4x - \int g^{\alpha\beta}R_{\alpha\beta}\delta(\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta})\text{d}^4x \\ &= \int \sqrt{-g}[R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}]\delta g^{\alpha\beta}\text{d}^4x. \end{aligned} \tag{3.98}$$

From the least action principle $\delta S = 0$

$$\begin{aligned} \delta S_{\text{EH}} &= \int \sqrt{-g}[R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}]\delta g^{\alpha\beta}\text{d}^4x = 0 \\ \frac{\delta S_{\text{EH}}}{\delta g^{\alpha\beta}} = &= \sqrt{-g}[R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}]\frac{\delta g^{\alpha\beta}}{\delta g^{\alpha\beta}} = 0 \\ \frac{1}{\sqrt{-g}}\frac{\delta S_{\text{EH}}}{\delta g^{\alpha\beta}} &= R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 0. \end{aligned} \tag{3.99}$$

This is Einstein's field equation in vacuum.

The full field equation

We assume that there is other Lagrangian presenting beside the gravitational field. The action is then

$$\delta S = \frac{1}{16\pi G/c^4}\delta S_{\text{EH}} + \delta S_{\text{M}} \tag{3.100}$$

S_{M} is the action for matter field,

$$\begin{aligned} \delta S &= \frac{1}{16\pi G/c^4}\sqrt{-g}[R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R]\delta g^{\alpha\beta} + \delta S_{\text{M}} \\ \frac{\delta S}{\delta g^{\alpha\beta}} &= \frac{1}{16\pi G/c^4}\sqrt{-g}[R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R]\frac{\delta g^{\alpha\beta}}{\delta g^{\alpha\beta}} + \frac{\delta S_{\text{M}}}{\delta g^{\alpha\beta}} \\ \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} &= \frac{1}{16\pi G/c^4}[R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R] + \frac{1}{\sqrt{-g}}\frac{\delta S_{\text{M}}}{\delta g^{\alpha\beta}} = 0 \end{aligned} \tag{3.101}$$

define the energy - momentum tensor

$$
\begin{aligned}
T_{\alpha\beta} &= -2\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{\alpha\beta}} \\
\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{\alpha\beta}} &= -\frac{1}{2}T_{\alpha\beta}
\end{aligned} \tag{3.102}
$$

we obtain

$$
\begin{aligned}
\frac{1}{16\pi G/c^4}[R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R] - \frac{1}{2}T_{\alpha\beta} &= 0 \\
R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R &= \left(16\pi G/c^4\right)\frac{1}{2}T_{\alpha\beta}.
\end{aligned}
$$

This allows us to recover the complete Einstein's equation

$$
R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \frac{8\pi G}{c^4}T_{\alpha\beta}. \tag{3.103}
$$

Chapter 4

Cosmological field equations

4.1 Schwarzschild metric

We discuss the exact solution to Einstein's equation here. The Schwarzschild solution represents the spacetime geometry outside a spherically symmetric matter distribution.

4.1.1 The general static isotropic metric

A static spacetime is one for which some timelike coordinate x^0 with following properties
(i) the metric component $g_{\mu\nu}$ are independent of x^0
(ii) line element $\mathrm{d}s^2$ is invariant under transformation $x^0 \to -x^0$.
Considering from the line element

$$\mathrm{d}s^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{4.1}$$

Which is invariant under $x^0 \to -x^0$. Hence the metric is static, and ds^2 depends only on **rotational invariant** of the spacelike coordinates x^i
Consider first the Minkowski interval in spherically coordinates (t, r, θ, ϕ)

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{4.2}$$

A general isotropic metric can be written as

$$\mathrm{d}s^2 = -A\mathrm{d}t^2 + B\mathrm{d}t\mathrm{d}r + C\mathrm{d}r^2 + D(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{4.3}$$

• Expect symmetry under $\phi \to -\phi, \theta \to \pi - \theta$.
• A, B, C and D cannot depend on θ or ϕ. (functions of r and t only)
Define a new radial coordinate by $\bar{r}^2 = D$

$$\mathrm{d}s^2 = -A\mathrm{d}t^2 + B\mathrm{d}t\mathrm{d}\bar{r} + C\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{4.4}$$

Introduce a new timelike coordinate $\bar{t}$ define by

$$\mathrm{d}\bar{t} = \Phi(t,\bar{r})\left[A(t,\bar{r})\mathrm{d}t - \frac{1}{2}B(t,\bar{r})\mathrm{d}\bar{r}\right], \tag{4.5}$$

where $\Phi(t, \bar{r}$ is an integrating factor and

$$\begin{aligned}
\mathrm{d}\bar{t}^2 &= \Phi^2\left(A^2\mathrm{d}t^2 - 2A\mathrm{d}t\frac{1}{2}B\mathrm{d}\bar{r} + \left(\frac{1}{2}B\mathrm{d}\bar{r}\right)^2\right) \\
\mathrm{d}\bar{t}^2 &= \Phi^2\left(A^2\mathrm{d}t^2 - AB\mathrm{d}t\mathrm{d}\bar{r} + \frac{1}{4}B^2\mathrm{d}\bar{r}^2\right) \\
-A^2\mathrm{d}t^2 + AB\mathrm{d}t\mathrm{d}\bar{r} &= -\frac{\mathrm{d}\bar{t}^2}{\Phi^2} + \frac{1}{4}B^2\mathrm{d}\bar{r}^2 \\
-A\mathrm{d}t^2 + B\mathrm{d}t\mathrm{d}\bar{r} &= -\frac{\mathrm{d}t^2}{\Phi^2 A} + \frac{1}{4}\frac{B^2}{A}\mathrm{d}\bar{r}^2 \qquad (4.6) \\
\mathrm{d}s^2 &= -\frac{\mathrm{d}\bar{t}^2}{\Phi^2 A} + \frac{B^2}{4A}\mathrm{d}\bar{r}^2 + C\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\
\mathrm{d}s^2 &= -\left(\frac{1}{\Phi^2 A}\right)\mathrm{d}\bar{t}^2 + \left(C + \frac{B^2}{4A}\right)\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\
\mathrm{d}s^2 &= -\bar{A}\mathrm{d}\bar{t}^2 + \bar{B}\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \qquad (4.7)
\end{aligned}$$

Where we define a new function $\bar{A} = \frac{1}{\Phi^2 A}$ and $\bar{B} = C + \frac{B^2}{4A}$
Dropping the bar we get

$$\mathrm{d}s^2 = -A(t,r)\mathrm{d}t^2 + B(t,r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2).$$

We require the metric function $g_{\mu\nu}$ to be independent of the timelike coordinate, i.e. A and B must be function of r only. Thus we have the general form of an isotropic metric.

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{4.8}$$

We are interested in the spacetime geometry outside a spherical mass distribution. The empty space field, the Ricci tensor vanishes,

$$R_{\mu\nu} = 0 \tag{4.9}$$

From the metric we get

$$g_{\mu\nu} = \begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2 \end{pmatrix} \tag{4.10}$$

and

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{A} & 0 & 0 & 0 \\ 0 & \frac{1}{B} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix} \tag{4.11}$$

The non-zero elements of $g_{\mu\nu}$ and $g^{\mu\nu}$ are

$$\begin{aligned} g_{00} &= -A \qquad g^{00} = -\frac{1}{A} \\ g_{11} &= B \qquad g^{11} = \frac{1}{B} \\ g_{22} &= r^2 \qquad g^{22} = \frac{1}{r^2} \\ g_{33} &= r^2\sin^2\theta \qquad g^{33} = \frac{1}{r^2\sin^2\theta}. \end{aligned}$$

Recall (3.26), the connection coefficients.

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right) \tag{4.12}$$

hence, non-zero component are

$$\begin{aligned} \Gamma^1_{00} &= \frac{1}{2}g^{11}(\partial_0 g_{10} + \partial_0 g_{10} - \partial_1 g_{00}) = \frac{1}{2}\left(\frac{1}{B}\right)\left(-\frac{d(-A)}{\mathrm{d}r}\right) = \frac{1}{2B}\frac{\mathrm{d}A}{\mathrm{d}r}, \\ \Gamma^0_{01} &= \frac{1}{2}g^{00}(\partial_0 g_{01} + \partial_1 g_{00} - \partial_0 g_{01}) = \frac{1}{2}\left(-\frac{1}{A}\right)\left(-\frac{\mathrm{d}A}{\mathrm{d}r}\right) = \frac{1}{2A}\frac{\mathrm{d}A}{\mathrm{d}r}, \\ \Gamma^1_{11} &= \frac{1}{2}g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = \frac{1}{2}\left(\frac{1}{B}\right)\left(\frac{\mathrm{d}B}{\mathrm{d}r}\right) = \frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r}, \\ \Gamma^1_{22} &= \frac{1}{2}g^{11}(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) = \frac{1}{2}\left(\frac{1}{B}\right)\left(-\frac{\mathrm{d}r^2}{\mathrm{d}r}\right) = -\frac{1}{2B}2r = -\frac{r}{B}, \\ \Gamma^1_{33} &= \frac{1}{2}g^{11}(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33}) = \frac{1}{2}\left(\frac{1}{B}\right)\left(-\frac{\mathrm{d}(r^2\sin^2\theta)}{\mathrm{d}r}\right) = -\frac{1}{2B}2r\sin^2\theta \\ &= -\frac{r\sin^2\theta}{B}, \\ \Gamma^2_{21} &= \frac{1}{2}g^{22}(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{21}) = \frac{1}{2}\left(\frac{1}{r^2}\right)\left(\frac{\mathrm{d}r^2}{\mathrm{d}r}\right) = \frac{1}{2r^2}(2r) = \frac{1}{r}, \end{aligned}$$

$$
\begin{aligned}
\Gamma^2_{33} &= \frac{1}{2}g^{22}(\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}) = \frac{1}{2}\left(\frac{1}{r^2}\right)\left(-\frac{\mathrm{d}(r^2\sin^2\theta)}{\mathrm{d}\theta}\right) \\
&= -\frac{1}{2}2\sin\theta cos\theta = -\sin\theta\cos\theta, \\
\Gamma^3_{31} &= \frac{1}{2}g^{33}(\partial_3 g_{31} + \partial_1 g_{33} - \partial_3 g_{31}) = \frac{1}{2}\left(\frac{1}{r^2\sin^2\theta}\right)\left(\frac{\mathrm{d}(r^2\sin^2\theta)}{\mathrm{d}r}\right) = \frac{1}{r}, \\
\Gamma^3_{32} &= \frac{1}{2}g^{33}(\partial_3 g_{32} + \partial_2 g_{33} - \partial_3 g_{32}) = \frac{1}{2}\left(\frac{1}{r^2\sin^2\theta}\right)\left(\frac{\mathrm{d}(r^2\sin^2\theta)}{\mathrm{d}\theta}\right) \\
&= \frac{1}{2\sin^2\theta}(2\sin\theta\cos\theta) = \frac{\cos\theta}{\sin\theta} = \cot\theta.
\end{aligned}
$$

The components of $R_{\mu\nu}$ Ricci tensor are non-zero

$$
R_{\mu\nu} = \partial_\sigma\Gamma^\sigma_{\mu\nu} - \partial_\nu\Gamma^\sigma_{\mu\sigma} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} \tag{4.13}
$$

Finding R_{00}

$$
R_{00} = \partial_\sigma\Gamma^\sigma_{00} - \partial_0\Gamma^\sigma_{0\sigma} + \Gamma^\rho_{00}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{0\sigma}\Gamma^\sigma_{\rho 0}
$$

The 1^{st} component, $\partial_\sigma\Gamma^\sigma_{00}$

$$
\sigma = 1, \qquad \partial_1\Gamma^1_{00} = \frac{d}{dr}\left(\frac{1}{2B}\frac{dA}{dr}\right) = \frac{A''}{2B} - \frac{A'B'}{2B^2}
$$

The 2^{nd} component

$$
-\partial_0\Gamma^\sigma_{0\sigma} = 0
$$

The 3^{rd} component, $\Gamma^\rho_{00}\Gamma^\sigma_{\rho\sigma}$

$$
\begin{aligned}
\rho = 1, \qquad \Gamma^1_{00}\Gamma^\sigma_{1\sigma} &= \Gamma^1_{00}\Gamma^0_{10} = \frac{A'}{2B}\frac{A'}{2A} = \frac{A'^2}{4AB} \\
&= \Gamma^1_{00}\Gamma^1_{11} = \frac{1}{2B}A\frac{1}{2B}B' = \frac{1}{4B^2}A'B' \\
&= \Gamma^1_{00}\Gamma^2_{12} = \frac{1}{2B}A'\frac{1}{r} = \frac{A'}{2Br} \\
&= \Gamma^1_{00}\Gamma^3_{13} = \frac{1}{2B}A'\frac{1}{r} = \frac{A'}{2Br}
\end{aligned}
$$

The 4^{th} component, -$\Gamma^\rho_{0\sigma}\Gamma^\sigma_{\rho 0}$

$$
\begin{aligned}
\sigma = 1, \qquad -\Gamma^\rho_{01}\Gamma^1_{\rho 0} &= -\Gamma^0_{01}\Gamma^1_{00} = -\frac{1}{2}\frac{1}{A}A'\frac{\acute{A}}{2B} = -\frac{A'^2}{4AB} \\
&= -\Gamma^1_{00}\Gamma^0_{10} = -\frac{A'^2}{4AB}
\end{aligned}
$$

We find the diagonal component are

$$\begin{aligned}
R_{00} &= \frac{1}{2B}A'' - \frac{1}{4AB}A'^2 + \frac{1}{4B^2}A'B' + \frac{2A'}{2Br} - \frac{A'B'}{2B^2} \\
&= \frac{A''}{2B} - \frac{A'^2}{4AB} + \frac{A'B'}{4B^2} + \frac{A'}{Br} - \frac{A'B'}{2B^2} \\
R_{00} &= \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{A'}{Br}. \qquad (4.14)
\end{aligned}$$

Finding R_{11}

Consider the the component, R_{11} ,

$$R_{11} = \partial_\sigma \Gamma^\sigma_{11} - \partial_1 \Gamma^\sigma_{1\sigma} + \Gamma^\rho_{11}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{1\sigma}\Gamma^\sigma_{\rho 1}.$$

Consider the 1^{st} term $\partial_\sigma \Gamma^\sigma_{11}$

$$\sigma = 1, \qquad \partial_1 \Gamma^1_{11} = \frac{d}{dr}\left(\frac{B'}{2B}\right) = \frac{2BB''}{4B^2} - \frac{2B'B'}{4B^2} = \frac{B''}{2B} - \frac{B'^2}{2B^2}.$$

The 2^{nd} term $-\partial_1 \Gamma^\sigma_{1\sigma}$

$$\begin{aligned}
-\partial_1 \Gamma^0_{10} &= -\frac{d}{dr}\left(\frac{A'}{2A}\right) = -\frac{2AA''}{4A^2} + \frac{2A'A'}{4A^2} = -\frac{A''}{2A} + \frac{A'^2}{2A^2}, \\
-\partial_1 \Gamma^1_{11} &= -\frac{d}{dr}\left(\frac{B'}{2B}\right) = -\frac{2BB''}{4B^2} + \frac{2B'B'}{4B^2} = -\frac{B''}{2B} + \frac{B'^2}{2B^2}, \\
-\partial_1 \Gamma^2_{12} &= -\frac{d}{dr}\left(\frac{1}{r}\right) = \frac{1}{r^2}, \\
-\partial_1 \Gamma^3_{13} &= -\frac{d}{dr}\left(\frac{1}{r}\right) = \frac{1}{r^2}.
\end{aligned}$$

The 3^{rd} term $\Gamma^\rho_{11}\Gamma^\sigma_{\rho\sigma}$

$$\begin{aligned}
\rho = 1, \qquad \Gamma^1_{11}\Gamma^\sigma_{1\sigma} &= \Gamma^1_{11}\Gamma^0_{10} = \frac{B'}{2B}\frac{A'}{2A} = \frac{A'B'}{4AB}, \\
&= -\Gamma^1_{11}\Gamma^1_{11} = \frac{B}{2B}\frac{B'}{2B} = \frac{B'^2}{4B^2}, \\
&= -\Gamma^1_{11}\Gamma^2_{12} = \frac{B'}{2B}\frac{1}{r} = \frac{B'}{2Br}, \\
&= -\Gamma^1_{11}\Gamma^3_{13} = \frac{B'}{2B}\frac{1}{r} = \frac{B'}{2Br},
\end{aligned}$$

The 4^{th} term, $-\Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1}$

$$\begin{aligned}
\sigma = 0, \qquad -\Gamma^{\rho}_{10}\Gamma^{0}_{\rho 1} &= -\Gamma^{0}_{10}\Gamma^{0}_{01} = \left(\frac{-A'}{2A}\right)^2 = \frac{-A'^2}{4A^2}, \\
\sigma = 1, \qquad -\Gamma^{\rho}_{11}\Gamma^{1}_{\rho 1} &= -\Gamma^{1}_{11}\Gamma^{1}_{11} = \left(\frac{-B'}{2B}\right)^2 = \frac{-B'^2}{4B^2}, \\
\sigma = 2, \qquad -\Gamma^{\rho}_{12}\Gamma^{2}_{\rho 1} &= -\Gamma^{2}_{12}\Gamma^{2}_{21} = -\frac{1}{r^2}, \\
\sigma = 3, \qquad -\Gamma^{\rho}_{13}\Gamma^{3}_{\rho 1} &= -\Gamma^{3}_{13}\Gamma^{3}_{31} = -\frac{1}{r^2}.
\end{aligned}$$

so we get the R_{11} component

$$\begin{aligned}
R_{11} &= -\frac{A''}{2A} + \frac{A'^2}{2A^2} - \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} + \frac{B'}{Br} \\
R_{11} &= -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{B'}{Br}. \qquad (4.15)
\end{aligned}$$

Finding R_{22}

Consider the Ricci tensor R_{22}

$$R_{22} = \partial_\sigma\Gamma^{\sigma}_{22} - \partial_2\Gamma^{\sigma}_{2\sigma} + \Gamma^{\rho}_{22}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\rho}_{2\sigma}\Gamma^{\sigma}_{\rho 2}.$$

The 1^{st} term $\partial_\sigma\Gamma^{\sigma}_{22}$

$$\sigma = 1, \qquad \partial_1\Gamma^{1}_{22} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{-r}{B}\right) = -\frac{B}{B^2} + \frac{rB'}{B^2} = -\frac{1}{B} + \frac{rB'}{B^2}.$$

The 2^{nd} term $-\partial_2\Gamma^{\sigma}_{2\sigma}$

$$\sigma = 3, \qquad -\partial_2\Gamma^{3}_{23} = -\frac{\mathrm{d}}{\mathrm{d}\theta}\cot\theta = -\sec^2\theta.$$

The 3^{rd} $\Gamma^{\rho}_{22}\Gamma^{\sigma}_{\rho\sigma}$

$$\begin{aligned}
\rho = 1, \qquad & \Gamma^{1}_{22}\Gamma^{\sigma}_{1\sigma} \\
\sigma = 0, \qquad \Gamma^{1}_{22}\Gamma^{0}_{10} &= \left(\frac{-r}{B}\right)\left(\frac{A'}{2A}\right) = -\frac{rA'}{2AB}, \\
\sigma = 1, \qquad \Gamma^{1}_{22}\Gamma^{1}_{11} &= \left(\frac{-r}{B}\right)\left(\frac{B'}{2B}\right) = -\frac{rB'}{2B^2}, \\
\sigma = 2, \qquad \Gamma^{1}_{22}\Gamma^{2}_{12} &= \left(\frac{-r}{B}\right)\left(\frac{1}{r}\right) = -\frac{1}{B}, \\
\sigma = 3, \qquad \Gamma^{1}_{22}\Gamma^{3}_{13} &= \left(\frac{-r}{B}\right)\left(\frac{1}{r}\right) = -\frac{1}{B}.
\end{aligned}$$

The 4^{th} term $-\Gamma^{\rho}_{2\sigma}\Gamma^{\sigma}_{\rho 2}$

$$\begin{aligned}
\sigma = 1, \qquad & -\Gamma^{\rho}_{21}\Gamma^{1}_{\rho 2} &=& -\Gamma^{2}_{21}\Gamma^{1}_{22} = -\left(\frac{1}{r}\right)\left(\frac{-r}{B}\right) = \frac{1}{B}, \\
\sigma = 2, \qquad & -\Gamma^{\rho}_{22}\Gamma^{2}_{\rho 2} &=& -\Gamma^{1}_{22}\Gamma^{2}_{12} = -\left(\frac{-r}{B}\right)\left(\frac{1}{r}\right) = \frac{1}{B}, \\
\sigma = 3, \qquad & -\Gamma^{\rho}_{23}\Gamma^{3}_{\rho 2} &=& -\Gamma^{3}_{23}\Gamma^{3}_{32} = -\cot^2\theta.
\end{aligned}$$

So we get Ricci tensor R_{22} component

$$\begin{aligned}
R_{22} &= 1 - \frac{1}{B} + \frac{rB'}{B^2} - \frac{rA'}{2AB} - \frac{rB'}{2B^2}, \\
&= 1 - \frac{1}{B} + \left(1 - \frac{1}{2}\right)\frac{rB'}{B^2} - \frac{rA'}{2AB}, \\
&= 1 - \frac{1}{B} + \frac{1}{2}\frac{rB'}{B^2} - \frac{rA'}{2AB}, \\
&= 1 - \frac{1}{B} + \frac{r}{2B}\left(\frac{B'}{B} - \frac{A'}{A}\right).
\end{aligned} \tag{4.16}$$

Finding R_{33}

Consider the Ricci tensor R_{33}

$$R_{33} = \partial_\sigma \Gamma^{\sigma}_{33} - \partial_3 \Gamma^{\sigma}_{3\sigma} + \Gamma^{\rho}_{33}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3}.$$

The 1^{st} term, $\partial_\sigma \Gamma^{\sigma}_{33}$

$$\begin{aligned}
\sigma = 1, \qquad & \partial_1 \Gamma^{1}_{33} &=& \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{-r\sin^2\theta}{B}\right) = -\sin^2\theta\left(\frac{B}{B^2} + \frac{rB'}{B^2}\right) = -\left(\frac{1}{B} + \frac{rB'}{B^2}\right)\sin^2\theta, \\
\sigma = 2, \qquad & \partial_2 \Gamma^{2}_{33} &=& \frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\sin\theta\cos\theta\right) = -\sin\theta(-\sin\theta) - \cos\theta(\cos\theta) = \sin^2\theta - \cos^2\theta.
\end{aligned}$$

The 2^{nd} term, $-\partial_3 \Gamma^{\sigma}_{3\sigma}$

$$-\partial_3 \Gamma^{\sigma}_{3\sigma} = 0.$$

The 3^{rd} term, $\Gamma^{\rho}_{33}\Gamma^{\sigma}_{\rho\sigma}$

$$\begin{aligned}
\rho = 1, \qquad \Gamma^{1}_{33}\Gamma^{\sigma}_{1\sigma} &= \Gamma^{1}_{33}\Gamma^{0}_{10} = \left(\frac{-r\sin^2\theta}{B}\right)\left(\frac{A'}{2A}\right) = -\frac{rA'\sin^2\theta}{2AB}, \\
&= \Gamma^{1}_{33}\Gamma^{1}_{11} = \left(\frac{-r\sin^2\theta}{B}\right)\left(\frac{B'}{2B}\right) = -\frac{rB'\sin^2\theta}{2B^2}, \\
&= \Gamma^{1}_{33}\Gamma^{2}_{12} = \left(\frac{-r\sin^2\theta}{B}\right)\left(\frac{1}{r}\right) = -\frac{\sin^2\theta}{B}, \\
&= \Gamma^{1}_{33}\Gamma^{3}_{13} = \left(\frac{-r\sin^2\theta}{B}\right)\left(\frac{1}{r}\right) = -\frac{\sin^2\theta}{B}, \\
\rho = 2, \qquad \Gamma^{2}_{33}\Gamma^{\sigma}_{2\sigma} &= \Gamma^{2}_{33}\Gamma^{3}_{23} = (-\sin\theta\cos\theta)(\cot\theta) = -\sin\theta\cos\theta\left(\frac{\cos\theta}{\sin\theta}\right) = -\cos^2\theta.
\end{aligned}$$

The 4^{th} term, $-\Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3}$

$$\begin{aligned}
\sigma = 1, \qquad -\Gamma^{\rho}_{31}\Gamma^{1}_{\rho 3} &= -\Gamma^{3}_{31}\Gamma^{1}_{33} = -\left(\frac{1}{r}\right)\left(\frac{-r\sin^2\theta}{B}\right) = \frac{\sin^2\theta}{B}, \\
\sigma = 2, \qquad -\Gamma^{\rho}_{32}\Gamma^{2}_{\rho 3} &= -\Gamma^{3}_{32}\Gamma^{2}_{33} = -\frac{\cos\theta}{\sin\theta}(-\sin\theta\cos\theta) = \cos^2\theta, \\
\sigma = 3, \qquad -\Gamma^{\rho}_{33}\Gamma^{3}_{\rho 3} &= -\Gamma^{1}_{33}\Gamma^{3}_{13} = -\left(\frac{-r\sin^2\theta}{B}\right)\left(\frac{1}{r}\right) = \frac{\sin^2\theta}{B}, \\
-\Gamma^{2}_{33}\Gamma^{3}_{23} &= -(-\sin\theta\cos\theta)\left(\frac{\cos\theta}{\sin\theta}\right) = \cos^2\theta.
\end{aligned}$$

We get the R_{33} component as

$$\begin{aligned}
R_{33} &= -\frac{\sin^2\theta}{B} + \sin^2\theta + \left(1 - \frac{1}{2}\right)\frac{rB'\sin^2\theta}{B^2} - \frac{rA'\sin^2\theta}{2AB}, \\
&= \left(-\frac{1}{B} + 1 + \frac{rB'}{2B^2} - \frac{rA'}{2AB}\right)\sin^2\theta, \\
&= \left(1 - \frac{1}{B} + \frac{r}{2B}\left(\frac{B'}{B} - \frac{A'}{A}\right)\right)\sin^2\theta \\
&= R_{22}\sin^2\theta. \qquad (4.17)
\end{aligned}$$

Remember the empty - space field

$R_{\mu\nu} = 0$ Hence $R_{00} = R_{11} = R_{22} = R_{33} = 0$

Multiply B/A to equation (4.14) - (R_{00} component)

$$\begin{aligned} \frac{A''}{2B}\frac{B}{A} - \frac{A'B}{4AB}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{A'B}{rBA} &= 0, \\ \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{A'}{rA} &= 0. \end{aligned} \tag{4.18}$$

Add equation (4.18) to (4.15) - (R_{11} component)

$$\begin{aligned} \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{A'}{rA} - \frac{A''}{2A} + \frac{A'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{B'}{Br} &= 0 \\ -\frac{A'^2}{4A^2} - \frac{A'B'}{4AB} + \frac{A'}{rA} + \frac{A'^2}{4A^2} + \frac{A'B'}{4AB} + \frac{B'}{rB} &= 0 \\ \frac{A'}{rA} + \frac{B'}{rB} &= 0 \\ \frac{1}{r}\left(\frac{A'}{A} + \frac{B'}{B}\right) &= 0 \\ \frac{A'}{A} + \frac{B'}{B} &= 0 \\ \frac{AB' + BA'}{AB} &= 0 \\ AB' + BA' &= 0 \end{aligned}$$

Which implies $AB =$ constant. Denote this constant with α, $AB = \alpha$. Hence

$$B = \frac{\alpha}{A}. \tag{4.19}$$

Substituting $B = \alpha/A$ into (4.16)-($R_{22} component$)

$$1 - \frac{A}{\alpha} + \frac{rA}{2\alpha}\left(\frac{B'A}{\alpha} - \frac{A'}{A}\right) = 0$$

from $A'B + AB' = 0, B' = -A'B/A = -A'/A\,(\alpha/A) = -A'\alpha/A^2$

$$\begin{aligned} 1 - \frac{A}{\alpha} + \frac{rA}{2\alpha}\left(\frac{-A'\alpha A}{\alpha A^2} - \frac{A'}{A}\right) &= 0 \\ \frac{A}{\alpha} + \frac{rA}{2\alpha}\left(\frac{2A'}{A}\right) &= 1 \\ A + \frac{rA2A'}{2A} &= \alpha \\ A + rA' &= \alpha \\ \frac{d(rA)}{dr} &= \alpha \\ \int d(rA) = \int \alpha dr \\ rA &= \alpha(r+k) \end{aligned}$$

where k is another integration constant.
Thus the functions $A(r)$ and $B(r)$ are given by

$$A(r) = \frac{\alpha}{r}(r+k) = \alpha\left(1 + \frac{k}{r}\right), \tag{4.20}$$

and

$$B(r) = \frac{\alpha}{A} = \frac{\alpha}{\alpha(1+k/r)} = \left(1 + \frac{k}{r}\right)^{-1}. \tag{4.21}$$

It can be seen that the integration constant k must in some way represent the mass of the object producing the gravitational field.
Consider the weak - field limit

$$\frac{A(r)}{c^2} \to 1 + \frac{2\Phi}{c^2}. \tag{4.22}$$

Where Φ is the Newtonian gravitational potential. We thus have $\Phi = -GM/r$
Consider

$$\begin{aligned} \frac{A(r)}{c^2} &= 1 + \frac{2\Phi}{c^2} = 1 - \frac{2GM}{c^2 r}, \\ A(r) &= c^2\left(1 - \frac{2GM}{c^2 r}\right), \\ A(r) &= \alpha\left(1 - \frac{k}{r}\right), \end{aligned} \tag{4.23}$$

thus we conclude that

$$\alpha = c^2 \qquad \text{and} \qquad k = \frac{-2GM}{c^2}. \tag{4.24}$$

Therefore the Schwarzschild metric for the empty spacetime outside a spherical body of mass M is

$$\mathrm{d}s^2 = -c^2\left(1-\frac{2GM}{c^2 r}\right)\mathrm{d}t^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2.$$

4.2 The Friedmann–Lemaître-Robertson–Walker metric and Friedmann equation

4.2.1 The cosmological principle

The universe is assumed to be homogeneous and isotropic in large scale. It is said to following the cosmological principle.

Homogeneity : at any particular time the universe looks the same everywhere at a particular time. The property of homogeneity is invariant under translational coordinate transformation.

Isotropy : all directions in space from any points are equivalent. The property of isotropy is invariant under rotational coordinate transformation.

The former demands that all points on particular spacelike hypersurface are equivalent. The spatial separation on the same hypersurface $t =$ constant of the two nearby inertial observers can be found from a root of

$$\mathrm{d}\sigma^2 = g_{ij}\Delta x^i \Delta x^j \tag{4.25}$$

Moreover, homogeneity requires that the magnification factor must be independent of the position in the 3-space so that the ratios of small distance are the same at everrywhere hence the metric must take the form

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + S^2(t)h_{ij}\mathrm{d}x^i\mathrm{d}x^j = -c^2\mathrm{d}t^2 + S^2(t)\mathrm{d}^2\sigma \tag{4.26}$$

where $S(t)$ is a time - dependent scale factor and h_{ij} are function of the coordinate (x^1, x^2, x^3)

4.2.2 The maximally symmetric 3-space

We require the 3-space spanned by the spacelike coordinates to be homogeneous and isotropic and also independent of time. This requires that curvature at any point must be a constant. This lead us to study the maximally symmetric 3-space. A maximally symmetric space is specified by just one number the curvature K, which is independent of the coordinate. such constant curvature space must be homogeneous and isotropic.

The symmetric of the Riemann tensor reveals that there is a unique possibility

$$R_{\alpha\beta\gamma\delta} \propto g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}$$

The simplest expression that satisfies the various symmetry properties and identities of $R_{\alpha\beta\gamma\delta}$ and contains just K is given by

$$R_{\alpha\beta\gamma\delta} = K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \tag{4.27}$$

where $g_{\beta\delta}$ is the metric tensor and K is a function called the Gaussian curvature. The Ricci tensor is given by.

$$\begin{aligned}
R_{\beta\delta} &= g^{\alpha\gamma}K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \\
R_{\beta\delta} &= K(g^{\alpha\gamma}g_{\alpha\gamma}g_{\beta\delta} - g^{\alpha\gamma}g_{\alpha\delta}g_{\beta\gamma}) \\
&= K(\delta^{\gamma}_{\gamma}g_{\beta\delta} - \delta^{\gamma}_{\delta}g_{\beta\gamma}) \\
&= K(3g_{\beta\delta} - g_{\beta\delta}) \qquad \because \delta^{\gamma}_{\gamma} = \delta^{1}_{1} + \delta^{2}_{2} + \delta^{3}_{3} = 1+1+1 = 3 \\
&= 2Kg_{\beta\delta} \qquad (4.28)
\end{aligned}$$

The curvature scalar is thus given by

$$\begin{aligned}
R &= g^{\beta\delta}R_{\beta\delta} = -2Kg^{\beta\delta}g_{\beta\delta} \\
R &= -2K\delta^{\beta}_{\beta} = -6K \qquad (4.29)
\end{aligned}$$

The metric of an isotropic 3-space must depend only on the rotational invariants define by.

$$\mathrm{d}\sigma^2 = B(r)\mathrm{d}r^2 + r^2\mathrm{d}^2\theta + r^2\sin^2\theta\mathrm{d}^2\phi \qquad (4.30)$$

where $B(r)$ is an arbitrary function of r .
From the metric we get

$$g_{\mu\nu} = \begin{pmatrix} B & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad (4.31)$$

and

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{B} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \qquad (4.32)$$

The non zero component are

$$g_{11} = B \qquad g_{22} = r^2 \qquad g_{33} = r^2\sin^2\theta$$
$$g^{11} = \frac{1}{B} \qquad g^{22} = \frac{1}{r^2} \qquad g^{33} = \frac{1}{r^2\sin^2\theta}$$

Recall the connection coefficients.

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_{\nu}g_{\rho\mu} + \partial_{\mu}g_{\rho\nu} - \partial_{\rho}g_{\mu\nu}\right)$$

We show that only non-zero connection coefficients are

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}g^{11}(\partial_1 g_{11}+\partial_1 g_{11}-\partial_1 g_{11}) = \frac{1}{2}\left(\frac{1}{B}\right)\left(\frac{\mathrm{d}B}{\mathrm{d}r}\right) = \frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r}\\
\Gamma^1_{22} &= \frac{1}{2}g^{11}(\partial_2 g_{12}+\partial_2 g_{12}-\partial_1 g_{22}) = \frac{1}{2}\left(\frac{1}{B}\right)\left(-\frac{\mathrm{d}r^2}{\mathrm{d}r}\right) = -\frac{r}{B}\\
\Gamma^1_{33} &= \frac{1}{2}g^{11}(\partial_3 g_{13}+\partial_3 g_{13}-\partial_1 g_{33}) = \frac{1}{2}\left(\frac{1}{B}\right)\left(-\frac{\mathrm{d}r^2\sin^2\theta}{\mathrm{d}r}\right) = -\frac{r\sin^2\theta}{B}\\
\Gamma^2_{33} &= \frac{1}{2}g^{22}(\partial_3 g_{23}+\partial_3 g_{23}-\partial_2 g_{33}) = \frac{1}{2}\left(\frac{1}{r^2}\right)\left(-\frac{\mathrm{d}r^2\sin^2\theta}{\mathrm{d}\theta}\right) = -\sin\theta\cos\theta\\
\Gamma^2_{12} &= \frac{1}{2}g^{22}(\partial_2 g_{21}+\partial_1 g_{22}-\partial_2 g_{12}) = \frac{1}{2}\left(\frac{1}{r^2}\right)\left(\frac{\mathrm{d}r^2}{\mathrm{d}r}\right) = \frac{1}{r}\\
\Gamma^3_{13} &= \frac{1}{2}g^{33}(\partial_3 g_{31}+\partial_1 g_{33}-\partial_3 g_{13}) = \frac{1}{2}\left(\frac{1}{r^2\sin^2\theta}\right)\left(\frac{\mathrm{d}r^2\sin^2\theta}{\mathrm{d}r}\right) = \frac{1}{r}\\
\Gamma^3_{32} &= \frac{1}{2}g^{33}(\partial_2 g_{33}+\partial_3 g_{33}-\partial_3 g_{32}) = \frac{1}{2}\left(\frac{1}{r^2\sin^2\theta}\right)\left(\frac{\mathrm{d}r^2\sin^2\theta}{\mathrm{d}\theta}\right) = \frac{\cos\theta}{\sin\theta} = \cot\theta
\end{aligned}$$

Since the Ricci tensor is given in term of the connection coefficient

$$R_{\mu\nu} = \partial_\sigma\Gamma^\sigma_{\mu\nu} - \partial_\nu\Gamma^\sigma_{\mu\sigma} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}$$

we fined that its non - zero components are

Finding R_{11}

Ricci tensor R_{11}

$$R_{11} = \partial_\sigma\Gamma^\sigma_{11} - \partial_1\Gamma^\sigma_{1\sigma} + \Gamma^\rho_{11}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{1\sigma}\Gamma^\sigma_{\rho 1}$$

The 1^{st} term $\partial_\sigma\Gamma^\sigma_{11}$ is

$$\partial_1\Gamma^1_{11} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r}\right)$$

The 2^{nd} term $-\partial_1\Gamma^\sigma_{1\sigma}$ is

$$\begin{aligned}
-\partial_1\Gamma^1_{11} &= -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2B}\right)\left(\frac{\mathrm{d}B}{\mathrm{d}r}\right) = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r}\right),\\
-\partial_1\Gamma^2_{12} &= -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right) = \frac{1}{r^2},\\
-\partial_1\Gamma^3_{13} &= -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right) = \frac{1}{r^2},
\end{aligned}$$

The 3^{rd} term $\Gamma^{\rho}_{11}\Gamma^{\sigma}_{\rho\sigma}$ is

$$\begin{aligned}
\rho = 1, \qquad & \Gamma^{1}_{11}\Gamma^{\sigma}_{1\sigma}, \\
\sigma = 1, \qquad & \Gamma^{1}_{11}\Gamma^{1}_{11} &=& \left(\frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r}\right)^2, \\
\sigma = 2, \qquad & \Gamma^{1}_{11}\Gamma^{2}_{12} &=& \left(\frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r}\right)\left(\frac{1}{r}\right), \\
\sigma = 3, \qquad & \Gamma^{1}_{11}\Gamma^{3}_{13} &=& \left(\frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r}\right)\left(\frac{1}{r}\right)
\end{aligned}$$

The 4^{th} term $-\Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1}$ is

$$\begin{aligned}
\rho = 1, \qquad -\Gamma^{\rho}_{11}\Gamma^{1}_{\rho 1} &=& -\Gamma^{1}_{11}\Gamma^{1}_{11} = -\left(\frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r}\right)^2, \\
\rho = 2, \qquad -\Gamma^{\rho}_{12}\Gamma^{2}_{\rho 1} &=& -\Gamma^{2}_{12}\Gamma^{2}_{21} = -\frac{1}{r^2}, \\
\rho = 3, \qquad -\Gamma^{\rho}_{13}\Gamma^{3}_{\rho 1} &=& -\Gamma^{3}_{13}\Gamma^{3}_{31} = -\frac{1}{r^2}
\end{aligned}$$

Combining all term therefore,

$$R_{11} = \frac{1}{Br}\frac{\mathrm{d}B}{\mathrm{d}r}. \tag{4.33}$$

Finding R_{22}

Ricci tensor R_{22} is

$$R_{22} = \partial\sigma\Gamma^{\sigma}_{22} - \partial_2\Gamma^{\sigma}_{2\sigma} + \Gamma^{\rho}_{22}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\rho}_{2\sigma}\Gamma^{\sigma}_{\rho 2}.$$

The 1^{st} term $\partial\sigma\Gamma^{\sigma}_{22}$ is

$$\partial_1\Gamma^{1}_{22} = \frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{r}{B}\right) = -\frac{B}{B^2} + \frac{r}{B^2}\frac{\mathrm{d}B}{\mathrm{d}r} = -\frac{1}{B} + \frac{r}{B^2}\frac{\mathrm{d}B}{\mathrm{d}r}.$$

The 2^{nd} term $-\partial_2\Gamma^{\sigma}_{2\sigma}$ is

$$-\partial_2\Gamma^{3}_{23} = -\frac{\mathrm{d}cot\theta}{\mathrm{d}\theta} = -\sec^2\theta.$$

The 3^{rd} term $\Gamma^{\rho}_{22}\Gamma^{\sigma}_{\rho\sigma}$ is

$$
\begin{aligned}
\rho = 1, \qquad & \Gamma^{1}_{22}\Gamma^{\sigma}_{1\sigma} \\
& \Gamma^{1}_{22}\Gamma^{1}_{11} \;=\; \frac{-r}{B}\left(\frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r}\right), \\
& \Gamma^{1}_{22}\Gamma^{2}_{12} \;=\; \frac{-r}{B}\left(\frac{1}{r}\right) = -\frac{1}{B}, \\
& \Gamma^{1}_{22}\Gamma^{3}_{13} \;=\; \frac{-r}{B}\left(\frac{1}{r}\right) = -\frac{1}{B}.
\end{aligned}
$$

The 4^{th} term $-\Gamma^{\rho}_{2\sigma}\Gamma^{\sigma}_{\rho 2}$ is

$$
\begin{aligned}
\sigma = 1, \qquad -\Gamma^{\rho}_{21}\Gamma^{1}_{\rho 2} &= -\Gamma^{2}_{21}\Gamma^{1}_{22} = -\frac{1}{r}\left(-\frac{r}{B}\right) = \frac{1}{B}, \\
\sigma = 2, \qquad -\Gamma^{\rho}_{22}\Gamma^{2}_{\rho 2} &= -\Gamma^{1}_{22}\Gamma^{2}_{12} = -\left(-\frac{r}{B}\right)\left(\frac{1}{r}\right) = \frac{1}{B}, \\
\sigma = 3, \qquad -\Gamma^{\rho}_{23}\Gamma^{3}_{\rho 2} &= -\Gamma^{3}_{23}\Gamma^{3}_{32} = -\cot^2\theta.
\end{aligned}
$$

Combining all term

$$
\begin{aligned}
R_{22} &= 1 - \frac{1}{B} + \frac{r}{B^2}\frac{\mathrm{d}B}{\mathrm{d}r} - \frac{r}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}r}, \\
&= 1 - \frac{1}{B} + \left(1 - \frac{1}{2}\right)\frac{r}{B^2}\frac{\mathrm{d}B}{\mathrm{d}r} = 1 - \frac{1}{B} + \frac{r}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}r}, \\
&= 1 - \frac{1}{B} + \frac{r}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}r}. \qquad (4.34)
\end{aligned}
$$

Finding R_{33}

Ricci tensor R_{33} is

$$R_{33} = \partial_\sigma\Gamma^{\sigma}_{33} - \partial_3\Gamma^{\sigma}_{3\sigma} + \Gamma^{\rho}_{33}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3}$$

The 1^{st} term $\partial_\sigma\Gamma^{\sigma}_{33}$is

$$
\begin{aligned}
\partial_1\Gamma^{1}_{33} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{-r\sin^2\theta}{B}\right) = -\sin^2\theta\left(\frac{1}{B} + \frac{r}{B^2}\frac{\mathrm{d}B}{\mathrm{d}r}\right), \\
\partial_2\Gamma^{2}_{33} &= \frac{\mathrm{d}}{\mathrm{d}\theta}(-\sin\theta\cos\theta) = -\sin\theta(-\sin\theta) - \cos\theta(\cos\theta) = \sin^2\theta - \cos^2\theta.
\end{aligned}
$$

The 2^{nd} term $-\partial_3 \Gamma^{\sigma}_{3\sigma} = 0$
The 3^{rd} term $\Gamma^{\rho}_{33}\Gamma^{\sigma}_{\rho\sigma}$ is

$$
\begin{aligned}
\rho = 1, \qquad \Gamma^{1}_{33}\Gamma^{\sigma}_{1\sigma} &= \Gamma^{1}_{33}\Gamma^{1}_{11} = -\frac{r\sin^2\theta}{B}\left(\frac{1}{2B}\frac{\mathrm{d}B}{\mathrm{d}r}\right), \\
&= \Gamma^{1}_{33}\Gamma^{2}_{12} = -\frac{r\sin^2\theta}{B}\left(\frac{1}{r}\right), \\
&= \Gamma^{1}_{33}\Gamma^{3}_{13} = -\frac{r\sin^2\theta}{B}\left(\frac{1}{r}\right). \\
\rho = 2, \qquad \Gamma^{2}_{33}\Gamma^{\sigma}_{2\sigma} &= \Gamma^{2}_{33}\Gamma^{3}_{23} = (-\sin\theta\cos\theta)\cot\theta.
\end{aligned}
$$

The 4^{th} term $-\Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3}$ is

$$
\begin{aligned}
\sigma = 1, \qquad -\Gamma^{\rho}_{31}\Gamma^{1}_{\rho 3} &= -\Gamma^{3}_{31}\Gamma^{1}_{33} = -\frac{1}{r}\left(\frac{-r\sin^2\theta}{B}\right) = \frac{\sin^2\theta}{B}, \\
\sigma = 2, \qquad -\Gamma^{\rho}_{32}\Gamma^{2}_{\rho 3} &= -\Gamma^{3}_{32}\Gamma^{2}_{33} = -\cot\theta(-\sin\theta\cos\theta) = \cot\theta(\sin\theta\cos\theta), \\
\sigma = 3, \qquad -\Gamma^{\rho}_{33}\Gamma^{3}_{\rho 3} &= -\Gamma^{1}_{33}\Gamma^{3}_{13} = -\left(\frac{-r\sin^2\theta}{B}\right)\left(\frac{1}{r}\right) = \frac{\sin^2\theta}{B}, \\
& \quad -\Gamma^{2}_{33}\Gamma^{3}_{23} = \cot\theta(\sin\theta\cos\theta).
\end{aligned}
$$

Combining together

$$
\begin{aligned}
R_{33} &= -\frac{1}{B}\sin^2\theta - \frac{r}{2B^2}\frac{\mathrm{d}B\theta}{\mathrm{d}r}\sin^2\theta + \frac{r}{B^2}\frac{\mathrm{d}B}{\mathrm{d}r}\sin^2\theta + \sin^2\theta, \\
&= -\frac{1}{B}\sin^2\theta + \sin^2\theta + \frac{1}{2}\frac{r}{B^2}\frac{\mathrm{d}B}{\mathrm{d}r}\sin^2\theta, \\
&= \left(1 - \frac{1}{B} + \frac{r}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}r}\right)\sin^2\theta, \\
&= R_{22}\sin^2\theta. \qquad (4.35)
\end{aligned}
$$

Remembering the Ricci tensor, we must have

$$
R_{\beta\delta} = 2K g_{\beta\delta}, \qquad (4.36)
$$

and

$$\begin{aligned}
R_{11} &= 2Kg_{11}, \\
\frac{1}{Br}\frac{\mathrm{d}B}{\mathrm{d}r} &= 2KB, \\
\frac{1}{B^2}\frac{\mathrm{d}B}{\mathrm{d}r} &= 2Kr, \\
\int \frac{1}{B^2}\mathrm{d}B &= \int 2Kr\mathrm{d}r, \\
-B^{-1} + A &= 2K\frac{r^2}{2} = Kr^2, \\
-\frac{1}{B} &= Kr^2 - A, \\
\frac{1}{B} &= A - Kr^2, \\
B &= \frac{1}{A - Kr^2},
\end{aligned}$$

where A is a constant of integration.

$$\begin{aligned}
R_{22} &= 2Kg_{22}, \\
1 - \frac{1}{B} + \frac{r}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}r} &= 2Kr^2, \\
1 - \frac{1}{B} + \frac{r}{2}\left(\frac{1}{B^2}\frac{\mathrm{d}B}{\mathrm{d}r}\right) &= 2Kr^2, \\
1 - \frac{1}{B} + \frac{r}{2}(2Kr) &= 2Kr^2, \\
1 - \frac{1}{B} + Kr^2 &= 2Kr^2, \\
1 - \frac{1}{B} &= 2Kr^2 - Kr^2 = Kr^2, \\
\frac{1}{B} &= 1 - Kr^2, \\
B &= \frac{1}{1 - Kr^2},
\end{aligned}$$

where

$$B = \frac{1}{A - Kr^2} = \frac{1}{1 - Kr^2}, \tag{4.37}$$

and

$$A = 1.$$

Thus

$$B = \frac{1}{1 - Kr^2}. \qquad (4.38)$$

Finally we have constructed the line element for maximally symmetric 3 - space.

$$\mathrm{d}\sigma^2 = \left(\frac{1}{1 - Kr^2}\right)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}^2\phi. \qquad (4.39)$$

4.2.3 Friedmann–Lemaître - Robertson–Walker metric

Combining for the maximally symmetric 3-space with the line element, we have

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + S^2(t)\left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}^2\phi)\right], \qquad (4.40)$$

assuming that $K \neq 0$. We define the variable $k = \frac{K}{|K|}$ in such the way that $k = \pm 1$ depending on K is positive or negative.

Introduce rescaled coordinate

$$\bar{r} = |K|^{\frac{1}{2}}r, \qquad (4.41)$$

we obtain

$$\begin{aligned}
\mathrm{d}s^2 &= -c^2\mathrm{d}t^2 + S^2(t)\left[\frac{|K|\mathrm{d}r^2}{|K|(1 - k|K|r^2)} + \frac{|K|r^2}{|K|}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}^2\phi)\right], \\
\mathrm{d}s^2 &= -c^2\mathrm{d}t^2 + \frac{S^2(t)}{|K|}\left[\frac{|K|\mathrm{d}r^2}{1 - k|K|r^2} + |K|r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}^2\phi)\right], \\
\mathrm{d}s^2 &= -c^2\mathrm{d}t^2 + \frac{S^2(t)}{|K|}\left[\frac{\mathrm{d}\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}^2\phi)\right].
\end{aligned}$$

Finally we define a rescaled scale function $R(t)$,

$$R(t) = \begin{cases} \frac{S(t)}{|K|^{\frac{1}{2}}}, & K \neq 0, \\ S(t), & K = 0. \end{cases} \qquad (4.42)$$

Dropping the bar on the radial coordinate, we obtain standard from of the FLRW metric line element,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + R^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}^2\phi)\right], \qquad (4.43)$$

where $k = -1, 1, 0$ depending on whether the spatial section has negative, zero or positive curvature.

4.2.4 The Friedmann equation

From the metric we get

$$g_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{R^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & R^2(t)r^2 & 0 \\ 0 & 0 & 0 & R^2(t)r^2\sin^2\theta \end{pmatrix} \tag{4.44}$$

and

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{c^2} & 0 & 0 & 0 \\ 0 & \frac{1-kr^2}{R^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{R^2(t)r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2(t)r^2\sin^2\theta} \end{pmatrix} \tag{4.45}$$

The non-zero elements of $g_{\mu\nu}$ and $g^{\mu\nu}$ are

$$g_{00} = -c^2 \qquad g_{11} = \frac{R^2(t)}{1-kr^2} \qquad g_{22} = R^2(t)r^2 \qquad g_{33} = R^2(t)r^2\sin^2\theta$$
$$g^{00} = -\frac{1}{c^2} \qquad g^{11} = \frac{1-kr^2}{R^2(t)} \qquad g^{22} = \frac{1}{R^2(t)r^2} \qquad g^{33} = \frac{1}{R^2(t)r^2\sin^2\theta}$$

Recall that,

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right).$$

The non - zero coefficients are

$$\begin{aligned}
\Gamma^0_{11} &= \frac{1}{2}g^{00}\left(\partial_1 g_{01} + \partial_1 g_{01} - \partial_0 g_{11}\right) = \frac{1}{2}\left(-\frac{1}{c^2}\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R^2(t)}{1-kr^2}\right)\right) = \frac{R\dot{R}}{c^2(1-kr^2)}, \\
\Gamma^0_{22} &= \frac{1}{2}g^{00}\left(\partial_2 g_{02} + \partial_2 g_{02} - \partial_0 g_{22}\right) = \frac{1}{2}\left(-\frac{1}{c^2}\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\left(R^2(t)r^2\right)\right) = \frac{R\dot{R}r^2}{c^2}, \\
\Gamma^0_{33} &= \frac{1}{2}g^{00}\left(\partial_3 g_{03} + \partial_3 g_{03} - \partial_0 g_{33}\right) = \frac{1}{2}\left(-\frac{1}{c^2}\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\left(R^2(t)r^2\sin^2\theta\right)\right) = \frac{R\dot{R}r^2\sin^2\theta}{c^2}, \\
\Gamma^1_{01} &= \frac{1}{2}g^{11}\left(\partial_1 g_{10} + \partial_0 g_{11} - \partial_1 g_{01}\right) = \frac{1}{2}\left(\frac{1-kr^2}{R^2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R^2}{1-kr^2}\right)\right) = \frac{\dot{R}}{R}, \\
\Gamma^1_{11} &= \frac{1}{2}g^{11}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right) = \frac{1}{2}\left(\frac{1-kr^2}{R^2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R^2}{1-kr^2}\right)\right) = \frac{kr}{1-kr^2},
\end{aligned}$$

$$
\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}g^{11}\left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}\right) = \frac{1}{2}\left(\frac{1-kr^2}{R^2}\right)\left(-\frac{\mathrm{d}}{\mathrm{d}r}\left(R^2r^2\right)\right) = -r(1-kr^2), \\
\Gamma^1_{33} &= \frac{1}{2}g^{11}\left(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33}\right) = \frac{1}{2}\left(\frac{1-kr^2}{R^2}\right)\left(-\frac{\mathrm{d}}{\mathrm{d}r}\left(R^2r^2\sin^2\theta\right)\right) \\
&= -r(1-kr^2)\sin^2\theta, \\
\Gamma^2_{02} &= \frac{1}{2}g^{22}\left(\partial_2 g_{20} + \partial_0 g_{22} - \partial_2 g_{02}\right) = \frac{1}{2}\left(\frac{1}{R^2r^2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(R^2r^2\right)\right) = \frac{\dot{R}}{R}, \\
\Gamma^2_{12} &= \frac{1}{2}g^{22}\left(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{12}\right) = \frac{1}{2}\left(\frac{1}{R^2r^2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(R^2r^2\right)\right) = \frac{1}{r}, \\
\Gamma^2_{33} &= \frac{1}{2}g^{22}\left(\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}\right) = \frac{1}{2}\left(\frac{1}{R^2r^2}\right)\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-R^2r^2\sin^2\theta\right)\right) = -\sin\theta\cos\theta, \\
\Gamma^3_{03} &= \frac{1}{2}g^{33}\left(\partial_0 g_{33} + \partial_3 g_{30} - \partial_3 g_{03}\right) = \frac{1}{2}\left(\frac{1}{R^2r^2\sin^2\theta}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(R^2r^2\sin^2\theta\right)\right) = \frac{\dot{R}}{R}, \\
\Gamma^3_{13} &= \frac{1}{2}g^{33}\left(\partial_1 g_{33} + \partial_3 g_{31} - \partial_3 g_{13}\right) = \frac{1}{2}\left(\frac{1}{R^2r^2\sin^2\theta}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(R^2r^2\sin^2\theta\right)\right) = \frac{1}{r}, \\
\Gamma^3_{23} &= \frac{1}{2}g^{33}\left(\partial_2 g_{33} + \partial_3 g_{32} - \partial_3 g_{23}\right) = \frac{1}{2}\left(\frac{1}{R^2r^2\sin^2\theta}\right)\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\left(R^2r^2\sin^2\theta\right)\right) = \frac{\cos\theta}{\sin\theta} = \cot\theta,
\end{aligned}
$$

since

$$R_{\mu\nu} = \partial_\sigma\Gamma^\sigma_{\mu\nu} - \partial_\nu\Gamma^\sigma_{\mu\sigma} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu},$$

Finding R_{00}

Consider the R_{00} component

$$R_{00} = \partial_\sigma\Gamma^\sigma_{00} - \partial_0\Gamma^\sigma_{0\sigma} + \Gamma^\rho_{00}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{0\sigma}\Gamma^\sigma_{\rho 0}.$$

The 1^{st} term $\partial_\sigma\Gamma^\sigma_{00} = 0$
The 2^{nd} term $-\partial_0\Gamma^\sigma_{0\sigma}$ is

$$
\begin{aligned}
-\partial_0\Gamma^1_{01} &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{R}}{R} = \frac{\dot{R}\dot{R}}{R^2} - \frac{R\ddot{R}}{R^2} = \frac{\dot{R}^2}{R^2} - \frac{\ddot{R}}{R}, \\
-\partial_0\Gamma^2_{02} &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{R}}{R} = \frac{\dot{R}\dot{R}}{R^2} - \frac{R\ddot{R}}{R^2} = \frac{\dot{R}^2}{R^2} - \frac{\ddot{R}}{R}, \\
-\partial_0\Gamma^3_{03} &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{R}}{R} = \frac{\dot{R}\dot{R}}{R^2} - \frac{R\ddot{R}}{R^2} = \frac{\dot{R}^2}{R^2} - \frac{\ddot{R}}{R}.
\end{aligned}
$$

The 3^{rd} term $\Gamma^{\rho}_{00}\Gamma^{\sigma}_{\rho\sigma} = 0$,
The 4^{th} term $-\Gamma^{\rho}_{0\sigma}\Gamma^{\sigma}_{\rho 0}$

$$\begin{aligned}
\sigma = 1, \qquad -\Gamma^{\rho}_{01}\Gamma^{1}_{\rho 0} &= -\Gamma^{1}_{01}\Gamma^{1}_{10} = -\left(\frac{\dot{R}}{R}\right)^2, \\
\sigma = 2, \qquad -\Gamma^{\rho}_{02}\Gamma^{2}_{\rho 0} &= -\Gamma^{2}_{02}\Gamma^{2}_{20} = -\left(\frac{\dot{R}}{R}\right)^2, \\
\sigma = 3, \qquad -\Gamma^{\rho}_{03}\Gamma^{3}_{\rho 0} &= -\Gamma^{3}_{03}\Gamma^{3}_{30} = -\left(\frac{\dot{R}}{R}\right)^2.
\end{aligned}$$

Combining all terms

$$R_{00} = -\frac{3\ddot{R}}{R} + 3\left(\frac{\dot{R}}{R}\right)^2 - 3\left(\frac{\dot{R}}{R}\right)^2 = -\frac{3\ddot{R}}{R} \tag{4.46}$$

Finding R_{11}

$$R_{11} = \partial_\sigma \Gamma^{\sigma}_{11} - \partial_1 \Gamma^{\sigma}_{1\sigma} + \Gamma^{\rho}_{11}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1},$$

the 1^{st} term $\partial_\sigma \Gamma^{\sigma}_{11}$ is

$$\begin{aligned}
\partial_0 \Gamma^{0}_{11} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R\dot{R}}{c^2(1-kr^2)}\right) = \frac{1}{c^2(1-kr^2)}\left(R\ddot{R} - \dot{R}\dot{R}\right), \\
\partial_1 \Gamma^{1}_{11} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{kr}{(1-kr^2)}\right) = \frac{k}{(1-kr^2)} - \frac{2k^2r^2}{(1-kr^2)^2},
\end{aligned}$$

The 2^{nd} term $-\partial_1 \Gamma^{\sigma}_{1\sigma}$ is

$$\begin{aligned}
-\partial_1 \Gamma^{1}_{11} &= -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{kr}{1-kr^2}\right) = \frac{kr(2kr) - (1-kr^2)k}{(1-kr^2)^2} = \frac{2k^2r^2}{(1-kr^2)^2} - \frac{k}{1-kr^2}, \\
-\partial_1 \Gamma^{2}_{12} &= -\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{r} = \frac{1}{r^2}, \\
-\partial_1 \Gamma^{3}_{13} &= -\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{r} = \frac{1}{r^2}.
\end{aligned}$$

The 3^{rd} term $\Gamma^{\rho}_{11}\Gamma^{\sigma}_{\rho\sigma}$ is

$$
\begin{aligned}
&\rho = 0, && \Gamma^{0}_{11}\Gamma^{\sigma}_{0\sigma},\\
&\quad\sigma = 1, && \Gamma^{0}_{11}\Gamma^{1}_{01} = \frac{\dot{R}}{R}\left(\frac{R\dot{R}}{c^2(1-kr^2)}\right) = \frac{\dot{R}\dot{R}}{c^2(1-kr^2)},\\
&\quad\sigma = 2, && \Gamma^{0}_{11}\Gamma^{2}_{02} = \frac{\dot{R}}{R}\left(\frac{R\dot{R}}{c^2(1-kr^2)}\right) = \frac{\dot{R}\dot{R}}{c^2(1-kr^2)},\\
&\quad\sigma = 3, && \Gamma^{0}_{11}\Gamma^{3}_{03} = \frac{\dot{R}}{R}\left(\frac{R\dot{R}}{c^2(1-kr^2)}\right) = \frac{\dot{R}\dot{R}}{c^2(1-kr^2)},\\
&\rho = 1, && \Gamma^{1}_{11}\Gamma^{\sigma}_{1\sigma},\\
&\quad\sigma = 1, && \Gamma^{1}_{11}\Gamma^{1}_{11} = \left(\frac{kr}{(1-kr^2)}\right)^2,\\
&\quad\sigma = 2, && \Gamma^{1}_{11}\Gamma^{2}_{12} = \frac{kr}{(1-kr^2)}\frac{1}{r} = \frac{k}{(1-kr^2)},\\
&\quad\sigma = 3, && \Gamma^{1}_{11}\Gamma^{3}_{13} = \frac{kr}{(1-kr^2)}\frac{1}{r} = \frac{k}{(1-kr^2)}.
\end{aligned}
$$

The 4^{th} term $-\Gamma^{\rho}_{1\sigma}\Gamma^{\sigma}_{\rho 1}$ is

$$
\begin{aligned}
\sigma = 0, \quad -\Gamma^{\rho}_{10}\Gamma^{0}_{\rho 1} &= -\Gamma^{1}_{10}\Gamma^{0}_{11} = -\frac{\dot{R}}{R}\left(\frac{R\dot{R}}{c^2(1-kr^2)}\right) = -\frac{\dot{R}\dot{R}}{c^2(1-kr^2)},\\
\sigma = 1, \quad -\Gamma^{\rho}_{11}\Gamma^{1}_{\rho 1} &= -\Gamma^{0}_{11}\Gamma^{1}_{01} = -\frac{\dot{R}}{R}\left(\frac{R\dot{R}}{c^2(1-kr^2)}\right) = -\frac{\dot{R}\dot{R}}{c^2(1-kr^2)},\\
&= -\Gamma^{1}_{11}\Gamma^{1}_{11} = \left(\frac{-kr}{c^2(1-kr^2)}\right)^2,\\
\sigma = 2, \quad -\Gamma^{\rho}_{12}\Gamma^{2}_{\rho 1} &= -\Gamma^{2}_{12}\Gamma^{2}_{21} = -\frac{1}{r^2},\\
\sigma = 3, \quad -\Gamma^{\rho}_{13}\Gamma^{3}_{\rho 1} &= -\Gamma^{3}_{13}\Gamma^{3}_{31} = -\frac{1}{r^2}.
\end{aligned}
$$

Combining all term,

$$
\begin{aligned}
R_{11} &= \frac{R\ddot{R}}{c^2(1-kr^2)} + \frac{\dot{R}\dot{R}}{c^2(1-kr^2)} + \frac{\dot{R}\dot{R}}{c^2(1-kr^2)} + \frac{2kc^2}{c^2(1-kr^2)},\\
R_{11} &= \frac{(R\ddot{R} + 2\dot{R}\dot{R} + 2c^2k)c^{-2}}{(1-kr^2)}. \qquad (4.47)
\end{aligned}
$$

Finding R_{22}

consider the component of R_{22},

$$R_{22} = \partial_\sigma \Gamma^\sigma_{22} - \partial_2 \Gamma^\sigma_{2\sigma} + \Gamma^\rho_{22}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{2\sigma}\Gamma^\sigma_{\rho 2}.$$

The 1^{st} term $\partial_\sigma \Gamma^\sigma_{22}$ is

$$\begin{aligned}
\partial_0 \Gamma^0_{22} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R\dot{R}r^2}{c^2}\right) = \frac{r^2}{c^2}(R\ddot{R} + \dot{R}\dot{R}), \\
\partial_1 \Gamma^1_{22} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(-r(1-kr^2)\right) = \frac{\mathrm{d}}{\mathrm{d}r}(r - kr^3) = -1 + 3kr^2.
\end{aligned}$$

The 2^{nd} term $-\partial_2 \Gamma^\sigma_{2\sigma}$ is

$$-\partial_2 \Gamma^3_{23} = -\frac{\mathrm{d}}{\mathrm{d}\theta}\cot\theta = -\sec^2\theta.$$

The 3^{rd} term $\Gamma^\rho_{22}\Gamma^\sigma_{\rho\sigma}$ is

$$\begin{aligned}
&\rho = 0, && \Gamma^0_{22}\Gamma^\sigma_{0\sigma}, \\
&\sigma = 1, && \Gamma^0_{22}\Gamma^1_{01} = \left(\frac{R\dot{R}r^2}{c^2}\right)\left(\frac{\dot{R}}{R}\right) = \frac{\dot{R}\dot{R}r^2}{c^2}, \\
&\sigma = 2, && \Gamma^0_{22}\Gamma^2_{02} = \left(\frac{R\dot{R}r^2}{c^2}\right)\left(\frac{\dot{R}}{R}\right) = \frac{\dot{R}\dot{R}r^2}{c^2}, \\
&\sigma = 3, && \Gamma^0_{22}\Gamma^3_{03} = \left(\frac{R\dot{R}r^2}{c^2}\right)\left(\frac{\dot{R}}{R}\right) = \frac{\dot{R}\dot{R}r^2}{c^2}, \\
&\rho = 0, && \Gamma^1_{22}\Gamma^\sigma_{1\sigma} \\
&\sigma = 1 && \Gamma^1_{22}\Gamma^1_{12} = \left(-r(1-kr^2)\right)\left(\frac{kr}{(1-kr^2)}\right) = -kr^2, \\
&\sigma = 2, && \Gamma^1_{22}\Gamma^2_{12} = \left(-r(1-kr^2)\right)\left(\frac{1}{r}\right) = -(1-kr^2), \\
&\sigma = 3, && \Gamma^1_{22}\Gamma^3_{13} = \left(-r(1-kr^2)\right)\left(\frac{1}{r}\right) = -(1-kr^2).
\end{aligned}$$

The 4^{th} term $-\Gamma^{\rho}_{2\sigma}\Gamma^{\sigma}_{\rho 2}$ is

$$\begin{aligned}
\sigma = 0, \qquad -\Gamma^{\rho}_{20}\Gamma^{0}_{\rho 2} &= -\Gamma^{2}_{20}\Gamma^{0}_{22} = -\frac{\dot{R}}{R}\left(\frac{R\dot{R}r^2}{c^2}\right) = \frac{-\dot{R}\dot{R}r^2}{c^2}, \\
\sigma = 1, \qquad -\Gamma^{\rho}_{21}\Gamma^{1}_{\rho 2} &= -\Gamma^{2}_{21}\Gamma^{1}_{22} = -\frac{1}{r}\left(-r(1-kr^2)\right) = (1-kr^2), \\
\sigma = 2, \qquad -\Gamma^{\rho}_{22}\Gamma^{2}_{\rho 2} &= -\Gamma^{0}_{22}\Gamma^{2}_{02} = \frac{-R\dot{R}r^2}{c^2}\left(\frac{\dot{R}}{R}\right) = \frac{-\dot{R}\dot{R}r^2}{c^2}, \\
& \quad -\Gamma^{1}_{22}\Gamma^{2}_{12} = -(1-kr^2)\left(\frac{1}{r}\right) = (1-kr^2), \\
\sigma = 3, \qquad -\Gamma^{\rho}_{23}\Gamma^{3}_{\rho 2} &= -\Gamma^{3}_{23}\Gamma^{3}_{32} = -\cot^2\theta.
\end{aligned}$$

Combining all term,

$$\begin{aligned}
R_{22} &= \frac{R\ddot{R}r^2}{c^2} + \frac{2\dot{R}\dot{R}r^2}{c^2} + 3kr^2 - kr^2, \\
R_{22} &= (R\ddot{R} + 2\dot{R}\dot{R} + 2kc^2)c^{-2}r^2. \qquad (4.48)
\end{aligned}$$

Finding R_{33}

consider the R_{33} component,

$$R_{33} = \partial_\sigma\Gamma^{\sigma}_{33} - \partial_3\Gamma^{\sigma}_{3\sigma} + \Gamma^{\rho}_{33}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3}.$$

The 1^{st} term $\partial_\sigma\Gamma^{\sigma}_{33}$ is

$$\begin{aligned}
\sigma = 1, \qquad \partial_0\Gamma^{0}_{33} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R\dot{R}r^2\sin^2\theta}{c^2}\right) = \frac{r^2\sin^2\theta}{c^2}\left(R\ddot{R} + \dot{R}\dot{R}\right), \\
\sigma = 1, \qquad \partial_1\Gamma^{1}_{33} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(-r(1-kr^2)\sin^2\theta\right) = -(1-3kr^2)\sin^2\theta, \\
\sigma = 2, \qquad \partial_2\Gamma^{2}_{33} &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\sin\theta\cos\theta\right) = \sin^2\theta - \cos^2\theta.
\end{aligned}$$

The 2^{nd} term $-\partial_3\Gamma^{\sigma}_{3\sigma} = 0$.

The 3^{rd} term $\Gamma^{\rho}_{33}\Gamma^{\sigma}_{\rho\sigma}$ is

$$\begin{aligned}
\rho = 0, \qquad \Gamma^0_{33}\Gamma^{\sigma}_{0\sigma} &= \Gamma^0_{33}\Gamma^1_{01} = \left(\frac{R\dot{R}r^2\sin^2\theta}{c^2}\right)\frac{\dot{R}}{R} = \frac{\dot{R}\dot{R}r^2\sin^2\theta}{c^2}, \\
& \Gamma^0_{33}\Gamma^2_{02} = \left(\frac{R\dot{R}r^2\sin^2\theta}{c^2}\right)\frac{\dot{R}}{R} = \frac{\dot{R}\dot{R}r^2\sin^2\theta}{c^2}, \\
& \Gamma^0_{33}\Gamma^3_{03} = \left(\frac{R\dot{R}r^2\sin^2\theta}{c^2}\right)\frac{\dot{R}}{R} = \frac{\dot{R}\dot{R}r^2\sin^2\theta}{c^2}, \\
\rho = 1, \qquad \Gamma^1_{33}\Gamma^{\sigma}_{1\sigma} &= \Gamma^1_{33}\Gamma^1_{11} = -r(1-kr^2)\sin^2\theta\left(\frac{kr}{1-kr^2}\right) = -kr^2\sin^2\theta, \\
& \Gamma^1_{33}\Gamma^2_{12} = -r(1-kr^2)\sin^2\theta\left(\frac{1}{r}\right) = -(1-kr^2)\sin^2\theta, \\
& \Gamma^1_{33}\Gamma^3_{13} = -r(1-kr^2)\sin^2\theta\left(\frac{1}{r}\right) = -(1-kr^2)\sin^2\theta, \\
\rho = 2, \qquad \Gamma^2_{33}\Gamma^{\sigma}_{2\sigma} &= \Gamma^2_{33}\Gamma^3_{23} = -(\sin\theta\cos\theta)\cot\theta = -\frac{\cos\theta}{\sin\theta}\sin\theta\cos\theta = -\cos^2\theta.
\end{aligned}$$

The 4^{th} term $-\Gamma^{\rho}_{3\sigma}\Gamma^{\sigma}_{\rho 3}$ is

$$\begin{aligned}
\sigma = 0, \qquad -\Gamma^{\rho}_{30}\Gamma^0_{\rho 3} &= -\Gamma^3_{30}\Gamma^0_{33} = -\frac{\dot{R}}{R}\left(\frac{R\dot{R}r^2\sin^2\theta}{c^2}\right) = \frac{-\dot{R}\dot{R}r^2\sin^2\theta}{c^2}, \\
\sigma = 1, \qquad -\Gamma^{\rho}_{31}\Gamma^1_{\rho 3} &= -\Gamma^3_{31}\Gamma^1_{33} = -\frac{\dot{R}}{R}\left(\frac{R\dot{R}r^2\sin^2\theta}{c^2}\right) = -\frac{1}{r}(-r(1-kr^2)\sin^2\theta) \\
& = (1-kr^2)\sin^2\theta, \\
\sigma = 2, \qquad -\Gamma^{\rho}_{32}\Gamma^2_{\rho 3} &= -\Gamma^3_{32}\Gamma^2_{33} = (\cos\theta)(-\sin\theta\cos\theta) = \cos^2\theta, \\
\sigma = 3, \qquad -\Gamma^{\rho}_{33}\Gamma^3_{\rho 3} &= -\Gamma^0_{33}\Gamma^3_{03} = -\left(\frac{R\dot{R}r^2\sin^2\theta}{c^2}\right)\left(\frac{\dot{R}}{R}\right) = \frac{-\dot{R}\dot{R}r^2\sin^2\theta}{c^2}, \\
&= -\Gamma^1_{33}\Gamma^3_{13} = (r(1-kr^2)\sin^2\theta)\left(\frac{1}{r}\right) = (1-kr^2)\sin^2\theta, \\
&= -\Gamma^2_{33}\Gamma^3_{23} = (\sin\theta\cos\theta)\cot\theta = \cos^2\theta.
\end{aligned}$$

Combining all terms,

$$\begin{aligned}
R_{33} &= \frac{R\ddot{R}r^2\sin^2\theta}{c^2} + 3kr^2\sin^2\theta + \frac{2\dot{R}\dot{R}r^2\sin^2\theta}{c^2} - kr^2\sin^2\theta, \\
R_{33} &= \frac{R\ddot{R}r^2\sin^2\theta}{c^2} + \frac{2\dot{R}\dot{R}r^2\sin^2\theta}{c^2} + 2kr^2\sin^2\theta, \\
R_{33} &= (R\ddot{R} + 2\dot{R}\dot{R} + 2kc^2)c^{-2}r^2\sin^2\theta. \qquad (4.49)
\end{aligned}$$

Considering the field equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

we contract the field equation with $g^{\mu\nu}$ then

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{8\pi G}{c^4}T_{\mu\nu}, \\
g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R &= \frac{8\pi G}{c^4}g^{\mu\nu}T_{\mu\nu}, \\
R^{\nu}_{\nu} - \frac{1}{2}\delta^{\nu}_{\nu}R &= \frac{8\pi G}{c^4}T^{\mu}_{\nu}, \\
R - \frac{1}{2}(4)R &= \frac{8\pi G}{c^4}T, \\
-R &= \frac{8\pi G}{c^4}T, \\
R &= -\frac{8\pi G}{c^4}T, \qquad (4.50)
\end{aligned}$$

therefore

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(-\frac{8\pi G}{c^4}T) &= \frac{8\pi G}{c^4}T_{\mu\nu}, \\
R_{\mu\nu} &= \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T). \qquad (4.51)
\end{aligned}$$

Recall the perfect fluid energy - momentum tensor,

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu + pg_{\mu\nu}.$$

Contracting the perfect fluid with $g^{\mu\nu}$,

$$\begin{aligned}
g^{\mu\nu}T_{\mu\nu} &= \left(\rho + \frac{p}{c^2}\right)g^{\mu\nu}u_\mu u_\nu + pg^{\mu\nu}g_{\mu\nu}, \\
T &= \left(\rho + \frac{p}{c^2}\right)u^\nu u_\nu + p\delta^\nu_\nu = \left(\rho + \frac{p}{c^2}\right)(-c^2) + 4p.
\end{aligned}$$

The dot product of two 4-velocity is $u^{\nu}u_{\nu} = -c^2$ hence,

$$T = -\rho c^2 + 3p.$$

Therefore we have the perfect fluid in comoving coordinate.
In our comoving coordinate system (t, r, θ, ϕ) the 4-velocity of the fluid is

$$u_{\mu} = (-c^2, 0, 0, 0) \tag{4.52}$$

the perfect fluid energy - momentum tensor components are

$$\begin{aligned}
T_{00} &= \left(\rho + \frac{p}{c^2}\right) u_0 u_0 + p g_{00} = \left(\rho + \frac{p}{c^2}\right)(-c^4) + pc^2, \\
T_{00} &= \rho c^4 - pc^2 + pc^2 = \rho c^4, && (4.53) \\
T_{11} &= \left(\rho + \frac{p}{c^2}\right) u_1 u_1 + p g_{11} = p\left(\frac{R^2}{1 - kr^2}\right), \\
T_{11} &= \frac{R^2 p}{1 - kr^2}, && (4.54) \\
T_{22} &= \left(\rho + \frac{p}{c^2}\right) u_2 u_2 + p g_{22} = p\left(R^2 r^2\right), \\
T_{22} &= pR^2 r^2, && (4.55) \\
T_{33} &= \left(\rho + \frac{p}{c^2}\right) u_2 u_2 + p g_{22} = p\left(R^2 r^2 \sin^2\theta\right), \\
T_{33} &= pR^2 r^2 \sin^2\theta. && (4.56)
\end{aligned}$$

Therefore we find the cosmological field equation with FLRW metric as

$$\begin{aligned}
R_{00} &= \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2} g_{00} T\right), \\
-\frac{3\ddot{R}}{R} &= \frac{8\pi G}{c^4}\left[\rho c^4 - \frac{1}{2}(-c^2)(-\rho c^2 + 3p)\right], \\
-\frac{3\ddot{R}}{R} &= \frac{8\pi G}{c^4}\left(\rho c^4 - \frac{1}{2}\rho c^4 + \frac{3c^2}{2} p\right), \\
-\frac{3\ddot{R}}{R} &= 8\pi G\rho - \frac{8}{2}\pi G\rho + \frac{3}{2c^2} 8\pi G p, \\
-\frac{3\ddot{R}}{R} &= 4\pi G\rho + 4\pi G\frac{3p}{c^2}, \\
-\frac{3\ddot{R}}{R} &= 4\pi G\left(\rho + \frac{3p}{c^2}\right),
\end{aligned}$$

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) R. \tag{4.57}$$

The 11 - component is

$$\begin{aligned}
R_{11} &= \frac{8\pi G}{c^4}\left(T_{11} - \frac{1}{2}g_{11}T\right), \\
\frac{(R\ddot{R} + 2\dot{R}^2 + 2c^2k)c^{-2}}{1-kr^2} &= \frac{8\pi G}{c^4}\left[\frac{R^2p}{1-kr^2} - \frac{1}{2}\left(\frac{R^2}{1-kr^2}\right)(-\rho c^2 + 3p)\right], \\
\frac{R\ddot{R} + 2\dot{R}^2 + 2c^2k}{c^2} &= \frac{8\pi G}{c^4}\left(R^2p + \frac{R^2pc^2}{2} - \frac{3pR^2}{2}\right), \\
R\ddot{R} + 2\dot{R}^2 + 2c^2k &= \frac{8\pi G}{c^2}\left(R^2p + \frac{R^2pc^2}{2} - \frac{3pR^2}{2}\right), \\
R\ddot{R} + 2\dot{R}^2 + 2c^2k &= \frac{8\pi G}{c^2}\left(\frac{-R^2p}{2} + \frac{R^2\rho c^2}{2}\right), \\
R\ddot{R} + 2\dot{R}^2 + 2c^2k &= \frac{8\pi G}{2c^2}(\rho c^2 - p)R^2.
\end{aligned}$$

The 22 - component is

$$\begin{aligned}
R_{22} &= \frac{8\pi G}{c^4}\left(T_{22} - \frac{1}{2}g_{22}T\right), \\
(R\ddot{R} + 2\dot{R}\dot{R} + 2kc^2)c^{-2}r^2 &= \frac{8\pi G}{c^4}\left[pR^2r^2 - \frac{1}{2}\left(R^2(t)r^2\right)(-\rho c^2 + 3p)\right], \\
\frac{R\ddot{R} + 2\dot{R}^2 + 2c^2k}{c^2} &= \frac{8\pi G}{c^4}\left(R^2p + \frac{R^2pc^2}{2} - \frac{3pR^2}{2}\right), \\
R\ddot{R} + 2\dot{R}^2 + 2c^2k &= \frac{8\pi G}{c^2}\left(R^2p + \frac{R^2pc^2}{2} - \frac{3pR^2}{2}\right), \\
R\ddot{R} + 2\dot{R}^2 + 2c^2k &= \frac{8\pi G}{c^2}\left(\frac{-R^2p}{2} + \frac{R^2\rho c^2}{2}\right), \\
R\ddot{R} + 2\dot{R}^2 + 2c^2k &= \frac{8\pi G}{2c^2}(\rho c^2 - p)R^2.
\end{aligned}$$

The 33 - component is

$$
\begin{aligned}
R_{33} &= \frac{8\pi G}{c^4}\left(T_{33} - \frac{1}{2}g_{33}T\right), \\
(R\ddot{R} + 2\dot{R}\dot{R} + 2kc^2)c^{-2}r^2\sin^2\theta &= \frac{8\pi G}{c^4}\left[pR^2r^2\sin^2\theta - \frac{1}{2}\left(R^2(t)r^2\sin^2\theta\right)(-\rho c^2 + 3p)\right], \\
\frac{R\ddot{R} + 2\dot{R}^2 + 2c^2k}{c^2} &= \frac{8\pi G}{c^4}\left(R^2p + \frac{R^2pc^2}{2} - \frac{3pR^2}{2}\right), \\
R\ddot{R} + 2\dot{R}^2 + 2c^2k &= \frac{8\pi G}{c^2}\left(R^2p + \frac{R^2pc^2}{2} - \frac{3pR^2}{2}\right), \\
R\ddot{R} + 2\dot{R}^2 + 2c^2k &= \frac{8\pi G}{c^2}\left(\frac{-R^2p}{2} + \frac{R^2\rho c^2}{2}\right), \\
R\ddot{R} + 2\dot{R}^2 + 2c^2k &= \frac{8\pi G}{2c^2}(\rho c^2 - p)R^2.
\end{aligned}
$$

Therefore, the three equations give the same result,

$$R\ddot{R} + 2\dot{R}^2 + 2c^2k = \frac{4\pi G}{c^2}(\rho c^2 - p)R^2. \tag{4.58}$$

Substituting $\ddot{R}$ gives

$$
\begin{aligned}
R\left[-\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R\right] + 2\dot{R}^2 + 2c^2k &= \frac{4\pi G}{c^2}(\rho c^2 - p)R^2, \\
\frac{-4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R^2 + 2\dot{R}^2 + 2c^2k &= \frac{4\pi G}{c^2}(\rho c^2 - p)R^2, \\
2\dot{R}^2 + 2c^2k &= \frac{4\pi G}{c^2}(\rho c^2 - p)R^2 + \frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R^2, \\
2\dot{R}^2 + 2c^2k &= 4\pi G R^2(\rho - \frac{p}{c^2} + \frac{\rho}{3} + \frac{p}{c^2}), \\
2\dot{R}^2 + 2c^2k &= 4\pi G R^2(\frac{4\rho}{3}), \\
2\dot{R}^2 + 2c^2k &= \frac{16}{3}\pi G\rho R^2, \\
2\dot{R}^2 &= \frac{16}{3}\pi G\rho R^2 - 2c^2k, \\
\dot{R}^2 &= \frac{8}{3}\pi G\rho R^2 - \frac{c^2kR^2}{R^2},
\end{aligned}
$$

$$\frac{\dot{R}^2}{R^2} = \frac{8}{3}\pi G\rho - \frac{c^2k}{R^2}. \tag{4.59}$$

Finally, we have derived the Friedmann equation.

Chapter 5

Conclusion

General relativity is able to explain gravity. GR based on the equivalence principle. Uses concept of curved space. Curved space is indicated by Riemann tensor. This theory attempts to explain gravity with geometry. The curvature of spacetime is directly related to the matter. The relation is specified by the Einstein field equation,

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

An exact solution to Einstein's equation is the Schwarzschild metric. We use rotational invariant or isotropy to derive schwarzschild metric. From these symmetry we got general isotropic metric, derive

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2).$$

To find variable, A and B. we rely on the fact that the Cristoffal Symbol is

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu} \right),$$

and the Ricci tensor is

$$R_{\mu\nu} = \partial_\sigma \Gamma^\sigma_{\mu\nu} - \partial_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\rho\nu}.$$

We get non - zero component of Ricci tensor. Finally we got the Schwarzschild metric.

In cosmology we use isotropic and homogeneity symmetries. From these symmetries we got FLRW metric. The method are the same with Schwarzschild metric, first find the metric component $g_{\mu\nu}$, second use the Cristoffel symbol $\Gamma^\sigma_{\mu\nu}$ to find Ricci tensor $R_{\mu\nu}$ and finally we got the FLRW metric.

Applying these metric to Einstein's field equation. We get the Friedmann equation.

References

[1] C. Kaeonikhom, *Variational Principle Approach to General Relativity*, B.S. (Physics), Independent Study Report, Naresuan University (2006).

[2] M. P. Hobson, G. Efstathiou and A. N. Lasenby, *General Relativity: An Introduction for Physicists*, Cambridge University Press, Cambridge (2007).

[3] R. d'Inverno, *Introducing Einstein's Relativity*, Oxford University Press (1996).

[4] S. M. Carroll, *An Introduction to General Relativity: Spacetime and Geometry*, Addison Wesley (2004).

[5] T. Marsh and E. Stanway, *PX436 General Relativity Lecture Notes*. Retrieved January 5, 2015, from http://www2.warwick.ac.uk /fac/sci/physics/ current/teach/module_home/px436/notes/students.pdf