ALTERNATIVE APPROACH TO SCALAR FIELD COSMOLOGY

CHONTICHA KRITPETCH

A Thesis Submitted to Graduate School of Naresuan University
in Partial Fulfillment of the Requirements
for the Doctor of Philosophy Degree in Theoretical Physics
July 2020

Thesis entitled “Alternative Approach to Scalar Field Cosmology”
by Chonticha Kritpetch
has been approved by the Graduate School as partial fulfillment of the requirements for the Doctor of Philosophy in Theoretical Physics of Naresuan University.

Oral Defense Committee

. . . . Chair
(Assistant Professor Seckson Sukhasena, Ph.D.)

. . . . Advisor
(Associate Professor Burin Gumjudpai, Ph.D.)

. . . . Co-Advisor
(Assistant Professor Pichet Vanichchapongjaroen, Ph.D.)

. . . . Examiner
(Nandan Roy, Ph.D.)

. . . . External Examiner
(Narakorn Kaewkhao, Ph.D.)

Approved

. . . .
(Professor Paisarn Muneesawang, Ph.D.)
Dean of the Graduate School

ACKNOWLEDGMENTS

I would like to take this opportunity to thank all those who have contributed in any way.

Firstly, I would like to express the deepest appreciation to my advisor Assoc. Prof. Burin Gumjudpai for being my inspiration in Theoretical Physics and valuing me anyway. He has been supporting my Ph.D study and related research, being patient and never lacking hope in me continuously. Moreover I would like to thank him and Prof. David Wands for the opportunity of visiting Institute of Cosmology and Gravitation, University of Portsmouth covered by Newton Advanced Fellowships, the Royal Society. I would deeply appreciate that.

I would like to express my gratitude to my co-advisor Assist. Prof. Pichet Vanichchapongjaroen for spending so much time to teach me Physics, Mathematics and what are needed in my journey of research. I also appreciate Dr. Mathew James Lake and Dr. Sheng-Lan Ko for their advice, training and encouragement.

I am so thankful to Assist. Prof. Seckson Sukhasena and Assist. Prof. Suchittra Sa-nguansin for supporting my study life during my Ph.D. I also thank all lecturers and staff in the institute for providing me knowledge and help.

I would like to thank School of Science, University of Phayao for study leave permission and financial support. Moreover, I am so grateful for physical and mental support from my family, my handsome father, my beautiful mother, Pim, Ring, Noon and Tuck. Thanks for warmth from their eyes, love from their smiles and security from their embraces. I also thank my spiritual family, brothers and sisters in Chist's vision church who always have me in their prayer, especially Aj. Chulalai's family.

Finally, I really thank God who created all things. He is always with me in hardship and happiness.

Chonticha Kritpetch

Title	ALTERNATIVE APPROACH TO SCALAR FIELD COSMOLOGY
Author	Chonticha Kritpetch
Advisor	Associate Professor Burin Gumjudpai, Ph.D.
Co-Advisor	Assistant Professor Pichet Vanichchapongjaroen, Ph.D.
Academic Paper	Thesis Ph.D. in Theoretical Physics, Naresuan University, 2019
Keywords	NLS, scalar field, NMDC, holographic

ABSTRACT

This thesis considers alternative approach to accelerating-expansion standard general relativistic cosmology in formal way and in theoretically modified way. The first one is to express the standard cosmology in form of non-linear Schrödinger-type (NLS) equation. The scenario is the non-zero curvature FLRW universe with canonical scalar field and two barotropic fluids. Hope in exploration of the NLS formulation is in searching for quantum cosmological formulation of the universe. Friedmann cosmological variables are derived in terms of NLS variables. Earlier seven exact scale factor solutions found by D'Ambroise in 2010 are analysed and a new solution is found here. We explored their cosmological validity and found that these solutions disagree with observations. The wave functions are non-normalizable and the total energy is negative. The solutions in D'Ambroise are solved under assumed NLS potential. Although Gumjudpai in 2008, showed that power-law, de-Sitter and super-acceleration expansions render non-normalizable wave functions. Observationally agreed solutions are not completely ruled out since the scale factor solution might come in form of quasi-de Sitter at present time with complicated form or has to be treated as an infinite series. There might exist some yet-to-know NLS potentials hence valid normalizable wave functions are not completely ruled out. The second approach is to modify standard canonical scalar field theory with

non-minimal derivative coupling to Einstein tensor term. The idea is allowed as a class of Horndeski theory which is the most generalized second order derivative in the metric tensor and scalar fields. The NMDC with positive coupling is viable in some conditions to give acceleration. The low-energy cut-off scale is imposed by incorporating of holographic idea of which dark energy density is bound by the cutoff scale.

LIST OF CONTENTS

LIST OF CONTENTS (CONT.)

Chapter **Page**

LIST OF TABLES

LIST OF FIGURES

Figure **Page**

CHAPTER I

INTRODUCTION

1.1 Background and Motivation

At present, observational data from cosmic microwave background [2, 3, 4, 5, 6], large scale structures [7, 8] and supernovae type Ia [9, 10, 11, 12, 13, 14, 15, 16] affirm that the universe is expanding accelerately [9, 10, 11, 17, 18]. It has been hypothesized that dynamical scalar field with time-dependent equation of state coefficient $w_\phi < -1/3$ called dark energy can be a source of repulsive gravity driving accelerating universe. Dark energy is considered as cosmological fluid type appearing in the matter term of the Einstein equation derived from FLRW cosmology. On the other hand at early time, the universe is needed to be in acceleration phase, the inflation [19, 20, 21], so that the horizon and flatness problems can be solved. The driving force of the early-universe inflationary phase are generated from the transforming of scalar potential energy to its kinetic energy. Scalar field is also believed to be the matter causing the inflation. Hence studying cosmology of scalar field could be interpreted as the situation in either (or both) early and late time. Observational constraints in the early time are the CMB data, e.g. power spectrum index and tensor-to-scalar ratio [22].

Hence, we have two eras of considerations of how we use scalar field in describing the cosmic dynamics. At early time, the scalar field is known as inflaton for the fluid that drives the early time acceleration. At late time, the scalar field is known as dark energy. Dark energy models are, for instance, quintessence (canonical scalar field), k-essence, cosmological constant and so forth [23]. We mainly consider the situation at late time. The quintessence field has scaling solution of which the scalar density can mimic the barotropic density hence solving the coincident problem.

The present universe also possesses the dark matter problem saying that there is exceeding amount of gravitational attraction to the total baryonic matter content of the universe. This has many hypothesized answers. One of this is that there is a large amount non-baryonic matter density at about 25% of the total matter-energy density. Whatever the dark matter is, we know that it must be non-relativistic, i.e. it is a dust fluid with $w = 0$. This is awaiting for explanation.

In the last decade, there has been an alternative mathematical approach to formulate the canonical scalar field FLRW cosmology. This alternative formulation is in the form of non-linear Schrödinger equation hence it is dubbed non-linear Schrödinger (NLS) formalism. The approach is invented from the Ermakov-Milne-Pinney (EMP) equation. The Ermakov system [24, 25], a pair of non-linear second-order ordinary differential equations, was noticed to have a connection to standard FLRW cosmology sourced by a barotropic perfect fluid and a self-interacting canonical scalar field minimally coupled to gravity, providing alternative analytical approach to the cosmological system [26]. This approach is hoped to provide quantum cosmological version of an quintessential cosmology which is in the regime of classical general relativistic cosmology and is at late time.

In order to consider the same scalar field as a single field to drive both inflation in the early universe and the present acceleration, we need a single picture of dynamical evolution of the universe from primordial time to present. Moreover it should also resolve dark matter problem. There are many proposals in order to unify the picture, i.e. dark energy-dark matter interaction, scalar-tensor theories and ideas of modification in geometrical sector of the action of the modified gravity [27]. One idea is to include a non-minimal derivative coupling (NMDC) to Einstein tensor into the Lagrangian. The NMDC term of this type is a sub-class of Horndeski theory [28, 29, 30] which is the most generalized second order derivative in the metric tensor and scalar fields. The NMDC with positive coupling is viable in some conditions to give acceleration phase and could be a candidate in describing dark energy and dark matter with some classes of potential [31, 32, 33, 34, 35, 36].

In this thesis we present our study on the NLS formulation of the quintessence and on the NMDC gravity with some proposed infrared cut-off energy scale. We hope to see cosmologically interesting results that may be of new description of contemporary cosmology.

1.2 Objectives

1.2.1 To derive NLS formulation of canonical scalar field with two barotropic fluids

1.2.2 To comment and check validity of the NLS solutions reported earlier in [1]

1.2.3 To review a status of NMDC gravity and its effects with holographic cutoff energy density

1.3 Frameworks

1.3.1 FLRW cosmology

1.3.2 Background evolution of the universe

1.3.3 Late universe

CHAPTER II

STANDARD COSMOLOGY

Here in this chapter, we give some necessary ideas of the standard Friedmann cosmology for further referencing in this thesis.

2.1 Standard Big Bang Theory

The standard Big Bang theory has been widely accepted to explain the origin of universe. However, there have been some puzzles which can not be answered by the theory: flatness problem, horizon problem, magnetic monopoles problem and structure problem. For this reason, the hypothesis of inflation is proposed to perfect the Big Bang theory by, e.g., A. A. Starobinsky (1979) [37], D. Kazanas (1980) [38], A. Guth (1981) [39] and K. Sato (1981) [40]. Inflation theory describes a duration when the universe was extremely rapidly expanding in early time. This phenomena occurs under $\ddot{a} > 0$ which corresponds to $w < -1/3$. The w providing a negative pressure relates to a scalar field, for example, a cosmological constant Λ whose pressure is $p_\Lambda = -\rho c^2$.

2.1.1 Standard hot Big Bang cosmology

In the early 20th century, astronomical observations After proposal of General relativity theory by Albert Einstein in 1915, Einstein field equation solved by Friedmann in 1915 expresses that the universe is not static. It can either expand or collapse. In 1929, Hubble discovered that there are not only Milky Way galaxy but also many galaxies in our universe. The observational data of galaxy specta in long distance have redshift. These evidences demonstrate that the further distant, the more redshifted. For this reason, Hubble proposed the empirical law to relate between velocity v and distance r of considered galaxies called Hubble's law

$$v = Hr, \tag{2.1}$$

where H is the Hubble parameter. The law indicates that the further galaxies will move away from us faster. In 1946, the idea that early universe should be extremely hot and dense was proposed by Gamow [41]. After it was predicted that the universe should be filled by microwave radiation with black-body spectrum, these ideas demonstrate that the universe originates from being very dense and hot. Now it is expanding and progressively cooling down. This idea has been known as the hot Big Bang theory.

Contemporary cosmological point of view bases on Copernican principle. The principle states that the Earth is not a center of the universe and there is no privilege location in the universe. This concept leads to the ideas of homogeneous and isotropic universe in the cosmological principle. In fact our universe is not homogeneous and isotropic in a small scale. Otherwise, structures such as star, galaxies, even human can not be generated. Approximately, the universe is considered to be homogeneous and isotropic at large scale. Smoothness of CMB data from COBE mission in 1995 supports this idea [42].

2.1.2 Equations of motion

In 1915, Albert Einstein introduced a set of 10 independent equations describing influence of mass and energy on a spacetime curvature. This proposal solves problems in Newtonian concept. Those are the divergence of Newtonian gravitational potential at infinite distance and the static property of this model which causes the universe collapses when setting gravitational field to be zero at some point in the universe but non-zero elsewhere. The Einstein field equations which originally described a non-expanding or non-contracting universe were given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.2}$$

where Greek letters $(\mu, \nu, ...)$ run over $0, 1, 2, 3,$. The Einstein tensor $G_{\mu\nu}$ represents a spacetime curvature, the energy-momentum tensor $T_{\mu\nu}$ identifies the matter content of the universe. Subsequent to Hubble's observation in 1920s, a modern cosmology and several observations have been investigated that the universe is acceleratedly expanding. Therefore the term with cosmological constant Λ in Eq.(2.2) needs to be reconsidered.

Then the Einstein field equations can be represented by

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.3}$$

Relation between the Einstien tensor and spacetime curvature is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.4}$$

For this reason, the equation describing a relation between spacetime curvature and enegy-momentum tensor of a barotropic fluid is given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.5}$$

Consider LHS of Eq.(2.5), Ricci tensor,

$$R_{\mu\nu} \equiv \partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\nu \Gamma^\lambda{}_{\mu\lambda} + \Gamma^\lambda{}_{\lambda\rho} \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\rho}, \tag{2.6}$$

and the Ricci scalar,

$$R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}, \tag{2.7}$$

need to be identified by introducing Christoffel symbol ($\Gamma^\rho{}_{\mu\nu}$). For the large-scale universe, the cosmological principle states that the universe is homogeneous and isotropic. Considering this statement with an expanding universe leads us to the Friedmann-Lemaitre-Roberson-Walker (FLRW) metric in term of,

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu & (2.8) \\ &= -c^2 \mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2 \right), & (2.9) \end{aligned}$$

where $a(t)$ is a scale factor, k identifies a spacetime curvature which $k = 0$ for a flat universe, $k = 1$ for a closed universe, $k = -1$ for an open universe, $x^0 \equiv ct$,
$x^1 \equiv r, x^2 \equiv \theta$, and $x^3 \equiv \phi$.

Non-vanishing elements of FLRW metric are

$$g_{00}=-1,\qquad g_{11}=\frac{a^2}{1-kr^2},\qquad g_{22}=a^2r^2,\qquad g_{33}=a^2r^2\sin^2\phi. \tag{2.10}$$

Their inverse elements are

$$g^{00}=-1,\qquad g^{11}=\frac{1-kr^2}{a^2},\qquad g^{22}=\frac{1}{a^2r^2},\qquad g^{33}=\frac{1}{a^2r^2\sin^2\phi}. \tag{2.11}$$

In order to determine spacetime curvature, Christoffel symbols need to be evaluated by using the formula

$$\Gamma^{\rho}{}_{\mu\nu}=\frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\mu\sigma}-\partial_\sigma g_{\mu\nu}\right). \tag{2.12}$$

All non-zero components of Christoffel symbols are

$$\Gamma^{0}{}_{ij} = \frac{a\dot{a}}{c}g_{ij}, \tag{2.13}$$

$$\Gamma^{i}{}_{0j} = \frac{\dot{a}}{ac}\delta^{i}{}_{j}, \tag{2.14}$$

$$\Gamma^{i}{}_{jk} = \frac{1}{2}g^{i\ell}(\partial_j g_{k\ell}+\partial_k g_{j\ell}-\partial_\ell g_{jk}), \tag{2.15}$$

where g_{ij} is an element in the metric, Latin letters $(i, j, k, \ell, ...)$ run over 1,2,3, $\delta^{i}{}_{j}=1$ when $i=j$ and $\delta^{i}{}_{j}=0$ when $i\neq j$. Substituting Christoffel symbols into the Eq.(2.6) produces

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a} \tag{2.16}$$

$$R_{ij} = \frac{1}{c^2}\left[\frac{\ddot{a}}{a}+2\left(\frac{\dot{a}}{a}\right)^2+2\frac{k}{a^2}\right]g_{ij}. \tag{2.17}$$

A curvature scalar called Ricci scalar which is evaluated by taking trace to the Ricci tensor as stated in the Eq.(2.7) performs

$$R = g^{00}R_{00}+g^{ij}R_{ij} \tag{2.18}$$

$$= 6\left[\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}\right]. \tag{2.19}$$

According to general relativity considers all types of matter which are interesting as perfect fluid, the energy-momentum tensor $T_{\mu\nu}$ can defined by

$$T_{\mu\nu}=\left(\rho+\frac{p}{c^2}\right)u_\mu u_\nu+pg_{\mu\nu}, \tag{2.20}$$

or

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p g^{\mu\nu}, \tag{2.21}$$

where $u^\mu = (c, 0, 0, 0)$ is a four-velocity, ρ is energy density, and p is pressure of the fluid.

Non-zero components of $T_{\mu\nu}$ are

$$T_{00} = \rho c^2, \tag{2.22}$$

$$T_{ij} = p g_{ij}. \tag{2.23}$$

To rewrite the Ricci tensor related to the energy-momentum tensor and its trace, we contract inverse metric $g^{\mu\nu}$ to the Eq.(2.5) which yields

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu}, \tag{2.24}$$

$$R^\mu{}_\mu - \frac{1}{2} R \delta^\mu{}_\mu = -\frac{8\pi G}{c^4} T^\mu{}_\mu, \tag{2.25}$$

$$R = -\frac{8\pi G}{c^4} T, \tag{2.26}$$

where R and T are traces of the Ricci tensor and the energy-momentum tensor respectively. The value of T is

$$T = g^{\mu\nu} T_{\mu\nu} \tag{2.27}$$

$$= -\rho c^2 + 3p. \tag{2.28}$$

The Eq.(2.5) is adopted by applying the Eq.(2.26). Therefore the Ricci tensor can be written in the form of

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(-\frac{8\pi G}{c^4} T\right) = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.29}$$

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right). \tag{2.30}$$

Consider the Eq.(2.30), the non-zero components provide results:

$$
\begin{aligned}
R_{00} &= \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2}g_{00}T\right), && (2.31)\\
-\frac{3}{c^2}\frac{\ddot{a}}{a} &= \frac{8\pi G}{c^4}\left[\rho c^2 + \frac{1}{2}(-\rho c^2 + 3p)\right], && (2.32)\\
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{c^2}\left(\rho + 3\frac{p}{c^2}\right), && (2.33)
\end{aligned}
$$

$$
\begin{aligned}
R_{11} &= \frac{8\pi G}{c^4}\left(T_{11} - \frac{1}{2}g_{11}T\right), && (2.34)\\
\frac{a\ddot{a} + 2\dot{a} + 2kc^2}{(1-kr^2)c^2} &= \frac{8\pi G}{c^4}\left[\left(\frac{a^2}{1-kr^2}\right)p - \frac{1}{2}\left(\frac{a^2}{1-kr^2}\right)(-\rho c^2 + 3p)\right], && (2.35)\\
\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{kc^2}{a^2} &= -\frac{4\pi G}{3}\left(p - \rho c^2\right), && (2.36)
\end{aligned}
$$

$$
\begin{aligned}
R_{22} &= \frac{8\pi G}{c^4}\left(T_{22} - \frac{1}{2}g_{22}T\right), && (2.37)\\
\frac{a\ddot{a} + 2\dot{a} + 2kc^2}{c^2}r^2 &= \frac{8\pi G}{c^4}\left[a^2r^2p - \frac{1}{2}a^2r^2(-\rho c^2 + 3p)\right], && (2.38)\\
\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{kc^2}{a^2} &= -\frac{4\pi G}{3}\left(p - \rho c^2\right), && (2.39)
\end{aligned}
$$

$$
\begin{aligned}
R_{33} &= \frac{8\pi G}{c^4}\left(T_{33} - \frac{1}{2}g_{33}T\right), && (2.40)\\
\frac{a\ddot{a} + 2\dot{a} + 2kc^2}{c^2}r^2\sin^2\theta &= \frac{8\pi G}{c^4}\left[a^2r^2\sin^2\theta p - \frac{1}{2}a^2r^2\sin^2\theta(-\rho c^2 + 3p)\right], && (2.41)\\
\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{kc^2}{a^2} &= -\frac{4\pi G}{3}\left(p - \rho c^2\right). && (2.42)
\end{aligned}
$$

As the results, it is noticed that Eq.(2.31) and (2.34) are non-linear second-order differential equations. To simplify the equations, we substitute Eq.(2.31) called the acceleration equation in Eq.(2.34) yields

$$
\begin{aligned}
-\frac{4\pi G}{c^2}\left(\rho + 3\frac{p}{c^2}\right) + 2\frac{\dot{a}^2}{a^2} + 2\frac{kc^2}{a^2} &= -\frac{4\pi G}{3}\left(p - \rho c^2\right), && (2.43)\\
\frac{\dot{a}^2}{a^2} &= \frac{\kappa^2}{3}\rho - \frac{kc^2}{a^2}. && (2.44)
\end{aligned}
$$

where $\kappa^2 = 8\pi G$. Then introduce Hubble parameter $H(t) = \dot{a}/a$, hence we have Friedmann equation

$$H^2 = \frac{\kappa^2}{3}\rho - \frac{kc^2}{a^2}. \tag{2.45}$$

We can include cosmological constant term in the Einstein field equation which is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{2.46}$$

This is a modification of initial condition which does not violate the energy conservation law. Therefore the Friedmann equation becomes

$$H^2 = \frac{\kappa^2}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}. \tag{2.47}$$

The energy density of the cosmological constant is given by

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G}. \tag{2.48}$$

The Friedmann and acceleration equations are hence

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{kc^2}{a^2}, \tag{2.49}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2} + \frac{\Lambda c^2}{3}\right), \tag{2.50}$$

respectively. According to consideration of objects in universe as fluids, they must obey the conservation law which is

$$\nabla_\nu T^{\mu\nu} = 0, \tag{2.51}$$

where ∇_ν is a covariant derivative. Substituting Eq.(2.21) in Eq.(2.52) performs

$$\nabla_\nu \left[\left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p g^{\mu\nu}\right] = 0. \tag{2.52}$$

Apply the standard form for the covariant derivative of the vector components

$$\nabla_\nu v^\mu = \partial_\nu v^\mu + \Gamma^\mu{}_{\nu\rho} v^\rho, \tag{2.53}$$

the vector $u^\mu = (c, 0, 0, 0)$ and $\partial_0 = 1/(c\partial t)$ to Eq.(2.52), calculation of the first term of LHS is

$$\nabla_\nu \left[\left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu\right] = \partial_\nu \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu \tag{2.54}$$

$$+ \left(\rho + \frac{p}{c^2}\right) \cancel{(\partial_\nu u^\mu)}^{0} u^\nu + \cancel{\Gamma^\mu_{\nu\sigma} v^\sigma}^{0} \tag{2.55}$$

$$+ \left(\rho + \frac{p}{c^2}\right) u^\mu \cancel{(\partial_\nu u^\nu)}^{0} + \Gamma^\nu_{\nu\sigma} v^\sigma, \tag{2.56}$$

$$= \dot{\rho} c + \frac{\dot{p}}{c} + 3\frac{\dot{a}}{a}. \tag{2.57}$$

The other term is evaluated by

$$\nabla_\nu (p g^{\mu\nu}) = p \cancel{\nabla_\nu g^{\mu\nu}}^{0} + g^{\mu\nu} \nabla_\nu p \tag{2.58}$$

$$= -\frac{\dot{p}}{c}. \tag{2.59}$$

The result of Eq.(2.52) is hence

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0 \tag{2.60}$$

which is dubbed a fluid equation which expresses energy conservation. The fluid equation can be determined from the Friedmann equation and the acceleration equation which means that the fluid equation depends on the Einstein field equation. In this case, the fluid is assumed to be a perfect barotropic fluid, its equation of state is in the form of

$$p = \rho c^2 w. \tag{2.61}$$

The Eq.(2.60) can be written as

$$\dot{\rho} + 3H\rho(1 + w) = 0, \tag{2.62}$$

and also be written in term of product rule as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho a^{3(1+w)}\right] = 0. \tag{2.63}$$

which provides the solution

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}. \tag{2.64}$$

Consider the universe as a perfect fluid. it contains a variety of ingredients which is classified into four species.

- Non-relativistic particles called matter or dust have $w = 0$.
- Relativistic particles called radiation have $w = 1/3$.
- Scalar field driving inflation in the early time called inflaton.
- Energy density driving accelerating expansion called dark energy.

Both inflaton and dark energy generally have $w < -1/3$. In the case of the vacuum energy, the cosmological constant has $w = -1$. From Eq.(2.45), another form of Friedmann equation can be written as

$$-k = a^2 H^2 \left(1 - \frac{8\pi G}{3H^2}\rho\right), \tag{2.65}$$

where ρ is the total energy density of the the universe after inflation which is

$$\rho{=}\rho_{\text{rad}} + \rho_{\text{mat}} + \rho_{\text{d.e.}}. \tag{2.66}$$

Consider flat universe, $k = 0$, we have

$$\rho = \frac{3H^2}{8\pi G} \equiv \rho_c. \tag{2.67}$$

For the closed universe,$k > 0$, the expansion will be suspended at some point before re-collapse. The other case, open universe $k < 0$, the universe will expand forever. In the very far future, dark energy in some from will dominate because matter and radiation will be diluted away. So the universe will be in acceleration phase eternally. Introducing total density parameter Ω, the Friedmann equation can be written by

$$1 - \Omega = -\frac{k}{a^2 H^2}, \tag{2.68}$$

where $\Omega \equiv \rho/\rho_c = 8\pi G\rho/3H^2$. The Eq.(2.68) demonstrates the relation between Ω and geometry of the universe: $\Omega = 1$ corresponds to $k = 0$, $\Omega > 1$ corresponds to $k = -1$, and $\Omega < 1$ corresponds to $k = 1$.

2.1.3 Puzzles in the hot Big Bang model Even though the Big Bang theory can describe several observations e.g. redshift, the existence of CMB and abundance of primordial nuclei, there have been some problems which the Big Bang can not explain.

- **Flatness problem**

In the radiation-dominated era and dust-dominated era, the Eq.(2.68) provides that

$$|1 - \Omega(t)| \propto a^2 \propto t, \tag{2.69}$$

$$|1 - \Omega(t)| \propto a^2 \propto t^{2/3}, \tag{2.70}$$

respectively. These imply time-evolution of the density parameter. If value of Ω is not exactly 1, the scale factor will increase. This means the universe is expanding. Due to the current observation showing that $|1 - \Omega_0| \sim 0.02$, we can evaluate that $|1 - \Omega| \sim 10^{-16}$ at nucleosynthesis and $\sim 10^{-27}$ at the electroweak scale. Those imply that the initial value of Ω should extremely close to 1, otherwise the universe would either re-collapse or expand vary rapidly. If then the present universe will be different than we are used to. The perfectly flat universe at the beginning can not be explained by the hot Big Bang theory.

- **Horizon problem**

It is assumed that the universe is approximately isotropic from Cosmic Microwave Background (CMB) observational data. This means the universe is thermalized. According to the Big Bang theory, the CMB was emitted after recombination when the universe was about 300,000 years old. Distance between that point to us is around 300,000 light year which is about one degree in the sky. For this reason, the thermal equilibrium will happen in one degree of observation.

- **Magnetic monopole problem**

In the early universe, it is predicted that there should be several kinds of exotic particles generated. Spontaneous symmetry breaking can cause plenty of magnetic monopoles and higher dimensional substances creation. However, many of them can not be detected at present. The Big Bang theory can not answer this question.

- **Origin of structure problem**

In 1992, COBE satellite observed the CMB anisotropies which shows how to form structures in the universe. This observation is still a mystery for the Big Bang theory. The data demonstrate the correspondence of CMB anisotropy and irregularities at last scattering surface. The irregular structure magnitude is equal to or larger than the particle horizon at decoupling era. This concept cannot provide explanation for this phenomenon.

2.2 Inflationary cosmology

In 1980s, it was introduced an accelerating phase of the universe by Guth, Srarobinsky, Sato, Albrecht, Steinhardt and Linde. This period called inflation corresponds to

$$\ddot{a} > 0. \tag{2.71}$$

This shows that $\dot{a}$ is increasing during the inflation phase which implies that the comoving Hubble length must be decreasing as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0. \tag{2.72}$$

In order to satisfy Eq.(2.71),

$$\rho c^2 + 3p < 0 \implies p < -\frac{\rho c^2}{3}. \tag{2.73}$$

As the result, the problems mentioned before can be solved by inflation. However inflation does not overthrow the hot Big Bang model, but perfects this model.

2.2.1 Solving the hot Big Bang problems

- **Flatness problem**

From Eq.(2.68) it is showed that if $1 - \Omega$ is approaching to zero, the aH must be increasing. This increase is influenced by the acceleration which is inflation. The reason why current density parameter is very close to 1 can be answered by this idea.

- **Horizon problem**

Before inflation, the universe had a very small size and was thermalised. During inflation, the universe grew extremely fast but still in thermal equilibrium.

- **Magnetic monopole problem**

The inflaton is dominant during inflation and its energy density is almost constant. In this period, its equation of state coefficient w approaches to -1. Under extremely fast expansion in inflationary phase, If there are other ingredients with $w > -1/3$, their densities decrease very fast because they are diluted away. This is the reason why they can not be detected today.

- **Structure problem**

During inflation, quantum fluctuations were stretching causing ripples in CMB temperature. At the end of inflation, the scalar fields form to be particles in standard model.

2.3 Scalar Field Cosmology with Barotropic fluids

In the previous section, it is roughly mentioned that the universe is considered to be filled by perfect fluids. Density and pressure of fluid are given by

$$\rho = \frac{D}{a^n}, \tag{2.74}$$
$$p = \left(\frac{n-3}{3}\right)\frac{D}{a^n}, \tag{2.75}$$

where D is a proportional constant. The equation of state coefficient (w) is defined by

$$w = \frac{n-3}{3}. \tag{2.76}$$

The value n identifies fluid species: $n = 3$ for dust, $n = 4$ for radiation, $n = 6$ for stiff fluid. To be more general, we consider FLRW universe containing a scalar field and two barotropic fluids. The energy density in Eq.(2.74) and pressure in Eq.(2.75) can be expanded in terms of two fluids as

$$\rho_1 = \frac{D_1}{a^n}, \qquad \rho_2 = \frac{D_2}{a^m}, \tag{2.77}$$
$$p_1 = \frac{(n-3)}{3}\frac{D_1}{a^n}, \qquad p_2 = \frac{(m-3)}{3}\frac{D_2}{a^m}, \tag{2.78}$$

where values of n and m identify types of fluid, 1 denotes a major fluid and 2 denotes a minor fluid, D_1 and D_2 denote proportional constant of major fluid and minor respectively. Whereas the density and pressure of scalar field are given by

$$\rho_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi), \tag{2.79}$$
$$p_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi), \tag{2.80}$$

where $\epsilon = 1$ for non-phantom case, $\epsilon = -1$ for phantom case. Because of the equation of state, the scalar field equation of state coefficient is hence

$$w_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi). \tag{2.81}$$

The scalar field satisfies by conservation equation

$$\epsilon\left(\ddot{\phi} + 3H\dot{\phi}\right) = -\frac{\mathrm{d}V}{\mathrm{d}\phi}. \tag{2.82}$$

Therefore the Friedmann and acceleration equations governing the scalar field with two fluids take the forms of

$$\begin{aligned} H^2 &= \frac{\kappa^2}{3}\rho_{\text{tot}} - \frac{k}{a^2} && (2.83)\\ &= \frac{\kappa^2}{3}\left(\frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi) + \frac{D_1}{a^n} + \frac{D_2}{a^m}\right) - \frac{k}{a^2}, && (2.84)\\ \frac{\ddot{a}}{a} &= -\frac{\kappa^2}{6}\left(\rho_{\text{tot}} + 3p_{\text{tot}}\right) && (2.85)\\ &= -\frac{\kappa^2}{6}\left[2\epsilon\dot{\phi}^2 - 2V + (n-2)\frac{D_1}{a^n} + (m-2)\frac{D_2}{a^m}\right], && (2.86) \end{aligned}$$

where c is set to be 1. Taking time-derivative in Eq.(2.84) performs

$$\begin{aligned} 2H\dot{H} &= \frac{\kappa^2}{3}\left[\epsilon\dot{\phi}\ddot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi}\dot{\phi} - \frac{nD_1}{a^n}H - \frac{mD_2}{a^m}H\right] - \frac{k}{a^2}H, && (2.87)\\ \frac{\kappa^2}{3}\left[\epsilon\dot{\phi}\ddot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi}\dot{\phi}\right] &= 2H\dot{H} + \frac{\kappa^2}{3}\left(\frac{nD_1}{a^n}H + \frac{mD_2}{a^m}H\right) + \frac{k}{a^2}H. && (2.88) \end{aligned}$$

Applying the conservation equation Eq.(2.82), dynamic of the field is

$$\epsilon\dot{\phi}(t)^2 = -\frac{2}{\kappa^2}\left(\dot{H} - \frac{k}{a^2}\right) - \frac{nD_1}{3a^n} - \frac{mD_2}{3a^m}. \qquad (2.89)$$

The term of $\epsilon\dot{\phi}(t)^2$ in Eq.(2.86) substituted by Eq.(2.89) equips potential equation of field

$$V(\phi) = \frac{3}{\kappa^2}\left(H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right) + \left(\frac{n-6}{6}\right)\frac{D_1}{a^n} + \left(\frac{m-6}{6}\right)\frac{D_2}{a^m}. \qquad (2.90)$$

If time-evolution of the scale factor is known, the scalar field velocity and potential can be performed as a function of time explicitly. Results in this chapter will be referred to in next chapter.

CHAPTER III

NON-LINEAR SCHRÖDINGER (NLS) EQUATION

In this chapter, the cosmological quantities in the scalar field with two barotropic fluids given in the previous chapter are represented in terms of NLS formalism. The NLS equation is developed from Ermakov-Pinney equation (EPE) related to Friedmann equation by mapping the scale factor in cosmological formulation to the solution in Schrödinger equation.

3.1 Reviews of the NLS Formulation

In 2002, Hawkins and Lidsey introduced an approach to analyse the dynamic of cosmology containing perfect fluids and self-interacting scalar fields. The approach shows a relevance of cosmologies to the non-linear Ermakov-Pinney system by redefinition of variables properly and reduction of the Friedmann equations in term of a single second-order linear ODE. This also implies that there exists a correspondence between a spatially flat FLRW universe containing a scalar field and a cosmology containing both a scalar field and a perfect fluid. To determine a solution of general solution of the Ermakov-Pinney equation, a particular solution to a homogeneous equation needs to be solved [26].

One-dimensional Ermakov system decouples to single equation dubbed the Ermakov-Pinney (or Milne-Pinney) equation [25, 43, 44],

$$\ddot{b} + Q(t)b \; = \; \frac{\lambda}{b^3} \tag{3.1}$$

where

$$Q(t) = \frac{\kappa^2 n}{4}\dot{\phi}^2 \quad \text{and} \quad \lambda = -\frac{Dn^2\kappa^2}{12}. \tag{3.2}$$

The system above is related to FLRW cosmology of the flat ($k = 0$) case of the system,

$$H^2 = \frac{\kappa^2}{3}\left(\rho_\phi + \frac{D}{a^n}\right), \quad (3.3)$$

$$\epsilon(\ddot{\phi} + 3H\dot{\phi}) = -\frac{\mathrm{d}V}{\mathrm{d}\phi}. \quad (3.4)$$

where the speed of light $c \equiv 1$, $\kappa^2 \equiv 8\pi G$, proportional constant $D \geq 0$, $\epsilon = 1$ or -1 for canonical or phantom field cases. The scalar field density is, $\rho_\phi = (1/2)\epsilon\dot{\phi}^2 + V(\phi)$, the scalar field pressure is, $p_\phi = (1/2)\epsilon\dot{\phi}^2 - V(\phi)$. Barotropic fluid pressure and density are, $p_\gamma = w_\gamma \rho_\gamma$ and $\rho_\gamma = D/a^n$ where $n = 3(1 + w_\gamma)$. With further reparameterization $x(t) = \int u\,\mathrm{d}t$, the Ermakov-Pinney equation (3.1) is expressed as one-dimensional linear Schrödinger equation,

$$u''(x) + [E - P(x)]u(x) = 0, \quad (3.5)$$

where $' \equiv \mathrm{d}/\mathrm{d}x$, $E = -(\kappa^2 n^2 D)/12$ and $P(x) = (\kappa^2 n/4)\epsilon(\mathrm{d}\phi/\mathrm{d}x)^2$. Hence flat FLRW cosmology with scalar field and a barotropic fluid can be described by a linear Schrödinger equation.

This relation is also applicable in case of RSII braneworld [45]. The connection between FLRW scalar field cosmologies to non-linear partial differential equations such as the Ermakov-Pinny equation in 2+1 dimensions which can be reduced to Ermakov-Pinney equation. They examined the method to some specific cosmological models. This approach allows us to construct novel solutions for a number of different scalar potentials [46]. In addition, 3+1 dimensions were further studied and blown-up solutions are found, giving hope to be relevant to non-linear quantum cosmology [47]. In the late of 2003, Lidsey constructed mapping between the equation of motion for positively curved, isotropic fluid cosmologies and quasi-two-dimensional, harmonically trapped Bose-Einstein condensates to the one-dimensional Ermakov formulation. He identified parameter of the latter, for instance width of condensate wavepacket, momentum and energy in term of the scale factor, Hubble expansion parameter and energy of the universe, respectively[48]. Non-flat ($k \neq 0$) case extension of the FLRW system is reported in [49] and Bianchi I and V extension of the approach are also made. It is also found that Bianchi I Einstein field equation with scalar field and a perfect fluid is equivalent to linear Schrödinger equation [50].

Perturbative scheme of the solution of the Ermakov-Pinney equation was developed in connection to generalized WKB method [51]. It was demonstrated that a generalized Ermakov-Milne-Pinney (EMP) equation is completely equivalent to the FLRW scalar field cosmology (including the non-flat case) [52]. It comfirms and generalizes the results in [53]. The generalized EMP equation later was found to be equivalent to the NLS equation,

$$u''(x) + [E - P(x)]\, u(x) = -\frac{nk}{2} u(x)^{(4-n)/n}\,, \tag{3.6}$$

providing alternative approach to the FLRW scalar field cosmology with quantum-mechanical formulation [49].

In the NLS-Friedmann correspondence, inputs are either assumed scale factor or scalar field function which enable us to obtain exact solutions for a non-flat Friedmann universe with a barotropic fluid and a scalar field [54]. Recently, parametric solutions of non-linear ordinary differential equation of which the special cases are homogeneous and inhomogenous cosmologies and Bose-Einstein condensation correspondence, are found [55]. The NLS formulation of Friedmann scalar field cosmology and its interpretations might fulfill the need of non-perturbative quantum description of gravity and cosmology since it establishes correspondence between quantum and gravitational systems [56]. These motivated consequential studies on the NLS formulation of scalar field cosmology assuming scale factors functions [57, 58, 59] and inflationary parameters [60]. Detail of the NLS formulation is presented in D'Ambroise's dissertation [1].

Here in this work, we investigate the NLS-Friedmann connection in the case of two barotropic fluids with a canonical scalar field. We also analyse solutions of the NLS system of the two-fluid case based on possible $u(x)$ solutions reported in [1] and give critics on physical interpretation of the solutions.

Since the generalized Ermakov-Pinney equation was known, A. Kamanshchik, M. Luzzi and G. Venturi established the connection between the generalized Ermakov-Pinney equation with a perturbative scheme and the generalized WKB method of comparison equations[51]. In 2007, D'Ambroise and Williams demonstrated an approach to determine a solution of the Einstein field equations by introducing the correspondence of the FLRW equations with scalar field and perfect fluid matter source to the non-linear Schrödinger(NLS) type equation. They proposed $a(t) \equiv u(\sigma(t))$ and $\psi'(x)^2 \equiv \frac{4}{nK^2}P(x)$ to be links between FLRW equation and NLS[49]. To consider standard cosmological quantities in the view of quantum mechanics, Gumjudpai obtained time-dependent scalar field potential by assuming power-law expansion, $a \sim t^q$ showing the corresponding quantities in Schrödinger-type formulation[57]. He also studied scalar field cosmology by applying slow-roll conditions, acceleration condition, the Big Rip and WKB approximation in the usual way and concluded them. Then he used NLS formulation to study the cosmology in the same conditions. Results of both approaches for flat FLRW universe containing scalar field and barotropic fluid are consistent. The simplified NLS in term of linear equation has the same form of time-dependent Schrödinger equation. Thus WKB approximation can be applied to its wave function. But it works only for very slowly-varying Schrödinger potential, because WKB approximation is proper to $n \gg 1$. For a flat universe with phantom, the NLS form of the Big Rip singularity in a final fate can be remove one infinite parameter out of three parameters[60]. Lidsey, in 2013, provided a link between cubic Schrödinger equation representing

radially symmetric and the Friedmann equations sourced by a self-interacting scalar field and barotropic perfect fluid representing dynamics. Consider NLS with a cubic U(1) invariant interaction term taking a form of $i\frac{\partial u}{\partial \tau} = -\frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{2}\lambda(\tau)r^2u + \nu|u|^2u$, integral quantities of NLS can be interpreted in term of moments of wavefunction I_1, I_2, I_3 and I_4 quantifying norm square, width, radial momentum and energy of quantum configuration, respectively. In the paper, a dictionary was established to interpret between quantities in quantum-mechanical system and in cosmological system. To apply the dictionary to Hamilton-Jacobi formalism of scalar fields in cosmology, dualities are needed to leave the Friedmann equation in term of the Hamilton-Jacobi form invariant.

We will discuss about the energy density of barotropic fluid with equation of state $p_{\text{mat}} \equiv [(n-3)/3]\rho_{\text{mat}}$ is defined by $\rho_{\text{mat}} \equiv Da^{-n}$, Hubble parameter is defined by $H = \dot{a}/a$, D is an arbitrary constant with $0 \leq n \leq 6$, $\kappa^2 \equiv 8\pi m_P^{-2}$. These equations together can be rewritten in the form of Ermakov-Pinney equation,

$$\frac{d^2b}{d\tau^2} + Q(\tau)b = \frac{\lambda}{b^3}, \tag{3.7}$$

by change of variables. An effective scale factor b constructed from the scale factor a is written by [43]

$$a \equiv b^{2/n}, \tag{3.8}$$

and new time parameter τ is defined by

$$\frac{d}{dt} \equiv b\frac{d}{d\tau}. \tag{3.9}$$

The method gives a non-linear Ermakov-Pinney equation,

$$\frac{d^2b}{d\tau^2} + \frac{n\kappa^2}{4}\left(\frac{d\phi}{d\tau}\right)^2 b = -\frac{Dn^2\kappa^2}{12}\frac{1}{b^3}. \tag{3.10}$$

Comparison to Eq.(3.7), the Eq.(3.10) implies that $Q = \frac{n\kappa^2}{4}\left(\frac{d\phi}{d\tau}\right)^2$ and $\lambda = -\frac{Dn^2\kappa^2}{12}$. Consider a time-independent non-linear Schrödinger(NLS) equation,

$$u''(x) + [E - P(x)]u(x) = -\frac{nk}{2}u(x)^{(4-n)/n}, \tag{3.11}$$

corresponding to the generalized Ermakov-Milne-Pinney equation,

$$Y'' + QY = \frac{\lambda}{Y^3} + \frac{nk}{2Y^{(n+4)/4}}, \tag{3.12}$$

where k is a curvature, E is a constant energy and $P(x)$ is a potential, the Friedmann equation can be demonstrated in term of NLS equation by choosing $\dot{\sigma}(t) = u(\sigma(t))$ and $\psi'(x)^2 = \frac{4}{n\kappa}P(x)$ and mapping between time-dependent variables in Cosmology and time-independent variables.

The links are

$$a(t) \equiv u(\sigma(t))^{-2/n}, \qquad \phi(t) \equiv \psi(\sigma(t)). \tag{3.13}$$

In the light of NLS, this might open opportunities to study quantum cosmology.

3.2 Correspondence between Non-linear Schrödinger (NLS) equation and Scalar field Cosmology

According to the generalized Ermakov-Milne-Pinney equation in Eq.(3.12), the equation takes a similar form as a non-linear second order differential equation in term of

$$u''(x) + [E - P(x)]u(x) = -\frac{n\kappa}{2}u(x)^{(4-n)/n}, \tag{3.14}$$

where $'$ denotes $\mathrm{d}/\mathrm{d}x$. The NLS is chosen to be an alternative formulation to solve cosmological problems. But it can not be used directly because the Ermakov-Pinney equation contains time-dependent functions while NLS contains time-independent function. For this reason, new quantities need be introduced for making links between them.

$$\dot{x}(t) \equiv u(x(t)) \quad , \quad \psi'(x)^2 \equiv \frac{4}{n\kappa^2}P(x) \tag{3.15}$$

where $\kappa^2 = 8\pi G$. The cosmological quantities corresponding to NLS take the forms

$$a(t) = u(x(t))^{-2/n}, \tag{3.16}$$

$$\phi(t) = \psi(x(t)) \tag{3.17}$$

The kinetic energy E is defined by

$$E = -\frac{\kappa^2 n^2}{12}D_1. \tag{3.18}$$

To determine $P(x)$, Eq.(3.16) and Eq.(3.18) are substituted in Eq.(3.14), so we have

$$P(x) = \frac{\kappa^2 n}{4}a(t)^n \epsilon\dot{\phi}(t)^2 + \frac{mD_2}{12}\kappa^2 n u^{2(m-n)/n}. \tag{3.19}$$

Hubble parameter and its time derivative are expressed by

$$\begin{aligned}
H &= \frac{\dot{a}}{a}, && (3.20)\\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}u^{-2/n}\frac{\mathrm{d}x}{\mathrm{d}t}}{u^{-2/n}}, && (3.21)\\
&= \frac{-\frac{2}{n}u'u^{-(2+n)/n}}{u^{-2/n}}, && (3.22)\\
&= -\frac{2}{n}u', && (3.23)\\
\dot{H} &= -\frac{2}{n}\frac{\mathrm{d}u'}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}, && (3.24)\\
&= -\frac{2}{n}uu''. && (3.25)
\end{aligned}$$

The kinetic and potential terms of the scalar field in Eq.(2.89) and Eq.(2.90) can consequently be expressed in terms of NLS quantities as

$$\begin{aligned}
\epsilon\dot{\phi}^2 &= \frac{4}{\kappa^2 n}uu'' + \frac{2k}{\kappa^2}u^{4/n} + \frac{4E}{\kappa^2 n}u^2 - \frac{mD_2}{3}u^{2m/n}, && (3.26)\\
V &= \frac{12}{\kappa^2 n^2}(u'^2) - \frac{2u^2 P}{\kappa^2 n} + \frac{12u^2 E}{\kappa^2 n^2} + \frac{3\kappa u^{4/n}}{\kappa^2}. && (3.27)
\end{aligned}$$

Furthermore, the NLS expressions of energy density in Eq.(2.79) and pressure in Eq.(2.80) of the scalar field read

$$\begin{aligned}
\rho_\phi &= \frac{12}{\kappa^2 n^2}(u')^2 + \frac{12E}{\kappa^2 n^2}u^2 + \frac{3k}{\kappa^2}u^{4/n} - D_2 u^{2m/n}, && (3.28)\\
p_\phi &= -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4P}{\kappa^2 n}u^2 - \frac{12E}{\kappa^2 n^2}u^2 - \frac{3k}{\kappa^2}u^{4/n} - \left(\frac{m-3}{3}\right)D_2 u^{2m/n}, && (3.29)
\end{aligned}$$

respectively. The total energy density and pressure of the universe in NLS form are presented by

$$\begin{aligned}
\rho_{\text{tot}} &= \frac{12}{\kappa^2 n^2}(u')^2 + \frac{3k}{\kappa^2}u^{4/n} - D_2 u^{2m/n}, && (3.30)\\
p_{\text{tot}} &= -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4}{\kappa^2 n}uu'' - \frac{k}{\kappa^2}u^{4/n}. && (3.31)
\end{aligned}$$

Some other cosmological quantities can be given in terms of NLS by

$$\begin{aligned}
\ddot{\phi} &= \pm\frac{P'u^2 + 2uu'\left(P - \frac{m^2 D_2\kappa^2 u' u^{2(m-n)/n}}{12}\right)}{\kappa\sqrt{n\epsilon}\sqrt{P - \frac{D_2 mnu^{2(m-n)/n}\kappa^2}{12}}}, && (3.32)\\
3H\dot{\phi} &= \mp\frac{12u'u\sqrt{P - \frac{1}{12}D_2\kappa^2 mnu^{\frac{2(m-n)}{n}}}}{n\kappa\sqrt{\epsilon n}}. && (3.33)
\end{aligned}$$

Using these relations, we recover the NLS equation (3.14) with the NLS potential,

$$P(x) = \frac{u''}{u} - \frac{kn}{2}u^{4/n} + E, \tag{3.34}$$

where the first term in RHS of (3.34) implies NLS kinetic energy. Consider relations between the energy of fluids and the critical density ρ_{c}. The critical energy defined by a value of the density which corresponds to the flat universe reads

$$\rho_{\text{tot}}(t) = \rho_{\text{c}}(t) = \frac{3H^2}{8\pi G}. \tag{3.35}$$

The NLS equation correlating to Eq.(3.35) is

$$\rho_{\text{c}} = \frac{12u'^2}{\kappa^2 n^2}. \tag{3.36}$$

The density parameters can be expressed in terms of NLS parameters as

$$\Omega_1 \equiv \frac{\rho_1}{\rho_c} = \frac{n^2 D_1 \kappa^2 u^2}{12u'^2}, \tag{3.37}$$

$$\Omega_2 \equiv \frac{\rho_2}{\rho_c} = \frac{m^2 D_2 \kappa^2 u^2}{12u'^2}. \tag{3.38}$$

To apply the definition of the density parameter, the Friedmann equation in Eq.(2.45) can be rewritten in term of

$$\Omega_{\text{tot}}(t) - 1 \quad = \quad \frac{kc^2}{a^2 H^2}, \tag{3.39}$$

Hence we define

$$\Omega_k \equiv -\frac{kc^2}{a^2 H^2} \tag{3.40}$$

which expresses in NLS form as

$$\Omega_k = -\frac{kc^2 n^2}{4u'^2 u^{-4/n}}, \tag{3.41}$$

where

$$\rho_k = -\frac{3ku^{4/n}}{\kappa^2}. \tag{3.42}$$

The Friedmann equation in the representation of density parameter which performs

$$\Omega_1 + \Omega_2 + \Omega_\phi + \Omega_k = 1 \tag{3.43}$$

provides

$$\Omega_\phi = \frac{\rho_\phi}{\rho} \tag{3.44}$$

3.3 NLS Exact Solutions under proposed NLS potential with two barotropic fluids

Following NLS equation given by D'Ambroise[49] and changing σ to x, the equation becomes

$$u''(x) + [E - P(x)]u(x) = \frac{F}{u(x)^C}, \tag{3.45}$$

where E, F and C are constants and

$$D_1 = -\frac{12E}{n^2\kappa^2}, \qquad F = -\frac{nk}{2}, \qquad C = \frac{n-4}{n}. \tag{3.46}$$

We manipulate the equation by substituting solutions satisfied the NLS equation. In this report, eight exact solutions are chosen. The results are shown in Table1

3.3.1 Solution 1: $u(x) = e_0x^2 + b_0x + c_0$

Assume that the solution of NLS equation is $u(x) = e_0x^2 + b_0x + c_0$. Conditions satisfied the equation are $E = 0 \implies D_1 = 0, F = -d_0 \implies n = 4$ and $C = 0 \implies k = d_0/n$. The conditions represent that we are considering the universe with curvature $k = d_0/2$ contains radiation fluid ($n = 4$). However $D_1 = 0$ show that the radiation density vanishes but D_2. This solution can be separated into two cases.

Table 1: The NLS exact solutions given by J. D'Ambroise [1]

	Solutions: $u(x)$	$P(x)$	E	F	C
1	$e_0x^2+b_0x+c_0$	$(2e_0+d_0)/(e_0x^2+b_0x+c_0)$	0	$-d_0$	0
2	$e_0\cos^2(b_0x)$	$2b_0^2\tan^2(b_0x)$	$2b_0^2$	0	arbitrary
		$4b_0^2\tan^2(b_0x)$	0	$-2b_0^2e_0$	0
3	$e_0\tanh(b_0x)$	c_0	$c_0+2b_0^2$	$2b_0^2/e_0$	-3
4	$e_0e^{(-x\sqrt{-c_0})}-b_0e^{x\sqrt{-c_0}}$	0	$c_0<0$	0	arbitrary
5	$(e_0/x)e^{c_0x^2/2}$	$c_0^2x^2+2/x^2+b_0$	c_0+b_0	0	arbitrary
6	$-e_0\cosh^2(b_0x)$	$2b_0^2\tanh^2(b_0x)+c_0$	$c_0-2b_0^2$	0	arbitrary
7	e_0/x^{b_0}	$\frac{b_0(b_0+1)}{x^2}+c_0$	c_0	0	arbitrary
8	$-e_0\sinh^2(b_0x)$	$2b_0^2\tanh^2(b_0x)+c_0$	$c_0-2b_0^2$	0	arbitrary

- Case 1.1: $e\neq 0$ The solution of this case is given by

$$u(x)\equiv\dot{x}(t)=e_0x^2+b_0x+c_0. \tag{3.47}$$

Taking integration of Eq.(3.47) and setting the integration constant be zero yields

$$x(t)=\frac{1}{2e_0}\left\{\sqrt{-\Delta}\tan\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}-\frac{b_0}{2e_0}, \tag{3.48}$$

where $\Delta=b_0^2-4e_0c_0<0$. In order to present u in term of t, the Eq.(3.48) is taken derivative with respect to t, the result is hence

$$u(t)=-\frac{\Delta}{4e_0}\sec^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]. \tag{3.49}$$

In this case, $\sqrt{-\Delta}$ has to be a real number which means e_0 and c_0 must take the same signs and $4e_0c_0$ must be greater than b_0^2. Following Eq.(3.16), scale factor takes the form of

$$a(t)=\left\{-\frac{4e_0}{\Delta}\cos^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}^{2/n}. \tag{3.50}$$

Relation of redshift(z) and scale factor a, $1+z = a(t_0)/a(t)$, leads to relation between redshift and time in terms of

$$z(t) = \left\{\sec^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}^{2/n} - 1, \tag{3.51}$$

$$t - t_0 = \frac{2}{\sqrt{-\Delta}}\left\{\text{arcsec}[(z+1)^{n/4}]\right\}. \tag{3.52}$$

Therefore, the expressions of Hubble rate are

$$H(t) = -\frac{2\sqrt{-\Delta}}{n}\tan\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right], \tag{3.53}$$

$$H(z) = \frac{2\sqrt{-\Delta}}{n}\tan\left\{\text{arcsec}\left[(z+1)^{n/4}\right]\right\}. \tag{3.54}$$

- Case 1.2: $e_0 = 0$ The given solution is reduced to $u(x) = b_0 x + c_0$ which provides

$$x(t) = \frac{1}{b_0}\left[e^{b_0(t-t_0)} - 1\right], \tag{3.55}$$

for $b_0 \neq 0$. Taking derivative of $x(t)$ yields

$$u(t) = e^{b_0(t-t_0)}. \tag{3.56}$$

Then

$$a(t) = c_0^{-2/n} e^{-2b_0(t-t_0)/n} \tag{3.57}$$

and

$$z(t) = e^{2b_0(t-t_0)/n} - 1. \tag{3.58}$$

3.3.2 Solution 2: $u(x) = e_0 \cos^2(b_0 x)$

This solution can be considered separately into two cases.

- Case 2.1: $E = 2b_0^2 \implies D_1 = -24b_0^2/n^2\kappa^2 < 0, F = 0 \implies -nk/2$ which can be either $n = 0$ with arbitrary k or $k = 0$ with arbitrary n. Those provide
  - $k = 0$ and n is arbitrary. This condition corresponds C and $D_1 < 0$ for $b_0 \neq 0$.
  - $n = 0$ and k is arbitrary. This condition corresponds C and $D_1 = -\infty$ for $b_0 \neq 0$.
  - Both n and k are zero. This condition corresponds C and $D_1 = -\infty$ for $b_0 \neq 0$.

Consider these conditions, the density proportional constant D_1 determined is nonphysical since its value could be only negative of infinity.

- Case 2.2: $E = 0 \implies D_1 = 0, F = -2b_0^2 e_0 \implies n = 4, C = 0 \implies k = b_0^2 e_0$. Whereas the major fluid of this condition is identified to be radiation fluid because $n = 4$, $D_1 = 0$ identifies that there is no major fluid. In this case, we have the solution with $k = b_0^2 e_0$. These cases produce the same solution satisfying NLS equation which correlates with

$$x(t) = \frac{1}{b_0} \arctan[e_0 b_0 (t - t_0)]. \tag{3.59}$$

Taking derivative with respect to time of Eq.(3.59) gives

$$u(t) = \frac{e_0}{1 + e_0^2 b_0^2 (t - t_0)^2}. \tag{3.60}$$

The scale factor which is a function of time can be evaluated as

$$a(t) = \left[\frac{1 + e_0^2 b_0^2 (t - t_0)^2}{e_0}\right]^{2/n}, \tag{3.61}$$

where $e_0 \neq 0$. Consider radiation case, Hubble parameter and time-redshift perform as

$$H(t) = \frac{e_0^2 b_0^2 (t - t_0)}{e_0^2 b_0^2 (t - t_0^2)^2 + 1}, \quad (3.62)$$

$$z(t) = \sqrt{\frac{1}{e_0^2 b_0^2 (t - t_0)^2 + 1}} - 1, \quad (3.63)$$

respectively. So the relation between H and z becomes

$$H(z) = e_0 b_0 (z + 1) \sqrt{-z(z + 2)}. \quad (3.64)$$

3.3.3 Solution 3: $u(x) = e_0 \tanh(b_0 x)$

This solution satisfies NLS equation under the condition of $C = -3, E = c_0 + 2b_0^2$ and $F = 2b_0^2/e_0$ which correspond to $n = 1, D_1 = -12(c_0 + 2b_0^2).\kappa^2$ and $k = -4b_0^2/e_0$ respectively. The value of n identifies $w_\gamma = -2/3$. The solution maps to

$$x(t) = \frac{1}{b_0} \operatorname{arcsinh}[e^{e_0 b_0 (t - t_0)}], \quad (3.65)$$

$$u(t) = \frac{e_0 e^{e_0 b_0 (t - t_0)}}{\sqrt{1 + e^{2 e_0 b_0 (t - t_0)}}}, \quad (3.66)$$

where $b_0 x > 0$. The scale factor induced by this solution can be written by

$$a(t) = \frac{1}{e_0^2} \left[1 + e_0^{-2 e_0 b_0 (t - t_0)} \right], \quad (3.67)$$

where $e_0 \neq 0$. The result produces time-redshift relations represented by

$$z(t) = \frac{2}{e^{-2 e_0 b_0 (t - t_0) + 1}} - 1, \quad (3.68)$$

$$t - t_0 = \frac{-1}{2e - 0 b_0} \ln \left(\frac{2}{z + 1} - 1 \right), \quad (3.69)$$

where $0 < z < 1$. As a result of time-redshift realations, the hubble rates in terms of time and redshift can be written as

$$H(t) = \frac{-2 e_0 b_0}{1 + e^{2 e_0 b_0 (t - t_0)}}, \quad (3.70)$$

$$H(z) = e_0 b_0 (z - 1). \quad (3.71)$$

3.3.4 Solution 4: $u(x) = e_0 e^{-x\sqrt{-c_0}} - b_0 e^{x\sqrt{-c_0}}$

Following the conditions of $E = c_0 < 0, F = 0$ and arbitrary C, D_1 is automatically set to be $-12c_0/n^2\kappa^2 > 0$. Then

$$x(t) = \frac{1}{\sqrt{-c_0}} \ln \left\{ \sqrt{\frac{e_0}{b_0}} \tanh \left[\sqrt{-e_0 b_0 c_0}(t - t_0) \right] \right\}. \tag{3.72}$$

Taking time derivative of $x(t)$ provides

$$u(t) = \frac{2\sqrt{e_0 b_0}}{\sinh \left[2\sqrt{-e_0 b_0 c_0}(t - t_0) \right]}, \tag{3.73}$$

which yield the scale factor,

$$a(t) = \left\{ \frac{\sinh \left[2\sqrt{-e_0 b_0 c_0}(t - t_0) \right]}{2\sqrt{e_0 b_0}} \right\}^{2/n}. \tag{3.74}$$

The scale factor provides a constant redshift,

$$z = -1, \tag{3.75}$$

and

$$H(t) = \frac{4}{n}\sqrt{-e_0 b_0 c_0} \coth \left[2\sqrt{-e_0 b_0 c_0}(t - t_0) \right]. \tag{3.76}$$

Because z is not time dependent, Hubble rate can not be written in term of $H(z)$.

3.3.5 Solution 5: $u(x) = \frac{e_0}{x} e^{c_0 x^2/2}$ In order to satisfy NLS equation, the solution must follow conditions of $E = c_0 + b_0 < 0, F = 0$ and arbitrary C. The conditions correlate to $D_1 = -12(c_0 + b_0)/n^2\kappa^2$. Hence

$$x(t) = \sqrt{\frac{-2}{c_0} \ln[-e_0 c_0 (t - t_0)]}, \tag{3.77}$$

where $c_0 < 0$.

The solution $u(t)$ performs

$$u(t) = \frac{-1}{(t - t_0)\sqrt{-2c_0 \ln[-e_0 c_0 (t - t_0)]}}, \tag{3.78}$$

which yield the scale factor in term of

$$a(t) = \left\{ (t - t_0)^2 (-2c_0 \ln[-e_0 c_0 (t - t_0)]) \right\}^{1/n}. \tag{3.79}$$

The Hubble rate is hence

$$H(t) = \frac{1}{n(t-t_0)}\left\{\frac{1}{\ln[-e_0c_0(t-t_0)]} + 2\right\}, \tag{3.80}$$

where $c-0<0, n\neq 0$. At $t=t_0$.

In this case, there is no time-redshift relation since

$$z = -1. \tag{3.81}$$

3.3.6 Solution 6: $u(x) = -e_0\cosh^2(b_0x)$ Consider $E = c_0 - 2b_0^2 < 0, F = 0$ and arbitrary C, the value of $D_1 = -12(c_0 - 2b_0^2)/n^2\kappa^2 > 0$. The solution $u(x)$ becomes

$$x(t) = \frac{1}{b_0}\,\text{arctanh}[-e_0b_0(t-t_0)], \tag{3.82}$$

and can be written in a function of time as

$$u(t) = \frac{-e_0}{1-e_0^2b_0^2(t-t_0)^2}. \tag{3.83}$$

The scale factor which is a function of t expresses as

$$a(t) = \left[\frac{1-e_0^2b_0^2(t-t_0)^2}{-e_0}\right]^{2/n}, \tag{3.84}$$

which yields Hubble rate as

$$H(t) = \frac{-4}{n}\left[\frac{e_0^2b_0^2(t-t_0)}{1-e_0^2b_0^2(t-t_0)^2}\right], \tag{3.85}$$

where $n\neq 0$. According to the scale factor and Hubble rate, $z(t)$ and $H(z)$ can be written in term of

$$z(t) = \frac{1}{[1-e_0^2b_0^2(t-t_0)^2]^{2/n}} - 1, \tag{3.86}$$

$$H(z) = \frac{-4}{n}|e_0||b_0|\sqrt{z(z+1)}. \tag{3.87}$$

Comparison between Taylor expansion of $u(x)$ performing

$$u(x) = -e_0[1 + b_0^2(x-x_0)^2 + \frac{b_0^4}{3}(x-x_0)^4 + ...], \tag{3.88}$$

and the power-law expansion solution $a \sim t^q$(constant q) following [57],

$$u(x)_{\text{power-law}} = \left[\frac{(2-qn)}{2}(x-x_0)\right]^{qn/(qn-2)}, \tag{3.89}$$

for dust($n = 3$) provides $b_0^2(x-x_0)^2$(2nd term) and $[b_0^2(x-x_0)^4]/3$(3rd term). In this case, the second and third terms of (3.88) correspond to $q = 4/3$ and $q = 8/9$ respectively. Density parameters are

$$\Omega_1(z) = \frac{-3D_1\kappa^2}{16b_0^2}\left[\frac{1}{(z+1)^{-3/2}-1}\right], \tag{3.90}$$

$$\Omega_2(z) = \frac{-3D_1\kappa^2}{16b_0^2}\left[\frac{e_0^{2/3}(z+1)}{(z+1)^{-3/2}-1}\right], \tag{3.91}$$

where $\Omega_\phi = 1 - \Omega_1 - \Omega_2$.

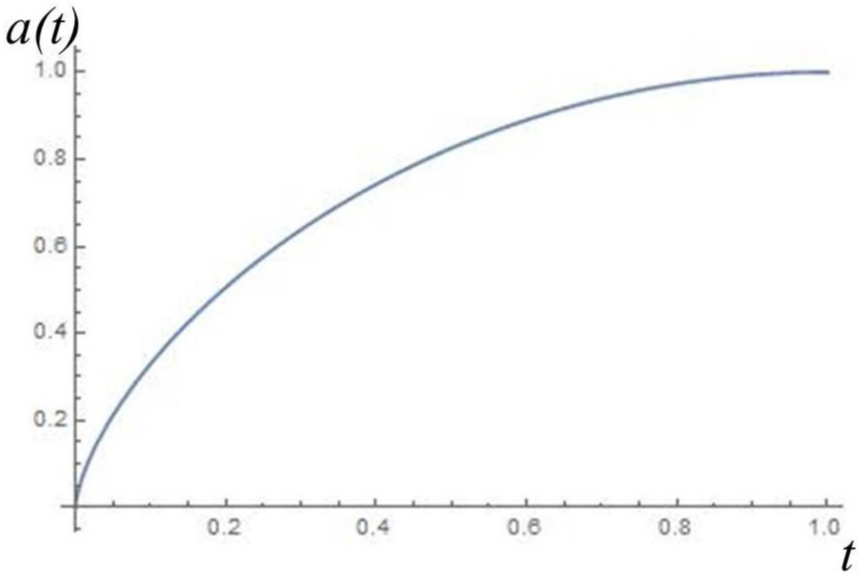

Figure 1: Scale factor $a(t)$ of the solution 6.

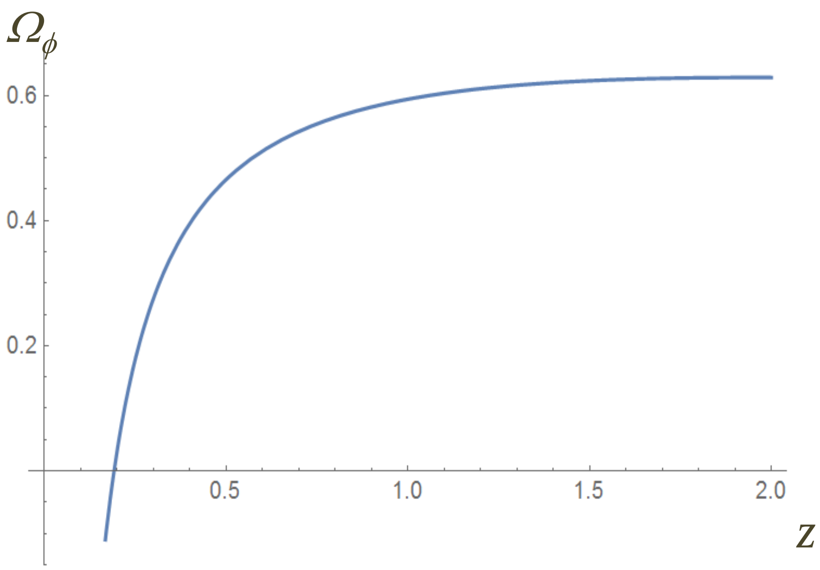

Figure 2: Scalar field density parameter $\Omega_\phi(z)$ of the solution 6 plotted versus redshift.

3.3.7 Solution 7: $u(x) = \frac{e_0}{x^{b_0}}$

Conditions of this solution are $E = c_0 < 0, F = 0$ and arbitrary C. In order to have $D_1 > 0$, c_0 must less than zero. So $D_1 = -12c_0/n^2\kappa^2 > 0$. These provide solutions in terms of $x(t)$ and $u(t)$ as

$$x(t) = [e_0(b_0+1)(t-t_0)]^{1/b_0+1}, \tag{3.92}$$

$$u(t) = e_0[e_0(b_0+1)(t-t_0)]^{-b_0/(b_0+1)}, \tag{3.93}$$

The solution $u(t)$ leads to the scale factor and Hubble rate which are

$$a(t) = \frac{1}{e_0^{2/n}}\left[e_0(b_0+1)(t-t_0)\right]^{2b_0/[n(b_0+1)]}, \tag{3.94}$$

$$H(t) = \frac{2b_0}{n(b_0+1)(t-t_0)}. \tag{3.95}$$

The calculation of redshift presents that

$$z(t) = -1 \tag{3.96}$$

The result can not show time-redshift relation which implies that it is not realistic.

3.3.8 Solution 8: $u(x) = -e_0 \sinh^2(b_0 x)$

Beside the seven solutions so far, we attempt to put new solution In order to satisfy NLS equation, $P(x)$ is set to be $2b_0^2 \tanh^2(b_0 x)$. Consequently, $E = c_0 - 2b_0^2 < 0, F = 0$ and C is arbitrary. Those imply $D_1 = -12(c_0 - 2b_0^2)/n^2\kappa^2$. Taylor expansion of $u(x)$ is

$$u(x) = -e_0 \left[b_0^2(x-x_0)^2 + \frac{b_0^4}{3}(x-x_0)^4 + ... \right]. \tag{3.97}$$

The coefficients of Eq.(3.89) are analogous to the coefficients of Eq.(3.97) which perform that $b_0^2(x-x_0)^2$ implies $q = 4/3$ and $\left[b_0^4(x-x_0)^4\right]/3$ implies $q = 8/9$ The solution $u(x)$ also provides $x(t)$ and $u(t)$ which are

$$x(t) = \frac{1}{b_0} \operatorname{arccoth}[e_0 b_0 (t-t_0)], \tag{3.98}$$

$$u(t) = u(t)\frac{e_0}{1 - e_0^2 b_0^2 (t-t_0)^2}. \tag{3.99}$$

Those yield the scale factor $a(t)$, redshift $z(t)$, and Hubble rate $H(t), H(z)$ the same as of solution 6.

3.4 Cosmological Validity of the Results

The results from the previous section are interpreted here. The solution 1.1 are considered into two cases. The case $e_0 \neq 0$ provides some finite value of $\tan$ function which Hubble rate can be either negative for contracting universe when $\tan[\sqrt{-\Delta}(t - t_0)/2] > 0$ or positive for expanding universe when $\tan[\sqrt{-\Delta}(t-t_0)/2] < 0$. The other case, $e = 0$, illustrates that the solution leads to a constant Hubble rate, $H = -2b_0/n$. The value of n is set to be 4, so $b_0 = -2H_0$. In this case if b_0 is negative, the expansion will be de Sitter type. However the given conditions that $D_1 = 0$ and $n = 1$. This seem contradict to have a major fluid $n = 4$ with zero density.

The solution 2 has two subcases. Those generate the same scale factor, redshift and Hubble rate. The determined redshift is valid when $z \in (-2, 0)$ which is not realistic.

The solution 3 refers to a major fluid $n = 1$ which implies $w = -2/3$. This fluid can not be identified. So the case is not our interest.

The solution 4 produces scale factor which is in form of $(\sinh)^{2/n}$. The first fluid can be any fluid but $n = 0$. In this case, the time-redshift relation is not calculate, since redshift is found to be constant. For this reason, $H(z)$ can not be evaluated. So it is not our interest.

The solution 5 and 6 express that the first fluid is arbitrary when $k = 0$, or vice versa. They can not be zero at the same time, if then $D_1 = \infty$ which is not needed. Consider at $t = t_0$. the scale factor of the solution 5 is indeterminate. Therefore time-redshift can not be evaluated. This case is not our interest. In case of solution 6, Taylor expansion of $u(x)$ corresponds to [54]. The graph in figure.2 illustrates relation between Ω_ϕ and redshift. The result is not relate to current observational data which should be around 0.7.

The solution 7 has the same F as in solution 5 and 6. The scale factor and Hubble rate $H(t)$ are able to be determined. The time-redshift is found to be constant. As the result, $H(z)$ can not be determined. This case is not realistic.

In addition, we found new solution apart from what are determined by D'Ambroise. It is showed in the solution 8. The scale factor a, redshift z, Hubble rate $H(t)$ and $H(z)$ are able to be evaluated and it provides the same results as of solution 6.

3.5 Chapter Conclusions and Discussions

In this work, all results which are determined are not agree with observations. We believe that there will be some $u(x)$ which provide physical cosmological quantities and agree with the observations. The NLS wave functions $u(x)$ are found to be non-normalizable. The kinetic energy E is set to be negative which is not physical. Consider each solution. Major fluid only effects to the wave function of NLS equation. On the other hand, minor fluid influences potential term.

CHAPTER IV

COSMOLOGY OF NON-MINIMAL DERIVATIVE COUPLING TO GRAVITY IN HOLOGRAPHIC DARK ENERGY

In this chapter, we consider ideas of modification of gravity as well as its scalar fluid by allowing the gravity sector to couple with the derivative of scalar field. Further we investigate this idea when incorporating with the holographic infrared cutoff. To avoid confusion, c in this chapter is not speed of light, but an arbitrary constant which is usually set to be 1. As well as, κ is a coupling constant, it is not $8\pi G$. Because we will play with $8\pi G$, it is going to be modified in term of a function of $\dot{\phi}$.

4.1 Reviews of the Non-Minimal Derivative Coupling to Gravity

As knowledge of inflation caused by scalar fields, there have been several models used for studying the early universe. Nonminimal coupling (NMC) to gravitation, one of candidates, generalized by involving derivative coupling term was first purposed by Amendola. Its effects in cosmology are investigated by introducing $f = f(\phi)$ for non-derivative terms and $f = f(\phi, \phi_{;\mu})$ for all terms including the derivative terms. Both derivative and non-derivative terms are coupled to the curvature scalar. This approach demonstrates that NMDC term provides an acceleration attractor whereas the other provide a graceful exit[31]. Furthermore, the NMDC to Ricci scalar(R) term appears in lower energy of higher dimensional theories, and in Weyl anomaly of $\mathcal{N} = 4$ disformal supergravity[61, 62]. In 1999, Capozziello, Lambiase and Schmidt studied effect of non-minimal derivative couplings in term of $R^{k\ell}\phi_{,k}\phi_{,\ell}$ in the Lagrangian. They found that other possible coupling terms being similar form may be ruled out[32]. In 2009, Sushkov introduced NMDC to curvature in terms of $\kappa_1 R\phi_{,\mu}\phi^{,\mu}$ and $\kappa_2 R_{\mu\nu}\phi^{,\mu}\phi^{,\nu}$. To consider in the case $-2\kappa_1 = \kappa_2 \equiv \kappa$. The derivative terms can be rewritten in composing term of the Einstein tensor $\kappa G_{\mu\nu}\phi^{,\mu}\phi^{,\nu}$. This approach leads to new exact cosmological solutions for spatially-flat FLRW model. The scale factor is set to be $a(t) = e^{\alpha(t)}$ which give the results: at large t, the results converge to standard case; $\kappa < 0$ and $a \sim (t - t_i)^{2/3}$. Moreover, the method also makes clear a unique manner both a quasi-de Sitter phase and an exit from it without any fine-tuned potential[63].

4.2 Holographic Dark Energy

So far, the observations of universe have confirmed that the universe is expanding acceleratedly. The cosmologists have made an attempt to describe this behavior. The dark energy theory is a widely acceptable concept providing explanation of accelerating expansion of the universe. There were two approaches to explain the accelerated expansion phase. Firstly, the dark energy is consider to be a fluid which has negative pressure [64, 65, 66, 67, 68, 69, 70, 71, 72]. Secondly, the modified gravity theories describe the accelerating expansion caused by geometrical effect [73]. Additionally, a new paradigm is invented form a proposal of quantum gravity in the context of black hole model by t' Hooft [74] and later extension of the idea to string theory by Suskind [75] named Holographic principle. The principle surprisingly reveals that the entropy instead of scaling by volume $V \sim L^3$, it is scaled by surface area $A \sim L^2$, where the reduce Plank mass $M_p^2 = 1/8\pi G = 1$. This implies that a spatial region's degrees of freedom does not exist in the bulk but at the region and is greater than 1.

In 1999, Cohen and others proposed to apply some cosmological ideas on the correlation between UV and IR cutoffs and the entropy of system by introducing the vacuum energy density $\rho_\Lambda \propto S/L^4$[76]. Applying this concept to the thermodynamics of black hole [77, 78, 79, 80, 81] leads to the Bekenstein-Hawking entropy bound $S_{BH} \sim M_p^2 L^2$. The result surprisingly shows that the entropy instead of scaling by volume $V \sim L^3$, it is scaled by area $A \sim L^2$, where the reduce Plank mass $M_p^2 = 1/8\pi G = 1$.[82, 83, 84, 85, 86, 87]

$$\rho_\Lambda = \frac{3c^2}{8\pi G L^2}, \tag{4.1}$$

where infrared cutoff scale L is considered as the size of H^{-1}, and c is a constant.

4.3 NMDC action and Equations of Motion

Consider spatially flat FLRW universe whose geometry is written by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2). \tag{4.2}$$

A scalar field with non-minimal derivative coupling to gravity is governed by the action

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{16\pi G} - \frac{1}{2}\left(\varepsilon g_{\mu\nu} + \kappa G_{\mu\nu}\right)\phi^{,\mu}\phi^{,\nu} - V(\phi)\right\} + S_m, \tag{4.3}$$

where $\varepsilon = 1$ for canonical case and $\varepsilon = -1$ for phantom case, κ is a coupling constant and the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. The Lagrangian is sub-class of Horndeski theory [88, 89, 29, 30] with $G_2 = -(\varepsilon/2)g_{\mu\nu}\phi^{,\mu}\phi^{,\nu}, G_3 = (16\pi G)^{-1}, G_5 = c_5\kappa/2$, with $c_5 \equiv \kappa/2$.

The variation of the action with respect to metric provides the Friedmann equation in term of

$$H^2 = \frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{2}(\varepsilon - 9\kappa H^2) + V(\phi) + \rho_{\text{tot}}\right], \tag{4.4}$$

and acceleration equation as

$$2\dot{H} + 3H^2 = 8\pi G\left\{-\frac{\dot{\phi}^2}{2}\left[\varepsilon + \kappa\left(2\dot{H} + 3H^2 + 4H\frac{\ddot{\phi}}{\dot{\phi}}\right)\right] + V(\phi) - p_{\text{tot}}\right\}. \tag{4.5}$$

Basically the components of the universe are matter and cosmological constant. Therefore the total energy density (ρ_{tot}) and the total pressure (p_{tot}) can be written as

$$\rho_{\text{tot}} = \rho_{\text{m}} + \rho_\Lambda, \tag{4.6}$$

$$p_{\text{tot}} = p_{\text{m}} + p_\Lambda, \tag{4.7}$$

where the subscripts m and Λ identify matter and cosmological constant components, respectively. To define $\dot{H}$, the Eq.(4.5) subtracted by Eq.(4.4) yields

$$\dot{H} = -4\pi G\left[\dot{\phi}^2\left(\varepsilon + \kappa\dot{H} - 3\kappa H^2 + 2\kappa H\frac{\ddot{\phi}}{\dot{\phi}}\right) + p_{\text{tot}} + \rho_{\text{tot}}\right] \tag{4.8}$$

Consider the Eq.(4.4) and (4.5), they are presented in the forms of kinetic term modification. On the one hand, their expressions can be represented by modifying the gravitational constant G. Using Eq.(4.4) and Eq.(4.1), the modified G performs

$$\left(\frac{3}{8\pi G} + \frac{9}{2}\dot{\phi}^2\kappa\right)H^2 = \frac{1}{2}\varepsilon\dot{\phi}^2 + V(\phi) + \rho_{\text{tot}}, \tag{4.9}$$

$$\left(\frac{6 + 72\pi G\dot{\phi}^2}{16\pi G}\right)H^2 = \frac{1}{2}\varepsilon\dot{\phi}^2 + V(\phi) + \rho_{\text{tot}}, \tag{4.10}$$

$$H^2 = G\left[\frac{8\pi}{3(1 + 12\pi G\dot{\phi}^2)}\right]\left[\frac{1}{2}\varepsilon\dot{\phi}^2 + V(\phi) + \rho_{\text{tot}}\right]. \tag{4.11}$$

Compare to the standard Friedmann equation in Eq.(2.83), the modification of G for is defined by

$$G_{\text{eff}}(\dot{\phi}) \equiv \frac{G}{1 + 12\pi G\dot{\phi}^2} \tag{4.12}$$

dubbed the effective cosmological constant. The Friedmann equation with modified gravitational constant takes the form of

$$H^2 = \frac{8\pi}{3}G_{\text{eff}}\left(\frac{\dot{\phi}^2}{2}\varepsilon + V + \rho_{\text{tot}}\right). \tag{4.13}$$

In the case of acceleration equation, its factoring out provides

$$2\dot{H} + 3H^2 = \frac{8\pi G}{(1 + 4\pi G\kappa\dot{\phi}^2)}\left(-\frac{\dot{\phi}^2}{2}\varepsilon + V - p_{\text{tot}} - 2\kappa H\dot{\phi}\ddot{\phi}\right), \tag{4.14}$$

$$\dot{H} = \frac{-4\pi G}{1 + 4\pi G\kappa\dot{\phi}^2}\left(\varepsilon\dot{\phi}^2 - 3\kappa\dot{\phi}^2H^2 + 2\kappa H\ddot{\phi}\dot{\phi} + p_{\text{tot}} + \rho_{\text{tot}}\right). \tag{4.15}$$

This shows that the attempt of writing the acceleration equation in term of G_{eff} cannot be done, since what in the round bracket in Eq.(4.14) and (4.15) cannot be written in the standard forms. Moreover, the coefficient $G/(1 + 4\pi G\kappa\dot{\phi}^2)$ is neither constructed from Lagrangian in Einstein frame nor the standard form of the Friedmann equation, we do not consider it as the effective gravitational constant. Furthermore, conservation of scalar field energy density explained by Klein-Gordon equation can be regarded in aspect of modification of field acceleration and field speed

$$(\varepsilon - 3\kappa H^2)\ddot{\phi} + (3\varepsilon H - 6\kappa H\dot{H} - 9kH^3)\dot{\phi} = -V_\phi, \tag{4.16}$$

where $V_\phi \equiv \mathrm{d}V/\mathrm{d}\phi$. The result is derived from taking time derivative to Eq.(4.4) which demonstrates by

$$\frac{3}{4\pi G}H\dot{H} = \dot{\phi}\ddot{\phi}(\varepsilon - 9\kappa H^2) - 9\kappa H\dot{H}\dot{\phi}^2 + V_\phi\dot{\phi} + \dot{\rho}_{\mathrm{m}} + \dot{\rho}_\Lambda, \tag{4.17}$$

$$0 = \ddot{\phi}(\varepsilon - 9\kappa H^2) - 9\kappa H\dot{H}\dot{\phi} + V_\phi \tag{4.18}$$

Another form of the Klein-Gordon equation written in term of modified scalar potential slope term is

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{V_\phi}{\varepsilon - 3\kappa H^2} + \frac{6\kappa H\dot{H}\dot{\phi}}{\varepsilon - 3\kappa H^2}. \tag{4.19}$$

The field derivative of the effective potential is defined by

$$V_{\mathrm{eff},\phi} = -\frac{V_\phi - 6\kappa H\dot{H}\dot{\phi}}{\varepsilon - 3\kappa H^2} \tag{4.20}$$

The potential slope term is a function expressed with choice of five variables $\phi, \dot{\phi}, \ddot{\phi}, H, \dot{H}$ of the system. Since there are three equations relating these variables, therefore there are only two degrees of freedom. The energy density of barotropic and cosmological constant are conserved separately as

$$\dot{\rho}_\Lambda + 3H(\rho_\Lambda + p_\Lambda) = 0\,, \tag{4.21}$$

$$\dot{\rho}_{\mathrm{m}} + 3H(\rho_{\mathrm{m}} + p_{\mathrm{m}}) = 0\,, \tag{4.22}$$

where ρ_Λ, p_Λ are density and pressure of the cosmological constant contribution.

4.4 Holographic Dark Energy with NMDC Gravity

In the previous subsection, the crucial equations in cosmology are introduced. Here they are regarded by including the holographic dark energy which scaled by the holographic cuttoff H^{-1} in Eq.(4.1) and the modified gravitational constant in Eq.(4.12). It leads to

$$\rho_\Lambda = \frac{3c^2H^2}{8\pi G_{\text{eff}}}, \tag{4.23}$$

$$= \frac{3c^2}{8\pi G}(1+12\pi G\kappa\dot{\phi}^2)H^2, \tag{4.24}$$

where c is an arbitrary constant which is set to be 1. Applying G_{eff} to the Friedmann equation yields

$$H^2 = \frac{8\pi}{3}G_{\text{eff}}\left(\frac{\varepsilon}{2}\dot{\phi}^2 + V + \rho_{\text{m}}\right). \tag{4.25}$$

At dark energy dominated era, the universe evolves under the scalar potential

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{4.26}$$

The solutions are supposed to be

$$a = a_0 e^{rt}, \tag{4.27}$$

$$\phi = \phi_0 e^{st} \tag{4.28}$$

The Friedmann equation with G_{eff} substituted by Eq.(4.26) and Eq(4.28) provides

$$H^2 = \frac{8\pi G_{\text{eff}}}{3}\left[\frac{\varepsilon\dot{\phi}^2}{2} + \frac{1}{2}m^2\phi^2 + \frac{3}{8\pi G_{\text{eff}}}H^2\right], \tag{4.29}$$

$$-\frac{1}{2}\varepsilon\dot{\phi}^2 = \frac{1}{2}m^2\phi^2, \tag{4.30}$$

$$\varepsilon\phi_0^2 s^2 e^{2st} = -\phi_0^2 e^{2st}, \tag{4.31}$$

$$\varepsilon s^2 = -m^2, \tag{4.32}$$

where $\dot{\phi} = \phi_0 s e^{st}$. We run the same process by using the Klein-Gordon equation instead.

This yield

$$\begin{aligned}
\ddot{\phi} + 3r\dot{\phi} &= -\frac{\mathrm{d}V/\mathrm{d}\phi}{\varepsilon - 3\kappa r^2}, && (4.33)\\
\phi_0 s^2 e^{st} + 3r\phi_0 s e^{st} &= -\frac{m^2\phi_0 e^{st}}{\varepsilon - 3\kappa r^2}, && (4.34)\\
s^2 + 3rs + \frac{m^2}{\varepsilon - 3\kappa r^2} &= 0, && (4.35)
\end{aligned}$$

where $H = r$ and $\dot{H} = 0$. Using the relation between s and m in Eq.(4.32), the Eq.(4.35) represents the relation between s and r by

$$\begin{aligned}
s^2 + 3rs + \frac{-\varepsilon s^2}{\varepsilon - 3\kappa r^2} &= 0, && (4.36)\\
\left(1 - \frac{\varepsilon}{\varepsilon - 3\kappa r^2}\right) s^2 + 3rs &= 0, && (4.37)\\
\frac{\varepsilon - 3\kappa r^2}{\kappa r} &= s. && (4.38)
\end{aligned}$$

As the result, the solution can be regarded in 3 cases according to the value of ε.

4.4.1 Case $\varepsilon = 0$: Holographic Dark Energy with Purely NMDC Kinetic Term

In this case, we study the kinetic term constructed by gravitation coupled to the derivative of field or NMDC term. Setting $\varepsilon = 0$ provides

$$\begin{aligned}
s &= \frac{-3\kappa r^2}{\kappa r}, && (4.39)\\
&= -3r. && (4.40)
\end{aligned}$$

The solutions are hence

$$\begin{aligned}
a &= a_0 e^{rt}, && (4.41)\\
\phi &= \phi_0 e^{-3rt}. && (4.42)
\end{aligned}$$

The ansatz for a manipulates

$$H = \frac{r a_0 e^{rt}}{a_0 e^{rt}} = r. \qquad (4.43)$$

Substituting the solutions in Eq.(4.41) and (4.42) to the energy density of the cosmological constant in Eq.(4.23) yields

$$\rho_\Lambda = \frac{3}{8\pi G}\left(1 + 108\pi G\kappa\phi_0^2 r^2 e^{-6rt}\right) r^2. \tag{4.44}$$

Taking time derivative to ρ_Λ results

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{3}{8\pi G}\left(1 + 108\pi G\kappa\phi_0^2 r^2 e^{-6rt}\right) r^2\right] = \frac{3}{8\pi G}\left(-648\pi G\kappa\phi_0^2 r^5\right). \tag{4.45}$$

Using the conservation equation in Eq.(4.21), the pressure is evaluated by

$$\frac{3}{8\pi G}\left(-648\pi G\kappa\phi_0^2 r^5 e^{-6rt}\right) = -3r\left[\frac{3r^2}{8\pi G}\left(1 + 108\pi G\kappa\phi_0^2 r^2 e^{-6rt}\right) + p_\Lambda\right] \tag{4.46}$$

$$\begin{aligned} p_\Lambda &= \frac{3}{8\pi G}\left(216\pi G\kappa\phi_0^2 r^4 e^{-6rt}\right) - \left[\frac{3r^2}{8\pi G}\left(1 + 108\pi G\kappa\phi_0^2 r^2 e^{-6rt}\right)\right] && (4.47)\\ &= -\frac{3}{8\pi G}(1 - 108\pi G\kappa\phi_0^2 r^2 e^{-6rt})r^2. && (4.48)\end{aligned}$$

Therefore the equation of state is given by

$$w_\Lambda = \frac{-1 + 108\pi G\kappa\phi_0^2 r^2 e^{-6rt}}{1 + 108\pi G\kappa\phi_0^2 r^2 e^{-6rt}}. \tag{4.49}$$

According to the time-redshift relation,

$$1 + z = \frac{a_0}{a}, \tag{4.50}$$

it is written in another form as

$$\begin{aligned} 1 + z &= \frac{a_0}{a_0 e^{rt}}, && (4.51)\\ 1 + z &= e^{-rt}, && (4.52)\\ -rt &= \ln(1 + z). && (4.53)\end{aligned}$$

At present time, $z = 0$ implies that $t = t_0 = 0$. If we look back in the past, $z = \infty$. provides $t = -\infty$. The equation of state which is a function of redshift reads

$$w_\Lambda = \frac{-1 + 108\pi G\kappa\phi_0^2 r^2 (1 + z)^6}{1 + 108\pi G\kappa\phi_0^2 r^2 (1 + z)^6}. \tag{4.54}$$

Considering r as the present Hubble parameter H_0, the w_Λ is examined in the condition of sub-Planckian values of ϕ_0 and of the coupling constant κ. Various values of these parameters are used and the evolution of $w_\Lambda(z)$ is demonstrated in Fig.3. The values of κ are chosen to be $-0.5, -0.25, 0.25, 0.5$ and 1.0 with fixing $r = 0.01$. In the cases $k < 0$, divergencies are predicted with singularity in Eq.(4.54) This case is interesting since it renders $w_\Lambda \longrightarrow -1$ at late time. However $\kappa < 0$ case is not favored because its w_Λ equations either goes out of the range $[-1, 1]$ or comes from outside the range $[-1, 1]$ for all evolution. In all plots to be presented, we set $8\pi G = 1, \phi_0 = 1$. Considering $r \sim H_0 \sim \sqrt{\Lambda/3}$ and $\Lambda/M_{\rm P}^2 \sim 10^{-121}$ (The reduced Planck mass $M_{\rm P} = (8\pi G)^{-1/2}$), hence $r \sim 10^{-60}$. The less r brings w_Λ to -1 earlier as shown in Fig. 4 in which two different values of r are used with other fixed variables. At present $z = 0$, singularity in w_Λ is when $\kappa = \kappa_{{\rm s},z=0} = -(108\pi G\phi_0^2 r)^{-1}$ which is $\kappa_{{\rm s},z=0} = -7.4 \times 10^{58}$ in Planck unit. Singularity in κ is negative, hence considering the favored positive κ case, it is no longer the problem. In this case, when there is no potential $V = 0$, i.e. $m = 0 = s$, we have $r = 0$ and the universe is static at $a = a_0$ all the time.

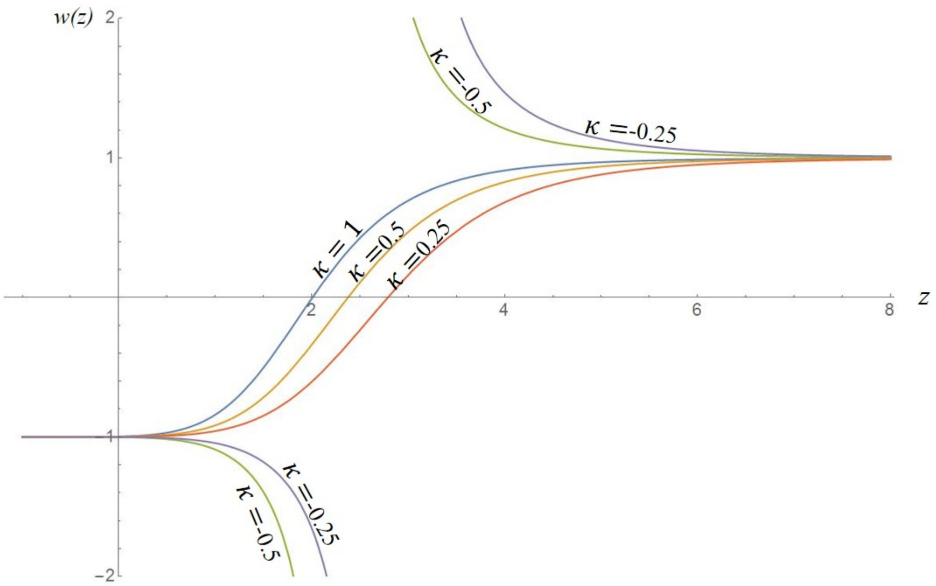

Figure 3: Relation between the equation of state for the holographic dark energy with purely NMDC kinetic term and z for various value of κ.

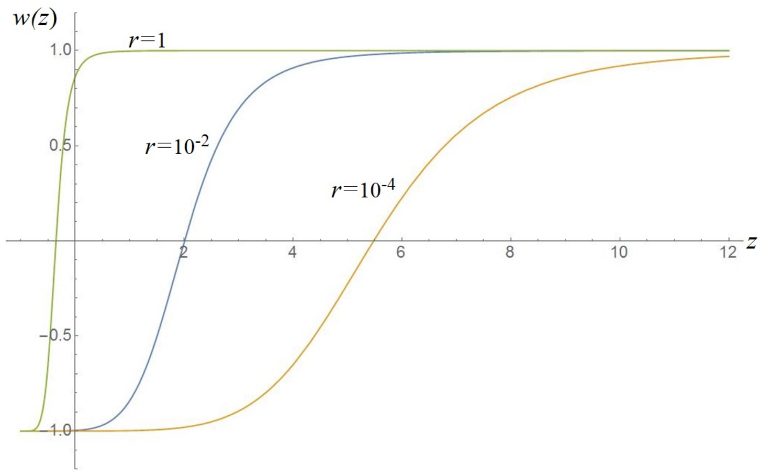

Figure 4: Relation between the equation of state for the holographic dark energy with purely NMDC kinetic term and z for various value of r.

4.5 Holographic dark energy with both NMDC and free kinetic terms

For $\varepsilon = \pm 1$, the expression of r can be given by using the relations in Eq.(4.32) and (4.36). These equations yield

$$
\begin{aligned}
s^2 + 3rs + \frac{-\varepsilon s^2}{\varepsilon - 3\kappa r^2} &= 0, && (4.55)\\
(\varepsilon - 3\kappa r^2)s^2 + 3rs(\varepsilon - 3\kappa r^2) - \varepsilon s^2 &= 0, && (4.56)\\
-3\kappa s^2 r^2 + 3\varepsilon s r - 9\kappa s r^3 &= 0, && (4.57)\\
3\kappa s r^2 + \kappa s^2 r - \varepsilon s &= 0 && (4.58)
\end{aligned}
$$

Apply the quadratic formula to Eq.(4.58) and substitute Eq.(4.38) to Eq.(4.58), the expression of r in term of κ and m is

$$
r = \pm \frac{\sqrt{\frac{(6-m^2\kappa)}{\varepsilon} \pm m\sqrt{\kappa(m^2\kappa - 12)}}}{\sqrt{18\kappa}}\,. \qquad (4.59)
$$

Energy density and its time derivative of the cosmological constant are

$$
\begin{aligned}
\rho_\Lambda &= \frac{3}{8\pi G}\left(1 + 12\pi G\kappa\phi_0^2 s^2 e^{2st}\right) r^2, && (4.60)\\
\dot{\rho_\Lambda} &= \frac{3}{8\pi G}\left(24\pi G\kappa\phi_0^2 s^3 e^{2st}\right) r^2. && (4.61)
\end{aligned}
$$

As stated in the fluid equation, the pressure is hence

$$p_\Lambda = -\frac{3}{8\pi G}\left(r^2 + 8\pi G\kappa\phi_0^2 s^3 e^{2st} r + 12\pi G\kappa\phi_0^2 s^2 e^{2st} r^2\right). \tag{4.62}$$

Therefore the equation of state can be given by

$$w_\Lambda = -\frac{(r + 8\pi G\kappa\phi_0^2 s^3 e^{2st} + 12\pi G\kappa\phi_0^2 s^2 e^{2st} r)}{r\,(1 + 12\pi G\kappa\phi_0^2 s^2 e^{2st})}. \tag{4.63}$$

When we use time-redshift relation in Eq.(4.53), the equation of state is also represented as function of redshift,

$$w_\Lambda = -\frac{\left[r + 8\pi G\kappa\phi_0^2 s^3 (1+z)^{-\frac{2s}{r}} + 12\pi G\kappa\phi_0^2 s^2 r (1+z)^{-\frac{2s}{r}}\right]}{r\left[1 + 12\pi G\kappa\phi_0^2 s^2 (1+z)^{-\frac{2s}{r}}\right]}. \tag{4.64}$$

In equation (4.32), $\varepsilon s^2 = -m^2$, the value of s is real if $\varepsilon = -1$, i.e. the kinetic term is phantom. On the other hand, if $\varepsilon = 1$, s is imaginary. The other interesting case is when the scalar potential is absent as below.

4.5.1 $V(\phi) = 0$

In a specific case where it is in absence of scalar potential, $V(\phi) = 0$, that is $m = 0$ and hence $s = 0$. From equation (4.59), we have

$$r = \pm\frac{1}{\sqrt{3\kappa\varepsilon}}. \tag{4.65}$$

The field is hence at $\phi = \phi_0$ (static solution) and the equation of state is $w_\Lambda = -1$. Positive root of r is chosen, since decaying solution is not realistic. The scale factor solution is de-Sitter,

$$a = a_0 e^{\pm t/\sqrt{3\kappa\varepsilon}}. \tag{4.66}$$

Effective cosmological constant λ can be defined as $\lambda \equiv (\kappa\varepsilon)^{-1}$ hence $a = a_0 \exp\left(\sqrt{\lambda/3}\,t\right)$. To keep λ real, we need either $\varepsilon = 1, \kappa > 0$ or $\varepsilon = -1, \kappa < 0$ at the same time.

4.5.2 Canonical kinetic term ($\varepsilon = 1$)

With free canonical kinetic term $\varepsilon = 1$, $s^2 = -m^2$ hence $s = \pm im$ which is imaginary and $s^3 = \mp im^3$. The equation of state as function of time is

$$w_\Lambda = -\frac{1}{C}(A + iB) \tag{4.67}$$

where A, B are real and imaginary parts,

$$\begin{aligned} A &\equiv 1 + 8\pi G\kappa\phi_0^2 m^2 \left[-3\cos(2mt) + \frac{m}{r}\sin(2mt) + 18\pi G\kappa\phi_0^2 m^2\right], &(4.68)\\ B &\equiv 8\pi G\kappa\phi_0^2 \frac{m^3}{r}\left[\pm 12\pi G\kappa\phi_0^2 m^2 \mp \cos(2mt)\right], &(4.69)\end{aligned}$$

and

$$C \equiv 1 + 24\pi G\kappa\phi_0^2 m^2\left[-\cos(2mt) + 6\pi G\kappa\phi_0^2 m^2\right]. \tag{4.70}$$

The $\pm$ and $\mp$ signs of equation (4.69) correspond to the sign of $s = \pm im$ consequently. There will be a singularity in w_Λ when $C = 0$. It occurs at time t_s which is determined as

$$1 + 24\pi G\kappa\phi_0^2 m^2\left[-\cos(2mt_s) + 6\pi G\kappa\phi_0^2 m^2\right] = 0, \tag{4.71}$$

$$\begin{aligned} -\cos(2mt_s) + 6\pi G\kappa\phi_0^2 m^2 &= \frac{-1}{24\pi G\kappa\phi_0^2 m^2}, &(4.72)\\ \cos(2mt_s) &= \frac{1}{24\pi G\kappa\phi_0^2 m^2} + 6\pi G\kappa\phi_0^2 m^2 &(4.73)\\ &= \frac{1 + 144\pi^2 G^2\kappa^2\phi_0^4 m^4}{24\pi G\kappa\phi_0^2 m^2}, &(4.74)\\ t_\mathrm{s} &= \frac{1}{2m}\arccos\left[\frac{1 + 144\pi^2 G^2\kappa^2\phi_0^4 m^4}{24\pi G\kappa\phi_0^2 m^2}\right], &(4.75)\end{aligned}$$

supposing fixed κ. If we need to know the range of κ that can give singularity at present epoch, we use $t = t_0 = 0$ at present time and assume $m = 1$ in equation (4.75). The result is $\kappa_{\mathrm{s},t_0} = 2/3$. As argument of $\arccos$ function in equation (4.75) must be in the range $[-1, 1]$, this puts the limits to κ_s to be in the range $[-2/3, 2/3]$ which limits t_s into $[0, \pi/2]$ in Planck unit. If considering uncarefully, we might think that present value w_{Λ,t_0} as a function κ in Planck unit (setting $8\pi G \equiv 1, m = 1, \phi_0 = 1$) can be found from the real part $-A/C$ in equation (4.67) as $w_{\Lambda,t_0} = [-1+3\kappa-(9/4)\kappa^2]/[1-3\kappa+(9/4)k^2] = -1$. However this is not correct. The coefficient r in equation (4.68) could take imaginary value and need to be taken into account. Hence considering equation (4.59) in our $\varepsilon = 1$ context,

$$r = \pm\frac{\sqrt{6-m^2\kappa\left(1\mp\sqrt{1-\frac{12}{m^2\kappa}}\right)}}{\sqrt{18\kappa}}. \tag{4.76}$$

The $\pm$ and $\mp$ signs in the expression of r come from solving quadratic equation. Here r always has imaginary value. Using relation $t = -r^{-1}\ln(1+z)$, we plot real part of $w_\Lambda(z)$ in all possible cases. Those are eight cases which are considered in the conditions of $m = 1$ and $\kappa = -5, -1, 1, 5$. To simplify, we define $D \equiv \sqrt{1-12/(m^2\kappa)}$, $E \equiv 8\pi G\kappa\phi_0^2 m^3/r$ and $F \equiv 12\pi G\kappa\phi_0^2 m^2$. Hence the eigth cases are when

- **case 1**:

  $r = \sqrt{6-m^2\kappa\,(1-D)}/\sqrt{18\kappa}\,,$

  $B = E\,[F-\cos(2mt)]\,,\ s = im$

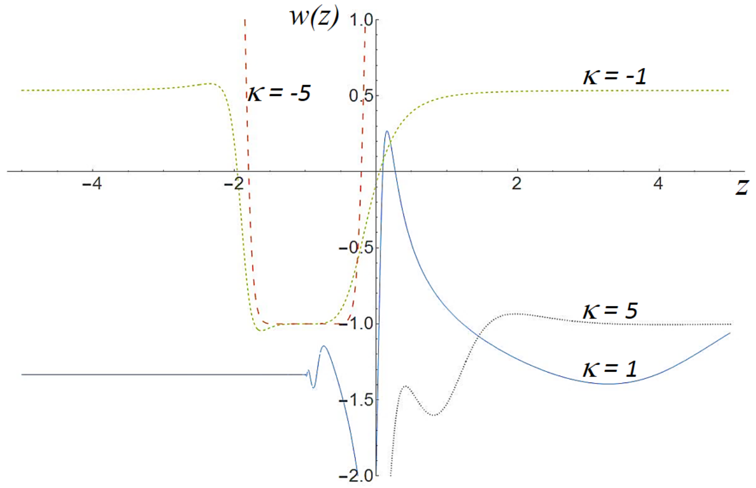

Figure 5: The plot of case 1.

- **case 2**:

  $r = \sqrt{6 - m^2\kappa\,(1-D)}/\sqrt{18\kappa}\,,$

  $B = E\,[-F + \cos(2mt)]\,,\ s = -im$

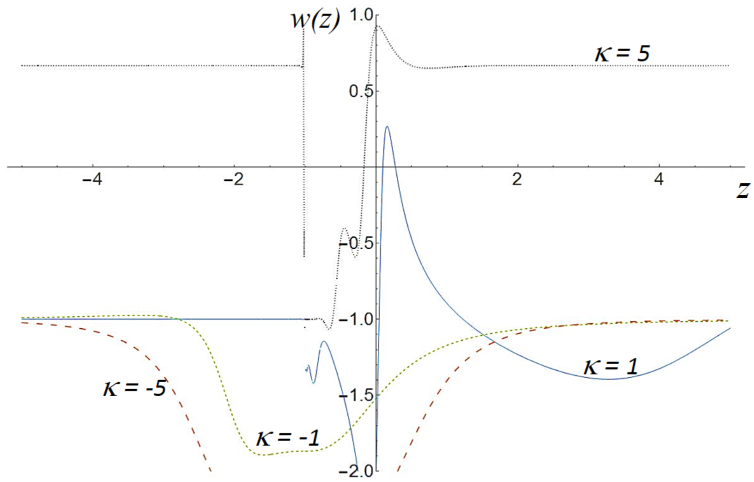

Figure 6: The plot of case 2.

- **case 3**:

  $r = \sqrt{6 - m^2\kappa\,(1+D)}/\sqrt{18\kappa}\,,$

  $B = E\,[F - \cos(2mt)]\,,\ s = im$

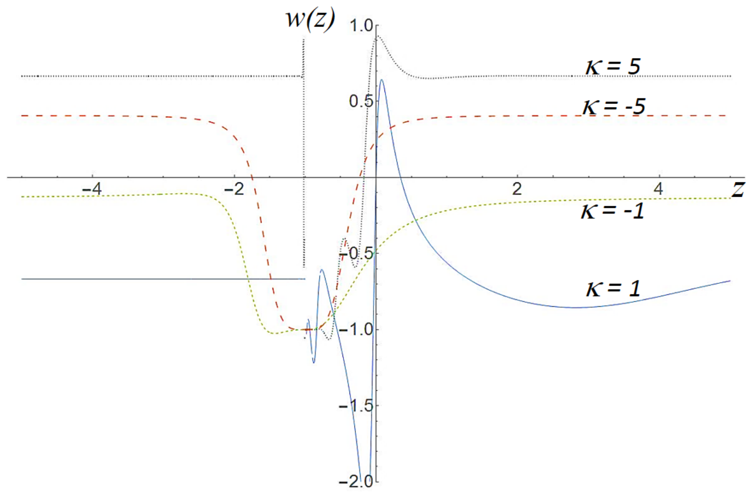

Figure 7: The plot of case 3.

- **case 4**:

$r = \sqrt{6 - m^2\kappa\,(1+D)}/\sqrt{18\kappa}\,,$

$B = E\,[-F + \cos(2mt)]\,,\ s = -im$

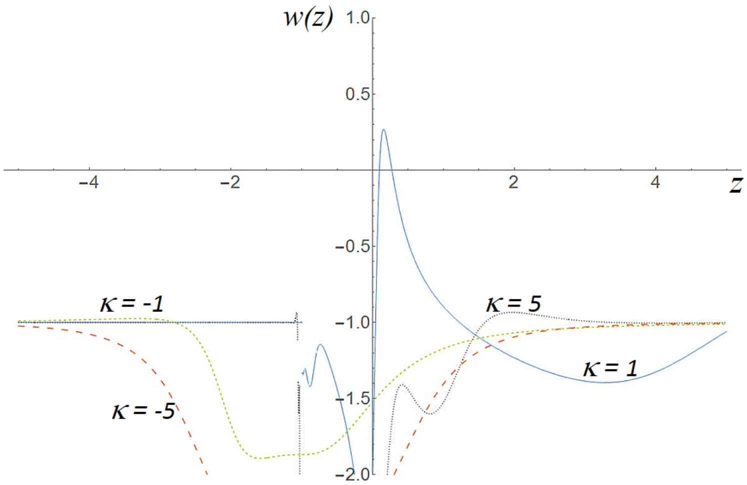

Figure 8: The plot of case 4.

- **case 5**:

$r = -\sqrt{6 - m^2\kappa\,(1-D)}/\sqrt{18\kappa}\,,$

$B = E\,[F - \cos(2mt)]\,,\ s = im$

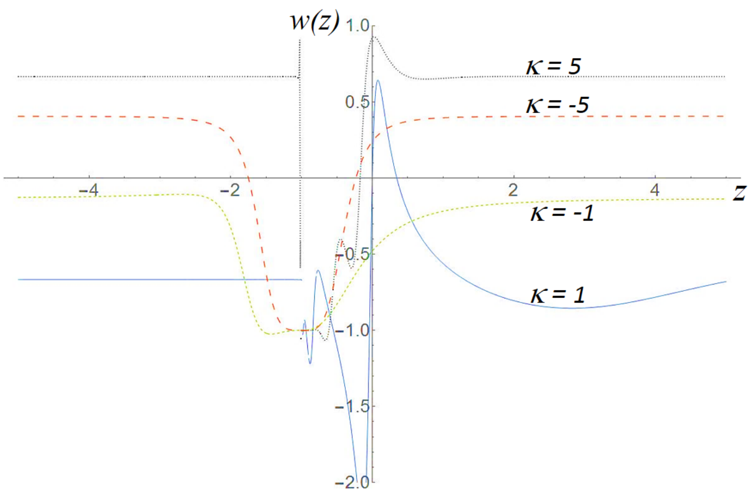

Figure 9: The plot of case 5.

- **case 6**:

  $r = -\sqrt{6 - m^2\kappa\,(1 - D)}/\sqrt{18\kappa}\,,$

  $B = E\,[-F + \cos(2mt)]\,,\ s = -im$

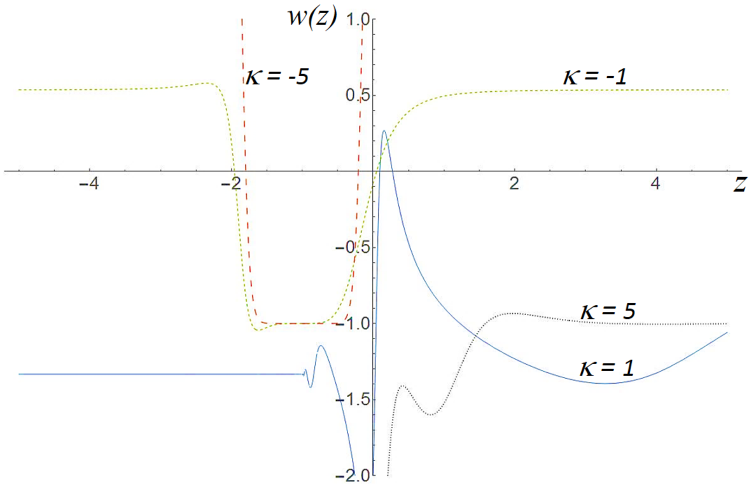

Figure 10: The plot of case 6.

- **case 7**:

  $r = -\sqrt{6 - m^2\kappa\,(1 + D)}/\sqrt{18\kappa}\,,$

  $B = E\,[F - \cos(2mt)]\,,\ s = im$

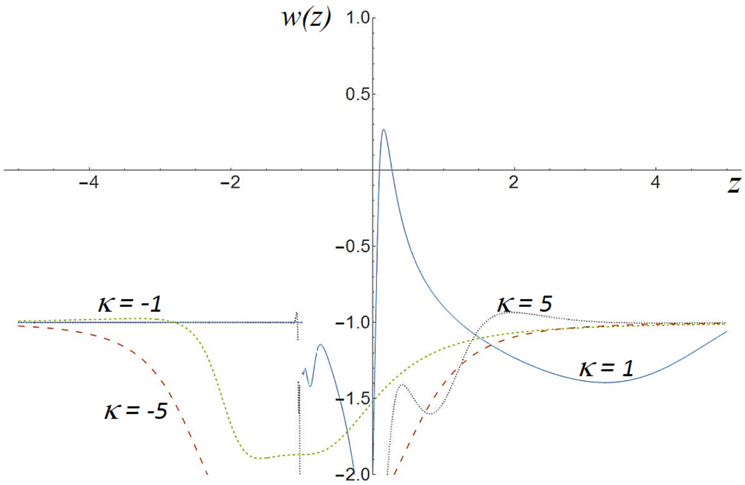

Figure 11: The plot of case 7.

- **case 8**:

  $r = -\sqrt{6 - m^2\kappa\,(1+D)}/\sqrt{18\kappa}\,,$

  $B = E\,[-F + \cos(2mt)]\,,\ s = -im\,.$

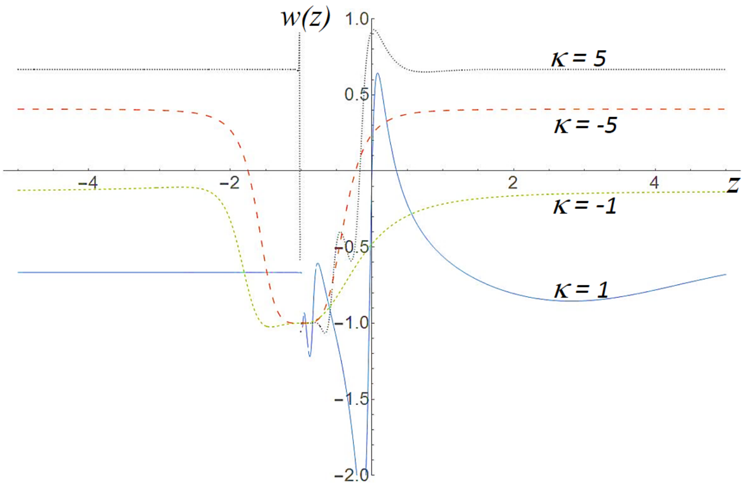

Figure 12: The plot of case 8.

Real part of the $w_\Lambda(z)$ for these cases are plotted in figures. Real parts of some cases are the same. These are (case 1 and case 6), (case 4 and case 7), (case 2 and case 5) and (case 3 and case 8). When $\kappa > 0$, real parts of $w_\Lambda(z)$ in Fig. 5, 8, 10 and Fig. 11 are the same. Moreover the real parts of $w_\Lambda(z)$ in Fig. 6, 7, 9 and Fig. 12 are also the same for $\kappa > 0$. This is because the distinct of each case appears in the imaginary parts. Cases 2 and 5 (Fig. 6,9) and cases 4 and 7 (Fig. 8,11) have -1 as late time value of w_Λ while the rests do not. Considering realistic character that w_Λ should be about -1 at present, and past evolution should not have $w_\Lambda < -1$ hence only reasonable cases are the portraits with $\kappa > 0$ in Fig. 6 and 9. These are of case 2 and case 5. Focusing on case 2, we take positive root of r with $s = -im$ while for case 5 we take negative root with $s = im$. These two cases result in the same real part of w_Λ. Physically, these are not different since all physical variables (m, ϕ_0, G) are real. The final point is that we need the expansion to be de-Sitter like and we should take positive root of r which does not match case 5 but case 2.

4.5.3 Phantom kinetic term ($\varepsilon = -1$)

Considering phantom kinetic term $\varepsilon = -1$, we have $s = \pm m$. The equation of state is

$$w_\Lambda = -\frac{\left[r \pm 8\pi G\kappa\phi_0^2 m^3 (1+z)^{\mp\frac{2m}{r}} + 12\pi G\kappa\phi_0^2 m^2 r(1+z)^{\mp\frac{2m}{r}}\right]}{r\left[1 + 12\pi G\kappa\phi_0^2 m^2 (1+z)^{\mp\frac{2m}{r}}\right]} \tag{4.77}$$

where the $\pm$ and $\mp$ sign correspond to $s = \pm m$ accordingly. However the $\pm$ signs in the expression of r (equation (4.59)) do not result from $s = \pm m$ but resulting from solving the equations of motion (4.13) and (4.16). Here we have

$$r = \frac{\sqrt{-6 + m^2\kappa\left(1 \pm \sqrt{1 - \frac{12}{m^2\kappa}}\right)}}{\sqrt{18\kappa}}\,. \tag{4.78}$$

Expanding solution is when r is real and is chosen to be positive, that is $\kappa > 0$ and

$$\pm\sqrt{1 - \frac{12}{m^2\kappa}} > \frac{6}{m^2\kappa} - 1\,. \tag{4.79}$$

The right-hand side requires that

$$m^2\kappa \geq 12 \quad \text{or} \quad m \geq \frac{2\sqrt{3}}{\sqrt{\kappa}}\,, \tag{4.80}$$

resulting that $\sqrt{1 - 12/(m^2\kappa)}$ falls into a range $[0, 1)$ for the positive branch of the left-hand side of equation (4.79). This also restricts the value of $\Delta \equiv (6/m^2\kappa) - 1$ to a range $(-1, -0.5]$. The negative branch of the left-hand side is restricted to $(-1, 0]$. The negative branch also agrees with the range of Δ, i.e. $(-1, -0.5]$. The situation requires both coupling $\kappa > 0$ and the scalar mass m to be in super-Planckian regime as we set $\phi_0 = 1$. Unless positive κ, scalar mass is imaginary. Fig. 13 and Fig. 14 present the plots of $w(z)$ for the positive branch of equation (4.79) and Fig. 15 and Fig. 16 presents the plots of $w(z)$ for the negative branch of equation (4.79). More or less value of scalar mass and of the coupling result in initial value of w_λ and how fast it changes. Since $\phi = \phi_0 e^{st}$, hence negative s is preferred otherwise the field evolves to super-Planckian regime. If considering that the present universe is expanding approximately like de-Sitter case, r should be very small ($\sim 10^{-60}$) in equation (4.78). Therefore if considering scalar mass $m \sim M_{\rm P} = 1$, we would need κ to be as large as 10^{60}.

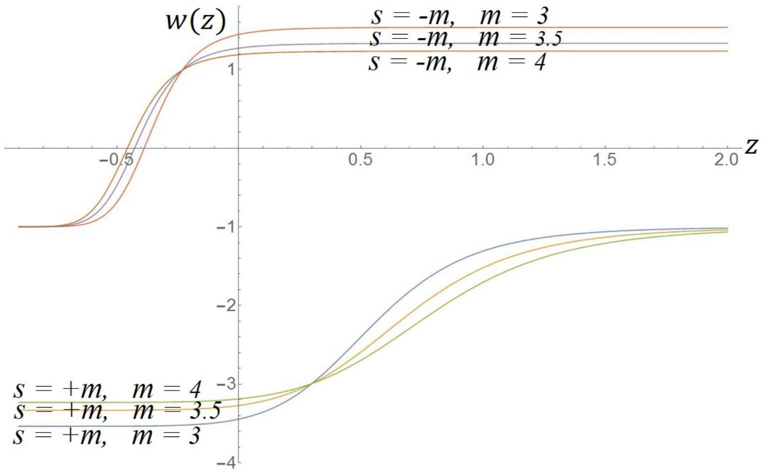

Figure 13: Equation of state of the positive branch of the equation (4.79) for $\kappa = 2$ and $m = 3, 3.5, 4$.

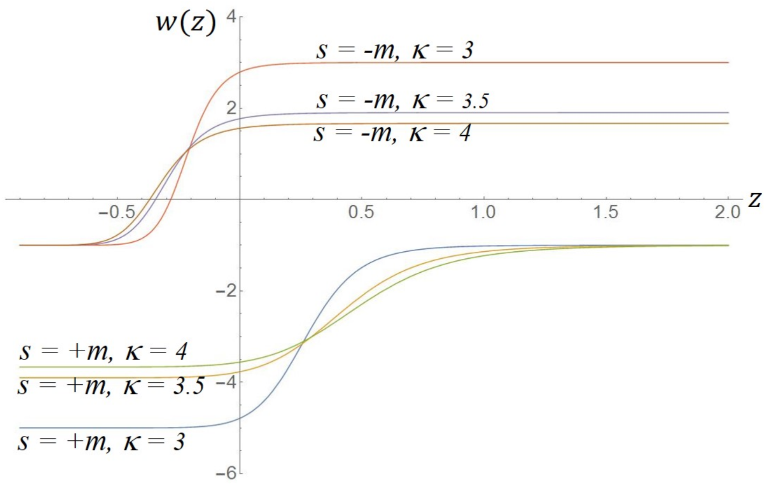

Figure 14: Equation of state of the positive branch of the equation (4.79) for $\kappa = 3, 3.5, 4$ and $m = 2$.

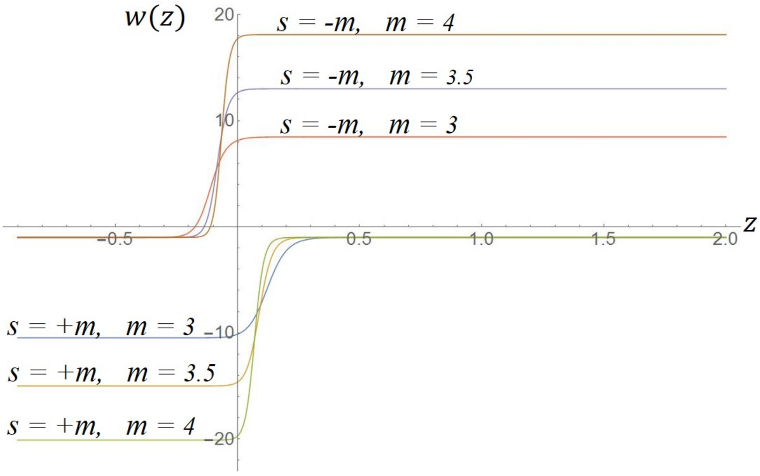

Figure 15: Equation of state of the negative branch of the equation (4.79) for $\kappa = 2$ and $m = 3, 3.5, 4$.

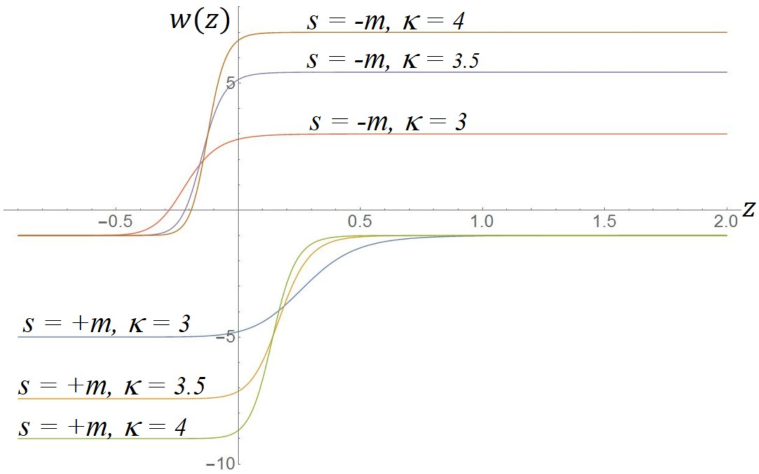

Figure 16: Equation of state of the negative branch of the equation (4.79) for $\kappa = 3, 3.5, 4$ and $m = 2$.

4.6 Variation of gravitational constant

Constraint of the model can be given by the measurement of gravitational constant variation. For example, the constraint with gravitational-wave standard sirens and supernovae is $\dot{G}/G|_{t_0} \lesssim 3 \times 10^{-12}$ year^{-1} for the ratio $\dot{G}_{\text{eff}}/G_{\text{eff}}$ at present time [90] and constraints of the same order ($\dot{G}/G|_{t_0} \lesssim 10^{-12}$ year^{-1}) given by observations of pulsars [91, 92], lunar laser ranging [93] and Big Bang nucleosynthesis [94, 95]. The variation in our model is

$$\frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} = \frac{-24\pi G\kappa\phi_0^2 s^3 e^{2st}}{1 + 12\pi G\kappa\phi_0^2 s^2 e^{2st}}\,. \tag{4.81}$$

We found that if without the scalar potential ($m = 0$), $\dot{G}_{\text{eff}}/G_{\text{eff}} = 0$, i.e. no variation of gravitational constant.

4.6.1 Variation of G: purely NMDC kinetic term ($\varepsilon = 0$)

If there is only purely NMDC kinetic term, $\varepsilon = 0$ and $s = -3r$. Using this relations in equation (4.81) and considering $t = t_0 = 0$ at present time,

$$\left[\frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}}\right]_{t_0} = \frac{648\pi G\kappa\phi_0^2 r^3}{1 + 108\pi G\kappa\phi_0^2 r^2}\,. \tag{4.82}$$

The variation diverges at singularity, $\kappa_{\text{s},t_0} = (-108\pi G\phi_0^2 r^2)^{-1} < 0$ which is $\kappa_{\text{s},t_0} \sim -7.4 \times 10^{118}$ (with $r \sim 10^{-60}$, $8\pi G \equiv 1$, $m = 1$ and $\phi_0 = 1$). The constraint $\dot{G}/G|_{t_0} \lesssim 10^{-12}$ year^{-1} limits the κ value to $-7.4 \times 10^{118} \lesssim \kappa \lesssim 7.4 \times 10^{118}$ in Planck unit. The allowed range includes the singularity value of κ. That is κ_{s,t_0} is very slightly greater than -7.4×10^{118}. Considering the equation of state of this case, $\kappa > 0$ is favored and this is not forbidden by the gravitational constant variation constraint. The other case is to consider purely NMDC term without potential, i.e. $m = 0$. Since $\varepsilon s^2 = -m^2$, $s = -3r$ and $\varepsilon = 0$, hence if $m = 0$, s can take any value. Therefore existence or absence of the potential does not make any difference for the $\dot{G}_{\text{eff}}/G_{\text{eff}}$.

4.6.2 Variation of G: canonical scalar field case ($\varepsilon = 1$)

For the case $s = im/\sqrt{\varepsilon}$, we have

$$\frac{\dot{G}_{\rm eff}}{G_{\rm eff}} = \frac{-\left(\frac{\alpha\kappa m^3}{\varepsilon\sqrt{\varepsilon}}\right)\sin\left(\frac{2mt}{\sqrt{\varepsilon}}\right) + i\left(\frac{\beta\kappa^2 m^5}{\sqrt{\varepsilon}}\right)\left[\frac{2\cos\left(\frac{2mt}{\sqrt{\varepsilon}}\right)}{\alpha\kappa m^2\varepsilon} - 1\right]}{1 - \left(\frac{\alpha\kappa m^2}{\varepsilon}\right)\cos\left(\frac{2mt}{\sqrt{\varepsilon}}\right) + \frac{\beta\kappa^2 m^4}{2}} \tag{4.83}$$

(4.84)

and for the case $s = -im/\sqrt{\varepsilon}$,

$$\frac{\dot{G}_{\rm eff}}{G_{\rm eff}} = \frac{-\left(\frac{\alpha\kappa m^3}{\varepsilon\sqrt{\varepsilon}}\right)\sin\left(\frac{2mt}{\sqrt{\varepsilon}}\right) + i\left(\frac{\beta\kappa^2 m^5}{\sqrt{\varepsilon}}\right)\left[-\frac{2\cos\left(\frac{2mt}{\sqrt{\varepsilon}}\right)}{\alpha\kappa m^2\varepsilon} + 1\right]}{1 - \left(\frac{\alpha\kappa m^2}{\varepsilon}\right)\cos\left(\frac{2mt}{\sqrt{\varepsilon}}\right) + \frac{\beta\kappa^2 m^4}{2}} \tag{4.85}$$

(4.86)

where $\alpha \equiv 24\pi G\phi_0^2$ and $\beta \equiv 288\pi^2 G^2\phi_0^4$. For $\varepsilon = 1$, we see that real parts of equations (4.84) and (4.86) are the same, i.e. the cases $s = im/\sqrt{\varepsilon}$ and $s = -im/\sqrt{\varepsilon}$ give the same real value of the equation of state. Considering present time, $t_0 = 0$, therefore

$$\mathrm{Re}\left[\frac{\dot{G}_{\rm eff}}{G_{\rm eff}}\right]_{t_0} = 0\,. \tag{4.87}$$

Hence in this case, at present time, there is no variation in the gravitational constant.

4.6.3 Variation of G: phantom scalar field case ($\varepsilon = -1$)

Considering $\varepsilon = -1$ case, for $s = im/\sqrt{\varepsilon} = m$, the variation is,

$$\frac{\dot{G}_{\rm eff}}{G_{\rm eff}} = \frac{-\alpha\kappa m^3\sinh(2mt) - \beta\kappa^2 m^5\left[\frac{2\cosh(2mt)}{\alpha\kappa m^2} + 1\right]}{1 + \alpha\kappa m^2\cosh(2mt) + \beta\kappa^2 m^4/2} \tag{4.88}$$

and for $s = -im/\sqrt{\varepsilon} = -m$,

$$\frac{\dot{G}_{\rm eff}}{G_{\rm eff}} = \frac{-\alpha\kappa m^3\sinh(2mt) + \beta\kappa^2 m^5\left[\frac{2\cosh(2mt)}{\alpha\kappa m^2} + 1\right]}{1 + \alpha\kappa m^2\cosh(2mt) + \beta\kappa^2 m^4/2}\,, \tag{4.89}$$

where the distinct is the signs of the second term in each case. At present time, we set $t_0 = 0$, hence

$$\left[\frac{\dot{G}_{\rm eff}}{G_{\rm eff}}\right]_{t_0} = \frac{\mp 288\pi^2 G^2\phi_0^4\kappa^2 m^5\left[\frac{2/3}{8\pi G\phi_0^2\kappa m^2} + 1\right]}{1 + 24\pi G\phi_0^2\kappa m^2 + 144\pi^2 G^2\phi_0^4\kappa^2 m^4}\,, \tag{4.90}$$

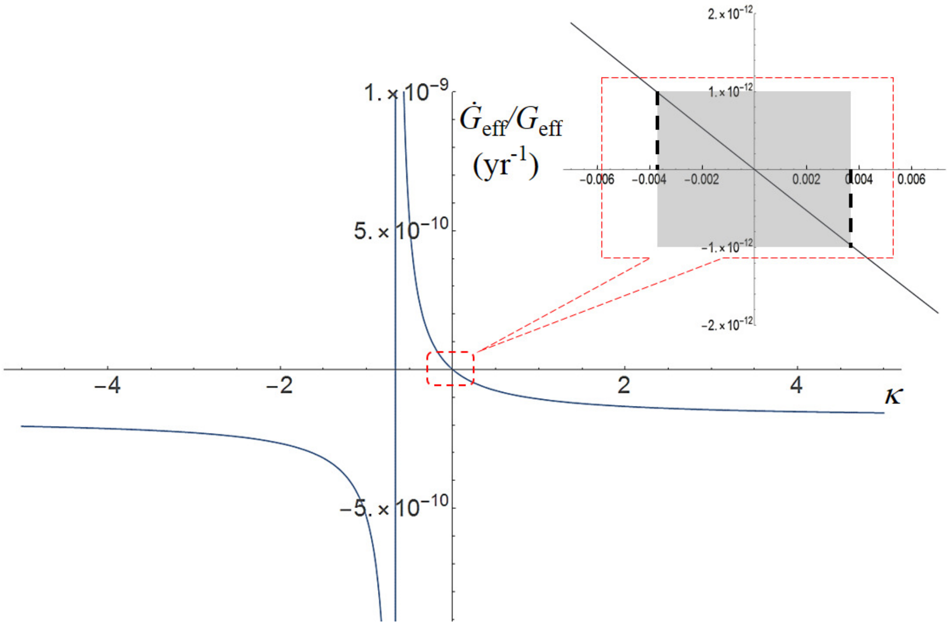

Figure 17: Variation of G_{eff} in unit of year^{-1} versus κ in Planck unit for the case of $\varepsilon = -1$ and $s = m$.

The $\mp$ sign denotes the case $s = m$ and $s = -m$ respectively. There is singularity at

$$\kappa_{\text{s},t_0} = \frac{-2/3}{8\pi G\phi_0^2 m^2} \,. \tag{4.91}$$

or in Planck unit, it is $\kappa_{\text{s},t_0} = -2/3$. The constraint $\dot{G}/G|_{t_0} \lesssim 10^{-12}$ year^{-1} limits the value of κ here. In the both cases ($s = m$ and $s = -m$) the constraints are the same, that is $-0.0038 \lesssim \kappa \lesssim 0.0038$. We see how the ratio $\dot{G}_{\text{eff}}/G_{\text{eff}}$ changes with κ for the phantom case in Fig. 17 and Fig. 18. The grey shade denotes the constraint on κ at present time. This contradicts to the results in section 4.5.3 which requires the coupling to be super-Planckian or very large.

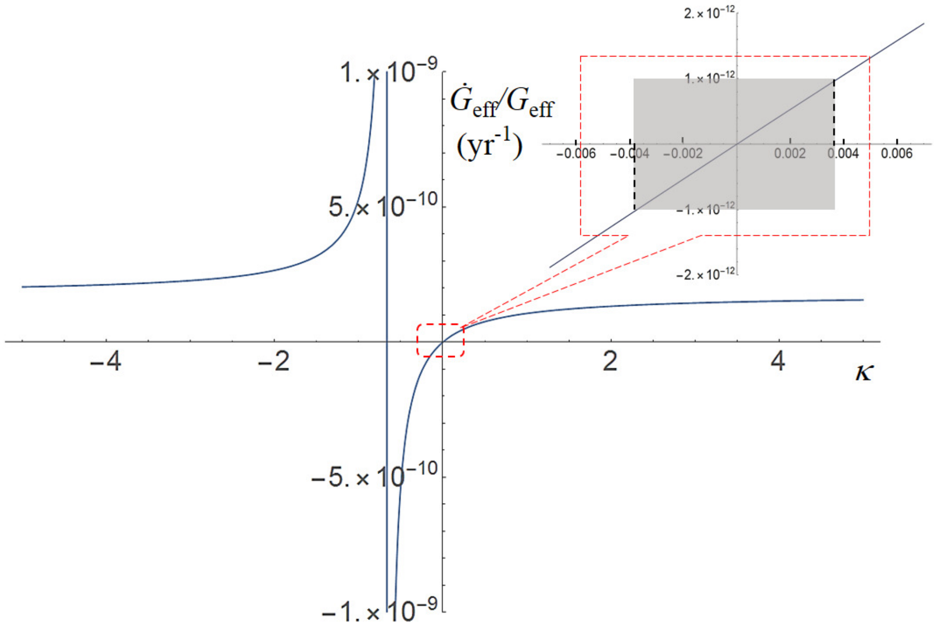

Figure 18: Variation of $G_{\rm eff}$ in unit of year^{-1} versus κ in Planck unit for the case of $\varepsilon = -1$ and $s = -m$.

4.7 Chapter Conclusions and Discussions

In this work, we study non-minimal derivative coupling (NMDC) to gravity in spatially flat FLRW universe in context of holographic dark energy. Having free kinetic term and NMDC term which is the kinetic term coupling to the Einstein tensor with constant coupling strength κ. The scalar potential in this study is $V = (1/2)m^2\phi^2$ and the field is allowed to be phantom, i.e. negative free kinetic term. In the NMDC gravity, gravitational constant is modified with the NMDC kinetic term. The limits of dark energy is introduced with the holographic IR cutoffs which takes cosmological scale, that is the Hubble horizon as the cutoff length scale of the theory. Hence dark energy density $\rho_\Lambda = 3c^2H^2/(8\pi G_{\rm eff})$ has combined NMDC and holographic modification effects. Assuming exact solution of the theory, we evaluate dark energy equation of state and the variation of gravitational constant of the theory in many possibilities of the solutions. We put some constraints such that we can rule out some cases of consideration. Conclusions for each possibility are,

- **Purely NMDC term with or without scalar potential:**

The purely NMDC case renders $w_\Lambda \longrightarrow -1$ at late time for $\kappa > 0$ while $\kappa < 0$ case is not favored since w_Λ either diverges from $[-1, 1]$ or approaches -1 from the region with $w_\Lambda < -1$. Hence $\kappa > 0$ case gives acceptable behavior. The scalar field evolve as $\phi = \phi_0 \exp(-3rt)$ with r could be as small as $\sim 10^{-60}$ In this case w_Λ bends to -1 earlier for less r as shown in Fig. 4. For purely NMDC case without $V(\phi)$, the universe is static at $a = a_0$ at all time. Gravitational constant variation constraint results that $-7.4 \times 10^{118} \lesssim \kappa \lesssim 7.4 \times 10^{118}$ in Planck unit. The allowed range includes the singularity value κ_{s,t_0}. The $\kappa > 0$ is hence allowed by the gravitational constant variation constraint. In purely NMDC case, exclusion of the scalar potential term does not affect the $\dot{G}_{\mathrm{eff}}/G_{\mathrm{eff}}$. Therefore the range $0 < \kappa \lesssim 7.4 \times 10^{118}$ is allowed for purely NMDC theory with potential $V = (1/2)m^2\phi^2$. Realisticly, the coupling can take the sub-Planckian value, $0 < \kappa < 1$.

- **Free kinetic and NMDC terms,** $V(\phi) = 0$**:**

When allowing free kinetic term in the dynamics, as in equations (4.19) and (4.20), the free kinetic term takes part in the damping and the NMDC term takes part in modification of the *force*, i.e. the modifying slope of the potential with an extra piece $(6\kappa H\dot{H}\dot{\phi}/(\varepsilon - 3\kappa H^2))$. Having both free kinetic and NMDC term but without $V(\phi)$ results in static solution, i.e. $\phi = \phi_0$ with $r = (\sqrt{3\kappa\varepsilon})^{-1}$, (positive root). The equation of state is $w_\Lambda = -1$. This corresponds to de-Sitter expansion $a = a_0 e^{\pm t/\sqrt{3\kappa\varepsilon}}$. with effective cosmological constant $\lambda \equiv (\kappa\varepsilon)^{-1}$. Either $\varepsilon = 1, \kappa > 0$ or $\varepsilon = -1, \kappa < 0$ must be chosen. There is no variation of the gravitational constant hence satisfying the constraint. Considering canonical scalar field and $\kappa > 0$ playing the role of cosmological constant, the coupling is required to be as large as about 10^{120}.

- **Free kinetic and NMDC terms with scalar potential (canonical field):**

The free kinetic term is considered in two possibilities, canonical or phantom fields. For the canonical field, some solutions of the theory (case 2) with $\kappa > 0$ are favored. The favored case is the one with $w_\Lambda \to -1$ at late time and w_Λ are not in the phantom region in the past history. This is with conditions, $s = -im$ with $r = [6 - m^2\kappa(1 - \sqrt{1 - 12/(m^2\kappa)})]^{1/2}/\sqrt{18\kappa}$ and other conditions as stated in case 2. As seen in Fig. 6 and 9, larger κ makes w_Λ approaching -1 sooner. The expansion function is $a = a_0 \exp(rt)$ with field oscillation, $\phi = \phi_0 \exp(-imt)$ Singularity of w_Λ at present is $\kappa_{\rm s,t_0} = 2/3$ and the singularity in w_Λ at any time t only happens in limited range $-2/3 \leq \kappa_{\rm s} \leq 2/3$. In this case, there is no variation in gravitational constant at present time. $k > 0$ is favored with oscillating scalar field.

- **Free kinetic and NMDC terms with scalar potential (phantom field):**

For phantom field, $s = -m$ is preferred as it results that equation of state approaches -1 at late time. Negative s prevents increasing scalar field function unless it evolves to super-Planckian regime. Positive coupling, $\kappa > 0$ is required to avoid imaginary r and imaginary mass m. However the shortcomings are that the scalar mass m and the coupling κ are required to be super-Planckian while the constraint on gravitational constant variation puts a very narrow viable range of κ as $-0.0038 \lesssim \kappa \lesssim 0.0038$.

In this scenario, in all cases $\kappa > 0$ is favored. The purely NMDC theory with the potential $V = (1/2)m^2\phi^2$ is favored with positive sub-Planckian coupling, satisfying $w_{\Lambda,t_0} \to -1$ and variation of gravitational constant constraint. The other viable case is the canonical field with NMDC term under the same potential with conditions stated in case 2 (section 4.5.2). In this model, it is noticed that the NMDC coupling, κ only affects r (the exponent of the scale factor) but it does not contribute to any modification of s which is the main character of scalar field evolution. Hence further study can be investigated on the adjusting of this point, that is, if the model can be modified such that the NMDC coupling could contribute to both r and s, interesting dynamics of the scalar field can be further studied.

CHAPTER V

THESIS CONCLUSION

In this work, we express NLS formulation of FLRW cosmology with canonical scalar field (evolving under unspecified potential) and two barotropic fluids. The first barotropic density (D_1) is related to NLS total energy (E) (see Eq. (3.16) and the second barotropic fluid density (D_2) contributes to additional term in $P(x)$ (see Eq. (3.34)). The choice of not adding D_2 term into definition of E is because E must be constant in deriving solutions. We give a lists of Friedmann formulation variables expressed in terms of NLS variables for two barotropic fluids case. The second part of this work is to explore seven solutions given in [1]. The solutions considered in this work base on top-down deducing derivation from the equation of motion (NLS equation). These are solutions of the system of scalar field with barotropic fluids under NLS potential ($P(x)$) listed in table 1. In addition, we found one new solution which gives the same result as of the sixth solution of [1]. Their cosmological expansions are checked and none is found to agree with realistic solution depicted by observation.

It is noticed that previous works in ([54], [57], [58], [59], [60]) assumed forms of the expansion functions, $a(t)$. These are power-law ($a \sim t^q$), de-Sitter ($a \sim \exp(t/\tau)$ and super-acceleration ($a \sim (t_a - t)^q$) (with constant q and τ). These expansion functions are converted to the explicit form of NLS solutions, $u(x)$. Although it is true that $u(x)$ are exact solutions but assuming the expansion forms is to force the problem to take the assumed answers in a bottom-up direction of reasoning. These alter the form of scalar potential $V = V(u, u') = V(a, \dot{a})$ to adjust so that the dynamics can accommodate the assumed expansions. Hence it is not a natural procedure. This is unlike conventional derivation of which at beginning step, $V(\phi)$ is taken from high energy physics motivation and as a result, solutions and Ω_ϕ are derived.

All solutions-the NLS wave functions $u(x)$ found here are non-normalizable (as it was previously claimed for a specific case of power-law expansion [57]). Hence it can not be probabilistically interpreted. The NLS total energy E is negative therefore it is not physical. The NLS formulation interpretation in quantum cosmology that $u(x)$ and E could be the wave function and total energy of the universe is not acceptable.

REFERENCES

REFERENCES

[1] D'Ambroise J. Generalized EMP and nonlinear Schrodinger-type reformulations of some scalar field cosmological models. Massachusetts U., Amherst; 2010.

[2] Masi S, et al. The BOOMERanG experiment and the curvature of the universe. Prog Part Nucl Phys. 2002 12;48(5):243–261.

[3] Spergel D, et al. Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology. Astrophys J Suppl. 2007;170:377.

[4] Hinshaw G, et al. Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Data Processing, Sky Maps, and Basic Results. Astrophys J Suppl. 2009 Sep;180(3):225–245.

[5] Dunkley J, et al. Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Likelihoods and Parameters from the WMAP data. Astrophys J Suppl. 2009 Sep;180(3):306–329.

[6] Komatsu E, et al. Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation. Astrophys J Suppl. 2009 Sep;180(3):330–376.

[7] Scranton R, et al. Physical evidence for dark energy. Astroparticle Physics. 2003 Aug;30(9):186–191.

[8] Tegmark M, et al. Cosmological parameters from SDSS and WMAP. Phys Rev. 2004 Apr;D69(11):103501.

[9] Astier P, et al. The Supernova Legacy Survey: Measurement of Ω_M, Ω_Λ and w from the first year data set. Astron Astrophys. 2006 Jan;447(11):31–48.

[10] Goldhaber G, et al. Timescale stretch parameterization of Type Ia supernova B-band light curves. Astrophys J. 2001 Jan;558:359–368.

[11] Perlmutter S, et al. Discovery of a supernova explosion at half the age of the Universe and its cosmological implications. Nature. 1998 Nov;391(11):51–54.

[12] Perlmutter S, et al. Measurements of Ω and Λ from 42 high redshift supernovae. Astrophys J. 1999 Apr;517(3):565–586.

[13] Riess A, et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. Astron J. 1998 jun;116(13):1009–1038.

[14] Riess A. Peculiar velocities from type ia supernovae. ASP Conf Ser. 2000 Aug;201(2):80.

[15] Riess A, et al. Type Ia supernova discoveries at $z > 1$ from the Hubble Space Telescope: Evidence for past deceleration and constraints on dark energy evolution. Astrophys J. 2004 Jan;607:665–687.

[16] Riess A, et al. New Hubble Space Telescope Discoveries of Type Ia Supernovae at $z >= 1$: Narrowing Constraints on the Early Behavior of Dark Energy. Astrophys J. 2007 Apr;659(11):98–121.

[17] Amanullah R, et al. Spectra and Light Curves of Six Type Ia Supernovae at$0.511 < z < 1.12$ and the Union2 Compilation. Astrophys J. 2010 12;716(5):712–738.

[18] Balland C, et al. Spectroscopy of twelve Type Ia supernovae at intermediate redshift. Astron Astrophys. 2006 12;445(5):387.

[19] Guth A. Inflation and eternal inflation. Physics Reports. 2000 Jan;333-334(5): 555 – 574.

[20] Starobinsky A. Inflation for large scale structure. In: 12th Potsdam Cosmology Workshop: Large Scale Structure: Tracks and Traces: International Workshop Potsdam, Germany, September 15-20, 1997. vol. 23. World Scientific Publishing Co. Pte. Ltd.; 1997. p. 347–356.

[21] Linde A. A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems. Physics Letters B. 1982 jan;108(6):389 – 393.

[22] Lyth D, Liddle AR. The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure. vol. 447. Cambridge University Press; 2009.

[23] Amendola L, Tsujikawa S. Dark Energy: Theory and Observations. vol. C930308. Cambridge University Press; 2010.

[24] Hartley J, Ray J. Ermakov systems and quantum-mechanical superposition laws. Phys Rev A. 1981 Dec;24(3):2873–2876.

[25] Ermakov P. Differential equations of the second order and the conditions of integrability in finite form. Univ Izv Kiev. 1880 Nov;C930308(11):284–296.

[26] Hawkins R, Lidsey J. Ermakov-Pinney equation in scalar field cosmologies. Phys Rev D. 2002 Jul;66(3):023523.

[27] Clifton T, Ferreira PG, Padilla A, Skordis C. Modified Gravity and Cosmology. Phys Rept. 2012 Apr;513(3):1–189.

[28] Horndeski G. Second-order scalar-tensor field equations in a four-dimensional space. Int J Theor Phys, v 10, no 6, pp 363-384. 1974 sep;2008(9):028.

[29] Kobayashi T, Yamaguchi M, Yokoyama J. Generalized G-Inflation: —Inflation with the Most General Second-Order Field Equations—. Progress of Theoretical Physics. 2011 09;126(3):511–529.

[30] Deffayet C, Gao X, Steer DA, Zahariade G. From k-essence to generalized Galileons. Phys Rev D. 2011 Sep;84(3):064039.

[31] Amendola L. Cosmology with nonminimal derivative couplings. Phys Lett. 1993 Apr;B301(11):175–182.

[32] Capozziello S, Lambiase G, Schmidt HJ. Nonminimal Derivative Couplings and Inflation in Generalized Theories of Gravity. Annalen der Physik. 2000 jan;9(1):39–48.

[33] Granda L. Non-minimal kinetic coupling and the phenomenology of dark energy. Class Quant Grav. 2011 jan;28(1):025006.

[34] Granda L. Inflation driven by scalar field with non-minimal kinetic coupling with Higgs and quadratic potentials. Journal of Cosmology and Astroparticle Physics. 2011 Jan;1104(5):016.

[35] Sushkov S. Exact cosmological solutions with nonminimal derivative coupling. Phys Rev. 2009 Nov;D80(3):103505.

[36] Burin G, Phongsaphat R. Non-minimal derivative coupling gravity in cosmology. Gen Rel Grav. 2015 jan;47(11):140.

[37] Starobinsky A. Spectrum of relict gravitational radiation and the early state of the universe. JETP Lett (USSR) (Engl Transl); (United States). 1979 12;30:11(5):243–261.

[38] Kazanas D. Dynamics of the universe and spontaneous symmetry breaking. The Astrophysical Journal Letters. 1980 Oct;241(11):L59–L63.

[39] Guth A. Inflationary universe: A possible solution to the horizon and flatness problems. Phys Rev D. 1981 Jan;23(5):347–356.

[40] Sato K. First-order phase transition of a vacuum and the expansion of the Universe. Monthly Notices of the Royal Astronomical Society. 1981 07;195(3):467–479.

[41] Gamow G. Expanding Universe and the Origin of Elements. Phys Rev. 1946 Oct;70(3):572–573.

[42] Bennett C, Banday A, Gorski K, Hinshaw G, Jackson P, Keegstra P, et al. Four year COBE DMR cosmic microwave background observations: Maps and basic results. Astrophys J. 1996 Sep;464(3):L1–L4.

[43] Pinney E. The Nonlinear Differential Equation $y'' + p(x)y + cy^{-3} = 0$. Proceedings of the American Mathematical Society. 1950 Jan;1(5):681–681.

[44] Milne WE. The Numerical Determination of Characteristic Numbers. Phys Rev. 1930 Apr;35:863–867.

[45] Ermakov V. Second-order differential equations: Conditions of complete integrability. Applicable Analysis and Discrete Mathematics. 2008 10;2(9):363–384.

[46] Williams F, Kevrekidis P. On (2+1)-dimensional Friedmann-Robertson-Walker universes: an Ermakov-Pinney equation approach. Classical and Quantum Gravity. 2003 jun;20(13):L177–L184.

[47] Williams F, G P, Christodoulakis T, Helias C, Papadopoulos G, Grammenos T. On 3+1 Dimensional Scalar Field Cosmologies. vol. 18. Cambridge University Press; 2004. p. 269–278.

[48] Lidsey J. Cosmic dynamics of Bose-Einstein condensates. Class Quant Grav. 2004 Apr;21(11):777–786.

[49] D'Ambroise J, Williams F. A Non-linear Schrodinger type formulation of FLRW scalar field cosmology. Modern Physics Letters A. 2006 jan;32(13):1750078.

[50] D'Ambroise J. A Schrodinger formulation of Bianchi I scalar field cosmology. Journal of Cosmology and Astroparticle Physics. 2007 jan;2020(1):053–053.

[51] Kamenshchik A, Venturi G. Remarks on the method of comparison equations (generalized WKB method) and the generalized Ermakov-Pinney equation. Russ Phys J. 2009 10;52(9):1339–1351.

[52] Williams F. Einstein field equations: An alternate approach towards exact solutions for an FRW universe. International Journal of Modern Physics A. 2005 Feb;20(11):2481–2484.

[53] Hawkins R, Lidsey J. The Ermakov-Pinney Equation in Scalar Field Cosmologies. Physical Review D. 2001 12;66(5):681–681.

[54] Burin G. Scalar field exact solutions for non-flat FLRW cosmology: A technique from non-linear Schrodinger-type formulation. Gen Rel Grav. 2009 10;41(9):249–265.

[55] D'Ambroise J, Williams F. Parametric solution of certain nonlinear differential equations in cosmology. Journal of Nonlinear Mathematical Physics. 2011 Aug;18(2):269–278.

[56] Lidsey J. Scalar Field Cosmologies Hidden Within the Nonlinear Schrodinger Equation. Gen Rel Grav. 2013 10;41(9):249–265.

[57] Gumjudpai B. Power-law expansion cosmology in Schrodinger-type formulation. Astroparticle Physics. 2008 Aug;30(9):186–191.

[58] Phetnora T, Sooksan R, Gumjudpai B. Phantom expansion with non-linear Schrödinger-type formulation of scalar field cosmology. General Relativity and Gravitation. 2010 Feb;42(2):225–240.

[59] Gumjudpai B. Scalar field cosmology: Its non-linear Schrodinger-type formulation. Gen Rel Grav. 2009 10;41(9):249–265.

[60] Gumjudpai B. Slow-roll, acceleration, the big rip and the Wentzel-Kramers-Brillouin approximation in the non-linear Schrödinger-type formulation of scalar field cosmology. Journal of Cosmology and Astroparticle Physics. 2008 sep;2008(9):028.

[61] Liu H, Tseytlin A. $D = 4$ Super Yang Mills, $D = 5$ gauged supergravity and $D = 4$ conformal supergravity. Nucl Phys B. 1998 Apr;533(11):88–108. 21 p.

[62] Nojiri S, Odintsov SD. Conformal anomaly for dilaton coupled theories from AdS/CFT correspondence. Physics Letters B. 1998 dec;444(10):92–97.

[63] Sushkov S. Exact cosmological solutions with nonminimal derivative coupling. Phys Rev D. 2009 Nov;80(3):103505.

[64] Wetterich C. Cosmology and the Fate of Dilatation Symmetry. Nucl Phys. 1988 Apr;B302(11):668–696.

[65] Caldwell R, Kamionkowski M, Weinberg N. Phantom Energy: Dark Energy with $w < -1$ Causes a Cosmic Doomsday. Phys Rev Lett. 2003 Aug;91(2):071301.

[66] Armendariz-Picon C, Mukhanov V, Steinhardt P. Dynamical Solution to the Problem of a Small Cosmological Constant and Late-Time Cosmic Acceleration. Phys Rev Lett. 2000 Nov;85(11):4438–4441.

[67] Cai R. A dark energy model characterized by the age of the Universe. Physics Letters B. 2007 jan;657(4):228 – 231.

[68] Wei H, Cai R. A new model of agegraphic dark energy. Physics Letters B. 2008 jan;660(3):113 – 117.

[69] Nojiri S, Odintsov S, Tsujikawa S. Properties of singularities in the (phantom) dark energy universe. Phys Rev D. 2005 Mar;71(9):063004.

[70] Ohta N. DARK ENERGY AND QCD GHOST. International Journal of Modern Physics: Conference Series. 2012 Nov;07(11):194–201.

[71] Moradpour H, Nunes R, Abreu E, Neto J. A note on the relations between thermodynamics, energy definitions and Friedmann equations. Modern Physics Letters A. 2017 jan;32(13):1750078.

[72] Bamba K, Capozziello S, Nojiri S, Odintsov S. Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests. Astrophys Space Sci. 2012 Nov;342(11):155–228.

[73] Faraoni V, Capozziello S. Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics. vol. 2020 of Fundamental Theories of Physics. Dordrecht: Springer; 2011.

[74] 't Hooft G. Dimensional reduction in quantum gravity. Conf Proc. 1993 Nov;C930308(11):284–296.

[75] Leonard S. The world as a hologram. Journal of Mathematical Physics. 1995 Jan;36(11):6377–6396.

[76] Cohen A, Kaplan D, Nelson A. Effective Field Theory, Black Holes, and the Cosmological Constant. Phys Rev Lett. 1999 Jun;82(11):4971–4974.

[77] Bekenstein J. Black Holes and Entropy. Phys Rev D. 1973 Apr;7(10):2333–2346.

[78] Bekenstein J. Generalized second law of thermodynamics in black-hole physics. Phys Rev D. 1974 Jun;9(5):3292–3300.

[79] Bekenstein J. Universal upper bound on the entropy-to-energy ratio for bounded systems. Phys Rev D. 1981 Jan;23:287–298.

[80] Bekenstein J. Entropy bounds and black hole remnants. Phys Rev D. 1994 Feb;49(1):1912–1921.

[81] Hawking S. Black holes and thermodynamics. Phys Rev D. 1976 Jan;13(10):191–197.

[82] Hořava P, Minic D. Probable Values of the Cosmological Constant in a Holographic Theory. Phys Rev Lett. 2000 Aug;85(9):1610–1613.

[83] Thomas S. Holography Stabilizes the Vacuum Energy. Phys Rev Lett. 2002 Jul;89(5):081301.

[84] Hsu S. Entropy bounds and dark energy. Physics Letters B. 2004 Dec;594(1): 13 – 16.

[85] Li M. A model of holographic dark energy. Physics Letters B. 2004 jan;603(1): 1 – 5.

[86] Guberina B, Horvat R, Nikolić H. Non-saturated holographic dark energy. Journal of Cosmology and Astroparticle Physics. 2007 jan;2007(1):012–012.

[87] Myung Y. Instability of holographic dark energy models. Physics Letters B. 2007 Apr;652(5):223 – 227.

[88] Charmousis C, Copeland EJ, Padilla A, Saffin PM. General Second-Order Scalar-Tensor Theory and Self-Tuning. Phys Rev Lett. 2012 Jan;108(3):051101.

[89] Horndeski G. Second-order scalar-tensor field equations in a four-dimensional space. Int J Theor Phys. 1974 sep;10(9):363–384.

[90] Zhao W, Wright B, Li B. Constraining the time variation of Newton's constant G with gravitational-wave standard sirens and supernovae. Journal of Cosmology and Astroparticle Physics. 2018 oct;2018(10):052–052.

[91] Kaspi V, Taylor J, Ryba M. High - precision timing of millisecond pulsars. 3: Long - term monitoring of PSRs B1855+09 and B1937+21. Astrophys J. 1994 Jul;428(5):713.

[92] Armaleo J, Nacir D, Urban F. Binary pulsars as probes for spin-2 ultralight dark matter. Journal of Cosmology and Astroparticle Physics. 2020 jan;2020(1):053–053.

[93] Williams J, Turyshev S, Boggs D. Progress in Lunar Laser Ranging Tests of Relativistic Gravity. Phys Rev Lett. 2004 Dec;93(9):261101.

[94] Copi C, Davis A, Krauss L. New Nucleosynthesis Constraint on the Variation of G. Phys Rev Lett. 2004 Apr;92(11):171301.

[95] Cyburt R, Fields B, Olive K, Skillman E. New BBN limits on physics beyond the standard model from 4He. Astroparticle Physics. 2005 Apr;23(3):313 – 323.

APPENDIX

APPENDIX A DERIVATION OF THE NLS EXACT SOLUTIONS

Following the table 1, we demonstrate the detail of calculation in section 3.3.

A.1 Solution 1 : $u(x) = e_0x^2 + b_0x + c_0$

A.1.1 Case 1.1: $e_0 \neq 0$

The mapping between x and $u(x)$ is

$$u(x) \equiv \dot{x}. \tag{A.1}$$

So

$$e_0x^2 + b_0x + c_0 = \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{A.2}$$

Using the integration formula

$$\int \frac{1}{ax^2+bx+c}\mathrm{d}x = \frac{2}{\sqrt{4ac-b^2}} \arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right), \tag{A.3}$$

in Eq.(A.2) yields

$$\begin{aligned}
\int_{t_0}^{t} dt &= \int_0^x \frac{1}{e_0x^2+b_0x+c_0}\mathrm{d}x && \text{(A.4)}\\
t - t_0 &= \frac{2}{\sqrt{-\Delta}} \arctan\left(\frac{2e_0x-b_0}{\sqrt{-\Delta}}\right) && \text{(A.5)}\\
\tan\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right] &= \frac{2e_0x-b_0}{\sqrt{-\Delta}} && \text{(A.6)}\\
x &= \frac{1}{2e_0}\left\{\sqrt{-\Delta}\tan\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right] - b_0\right\}, && \text{(A.7)}
\end{aligned}$$

where $\Delta = b_0^2 - 4e_0c_0$. Due to $u = \dot{x}$, Taking time derivative to Eq.(A.7) provides

$$\begin{aligned}
u(t) &= \frac{1}{2e_0}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sqrt{-\Delta}\tan\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right] - b_0\right\}, && \text{(A.8)}\\
&= \frac{-\Delta}{4e_0}\left\{\sec^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}. && \text{(A.9)}
\end{aligned}$$

Then the scale factor evaluated from the relation

$$u(t) = a(t)^{-n/2}, \tag{A.10}$$

takes the form of

$$a(t)^{-n/2} = \frac{-\Delta}{4e_0}\left\{\sec^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}, \tag{A.11}$$

$$a(t) = \left\{\frac{-\Delta}{4e_0}\sec^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}^{-2/n}, \tag{A.12}$$

$$= \left\{\frac{4e_0}{-\Delta}\cos^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}^{2/n}. \tag{A.13}$$

By the definition of time-redshift relation,$z = a(t_0)/a(t) - 1$, therefore

$$z = \left\{\frac{\frac{4e_0}{-\Delta}}{\frac{4e_0}{-\Delta}\cos^2(t-t_0)}\right\}^{2/n} - 1, \tag{A.14}$$

$$= \left\{\sec^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}^{2/n} - 1, \tag{A.15}$$

$$\tag{A.16}$$

and

$$z+1 = \left\{\sec^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}^{2/n}, \tag{A.17}$$

$$(z+1)^{n/2} = \sec^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right], \tag{A.18}$$

$$(z+1)^{n/4} = \sec\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right], \tag{A.19}$$

$$\operatorname{arcsec}\left[(z+1)^{n/4}\right] = \frac{\sqrt{-\Delta}}{2}(t-t_0), \tag{A.20}$$

$$t - t_0 = \frac{2}{\sqrt{-\Delta}}\left\{\operatorname{arcsec}\left[(z+1)^{n/4}\right]\right\}. \tag{A.21}$$

The Hubble rate can be derive by $H = \dot{a}/a$. Hence

$$\dot{a} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{4e_0}{-\Delta}\cos^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}^{2/n}, \quad (A.22)$$

$$= \frac{\not{2}}{n}\frac{\sqrt{-\Delta}}{\not{2}}\frac{4e_0}{-\Delta}\left\{\frac{4e_0}{-\Delta}\cos^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}^{\frac{2}{n}-1} \quad (A.23)$$

$$\times 2\cos\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\left\{-\sin\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\} \quad (A.24)$$

$$H(t) = \frac{\dot{a}}{a} = \frac{-\frac{8e_0}{n\sqrt{-\Delta}}\left\{\frac{4e_0}{-\Delta}\cos^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}^{\frac{2}{n}-1}}{\left\{\frac{4e_0}{-\Delta}\cos^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}^{\frac{2}{n}}} \quad (A.25)$$

$$\times\cos\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\left\{\sin\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\right\}, \quad (A.26)$$

$$= \frac{-\frac{8e_0}{n\sqrt{-\Delta}}\frac{(-\Delta)}{4e_0}\cos\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]\sin\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]}{\cos^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]}, \quad (A.27)$$

$$= -\frac{2\sqrt{-\Delta}}{n}\tan\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right]. \quad (A.28)$$

According to Eq(A.21), the Hubble parameter related to redshift representation is hence

$$H(z) = -\frac{2\sqrt{-\Delta}}{n}\tan\left[\frac{\sqrt{-\Delta}}{2}\left(\frac{2}{\sqrt{-\Delta}}\left\{\mathrm{arcsec}\left[(z+1)^{n/4}\right]\right\}\right)\right], \quad (A.29)$$

$$= -\frac{2\sqrt{-\Delta}}{n}\tan\left\{\mathrm{arcsec}\left[(z+1)^{n/4}\right]\right\}. \quad (A.30)$$

A.1.2 Case 1.2: $e_0 = 0$ The solution becomes

$$u(x) = b_0 x + c_0. \quad (A.31)$$

The solution in term of x can be solved by

$$\int_{t_0}^{t} dt = \int_0^x \frac{1}{b_0 x + c_0}dx, \quad (A.32)$$

$$t - t_0 = \frac{1}{b_0}\ln|b_0 x + c_0|, \quad (A.33)$$

$$e^{b_0(t-t_0)} = |b_0 x + c_0|, \quad (A.34)$$

$$x(t) = \frac{e^{b_0(t-t_0)} - c_0}{b_0}, \quad (A.35)$$

for $b_0 \neq 0$ and $b_0 x > c_0$. Then $u(t)$ is hence

$$\dot{x}(t) = u(t) = e^{b_0(t-t_0)}. \quad (A.36)$$

The scale factor and time-redshift perform as

$$a(t) = e^{-2b_0(t-t_0)/n}, \tag{A.37}$$

$$z(t) = \frac{e^{-2b_0(t_0-t_0)/n}}{e^{-2b_0(t-t_0)/n}} - 1, \tag{A.38}$$

$$= e^{2b_0(t-t_0)/n} - 1. \tag{A.39}$$

Hubble rate is hence

$$H(t) = \frac{\dot{a}}{a}, \tag{A.40}$$

$$= \frac{-\frac{2b_0}{n}\cancel{e^{-2b_0(t-t_0)}}}{\cancel{e^{-2b_0(t-t_0)}}}, \tag{A.41}$$

$$= -\frac{2b_0}{n}. \tag{A.42}$$

A.2 Solution 2: $u(x) = e_0\cos^2(b_0x)$

From this solution, $x(t)$ can be evaluated by

$$\mathrm{d}t = \frac{\mathrm{d}x}{e_0\cos^2(b_0x)}, \tag{A.43}$$

$$e_0\int_{t_0}^{t} dt = \int_0^x \sec^2(b_0x)\mathrm{d}x, \tag{A.44}$$

$$e_0(t-t_0) = \frac{1}{b_0}\tan(b_0x), \tag{A.45}$$

$$b_0x = \arctan\left[e_0b_0(t-t_0)\right], \tag{A.46}$$

$$x = \frac{1}{b_0}\arctan\left[e_0b_0(t-t_0)\right]. \tag{A.47}$$

The taking time-derivative of $x(t)$ provides

$$\dot{x} = u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{b_0}\arctan\left[e_0b_0(t-t_0)\right], \tag{A.48}$$

$$= \frac{1}{\cancel{b_0}}\frac{e_0\cancel{b_0}}{1+\left[e_0b_0(t-t_0)\right]^2}, \tag{A.49}$$

$$= \frac{e_0}{1+\left[e_0b_0(t-t_0)\right]^2}. \tag{A.50}$$

The scale factor is determined as

$$a = u(t)^{-2/n}, \tag{A.51}$$

$$= \left\{\frac{e_0}{1+\left[e_0b_0(t-t_0)\right]^2}\right\}^{-2/n}. \tag{A.52}$$

The time-redshift is calculate by

$$
\begin{aligned}
z &= \left\{ \frac{1+[e_0 b_0 (t_0 - t_0)]^2}{e_0} \times \frac{e_0}{1+[e_0 b_0 (t - t_0)]^2} \right\}^{2/n} - 1, && \text{(A.53)} \\
&= \left\{ \frac{1}{1+[e_0 b_0 (t - t_0)]^2} \right\}^{2/n} - 1. && \text{(A.54)} \\
& && \text{(A.55)}
\end{aligned}
$$

In the case of $n = 4$ and $e_0 \neq 0$, the redshift becomes

$$
\begin{aligned}
z &= \left\{ \frac{1}{1+[e_0 b_0 (t - t_0)]^2} \right\}^{2/4} - 1, && \text{(A.56)} \\
&= \sqrt{\frac{1}{1+[e_0 b_0 (t - t_0)]^2}} - 1, && \text{(A.57)}
\end{aligned}
$$

and also

$$
\begin{aligned}
(z+1)^2 &= \frac{1}{1+[e_0 b_0 (t - t_0)]^2}, && \text{(A.58)} \\
(t - t_0) &= \frac{1}{e_0 b_0} \sqrt{\frac{1}{(z+1)^2} - 1}. && \text{(A.59)}
\end{aligned}
$$

The Hubble parameters $H(t)$ and $H(z)$ are

$$
\begin{aligned}
H(t) = \frac{\dot{a}}{a} &= e_0 b_0^2 (t - t_0) \sqrt{\frac{e_0}{1+[e_0 b_0 (t - t_0)]^2}} \sqrt{\frac{e_0}{1+[e_0 b_0 (t - t_0)]^2}}, && \text{(A.60)} \\
&= \frac{e_0^2 b_0^2 (t - t_0)}{1 + e_0^2 b_0^2 (t - t_0)^2}, && \text{(A.61)} \\
H(z) &= \frac{e_0^2 b_0^2 \left(\frac{1}{e_0 b_0} \sqrt{\frac{1}{(z+1)^2} - 1} \right)}{1 + e_0^2 b_0^2 \left(\frac{1}{e_0 b_0} \sqrt{\frac{1}{(z+1)^2} - 1} \right)^2}, && \text{(A.62)} \\
&= e_0 b_0 (z+1) \sqrt{-z(z+2)}. && \text{(A.63)}
\end{aligned}
$$

A.3 Solution 3: $u(x) = e_0 \tanh(b_0 x)$

The solution $x(t)$ which map to $u(x)$ is calculated by

$$\int_{t_0}^{t} \mathrm{d}t = \int_0^x \frac{\mathrm{d}x}{e_0 \tanh(b_0 x)}, \tag{A.64}$$

$$(t - t_0) = \frac{1}{e_0 b_0} \ln|\sinh(b_0 x)|, \tag{A.65}$$

$$e^{e_0 b_0 (t-t_0)} = \sinh(b_0 x), \tag{A.66}$$

$$x(t) = \frac{1}{b_0} \operatorname{arcsinh}\left(e^{e_0 b_0 (t-t_0)}\right). \tag{A.67}$$

Following the identity $\mathrm{d}[\operatorname{arcsinh}(x)] = 1/\sqrt{(x^2+1)}\mathrm{d}x$ Therefore $u(t)$ is written in term of

$$u(t) = \dot{x} = \frac{1}{b_0} \frac{1}{\sqrt{e^{2e_0 b_0 (t-t_0)} + 1}} \frac{\mathrm{d}e^{e_0 b_0 (t-t_0)}}{\mathrm{d}t}, \tag{A.68}$$

$$= \frac{e_0 e^{e_0 b_0 (t-t_0)}}{\sqrt{e^{2e_0 b_0 (t-t_0)} + 1}}. \tag{A.69}$$

This solution is under the condition of $n = 1$, the scale factor is evaluated by

$$a(t) = u(t)^{-2/n} = \left[\frac{e_0 e^{e_0 b_0 (t-t_0)}}{\sqrt{e^{2e_0 b_0 (t-t_0)} + 1}}\right]^{-2}, \tag{A.70}$$

$$= \left[\frac{e^{2e_0 b_0 (t-t_0)} + 1}{e_0^2 e^{2e_0 b_0 (t-t_0)}}\right], \tag{A.71}$$

$$= \frac{1}{e_0^2}\left[1 + e^{-2e_0 b_0 (t-t_0)}\right] \tag{A.72}$$

The calculation of $z(t)$ and $t - t_0$ performs as

$$z(t) = \frac{\cancel{\frac{1}{e_0^2}}\left[1 + e^{-2e_0 b_0 (t_0-t_0)}\right]}{\cancel{\frac{1}{e_0^2}}\left[1 + e^{-2e_0 b_0 (t-t_0)}\right]} - 1, \tag{A.73}$$

$$= \frac{2}{1 + e^{-2e_0 b_0 (t-t_0)}} - 1, \tag{A.74}$$

$$z(t) + 1 = \frac{2}{1 + e^{-2e_0 b_0 (t-t_0)}}, \tag{A.75}$$

$$e^{-2e_0 b_0 (t-t_0)} = \frac{2}{z(t) + 1} - 1, \tag{A.76}$$

$$-2e_0 b_0 (t - t_0) = \ln\left(\frac{2}{z+1} - 1\right), \tag{A.77}$$

$$t - t_0 = -\frac{1}{2e_0 b_0} \ln\left(\frac{2}{z+1} - 1\right). \tag{A.78}$$

Using $H(t) = \dot{a}/a$, the Hubble rate calculation presented by

$$H(t) = -\frac{\cancel{\frac{1}{e_0^2}}2e_0b_0e^{-2e_0b_0(t-t_0)}}{\cancel{\frac{1}{e_0^2}}\left[1+e^{-2e_0b_0(t-t_0)}\right]}, \tag{A.79}$$

$$= -\frac{2e_0b_0}{1+e^{2e_0b_0(t-t_0)}}. \tag{A.80}$$

Substituting Eq.(A.78) to $H(t)$ yields

$$H(z) = -\frac{2e_0b_0}{1+e^{2e_0b_0\left[-\frac{1}{2e_0b_0}\ln\left(\frac{2}{z+1}-1\right)\right]}}, \tag{A.81}$$

$$= e_0b_0(z-1). \tag{A.82}$$

A.4 Solution 4: $u(x) = e_0e^{-x\sqrt{-c_0}} - b_0e^{x\sqrt{-c_0}}$

The solution $x(t)$ can be constructed from

$$\int_{t_0}^{t} \mathrm{d}t = \int_0^x \frac{1}{e_0e^{-x\sqrt{-c_0}} - b_0e^{x\sqrt{-c_0}}}\mathrm{d}x, \tag{A.83}$$

$$(t-t_0) = \frac{\operatorname{arctanh}\left(\frac{\sqrt{b_0}e^{\sqrt{c_0}x}}{\sqrt{e_0}}\right)}{\sqrt{e_0b_0c_0}}, \tag{A.84}$$

$$\sqrt{-e_0b_0c_0}(t-t_0) = \operatorname{arctanh}\left(\frac{\sqrt{b_0}e^{\sqrt{-c_0}x}}{\sqrt{e_0}}\right), \tag{A.85}$$

$$e^{\sqrt{-c_0}x} = \sqrt{\frac{e_0}{b_0}}\tanh(\sqrt{-e_0b_0c_0}(t-t_0)), \tag{A.86}$$

$$x = \frac{1}{\sqrt{-c_0}}\ln\left\{\sqrt{\frac{e_0}{b_0}}\tanh[\sqrt{-e_0b_0c_0}(t-t_0)]\right\}. \tag{A.87}$$

Taking time derivative of x provides

$$u(t) = \sqrt{\frac{b_0}{-e_0c_0}}\frac{\operatorname{sech}^2[\sqrt{-e_0b_0c_0}(t-t_0)]}{\tanh[\sqrt{-e_0b_0c_0}(t-t_0)]}(-e_0b_0c_0) \tag{A.88}$$

$$= \frac{2\sqrt{e_0b_0}}{\sinh[2\sqrt{-e_0b_0c_0}(t-t_0)]} \tag{A.89}$$

The scale factor and its time derivative are

$$a(t) = u^{-2/n} = \left\{\frac{\sinh[2\sqrt{-e_0b_0c_0}(t-t_0)]}{2\sqrt{e_0b_0}}\right\}^{2/n}, \tag{A.90}$$

$$\dot{a} = \frac{2}{n}\left\{\frac{\sinh[2\sqrt{-e_0b_0c_0}(t-t_0)]}{2\sqrt{e_0b_0}}\right\}^{2/n} \times 2\sqrt{-e_0b_0c_0}\left\{\frac{\cosh[2\sqrt{-e_0b_0c_0}(t-t_0)]}{\sinh[2\sqrt{-e_0b_0c_0}(t-t_0)]}\right\}. \tag{A.91}$$

Therefore the Hubble parameter is

$$H(t)=\frac{\dot{a}}{a} \quad = \quad \frac{4\sqrt{-e_0b_0c_0}}{n}\coth[2\sqrt{-e_0b_0c_0}(t-t_0)] \tag{A.92}$$

The calculation of the scale factor at $t=t_0$ is

$$a(t_0) \quad = \quad \frac{2}{n}\left\{\frac{\sinh[2\sqrt{-e_0b_0c_0}(t_0-t_0)]}{2\sqrt{e_0b_0}}\right\}^{2/n} \tag{A.93}$$

$$= \quad \frac{2}{n}\left\{\frac{\sinh(0)}{2\sqrt{e_0b_0}}\right\}^{2/n} \quad = \quad 0. \tag{A.94}$$

So

$$z(t) \quad = \quad \frac{a(t_0)}{a(t)}-1 \quad = \quad -1. \tag{A.95}$$

A.5 Solution 5: $u(x)=(e_0/x)e^{c_0x^2/2}$

Following the formulation $\int xe^{-ax^2}dx=-\frac{1}{2a}e^{-ax^2}$, derivations of $x(t)$ and $u(t)$ are

$$\int_{t_0}^{t}\mathrm{d}t \quad = \quad \frac{1}{e_0}\int_0^x xe^{-\frac{c_0}{2}x^2}\mathrm{d}x, \tag{A.96}$$

$$(t-t_0) \quad = \quad \frac{1}{e_0c_0}e^{\frac{c_0x^2}{2}}, \tag{A.97}$$

$$-e^{\frac{c_0x^2}{2}} \quad = \quad e_bc_0(t-t_0), \tag{A.98}$$

$$x(t) \quad = \quad \sqrt{-\frac{2}{c_0}\ln\left[-e_0c_0(t-t_0)\right]}, \tag{A.99}$$

$$u(t) \quad = \quad \frac{\mathrm{d}x(t)}{\mathrm{d}t} \quad = \quad \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{-\frac{2}{c_0}\ln\left[-e_0c_0(t-t_0)\right]} \tag{A.100}$$

$$= \quad \frac{-1}{(t-t_0)\sqrt{-2c_0\ln\left[-e_0c_0(t-t_0)\right]}}. \tag{A.101}$$

The scale factor, its time derivative and Hubble parameter are evaluated by

$$a(t) \quad = \quad -\left\{(t-t_0)\sqrt{-2c_0\ln\left[-e_0c_0(t-t_0)\right]}\right\}^{2/n}, \tag{A.102}$$

$$\dot{a}(t) \quad = \quad -\frac{2}{n}\frac{\left\{(t-t_0)\sqrt{-2c_0\ln\left[-e_0c_0(t-t_0)\right]}\right\}^{2/n}}{(t-t_0)\sqrt{-2c_0\ln\left[-e_0c_0(t-t_0)\right]}}\times$$
$$\left\{\frac{-c_0}{\sqrt{-2c_0\ln[-e_0c_0(t-t_0)]}}+\sqrt{-2c_0\ln[-e_0c_0(t-t_0)]}\right\} \tag{A.103}$$

$$H = \frac{\dot{a}}{a} \quad = \quad \frac{1}{n(t-t_0)}\left\{\frac{1}{\ln[-e_0c_0(t-t_0)]}+2\right\} \tag{A.104}$$

At $t = t_0$, the scale factor is undetermined because $\ln 0$ is undefined. Therefore the time-redshift relation cannot be evaluated for this case.

A.6 Solution 6: $u(x) = -e_0 \cosh^2(b_0 x)$

This solution provides $x(t)$ which is determined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -e_0 \cosh^2(b_0 x), \tag{A.105}$$

$$\int_{t_0}^{t} \mathrm{d}t = \int_0^x -\frac{1}{e_0}\,\mathrm{sech}^2(b_0 x)\mathrm{d}x, \tag{A.106}$$

$$t - t_0 = -\frac{1}{e_0 b_0}\tanh(b_0 x), \tag{A.107}$$

$$-e_0 b_0(t-t_0) = \tanh(b_0 x), \tag{A.108}$$

$$x = \frac{1}{b_0}\,\mathrm{arctanh}[-e_0 b_0(t-t_0)]. \tag{A.109}$$

Due to the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{arctanh}(x) = \frac{1}{1-x^2}, \tag{A.110}$$

the calculation of $\dot{x}$ performs

$$\dot{x} = u(t) \quad = \quad \frac{-e_0}{1-[e_0^2 b_0^2 (t-t_0)^2]}. \tag{A.111}$$

This provides the scale factor, time-derivative of the scale factor, Hubble parameter and time-redshift relation as

$$a = u^{-2/n} = \left\{ \frac{1 - e_0^2 b_0^2 (t - t_0)^2}{-e_0} \right\}^{2/n}, \tag{A.112}$$

$$\dot{a} = \frac{2}{n} \left\{ \frac{1 - e_0^2 b_0^2 (t - t_0)^2}{-e_0} \right\}^{2/n} \left\{ \frac{e_0}{1 - e_0^2 b_0^2 (t - t_0)^2} \right\} \times \left\{ -2 e_0 b_0^2 (t - t_0)^2 \right\}, \tag{A.113}$$

$$H = \frac{\dot{a}}{a} = -\frac{4}{n} \left[\frac{e_0^2 b_0^2 (t - t_0)}{1 - e_0^2 b_0^2 (t - t_0)^2} \right], \tag{A.114}$$

$$z = \frac{a(t_0)}{a(t)} - 1 = \left\{ -\frac{1}{e_0} \left[\frac{-e_0}{1 - e_0^2 b_0^2 (t - t_0)^2} \right] \right\}^{2/n} - 1 \tag{A.115}$$

$$= \frac{1}{[1 - e_0^2 b_0^2 (t - t_0)^2]^{2/n}} - 1 \tag{A.116}$$

The time-reshift relation can be written in another form as

$$z + 1 = \frac{1}{[1 - e_0^2 b_0^2 (t - t_0)^2]^{2/n}}, \tag{A.117}$$

$$(z + 1)^{-n/2} = 1 - e_0^2 b_0^2 (t - t_0)^2, \tag{A.118}$$

$$t - t_0 = \pm \sqrt{\frac{1}{e_0^2 b_0^2 (z + 1)^{n/2}} - 1}. \tag{A.119}$$

The scalar field and Hubble parameter as a function of z are given by

$$a(z) = \left\{ \frac{1 - e_0^2 b_0^2 \left(\frac{1}{e_0^2 b_0^2 (z+1)^{n/2}} - 1 \right)}{-e_0} \right\}^{2/n}, \tag{A.120}$$

$$= \frac{1}{e_0^{2/n} (z + 1)}, \tag{A.121}$$

$$H(z) = -\frac{4}{n} \left[\frac{e_0^2 b_0^2 \left(\sqrt{\frac{1}{e_0^2 b_0^2 (z+1)^{n/2}}} - 1 \right)}{1 - e_0^2 b_0^2 \left(\frac{1}{e_0^2 b_0^2 (z+1)^{n/2}} - 1 \right)} \right], \tag{A.122}$$

$$= -\frac{4}{n} \left[\frac{e_0^2 b_0^2 \sqrt{\frac{1 - e_0^2 b_0^2 (z+1)^{n/2}}{e_0^2 b_0^2 (z+1)^{n/2}}}}{1 - e_0^2 b_0^2 \left(\frac{1 - e_0^2 b_0^2 (z+1)^{n/2}}{e_0^2 b_0^2 (z+1)^{n/2}} \right)} \right] \tag{A.123}$$

A.7 Solution 7: $u(x) = e_0/x^{b_0}$

The solution written in term of $x(t)$ is

$$\begin{aligned} \dot{x} &= \frac{e_0}{x^{b_0}}, &\text{(A.124)}\\ \int_{t_0}^{t} \mathrm{d}t &= \int_0^x \frac{x^{b_0}}{e_0}\mathrm{d}x, &\text{(A.125)}\\ t - t_0 &= \frac{1}{e_0(b_0+1)} x^{b_0+1}, &\text{(A.126)}\\ x &= [e_0(b_0+1)(t-t_0)]^{1/(b_0+1)}. &\text{(A.127)}\\ & &\text{(A.128)} \end{aligned}$$

Hence $u(t)$ can be determined by

$$\begin{aligned} u(t) = \dot{x} &= \frac{e_0(b_0+1)}{b_0+1}[e_0(b_0+1)(t-t_0)]^{-b_0/(b_0+1)} &\text{(A.129)}\\ &= e_0[e_0(b_0+1)(t-t_0)]^{-b_0/(b_0+1)}. &\text{(A.130)} \end{aligned}$$

This leads to

$$\begin{aligned} a(t) = u^{-2/n} &= \frac{1}{e_0^{2/n}}[e_0(b_0+1)(t-t_0)]^{2b_0/n(b_0+1)}, &\text{(A.131)}\\ \dot{a} &= \frac{2b_0 e_0}{n e_0^{2/n}} \frac{[e_0(b_0+1)(t-t_0)]^{2b_0/n(b_0+1)}}{e_0(b_0+1)(t-t_0)}, &\text{(A.132)}\\ H(t) &= \frac{2b_0}{n(b_0+1)(t-t_0)}. &\text{(A.133)} \end{aligned}$$

The calculation of time-redshift demonstrates by

$$\begin{aligned} z(t) = \frac{a(t_0)}{a(t)} - 1 &= \left[\frac{\frac{1}{e_0^{2/n}}[e_0(b_0+1)\cancel{(t_0-t_0)}^0]^{2b_0/n(b_0+1)}}{\frac{1}{e_0^{2/n}}[e_0(b_0+1)(t-t_0)]^{2b_0/n(b_0+1)}} \right] - 1 &\text{(A.134)}\\ &= -1. &\text{(A.135)} \end{aligned}$$

A.8 Solution 8: $u(x) = -e_0 \sinh^2(b_0 x)$

This solution corresponds to $x(t)$ and $u(t)$ which are evaluated by

$$\int_{t_0}^{t} = -\int_0^x \frac{\mathrm{d}x}{e_0 \sinh^2(b_0 x)}, \tag{A.136}$$

$$t - t_0 = \frac{\coth(b_0 x)}{e_0 b_0}, \tag{A.137}$$

$$x(t) = \frac{1}{b_0} \operatorname{arccoth}[e_0 b_0 (t - t_0)], \tag{A.138}$$

$$\tag{A.139}$$

and

$$u(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{1}{b_0} \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{arccoth}[e_0 b_0 (t - t_0)], \tag{A.140}$$

$$= \frac{e_0 b_0}{b_0 (1 - [e_0 b_0 (t - t_0)]^2)}, \tag{A.141}$$

$$= \frac{e_0}{(1 - [e_0 b_0 (t - t_0)]^2)}, \tag{A.142}$$

where $|e_0 b_0 (t - t_0)| \neq 1$. The scale factor, its time-derivative and Hubble parameter can be determined by

$$a(t) = \left[\frac{1 - [e_0 b_0 (t - t_0)]^2}{e_0} \right]^{2/n} \tag{A.143}$$

$$\dot{a}(t) = \frac{-4 e_0^2 b_0^2 (t - t_0)}{n e_0^{2/n}} \left[1 - [e_0 b_0 (t - t_0)]^2 \right]^{(2/n) - 1}, \tag{A.144}$$

and

$$H(t) = \frac{\dot{a}}{a} = -\frac{4}{n} \left[\frac{e_0^2 b_0^2 (t - t_0)}{1 - e_0^2 b_0^2 (t - t_0)^2} \right]. \tag{A.145}$$

We use $a(t)$ to determine the time-redshift relation which perform

$$z(t) = \left[\frac{1 - [e_0^2 b_0^2 (t_0 - t_0)]^2}{e_0} \times \frac{e_0}{1 - [e_0^2 b_0^2 (t - t_0)]^2} \right]^{2/n} - 1, \tag{A.146}$$

$$= \left[\frac{1}{1 - [e_0^2 b_0^2 (t - t_0)]^2} \right]^{2/n} - 1. \tag{A.147}$$

or

$$z + 1 = \left[\frac{1}{1 - [e_0^2 b_0^2 (t - t_0)]^2} \right]^{2/n}, \tag{A.148}$$

$$t - t_0 = \sqrt{\frac{1}{e_0^2 b_0^2} \left[\left(\frac{1}{z + 1} \right)^{n/2} - 1 \right]}. \tag{A.149}$$

APPENDIX B DERIVATION OF THE KLEIN-GORDON EQUATION

The Klein-Gordon equation can be evaluated from the variation of the action in Eq.(4.3) with respect to ϕ demonstrated by

$$\delta S = \int d^4x\sqrt{-g}\delta\left\{\frac{R}{16\pi G} - \frac{1}{2}\left(\varepsilon g_{\mu\nu} + \kappa G_{\mu\nu}\right)\nabla^\mu\phi\nabla^\nu\phi - V(\phi)\right\} + \cancel{\delta S_m} \tag{B.1}$$

$$= \int d^4x\sqrt{-g}\left\{-\frac{1}{2}\left(\varepsilon g_{\mu\nu} + \kappa G_{\mu\nu}\right)\left(2\nabla^\mu\phi\delta\nabla^\nu\phi\right) - \frac{\partial V}{\partial\phi}\delta\phi\right\} \tag{B.2}$$

$$= \int d^4x\sqrt{-g}\left\{\underbrace{-\varepsilon g_{\mu\nu}\nabla^\mu\phi\delta\nabla^\nu\phi}_{1st} - \underbrace{\kappa G_{\mu\nu}\nabla^\mu\phi\delta\nabla^\nu\phi}_{2nd} - \frac{\partial V}{\partial\phi}\delta\phi\right\}. \tag{B.3}$$

Consider the first and second term, the results are

$$-\varepsilon g_{\mu\nu}\nabla^\mu\phi\delta\nabla^\nu\phi = -\nabla_\mu(\varepsilon\nabla^\mu\phi\delta\phi) + \nabla_\mu(\varepsilon\nabla^\mu\phi)\delta\phi \tag{B.4}$$

and

$$-\kappa G_{\mu\nu}\nabla^\mu\phi\delta\nabla^\nu\phi = -\nabla^\nu(\kappa G_{\mu\nu}\nabla^\mu\phi\delta\phi) + \nabla^\nu(\kappa G_{\mu\nu}\nabla^\mu\phi)\delta\phi \tag{B.5}$$

$$= -\nabla^\nu(\kappa G_{\mu\nu}\nabla^\mu\phi\delta\phi) + \nabla^\nu[\kappa(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\nabla^\mu\phi]\delta\phi. \tag{B.6}$$

Consider the first terms on the RHS of Eq.(B.4) and (B.6), if we substitute them back to the Eq.(B.3), they can be manipulated by the divergence theorem to be considered as an integral over the boundary which is vanished. The second term on the RHS of Eq.(B.4) is evaluated by

$$\nabla_\mu(\varepsilon\nabla^\mu\phi)\delta\phi = \varepsilon\nabla_\mu(\partial^\mu\phi)\delta\phi \tag{B.7}$$

$$= \varepsilon g^{\mu\nu}[\partial_\mu(\partial_\nu\phi) - \Gamma^\sigma_{\nu\mu}\partial_\sigma\phi]\delta\phi \tag{B.8}$$

$$= \varepsilon[-\ddot{\phi} - (g^{rr}\Gamma^t_{rr} + g^{\theta\theta}\Gamma^t_{\theta\theta} + g^{\phi\phi}\Gamma^t_{\phi\phi})\dot{\phi}]\delta\phi \tag{B.9}$$

$$= -\varepsilon(\ddot{\phi} + 3H\dot{\phi})\delta\phi. \tag{B.10}$$

Calculation of the second term on RHS of Eq.((B.6)) performs

$$\begin{aligned}
&\nabla^\nu \left[\kappa\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)\nabla^\mu\phi\right]\delta\phi \\
=\;& \kappa\left[g^{\sigma\nu}g^{\rho\mu}\nabla_\sigma\left(R_{\mu\nu}\nabla_\rho\phi\right)-\frac{1}{2}\nabla_\mu\left(R\nabla^\mu\phi\right)\right]\delta\phi \qquad \text{(B.11)}\\
=\;& \kappa\Bigg\{\underbrace{g^{\sigma\nu}g^{\rho\mu}(\nabla_\sigma R_{\mu\nu})(\nabla_\rho\phi)}_{1st}+\underbrace{R_{\mu\nu}g^{\sigma\nu}g^{\rho\mu}(\nabla_\sigma\nabla_\rho\phi)}_{2nd}\\
&\underbrace{-\frac{1}{2}(\nabla_\mu R)\nabla^\mu\phi}_{3rd}\underbrace{-\frac{1}{2}R\nabla_\mu\nabla^\mu\phi}_{4th}\Bigg\}\delta\phi. \qquad \text{(B.12)}
\end{aligned}$$

The 1st term of (B.12) results

$$\begin{aligned}
&g^{\sigma\nu}g^{\rho\mu}(\nabla_\sigma R_{\mu\nu})(\nabla_\rho\phi)\\
=\;& g^{\sigma\nu}g^{\rho\mu}(\partial_\sigma R_{\mu\nu}-\Gamma^\alpha_{\ \mu\sigma}R_{\alpha\nu}-\Gamma^\alpha_{\ \sigma\nu}R_{\alpha\mu})\partial_\rho\phi \qquad \text{(B.13)}\\
=\;& -\dot\phi\Bigg\{g^{tt}\partial_t R_{tt}-g^{rr}\Gamma^r_{\ tr}R_{rr}-g^{\theta\theta}\Gamma^\theta_{\ t\theta}R_{\theta\theta}-g^{\phi\phi}\Gamma^\phi_{\ t\phi}R_{\phi\phi}\\
&-\left[g^{rr}\Gamma^t_{\ rr}+g^{\theta\theta}\Gamma^t_{\theta\theta}+g^{\phi\phi}\Gamma^t_{\ \phi\phi}\right]R_{tt}\Bigg\} \qquad \text{(B.14)}\\
=\;& -\dot\phi\left\{3\partial_t\left(\frac{\ddot a}{a}\right)-3\frac{\dot a}{a}\left(\frac{\ddot a}{a}+2\frac{\dot a^2}{a^2}\right)+9\left(\frac{\dot a}{a}\right)\frac{\ddot a}{a}\right\} \qquad \text{(B.15)}\\
=\;& -3\dot\phi\ddot H-6\dot\phi H\dot H+3\dot\phi H\dot H+9\dot\phi H^3-9\dot\phi H\dot H-9\dot\phi H^3. \qquad \text{(B.16)}\\
=\;& -3\dot\phi\ddot H-12\dot\phi H\dot H \qquad \text{(B.17)}
\end{aligned}$$

The 2nd term of (B.12) is evaluated by

$$\begin{aligned}
&R_{\mu\nu}g^{\sigma\nu}g^{\rho\mu}(\nabla_\sigma\nabla_\rho\phi)\\
=\;& R_{\mu\nu}g^{\sigma\nu}g^{\rho\mu}(\partial_\sigma\partial_\rho\phi-\Gamma^\alpha_{\ \rho\sigma}\partial_\alpha\phi) \qquad \text{(B.18)}\\
=\;& R_{tt}\ddot\phi-\left[R_{rr}g^{rr}g^{rr}\Gamma^t_{\ rr}+R_{\theta\theta}g^{\theta\theta}g^{\theta\theta}\Gamma^t_{\theta\theta}+R_{\phi\phi}g^{\phi\phi}g^{\phi\phi}\Gamma^t_{\ \phi\phi}\right]\dot\phi \qquad \text{(B.19)}\\
=\;& -3\frac{\ddot a}{a}\ddot\phi-3\dot\phi\left(\frac{\ddot a}{a}+2\frac{\dot a^2}{a^2}\right)\frac{\dot a}{a} \qquad \text{(B.20)}\\
=\;& -3\dot H\ddot\phi-3H^2\ddot\phi-3\dot\phi\dot H H-9\dot\phi H^3. \qquad \text{(B.21)}
\end{aligned}$$

The calculation for the 3rd term of (B.12) performs

$$-\frac{1}{2}(\nabla_\mu R)\nabla^\mu\phi = 3\dot{\phi}\partial_t\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a}\right) \tag{B.22}$$

$$= 3\dot{\phi}\partial_t(\dot{H}+2H^2) \tag{B.23}$$

$$= 3\dot{\phi}(\ddot{H}+4H\dot{H}). \tag{B.24}$$

$$\tag{B.25}$$

The 4th term of (B.12) is determined by

$$-\frac{1}{2}R\nabla_\mu\nabla^\mu\phi = 3(\dot{H}+2H^2)(\ddot{\phi}+3H\dot{\phi}) \tag{B.26}$$

$$= 3\dot{H}\ddot{\phi}+6\ddot{\phi}H^2+9\dot{\phi}H\dot{H}+18\dot{\phi}H^3. \tag{B.27}$$

The Eq.(B.12) is hence

$$\nabla^\nu\left[\kappa\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)\nabla^\mu\phi\right]\delta\phi = \kappa\Big\{3\ddot{\phi}H^2+6\dot{\phi}H\dot{H}+9\dot{\phi}H^3\Big\}\delta\phi. \tag{B.28}$$

The variation of the action with respect to the field is hence

$$\delta S = \int d^4x\sqrt{-g}\left\{-\varepsilon(\ddot{\phi}+3H\dot{\phi})+\kappa(3\ddot{\phi}H^2+6\dot{\phi}H\dot{H}+9\dot{\phi}H^3)-\frac{\partial V}{\partial\phi}\right\}\delta\phi. \tag{B.29}$$

This variation provides the Klein-Gordon equation in the form of

$$0 = -\varepsilon(\ddot{\phi}+3H\dot{\phi})+\kappa(3\ddot{\phi}H^2+6\dot{\phi}H\dot{H}+9\dot{\phi}H^3)-\frac{\partial V}{\partial\phi}, \tag{B.30}$$

$$-\frac{\partial V}{\partial\phi} = (\varepsilon-3\kappa H^2)\ddot{\phi}+(3\varepsilon H-6\kappa H\dot{H}-9\kappa H^3)\dot{\phi}. \tag{B.31}$$

BIOGRAPHY

BIOGRAPHY

Name-Surname	Chonticha Kritpetch
Date of Birth	October 11, 1977
Address	9/41 Mitraparp Road, Tambon Nai Mueang, Amphoe Mueang, Phitsanulok Province, Thailand 65000

Work Experiences

2006	Lecturer, University of Phayao, Phayao, Thailand
2000	Scientist, Naresuan Unversity Phayao campus, Phayao, Thailand 56000

Education Background

2014	H.G.D. (Quantum fields, Gravitation and Cosmology) Naresuan University, Phitsanulok, Thailand 65000
2006	M.Sc.(Nuclear Technology) Chulalongkorn University, Bangkok, Thailand 10330
2000	B.S.(Physics) Naresuan University, Phitsanulok, Thailand 65000