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# 45° Relative Orientations of Planes of Polarizations States of Gravitational Waves and the Graviton

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**Abstract** *The recent detection of gravitational waves* calls for, not just in words or by plausible arguments, of an explicit *derivation* of polarization aspects of gravitational waves with emphasis, especially, on the non-trivial aspect of the *relative* 45° *orientations of the planes of polarization states* of gravitation in the same way as has been done over the years for the far simpler case involving electromagnetic wave propagation with the well known relative 90° between its polarization states. The purpose of this communication is to carry out in a *covariant* description as well as by giving special attention to the underlying *gauge* problem these polarization aspects via a direct consideration of the graviton *propagator* in a quantum field theory setting from which fundamental properties of polarizations are readily extracted.

**Keywords** General relativity · 45° relative orientations of the planes of polarizations states · Propagator theory · The graviton

## 1 Introduction

The detection [1, 2] of gravitational waves from the merger of two black holes 1.3 billion light-years from the Earth via the Laser Interferometer Gravitational Wave Observatory (LIGO) (see also [3]), has been well publicized in the literature, see, e.g., [4]. In the seventies there was also indirect evidence of their existence through the discovery [5] of the Hulse-Taylor Pulsar PSR B193+16. Credit should be also given to Joseph Weber for his heroic pioneering attempts [6, 7] on the detection of gravitational waves. The existence of gravitational waves were of course predicted by Einstein [8] himself and are consequences

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of general relativity in which, unlike in Newton' theory, gravitation propagates with a finite speed. Although the LIGO experiment gives support of the existence of black holes as well, Einstein did not believe in black holes. For details on the underlying theory of gravitational waves, see, e.g., [9–11].

The particle associated with the gravitational field, the elusive graviton, is massless in the same way as the particle associated with the electromagnetic wave, the photon, is massless. The mere existence of gravitational waves calls for a careful study of its polarizations aspects emphasizing especially a non-trivial property of a 45° of relative orientations of planes of polarizations states. The tensor character of gravitation makes such a study much harder in comparison to the vector character of the photon. The vector character of the photon implies the existence of two perpendicular polarization states which, in turn, are perpendicular to the direction of propagation of the electromagnetic wave. On the other hand, the tensor character of gravitation implies the existence of two polarization states, with one described in a plane, perpendicular to the direction of propagation of gravitation, and another state also in a plane with the axes rotated by 45° relative to the axes of oscillations in the plane of the other state. The corresponding details and the underlying intricacies are *derived* and spelled out below in a *quantum field theory* description via the graviton propagator. The advantage of working directly with the propagator is that polarization aspects are readily and simply extracted [12]. One may also consider polarization aspects as done in standard classical electrodynamics [13] through a study of propagation of plane-waves, e.g. as in [14] which is not, however, involved with the relative orientations of planes of polarizations states considered in the present communication. For completeness we begin with the Lagrangian density of general relativity and show how the above properties of polarizations associated with gravitation emerge. We also pay special attention to the gauge problem in the derivation which naturally arises in theories involving massless particles. Also for the convenience of the reader the well situation involved with the electromagnetic case is worked out in an Appendix for comparison.

## 2 Relative Orientations of Planes of Polarizations States and the Graviton

We consider the action of general relativity,

$$W = \int (\mathrm{d}x)\sqrt{-g}g^{\mu\nu}R_{\mu\nu}, \qquad g = \det[g_{\mu\nu}], \quad (\mathrm{d}x) \equiv \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \tag{1}$$

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\nu\Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\rho}, \tag{2}$$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right), \tag{3}$$

where for the simplicity of the notation we have set the parameter, involving Newton's gravitational constant, $\kappa = 1$. The fluctuation about the Minkowski metric is obtained by writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4}$$

where, for the Minkowski metric, we use $[\eta_{\mu\nu}] = \mathrm{diag}[-1, 1, 1, 1]$. One may then eventually carry out an expansion of the action in "powers" of $h_{\mu\nu}$ by using, in the process, the Minkowski metric to lower and raise indices. Details of such expansions are given below.

We may write $\det[g_{\mu\nu}]$ as $\det[\eta_{\alpha\beta}] \times \det[\delta^\mu_\nu + \eta^{\mu\sigma}h_{\sigma\nu}]$, and note that $\eta^{\mu\sigma}h_{\sigma\nu} = h^\mu_\nu$, where we have used the relation $\eta^{\mu\nu}\eta_{\nu\lambda} = \delta^\mu_\lambda$. The expansion of $\sqrt{-\det[g_{\mu\nu}]}$ is easily obtained upon expanding the logarithm in the following expression and taking the trace of the resulting matrix

$$\sqrt{-\det[g_{\mu\nu}]} = \exp\frac{1}{2}(\mathrm{Tr}\ln[\delta^\mu_\nu + h^\mu_\nu]) = \exp\frac{1}{2}\left(h^\mu_\mu - \frac{1}{2}h^\mu_\nu h^\nu_\mu + \mathcal{O}(h^3)\right). \tag{5}$$

Useful expansions, in "powers" of $h_{\mu\nu}$, are readily generated as follows:

$$\begin{aligned}
\sqrt{-g} &= 1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3),\\
g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}h^\nu_\lambda + \mathcal{O}(h^3),\\
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}(\partial_\mu h^\rho_\nu + \partial_\nu h^\rho_\mu - \partial^\rho h_{\mu\nu}) + \mathcal{O}(h^2),\\
R_{\mu\nu} &= R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \mathcal{O}(h^3), \quad \partial_\mu\Gamma^\rho_{\nu\rho} = \frac{1}{2}\partial_\mu\partial_\nu h + \mathrm{O}(h^2),\\
R^{(1)}_{\mu\nu} &= \frac{1}{2}\left(\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right),\\
\eta^{\mu\nu}R^{(2)}_{\mu\nu} &= \frac{1}{2}\partial_\lambda h\partial_\rho h^{\rho\lambda} - \frac{1}{4}\partial_\rho h\partial^\rho h - \frac{1}{2}\partial_\rho h_{\sigma\lambda}\partial^\sigma h^{\rho\lambda} + \frac{1}{4}\partial^\rho h_{\sigma\lambda}\partial_\rho h^{\sigma\lambda},
\end{aligned}$$

with the latter equation satisfied up to a total derivative, where $h = h^\mu_\mu$, and we have used the fact that $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$. The following expression then emerges for the integrand in (1):

$$\sqrt{-g}g^{\mu\nu}R_{\mu\nu} = -\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right)R^{(1)}_{\mu\nu} + \eta^{\mu\nu}R^{(2)}_{\mu\nu} + \mathrm{O}(h^3), \tag{6}$$

up to a total derivative. This leads to

$$\sqrt{-g}g^{\mu\nu}R_{\mu\nu} = \frac{1}{2}\left(-\frac{1}{2}\partial^\sigma h_{\mu\nu}\partial_\sigma h^{\mu\nu} + \partial_\mu h^{\mu\sigma}\partial_\nu h^\nu_\sigma - \partial_\sigma h^{\sigma\mu}\partial_\mu h + \frac{1}{2}\partial^\mu h\partial_\mu h\right), \tag{7}$$

satisfied to second order, again up to a total derivative. The expression between the round brackets in (7) defines the Lagrangian density of a massless spin 2-particle.

The Lagrangian density in (7) is, up to a total derivative, invariant under the transformation

$$h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu, \tag{8}$$

for arbitrary $\xi^\mu$, or equivalently under the transformation of the combination

$$\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right) \to \left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right) + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu - \eta^{\mu\nu}\partial\xi. \tag{9}$$

Accordingly, upon taking the $\partial_\mu$-derivative of the above equation, we may infer that one may always choose $\Box\xi^\nu = -\partial_\mu(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h)$ so that the *new* transformed combination satisfies the gauge condition

$$\partial_\mu\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right) = 0, \tag{10}$$

referred often as the harmonic gauge.

We provide a covariant description of the graviton and work with general covariant gauge constraints:

$$\partial_\mu\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right) = \lambda\chi^\nu, \tag{11}$$

for some vector field $\chi^{\nu}$, and $\lambda$ is an arbitrary real parameter specifying arbitrary covariant gauges. This constraint may be derived directly from a Lagrangian density having the following structure

$$\mathcal{L} = \mathcal{L}_G + h^{\mu\nu}T_{\mu\nu} - 2\chi_{\nu}\partial_{\mu}\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right) + \lambda\chi_{\mu}\chi^{\mu}, \tag{12}$$

where $\mathcal{L}_G$ is the Lagrangian density *within* the round brackets in (7) for a spin 2 massless particle. We have added an external source coupling to the field $h^{\mu\nu}$ given by $h^{\mu\nu}T_{\mu\nu}$, with a symmetric source $T_{\mu\nu} = T_{\nu\mu}$. With the gauge constraint now included in the Lagrangian density this allows us to vary all the components of the field $h^{\mu\nu}$ *independently* thus giving

$$\begin{aligned} -\Box h_{\mu\nu} + \partial_{\mu}\partial^{\lambda}h_{\lambda\nu} + \partial_{\nu}\partial^{\lambda}h_{\lambda\mu} - \partial_{\mu}\partial_{\nu}h + \eta_{\mu\nu}(\Box h - \partial_{\sigma}\partial_{\lambda}h^{\sigma\lambda}) \\ = T_{\mu\nu} + (\partial_{\mu}\chi_{\nu} + \partial_{\nu}\chi_{\mu} - \eta_{\mu\nu}\partial_{\sigma}\chi^{\sigma}). \end{aligned} \tag{13}$$

Upon taking the $\partial^{\mu}$-derivative of the later equation gives

$$\partial^{\mu}(\partial_{\mu}\chi_{\nu} + \partial_{\nu}\chi_{\mu} - \eta_{\mu\nu}\partial_{\sigma}\chi^{\sigma}) = \Box\chi_{\nu} = -\partial^{\mu}T_{\mu\nu}, \tag{14}$$

On the other hand, a variation of the Lagrangian density $\mathcal{L}$ with respect to $\chi_{\nu}$, gives (11) as a derived constraint. Upon replacing $\lambda\chi^{\nu}$ by $\partial_{\mu}(h^{\mu\nu} - \eta^{\mu\nu}h/2)$ everywhere in (13), leads to the basic equation

$$-\Box h^{\mu\nu} = \frac{(\eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\sigma\rho})}{2}T_{\sigma\rho} + (\lambda - 1)\left(\frac{\partial^{\mu}\partial^{\sigma}}{\Box}\eta^{\nu\rho} + \frac{\partial^{\nu}\partial^{\sigma}}{\Box}\eta^{\mu\rho}\right)T_{\sigma\rho}. \tag{15}$$

In a quantum field theory description, the graviton propagator may be simply read from this equation to be

$$\Delta^{\mu\nu\sigma\rho}(x - x') = \int \frac{(\mathrm{d}p)}{(2\pi)^4}\mathrm{e}^{\mathrm{i}p(x-x')}\Delta^{\mu\nu\sigma\rho}(p), \qquad (\mathrm{d}p) \equiv \mathrm{d}p^0\mathrm{d}p^1\mathrm{d}p^3\mathrm{d}p^3, \tag{16}$$

$$\Delta^{\mu\nu\sigma\rho}(p) = \frac{(\eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\sigma\rho})}{2(p^2 - \mathrm{i}\epsilon)} + \frac{(\lambda - 1)}{2(p^2 - \mathrm{i}\epsilon)}\left(\frac{p^{\mu}\,(p^{\sigma}\eta^{\nu\rho} + p^{\rho}\eta^{\nu\sigma})}{p^2} + \frac{p^{\nu}\,(p^{\sigma}\eta^{\mu\rho} + p^{\rho}\eta^{\mu\sigma})}{p^2}\right), \tag{17}$$

$\epsilon \to +0$, with the usual boundary condition specified by the i$\epsilon$ factor.

For a conserved $T^{\mu\nu}$, and for $x^0 > x'^0$, for all $x'^0$ in the support of $T_{\mu\nu}(x')$, with all time dependence explicitly absorbed in $T_{\mu\nu}(x')$, we may from (15) solve for the vacuum expectation value of $h^{\mu\nu}(x)$ to obtain

$$\frac{\langle 0_+ \mid h^{\mu\nu}(x) \mid 0_-\rangle}{\langle 0_+ \mid 0_-\rangle} = \int \frac{\mathrm{d}^3\mathbf{p}}{2p^0(2\pi)^3}\mathrm{e}^{\mathrm{i}px}\frac{(\eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\sigma\rho})}{2}[\mathrm{i}T_{\sigma\rho}(p)], \tag{18}$$

$p^0 = |\mathbf{p}|$, where $\mid 0_-\rangle$ is the vacuum state before the emission source is in operation, and $\mid 0_+\rangle$ is the vacuum state after all the sources are retrieved [12] and the graviton has been detected so that the system ends up again in the vacuum state. The $\lambda$ independence of (18) establishes the gauge invariance of the formalism.

To find out the nature of the polarizations that may be detected, we introduce a causally arranged detection source $\widetilde{T}^{\mu\nu}(x)$, with $x^0 > x'^0$, and $T_{\sigma\rho}(x')$ denoting the emission source. We may then use (18) to write

$$\mathrm{i}\int(\mathrm{d}x)\widetilde{T}_{\mu\nu}(x)\frac{\langle 0_+ \mid h^{\mu\nu}(x) \mid 0_-\rangle}{\langle 0_+ \mid 0_-\rangle} = \int \frac{\mathrm{d}^3\mathbf{p}}{2p^0(2\pi)^3}[\mathrm{i}\widetilde{T}^*_{\mu\nu}(p)]\frac{(\eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\sigma\rho})}{2}[\mathrm{i}T_{\sigma\rho}(p)], \tag{19}$$

$p^0 = |\mathbf{p}|$, where $\mid 0_-\rangle$ is the vacuum state before the source is in operation, and $\mid 0_+\rangle$ is the vacuum state after all the sources are retrieved [12] and the graviton has been detected so that the system ends up *again* in the vacuum state. To extract the information on polarization aspects of gravitational waves from the above equation, we further introduce the following completeness relation ([12], p.15) in 4 dimensions:

$$\eta^{\mu\nu} = \frac{(p+\bar{p})^\mu (p+\bar{p})^\nu}{(p+\bar{p})^2} + \frac{(p-\bar{p})^\mu (p-\bar{p})^\nu}{(p-\bar{p})^2} + \sum_{\lambda=1,2} \mathrm{e}^\mu_\lambda \mathrm{e}^\nu_\lambda, \qquad p^2 = 0, \tag{20}$$

expanded in terms of the orthogonal system $\{(p+\bar{p})^\mu, (p-\bar{p})^\mu, \mathrm{e}^\mu_1, \mathrm{e}^\mu_2\}$, with $p = (p^0, \mathbf{p})$, $\bar{p} = (p^0, -\mathbf{p})$, and with $\mathrm{e}^\mu_1, \mathrm{e}^\mu_2$ real orthonormal vectors. Equation (20) may be simplified further to

$$\eta^{\mu\nu} = \frac{p^\mu \bar{p}^\nu + \bar{p}^\mu p^\nu}{p\bar{p}} + \sum_{\lambda=1,2} \mathrm{e}^\mu_\lambda \mathrm{e}^\nu_\lambda. \tag{21}$$

For conserved sources, i.e., for $p^\mu\, T_{\mu\nu}(p) = 0$, $p^\nu\, T_{\mu\nu}(p) = 0$, and $p^\mu\, \widetilde{T}_{\mu\nu}(p) = 0$, $p^\nu\, \widetilde{T}_{\mu\nu}(p) = 0$, we may replace the metric in (19) by its equivalent expression in (21), and effectively make the substitution

$$\frac{(\eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\sigma\rho})}{2} \to \sum_{\lambda,\lambda'=1,2} \mathrm{e}^{\mu\nu}_{\lambda\lambda'} \mathrm{e}^{\sigma\rho}_{\lambda\lambda'}, \tag{22}$$

where

$$\mathrm{e}^{\mu\nu}_{\lambda\lambda'} = \frac{1}{2}\left[ \mathrm{e}^\mu_\lambda \mathrm{e}^\nu_{\lambda'} + \mathrm{e}^\mu_{\lambda'} \mathrm{e}^\nu_\lambda - \delta_{\lambda\lambda'} \sum_{\gamma=1,2} \mathrm{e}^\mu_\gamma \mathrm{e}^\nu_\gamma \right], \tag{23}$$

as obtained by using, in the process, (21). On the other hand, simple algebra allows us to write

$$\sum_{\lambda,\lambda'=1,2} \mathrm{e}^{\mu\nu}_{\lambda\lambda'} \mathrm{e}^{\sigma\rho}_{\lambda\lambda'} = \sum_{\xi=1,2} \varepsilon^{\mu\nu}_\xi \varepsilon^{\sigma\rho}_\xi, \tag{24}$$

$$\varepsilon^{\mu\nu}_1 = \frac{1}{\sqrt{2}}(\mathrm{e}^{\mu\nu}_{11} - \mathrm{e}^{\mu\nu}_{22}) = \sqrt{2}\mathrm{e}^{\mu\nu}_{11} = -\sqrt{2}\mathrm{e}^{\mu\nu}_{22}, \tag{25}$$

$$\varepsilon^{\mu\nu}_2 = \frac{1}{2}(\mathrm{e}^{\mu\nu}_{12} + \mathrm{e}^{\mu\nu}_{21}) = \sqrt{2}\mathrm{e}^{\mu\nu}_{12} = \sqrt{2}\mathrm{e}^{\mu\nu}_{21}. \tag{26}$$

Thus the following expression emerges from (19)

$$\mathrm{i}\int (\mathrm{d}x)\widetilde{T}_{\mu\nu}(x) \frac{\langle 0_+ \mid h^{\mu\nu}(x) \mid 0_-\rangle}{\langle 0_+ \mid 0_-\rangle} = \int \sum_{\xi=1,2} \frac{\mathrm{d}^3\mathbf{p}}{2p^0(2\pi)^3} [\mathrm{i}\varepsilon^{\mu\nu}_\xi \widetilde{T}^*_{\mu\nu}(p)][\mathrm{i}\,\varepsilon^{\sigma\rho}_\xi T_{\sigma\rho}(p)],\ p^0 = |\mathbf{p}|, \tag{27}$$

described in a Lorentz and gauge *invariant* manner. Here $[\,\mathrm{i}\,\mathrm{e}^{\sigma\rho}_\xi T_{\sigma\rho}(p)]$ is an amplitude that the source has emitted a graviton of momentum $p$ and polarization specified by $\xi$, and $[\,\mathrm{i}\,\varepsilon^{\mu\nu}_\xi \widetilde{T}^*_{\mu\nu}(p)\,]$ denotes an amplitude for detection of a graviton with the same attributes [12]. For $\partial^\mu T_{\mu\nu} = 0$, we note that $\chi_\nu$ in (14) satisfies the equation of a free field.
In the observation frame, let $\mathbf{p} = \mathbf{k} = |\mathbf{k}|(0, 0, 1)$, denote the gravitational wave propagation vector, and $\mathrm{e}^\mu_1 = (0, 1, 0, 0)$, $\mathrm{e}^\mu_2 = (0, 0, 1, 0)$. Then, (23)–(26), with $\xi = 1, 2$, give

$$\varepsilon^{\,11}_{\,1} = \frac{1}{\sqrt{2}} = -\,\varepsilon^{\,22}_{\,1}, \qquad \varepsilon^{\,12}_{\,2} = \frac{1}{\sqrt{2}} = \varepsilon^{\,21}_{\,2}, \tag{28}$$

with all the other components of $\varepsilon^{\,\mu\nu}_{\,\xi}$ equal to zero.

The components of the polarization tensors $\varepsilon^{\,a\,b}_{\,2}$, $a, b = 1, 2$, may be considered in reference to a coordinate system rotated clockwise by an angle $45°$ of the original coordinate system, about the $x^3$-axis, defined in the 1-2, i.e., $x^{\,1} - x^{\,2}$, plane. Remembering

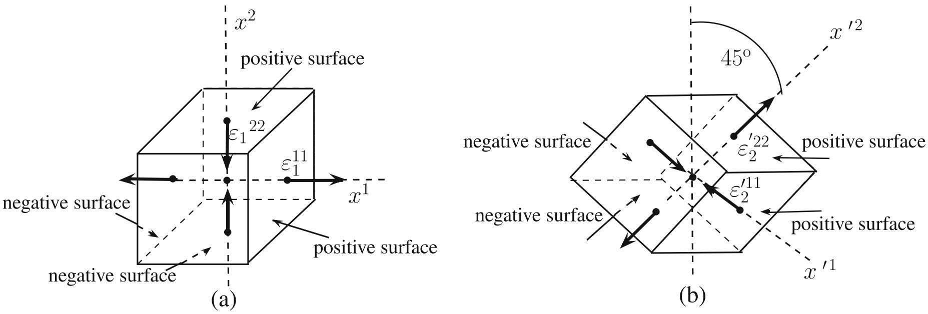


**Fig. 1** $\left(\varepsilon_1^{11} = 1/\sqrt{2} = -\varepsilon_1^{22}\right)$, $\left(\varepsilon_2'^{11} = -1/\sqrt{2} = -\varepsilon_2'^{22}\right)$ define the two polarization states of a gravitation field propagating along the $x^3$ - direction. The polarizations $\left(\varepsilon_2'^{11}, \varepsilon_2'^{22}\right)$ may be considered to be described in a coordinate system $x'^1 - x'^2$ in (**b**) obtained by a 45° c.w. rotation of the coordinate system $x^1 - x^2$ in (**a**). Note, for example, that since $\varepsilon_1^{11}$ in (**a**) is positive, then at a negative surface the direction of the arrow must be in the negative $x^1$- direction to ensure that $\varepsilon_1^{11}$ is positive, and so on for the directions of the other arrows

that a second rank tensor transforms as the product of two vectors, then the corresponding components are given by the following expressions in this new coordinate system $(\sin\theta = \cos\theta = 1/\sqrt{2})$

$$\varepsilon_2'^{11} = \varepsilon_2^{11}\cos^2\theta + \varepsilon_2^{22}\sin^2\theta - \varepsilon_2^{12}\sin\theta\,\cos\theta - \varepsilon_2^{21}\sin\theta\,\cos\theta = -1/\sqrt{2}, \quad (29)$$

$$\varepsilon_2'^{22} = \varepsilon_2^{11}\sin^2\theta + \varepsilon_2^{22}\cos^2\theta + \varepsilon_2^{12}\sin\theta\,\cos\theta + \varepsilon_2^{21}\sin\theta\,\cos\theta = +1/\sqrt{2}, \quad (30)$$

$$\varepsilon_2'^{12} = \varepsilon_2^{11}\sin\theta\,\cos\theta - \varepsilon_2^{22}\sin\theta\,\cos\theta + \varepsilon_2^{12}\cos^2\theta - \varepsilon_2^{21}\sin^2\theta = 0, \quad (31)$$

$$\varepsilon_2'^{21} = \varepsilon_2^{11}\sin\theta\,\cos\theta - \varepsilon_2^{22}\sin\theta\,\cos\theta - \varepsilon_2^{12}\sin^2\theta + \varepsilon_2^{21}\cos^2\theta = 0. \quad (32)$$

Thus,

$$\left(\varepsilon_1^{11} = \frac{1}{\sqrt{2}} = -\varepsilon_1^{22}\right), \quad \left(\varepsilon_2'^{11} = -\frac{1}{\sqrt{2}} = -\varepsilon_2'^{22}\right), \quad (33)$$

define the two polarization states of gravitons. These are pictorially represented in Fig. 1.

## Appendix: Brief Account of the Electromagnetic Case

To work in covariant gauges, we modify the Maxwell Lagrangian density to read

$$L_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu - \chi\partial_\mu A^\mu + \frac{\lambda}{2}\chi^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (34)$$

where the scalar field $\chi$ and the arbitrary real parameter $\lambda$ introduced give rise to arbitrary covariant gauges for the Maxwell field and allow us, in turn, to vary the components of the latter independently. Variations of $A_\mu$, and $\chi$ lead to

$$-\Box A^\mu = J^\mu + (1-\lambda)\partial^\mu\chi, \quad (35)$$

$$\partial_\mu A^\mu = \lambda\chi, \quad (36)$$

involving covariant gauges which are now derived. Upon taking the partial derivative $\partial_\mu$ of (35), and using (36), we obtain

$$-\Box\chi = \partial_\mu J^\mu. \quad (37)$$

We take the vacuum expectation value of (35) and use (36) to obtain

$$\frac{\langle 0_+ \mid A^\mu(x) \mid 0_- \rangle}{\langle 0_+ \mid 0_- \rangle} = \int (\mathrm{d}x') D^{\mu\nu}(x-x') J_\nu(x'), \tag{38}$$

where the propagator is given by

$$D^{\mu\nu}(x-x') = \int \frac{(\mathrm{d}k)}{(2\pi)^4} \left[ \eta^{\mu\nu} - (1-\lambda) \frac{k^\mu k^\nu}{k^2} \right] \frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2 - \mathrm{i}\epsilon}. \tag{39}$$

To find out the nature of the polarizations that may be detected, we introduce a causally arranged detection source $\widetilde{J}^\mu(x)$, with $x^0 > x'^0$, and $J_\nu(x')$ denoting an emission source both conserved. Hence upon using the completeness relation in (21), we may write

$$\mathrm{i} \int (\mathrm{d}x)\, \widetilde{J}_\mu(x) \frac{\langle 0_+ \mid A^\mu(x) \mid 0_- \rangle}{\langle 0_+ \mid 0_- \rangle} = \int \frac{\mathrm{d}^3\mathbf{k}}{2k^0 (2\pi)^3} \sum_{\lambda=1,2} [\, \mathrm{i} \widetilde{J}^*_\mu(k)\, \mathrm{e}^\mu_\lambda \,][\, \mathrm{i}\, \mathrm{e}^\nu_\lambda J_\nu(k)], \; k^0 = |\mathbf{k}\,|. \tag{40}$$

let $\mathbf{p} = \mathbf{k} = |\mathbf{k}|(0,0,1)$, denote the wave propagation vector, and $\mathrm{e}^\mu_1 = (0,1,0,0)$, $\mathrm{e}^\mu_2 = (0,0,1,0)$, with $(1,0,0)$ & $(0,1,0)$ defining two orthonormal polarization states, i.e., with a 90° degree between them.

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