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Electromagnetic propagation across a metal: the causal propagator

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ABSTRACT

The very general expression of the Schwinger–Feynman causal *propagator* is explicitly derived for the transmission of a photon through a metal from air, as a generalization of our earlier work dealing with dielectrics. In a very general context, the corresponding transition amplitudes for crossing through the metal of a polarized or unpolarized photon are obtained. Inspired by Feynman's intuitive and well-publicized non-technical treatment, we consider, from a quantum viewpoint, the fate of a red and a blue photon in determining their transmission *probabilities*, as an application, through a thin layer of silver whose complex dielectric function, in particular, has been experimentally carefully determined in recent years. Particular emphasis is also put in assessing the accuracy of the phenomenological expression of exponential damping used in considering transmissions through metals in comparison to the exact expressions obtained in the bulk of the paper.

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1. Introduction

There has been much interest in recent years on optical properties of metals (see, e.g. [1–13]). For example, enhanced transmission through holey metal films have been observed (see, e.g. [6,10]). The possibility of so-called cloaking has been considered [8], where optical waves are guided around objects without being affected by the object itself and shielding it from view, as well as the application of thin metal layers to play the role of perfect lenses [7]. Of particular interest was the accurate experimental measurements of the complex dielectric functions of various metals such as aluminum [4] and silver [5] (see also [1–3]). Of great importance, it was noted [5] that silver forms a silver sulfide layer [14] at the metal boundary which affects the measured dielectric function and had to be taken into account in further challenging experiments. As a result, earlier related experiments, e.g. [15], were not reliable as the silver surface were contaminated. These considerations lead us to infer that both experimentally and theoretically, electromagnetic transmissions through metals certainly deserve further investigations.

With the aim of providing a physically appealing approach, as well as a mathematically rigorous treatment, of the transmission of electromagnetic waves from air through metals, generalizing our earlier work [16] dealing with the far simpler problem involving dielectric media, we explicitly derive the so-called Schwinger–Feynman causal *propagator* for

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transmission through a layer of metal, for the first time, to describe the fate of photons to cross a layer of metal of parallel surfaces. The underlying probabilistic interpretation that emerges for describing the fate of photons in crossing a metal from a quantum field theory analysis, is much in the spirit of a fascinating, intuitive and highly non-technical well-publicized treatment by Feynman [17]. Particular emphasis is given on polarization aspects of photons as they travel across the layer of metal from an emission source to a detection source above and below the metal as a scattering process. The sources thus described are localized in air and are not point-like as one might expect. They are simply introduced as a simple way of generating amplitudes, in a quantum mechanical setting, and are withdrawn or switched off once they create or introduce photons into the system, to be analyzed, and are finally detected. This method of generating amplitudes by introducing sources was developed and applied for years by Julian Schwinger culminating in his monumental work described in one of his books [18]. Inspired by Feynman's study of the fate of red and blue photons, one, in a quantum viewpoint, is interested in the explicit *probabilities*, associated with quantum amplitudes. The causal propagator has the distinct property as describing electromagnetic propagation between any two points, not necessarily in the same medium, as a time evolution process. This is in contrast, for example, to an oscillating electric field at a given one point in a medium, making the actual physical process of transmission certainly much clearer in terms of propagators. It is worth mentioning that several interesting studies have been carried out in the literature (see, e.g. [19–22]), in general, in describing quantum particles, as the photon, in general classical situations. We should also mention an interesting work [23] (see also [24]) based on linear response theory which is not, however, developed as a time evolution process. This method, being linear, does not allow any non-linearities to be introduced into a theory that may be generated as encountered in quantum electrodynamics. Needless to say, our formalism as a time evolution process, in terms of a photon propagator, is just what is required for applications in quantum field theories with built-in non-linearities embodied in them which certainly go beyond application only of Maxwell's equations as such. The present method is also expected to have applications for the propagation of EM waves in plasmas (see, e.g. [25]) with possible non-linearities present in the formalism.

In Section 2, we set up the evolution equations and spell out the boundary conditions to be satisfied by the propagator. Section 3 deals with the general structure of the propagator as described in the various regions in reference to the region of emission of photons. The boundary conditions are then applied in Section 4. The explicit solution of the propagator for crossing the layer of metal is obtained in Section 5. In the final section (Section 6), applications to Feynman's red and blue photons are given for silver for which accurate (complex) numerical values have been determined experimentally for the dielectric function.[5]

2. Evolution equations and boundary conditions

To simplicity the notation, we work with vector quantities throughout, while the propagator, describing the propagation of the electromagnetic wave, is simply given by matrix elements D_{ij}, $i, j = 1, 2, 3$, of 3×3 matrices and, as we will encounter in some cases, are given by matrix elements D_{ab}, $a, b = 1, 2$ of 2×2 matrices. The Lagrangian density of electrodynamics in a medium of conductivity σ, permeability μ, and permittivity ε, in the celebrated temporal gauge for the vector potential $A^0 = 0$, in the presence of an external current source

$\mathbf{J} = (J^i)$, $i = 1, 2, 3$ may be defined by

$$\mathcal{L} = \left[-\frac{1}{2\mu}\,\mathbf{B} \cdot \mathbf{B} + \frac{\varepsilon}{2}\,\mathbf{E} \cdot \mathbf{E} + \mathbf{J} \cdot \mathbf{A} \right] \mathrm{e}^{4\pi\sigma x^0/\varepsilon}, \tag{1}$$

and where

$$\mathbf{E} = -\partial_0 \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \qquad x = (x^0, \mathbf{x}), \quad \partial_0 \equiv \frac{\partial}{\partial x^0}, \tag{2}$$

expressed in terms of the vector potential $\mathbf{A} = (A^i)$, $i = 1, 2, 3$. The resulting field equations obtained by varying the vector potential $\mathbf{A} = (A^i)$, $i = 1, 2, 3$ are given by

$$-\nabla^2 A^i(x) + \partial^i \nabla \cdot \mathbf{A}(x) + \varepsilon\mu\, \partial_0^2 A^i(x) + 4\pi\sigma\mu\, \partial_0 A^i(x) = \mu\, J^i(x), \quad i = 1, 2, 3. \tag{3}$$

Upon multiplying this equation by ∂_i and summing over $i = 1, 2, 3$, it may be rewritten as follows:

$$\left[-\nabla^2 + \varepsilon\mu\, \partial_0^2 + 4\pi\sigma\mu\, \partial_0 \right] A^i(x) = \mu \left[\delta^{ij} - \frac{\partial^i \partial^j}{\varepsilon\mu\, \partial_0^2 + 4\pi\sigma\mu\, \partial_0} \right] J^j(x). \tag{4}$$

We may thus introduce the propagator as a 3×3 matrix with elements $D^{ij}(x', x)$, satisfying equation

$$\left[-\nabla'^2 + \varepsilon\mu\, \partial_0'^2 + 4\pi\sigma\mu\, \partial_0' \right] D^{ij}(x', x) = \left[\delta^{ij} - \frac{\partial'^i \partial'^j}{\varepsilon\mu\, \partial_0'^2 + 4\pi\sigma\mu\, \partial_0'} \right] \delta^{(4)}(x', x). \tag{5}$$

Upon setting, $\mathbf{x} = (\mathbf{x}_\mathrm{T}, z)$, we note that because translational invariance is broken along the z-axis, due to the presence of a layer of metal, with the upper surface at $z = 0$, and the lower one at $z = -D$, as shown below, we have written the arguments of D^{ij} and of $\delta^{(4)}$ as (x', x) rather than as $(x' - x)$:

$z > 0$

$- - - - - - - - z = 0$

ε, μ, σ

$- - - - - - - - z = -D$

For emission of a photon from air in the $z > 0$ region, to consider transmission through the metal, we must clearly choose $z > 0$, and eventually choose $z' < -D$. Through complex integration in the energy plane, the causal propagator for time evolution from x^0 to x'^0, for $x^0 \leq x'^0$, implies that we may replace the denominator term $(\varepsilon\mu\, \partial_0'^2 + 4\pi\sigma\mu\, \partial_0')$ on the right-hand side of (5) by ∇'^2, giving rise to equation

$$\left[-\nabla'^2 + \varepsilon\mu\, \partial_0'^2 + 4\pi\sigma\mu\, \partial_0' \right] D^{ij}(x', x) = \left[\delta^{ij} - \frac{\partial'^i \partial'^j}{\nabla'^2} \right] \delta^{(4)}(x', x), \tag{6}$$

satisfying the transversality conditions

$$\partial'^i D^{ij}(x', x) = 0, \qquad \partial^j D^{ij}(x', x) = 0. \tag{7}$$

With $a = 1, 2, j = 1, 2, 3$, and $(\varepsilon(z'), \mu(z')) = (\varepsilon, \mu)$, for $-D < z' < 0$, and $(\varepsilon(z'), \mu(z')) = (1, 1)$ otherwise, we may spell out the boundary conditions of the problem as follows:

$$D^{aj}(x', x), \quad \varepsilon(z')\, D^{3j}(x', x), \quad \frac{\left[\partial'^3 D^{aj}(x', x) - \partial'^a D^{3j}(x', x)\right]}{\mu(z')}, \quad \left[\partial'^1 D^{2j}(x', x) - \partial'^2 D^{1j}(x', x)\right], \tag{8}$$

are continuous at $z' = 0$ and at $z' = -D$.

3. Expression of the causal propagator as it emerges in various regions

Region $z' > 0$:

With the causal arrangement $x'^0 > x^0$, we look for a particular as well as for a homogeneous solution of (6). To this end, a particular solution, for $a, b = 1, 2$, is elementary and, by complex integration in the energy plane, is given by

$$D_{\mathrm{p}}^{ab}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \int \frac{\mathrm{d}q}{2\,|\mathbf{k}|2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{K}\cdot(\mathbf{x}'_{\mathrm{T}} - \mathbf{x}_{\mathrm{T}})} \mathrm{e}^{\mathrm{i}q(z'-z)} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0 - x^0)} \left(\delta^{ab} - \frac{K^a K^b}{\mathbf{k}^2}\right), \tag{9}$$

representing the elements of a 2×2 matrix, with $a, b = 1, 2$. On the other hand, a solution of the homogeneous equation

$$\left(-\bar{\partial}'^2 + \partial_0'^2\right) D_{\mathrm{h}}^{jk}(x', x) = 0, \tag{10}$$

which, in the variable x, also satisfies

$$\left(-\bar{\partial}^2 + \partial_0^2\right) D_{\mathrm{h}}^{jk}(x', x) = 0, \tag{11}$$

is of the form

$$D_{\mathrm{h}}^{ab}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \int \frac{\mathrm{d}q'\mathrm{d}q}{2\,|\mathbf{k}|2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{K}\cdot(\mathbf{x}'_{\mathrm{T}} - \mathbf{x}_{\mathrm{T}})} \mathrm{e}^{\mathrm{i}q'z'} \mathrm{e}^{-\mathrm{i}qz} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0 - x^0)} \tilde{D}_{\mathrm{h}}^{ab}, \tag{12}$$

where the overall i multiplicative factor is chosen for convenience, and

$$\mathbf{K}^2 + q'^2 = \mathbf{k}^2, \tag{13}$$

while (11) gives,

$$\mathbf{K}^2 + q^2 = \mathbf{k}^2. \tag{14}$$

From the last two equations, we *derive that* $q' = \pm q$. In the reflection region $z' > 0$, we must choose $q' = -q$, $q < 0$, with the latter negativity condition achieved by the restriction set by a detection source $J^i(x')$ in region $z' > 0$. Equation (15) becomes

$$D_{\mathrm{h}}^{ab}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \int \frac{\mathrm{d}q}{2\,|\mathbf{k}|2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{K}\cdot(\mathbf{x}'_{\mathrm{T}} - \mathbf{x}_{\mathrm{T}})} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0 - x^0)} \mathrm{e}^{-\mathrm{i}q(z'+z)} A_{>}^{ab}, \tag{15}$$

where $A_{>}^{ab}$ will be determined from the boundary conditions.

We may thus write the solution in question for this region as follows:

$$D^{ab}_{>}(x',x) = \mathrm{i}\int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\int \frac{\mathrm{d}q}{2\,|\mathbf{k}|2\pi}\,\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot(\mathbf{x}'_{\mathrm{T}}-\mathbf{x}_{\mathrm{T}})}\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^{0}-x^{0})}\tilde{D}^{ab}_{>}, \tag{16}$$

and

$$\tilde{D}^{ab}_{>} = \mathrm{e}^{\mathrm{i}q(z'-z)}\left(\delta^{ab} - \frac{K^a K^b}{\mathbf{k}^2}\right) + \mathrm{e}^{-\mathrm{i}q(z'+z)}A^{ab}_{>}. \tag{17}$$

From (11), we may readily obtain the corresponding expressions for D^{3b}, D^{a3}, and D^{33}. Accordingly, for this region, with $j,k = 1,2,3$, we have

$$D^{jk}_{>}(x',x) = \mathrm{i}\int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\int \frac{\mathrm{d}q}{2\,|\mathbf{k}|2\pi}\,\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot(\mathbf{x}'_{\mathrm{T}}-\mathbf{x}_{\mathrm{T}})}\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^{0}-x^{0})}\tilde{D}^{jk}_{>}, \tag{18}$$

and, with $a = 1,2$,

$$\tilde{D}^{a3}_{>} = \mathrm{e}^{\mathrm{i}q(z'-z)}\left(-\frac{K^a q}{\mathbf{k}^2}\right) + \mathrm{e}^{-\mathrm{i}q(z'+z)}\left(-\frac{A^{ab}_{>}K^b}{q}\right), \tag{19}$$

$$\tilde{D}^{33}_{>} = \mathrm{e}^{\mathrm{i}q(z'-z)}\left(\frac{\mathbf{K}^2}{\mathbf{k}^2}\right) + \mathrm{e}^{-\mathrm{i}q(z'+z)}\left(-\frac{K^a A^{ab}_{>}K^b}{q^2}\right), \tag{20}$$

$$\tilde{D}^{3a}_{>} = \mathrm{e}^{\mathrm{i}q(z'-z)}\left(-\frac{K^a q}{\mathbf{k}^2}\right) + \mathrm{e}^{-\mathrm{i}q(z'+z)}\left(\frac{K^b A^{ba}_{>}}{q}\right). \tag{21}$$

Region $z' < -D$:

The structure of D^{jk} may be essentially written down from the considerations of the above region, except now the question of a particular solution does not arise ($z' \neq z$). Also, we have $q' = q$, $q < 0$. Accordingly

$$D^{jk}_{<}(x',x) = \mathrm{i}\int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\int \frac{\mathrm{d}q}{2\,|\mathbf{k}|2\pi}\,\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot(\mathbf{x}'_{\mathrm{T}}-\mathbf{x}_{\mathrm{T}})}\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^{0}-x^{0})}\tilde{D}^{jk}_{<}, \tag{22}$$

$$\tilde{D}^{ab}_{<} = \mathrm{e}^{\mathrm{i}q(z'-z)}A^{ab}_{<}, \tag{23}$$

$$\tilde{D}^{a3}_{<} = \mathrm{e}^{\mathrm{i}q(z'-z)}\left(-\frac{A^{ab}_{<}K^b}{q}\right), \tag{24}$$

$$\tilde{D}^{33}_{<} = \mathrm{e}^{\mathrm{i}q(z'-z)}\left(\frac{K^a A^{ab}_{<}K^b}{q^2}\right), \tag{25}$$

$$\tilde{D}^{3a}_{<} = \mathrm{e}^{\mathrm{i}q(z'-z)}\left(-\frac{K^b A^{ba}_{<}}{q}\right), \tag{26}$$

with $a = 1,2$. The unknown $A^{ba}_{<}$ will be also determined from the boundary conditions.

Region $-D < z' < 0$:

Here, we have the homogeneous equation:

$$\left(-\bar{\partial}'^2 + \varepsilon\mu\,\partial'^2_0 + 4\pi\sigma\mu\partial'_0\right)D^{jk}(x',x) = 0, \tag{27}$$

while in terms of the variable x,

$$\left(-\bar{\partial}^2 + \partial_0^2\right) D^{jk}(x', x) = 0. \tag{28}$$

Thus, upon writing

$$D^{ab}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \int \frac{\mathrm{d}q'\mathrm{d}q}{2\,|\mathbf{k}|2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{K}\cdot(\mathbf{x}'_{\mathrm{T}}-\mathbf{x}_{\mathrm{T}})} \mathrm{e}^{\mathrm{i}q'z'} \mathrm{e}^{-\mathrm{i}qz} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0-x^0)} \tilde{D}^{ab}, \tag{29}$$

with $a, b = 1, 2$, we have from (27)

$$\mathbf{K}^2 + q'^2 = \varepsilon\mu \left[\mathbf{k}^2 + \mathrm{i}\frac{4\pi\sigma}{\varepsilon}\right], \tag{30}$$

while (28) gives

$$\mathbf{K}^2 + q^2 = \mathbf{k}^2. \tag{31}$$

From the last two equations, we obtain

$$q' = \pm Q, \qquad \text{with} \quad Q = \sqrt{(\varepsilon\mu - 1)\mathbf{K}^2 + \varepsilon\mu q^2 + \mathrm{i}\,4\pi\sigma\mu|\mathbf{k}|}. \tag{32}$$

Hence, the general solution for $a, b = 1, 2$ is given by

$$D^{ab}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \int \frac{\mathrm{d}q}{2\,|\mathbf{k}|2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{K}\cdot(\mathbf{x}'_{\mathrm{T}}-\mathbf{x}_{\mathrm{T}})} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0-x^0)} \underline{D}^{ab}, \tag{33}$$

$$\underline{D}^{ab} = \mathrm{e}^{\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} M_1^{ab} + \mathrm{e}^{-\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} M_2^{ab}, \tag{34}$$

and we must consider the two solutions corresponding to $\pm Q$.

From (33) and (34), we then have, for the general solution for this region

$$D^{jk}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \int \frac{\mathrm{d}q}{2\,|\mathbf{k}|2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\mathrm{T}}-\mathbf{x}_{\mathrm{T}})} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0-x^0)} \underline{D}^{jk}, \tag{35}$$

with

$$\underline{D}^{a3} = \mathrm{e}^{\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \left(-\frac{M_1^{ab}K^b}{q}\right) + \mathrm{e}^{-\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \left(-\frac{M_2^{ab}K^b}{q}\right), \tag{36}$$

$$\underline{D}^{33} = \mathrm{e}^{\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \left(\frac{K^a M_1^{ab} K^b}{qQ}\right) + \mathrm{e}^{-\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \left(-\frac{K^a M_2^{ab} K^b}{qQ}\right), \tag{37}$$

$$\underline{D}^{3a} = \mathrm{e}^{\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \left(-\frac{K^b M_1^{ba}}{Q}\right) + \mathrm{e}^{-\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \left(\frac{K^b M_2^{ba}}{Q}\right). \tag{38}$$

4. Satisfying the boundary conditions

Boundary Conditions at $z' = 0$:

The boundary conditions may be read directly from the previous section, and according to the continuity conditions spelled out in (8), respectively, are given by the constraints

$$\left(\delta^{ab} - \frac{K^a K^b}{\mathbf{k}^2}\right) + A_>^{ab} = M_1^{ab} + M_2^{ab}, \tag{39}$$

$$-\frac{K^b q}{\mathbf{k}^2} + \frac{K^a A_>^{ab}}{q} = \varepsilon\left(-\frac{K^a M_1^{ab}}{Q} + \frac{K^a M_2^{ab}}{Q}\right), \tag{40}$$

$$q\left(\delta^{ab} - A_>^{ab}\right) - K^a\frac{K^c A_>^{cb}}{q} = \frac{1}{\mu}\left[Q\left(M_1^{ab} - M_2^{ab}\right) + K^a\left(\frac{K^c M_1^{cb}}{Q} - \frac{K^c M_2^{cb}}{Q}\right)\right]. \tag{41}$$

One may then solve for M_1^{ab}, M_2^{ab} to obtain

$$2M_1^{ab} = \left(\frac{Q+\mu\, q}{Q}\right)\delta^{ab} - \left(1 + \frac{q}{\varepsilon\, Q}\right)\frac{K^a k^b}{\mathbf{k}^2} + \left(\frac{Q-\mu\, q}{Q}\right)A_>^{ab} + \mu\left(\frac{1}{\varepsilon\mu} - 1\right)K^a\frac{K^c A_>^{cb}}{Qq}, \tag{42}$$

$$2M_2^{ab} = \left(\frac{Q-\mu\, q}{Q}\right)\delta^{ab} - \left(1 - \frac{q}{\varepsilon\, Q}\right)\frac{K^a k^b}{\mathbf{k}^2} + \left(\frac{Q+\mu\, q}{Q}\right)A_>^{ab} - \mu\left(\frac{1}{\varepsilon\mu} - 1\right)K^a\frac{K^c A_>^{cb}}{Qq}. \tag{43}$$

Boundary Conditions at $z' = -D$:
According to the expressions in the previous section and the continuity conditions spelled out in (8) we have, respectively,

$$\mathrm{e}^{-\mathrm{i}QD}M_1^{ab} + \mathrm{e}^{\mathrm{i}QD}M_2^{ab} = \mathrm{e}^{-\mathrm{i}qD}A_<^{ab}, \tag{44}$$

$$\mathrm{e}^{-\mathrm{i}qD}\left(-\frac{K^a A_<^{ab}}{q}\right) = \varepsilon\left[\mathrm{e}^{-\mathrm{i}QD}\left(-\frac{K^a M_1^{ab}}{Q}\right) + \mathrm{e}^{\mathrm{i}QD}\left(\frac{K^a M_2^{ab}}{Q}\right)\right], \tag{45}$$

$$\mathrm{e}^{-\mathrm{i}qD}\left(qA_<^{ab} + K^a\frac{K^c A_<^{cb}}{q}\right) = \frac{1}{\mu}\left[Q\left(\mathrm{e}^{-\mathrm{i}QD}M_1^{ab} - \mathrm{e}^{\mathrm{i}QD}M_2^{ab}\right) + K^a K^c\left(\mathrm{e}^{-\mathrm{i}QD}\frac{M_1^{cb}}{Q} - \mathrm{e}^{\mathrm{i}QD}\frac{M_2^{cb}}{Q}\right)\right]. \tag{46}$$

These set of equation lead in turn to

$$2M_1^{ab} = \mathrm{e}^{+\mathrm{i}(Q-q)D}\left[\left(\frac{Q+\mu\, q}{Q}\right)A_<^{ab} + \mu\left(1 - \frac{1}{\varepsilon\mu}\right)K^a\frac{K^c A_<^{cb}}{qQ}\right], \tag{47}$$

$$2M_2^{ab} = \mathrm{e}^{-\mathrm{i}(Q+q)D}\left[\left(\frac{Q-\mu\, q}{Q}\right)A_<^{ab} - \mu\left(1 - \frac{1}{\varepsilon\mu}\right)K^a\frac{K^c A_<^{cb}}{qQ}\right]. \tag{48}$$

Upon setting, in particular,

$$A_<^{ab} = \delta^{ab}\, a_< + \frac{K^a K^b}{\mathbf{K}^2} b_<, \tag{49}$$

and comparing (42) with (47), and (43) with (48), a fairly tedious analysis gives the following solutions relevant to a photon crossing the layer of metal:

$$a_< = \frac{[(Q+\mu q)^2 - (Q-\mu q)^2]\,\mathrm{e}^{\mathrm{i}qD}}{(Q+\mu q)^2\mathrm{e}^{\mathrm{i}QD} - (Q-\mu q)^2\mathrm{e}^{-\mathrm{i}QD}}. \tag{50}$$

and the combination

$$a_< + b_< = \frac{q^2}{\mathbf{k}^2}\frac{\left[(\varepsilon\mu\,\mathbf{k}^2 - \mathbf{K}^2 + \varepsilon qQ)^2 - (\varepsilon\mu\,\mathbf{k}^2 - \mathbf{K}^2 - \varepsilon qQ)^2\right]\,\mathrm{e}^{\mathrm{i}qD}}{(\varepsilon\mu\,\mathbf{k}^2 - \mathbf{K}^2 + \varepsilon qQ)^2\mathrm{e}^{\mathrm{i}QD} - (\varepsilon\mu\,\mathbf{k}^2 - \mathbf{K}^2 - \varepsilon qQ)^2\mathrm{e}^{-\mathrm{i}QD}}. \tag{51}$$

5. Explicit solution of the causal propagator crossing the metal

From (22)–(26), the causal propagator corresponding to a photon crossing the piece of metal may be now written in an explicit unified notation as follows:

$$D^{ij}_<(x',x) = \mathrm{i}\int\frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\int_{q<0}\frac{\mathrm{d}q}{2\,|\mathbf{k}|2\pi}\,\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot(\mathbf{x}'_\mathrm{T}-\mathbf{x}_\mathrm{T})}\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0-x^0)}\mathrm{e}^{\mathrm{i}q(z'-z)}A^{ij}_<, \tag{52}$$

where a tedious analysis shows that the expressions for $A^{ab}_<$, $A^{3a}_<$, $A^{a3}_<$, $A^{33}_<$, may be expressed in a unified manner as follows:

$$A^{ij}_< = \left(\delta^{ij}a_< + \frac{k^ik^j}{\mathbf{K}^2}b_<\right) + \left(a_< + \frac{\mathbf{k}^2}{\mathbf{K}^2}b_<\right)\left[\frac{\mathbf{k}^2}{q^2}\delta^{i3}\delta^{j3} - \frac{k^i\delta^{j3}+\delta^{i3}k^j}{q}\right],\quad \mathbf{k} = (\mathbf{K},q), \tag{53}$$

where $a<$, $b_<$ are given through (50) and (51). The important transversality conditions are also readily verified

$$k^i A^{ij}_< = 0, \qquad A^{ij}_< k^j = 0. \tag{54}$$

6. Transition probability of a photon crossing the layer of metal

The transition amplitude for a photon crossing the layer of metal is obtained from the expression

$$J^{i*}_3(k)\; A^{ij}_>\; J^j_1(k), \tag{55}$$

where $J^i_1(k), J^j_3(k)$ are Fourier transforms of an emission source and a detection source, respectively, set causally, in the $z > 0$ and $z' < -D$ regions, and $A^{ij}_>$ is given in (53). We introduce polarization vectors $\mathbf{e}_\alpha, \alpha, \beta = 1, 2$, and introduce the completeness relations:

$$\delta^{ij} = \frac{k^i\,k^j}{\mathbf{k}^2} + \sum_{\alpha=1,2}\mathrm{e}^{i*}_\alpha\,\mathrm{e}^j_\alpha = \frac{k^i\,k^j}{\mathbf{k}^2} + \sum_{\alpha=1,2}\mathrm{e}^i_\alpha\,\mathrm{e}^{j*}_\alpha, \tag{56}$$

$$\mathbf{k}\cdot\mathbf{e}_\alpha = 0,\quad \mathbf{k}\cdot\mathbf{e}^*_\alpha = 0,\qquad \mathbf{e}_\alpha\cdot\mathbf{e}^*_\beta = \delta_{\alpha\beta},\qquad \mathbf{e}_\alpha = (e_{i\alpha}),\ i = 1,2,3. \tag{57}$$

Upon rewriting (55) as $J^{\ell *}_3(k)\delta^{\ell i}A^{ij}_<\delta^{j\ell'}J^{\ell'}_1(k)$, using the completeness relations in (56), and the important transversality conditions in (57), Equation (55), may be rewritten as follows:

$$\sum_{\alpha,\beta=1,2}\left(\mathbf{J}^*_3(k)\cdot\mathbf{e}_\beta\right)[\,\mathrm{e}^{i*}_\beta\,A^{ij}_<\,\mathrm{e}^j_\alpha\,]\left(\mathbf{e}^*_\alpha\cdot\mathbf{J}_1(k)\right),\quad \mathbf{J}_1 = (J^i_1),\ \mathbf{J}^*_3 = (J^{*i}_3),\ i = 1,2,3, \tag{58}$$

from which the amplitude of a photon with polarization $\mathbf{e}_\alpha$ being transmitted through the layer of metal and ending up with polarization $\mathbf{e}_\beta$, is given by

$$(\mathbf{k},\beta|\mathbf{k},\alpha) \;=\; \mathrm{e}^{i*}_{\beta}\, A^{ij}_{<}\, \mathrm{e}^{j}_{\alpha}, \quad \mathbf{k}=(\mathbf{K},q), \quad q<0, \tag{59}$$

with $\mathbf{e}^*_\alpha \cdot \mathrm{i}\mathbf{J}_1(k)$, $\mathrm{i}\mathbf{J}^*_3(k)\cdot\mathbf{e}_\beta$, denoting, respectively, amplitudes for emitting and detecting a photon with such attributes, by their respective sources.

We may choose momenta and real polarization vectors as follows ($q<0, K_2\equiv K$)

$$\mathbf{k}=(0,K,q), \qquad \mathbf{e}_1=(1,0,0), \quad \mathbf{e}_2=\frac{1}{|\mathbf{k}|}(0,q,-K), \tag{60}$$

and rewrite (59) as follows:

$$(\mathbf{k},\beta|\mathbf{k},\alpha)=\delta_{\alpha 1}\delta_{\beta 1}\,a_{<}+\frac{\mathbf{k}^2}{q^2}\,\big(a_{<}+\,b_{<}\big)\,\delta_{\alpha 2}\,\delta_{\beta 2}, \tag{61}$$

with $a_<$, $[(\mathbf{k}^2/q^2)\,(\,a_<+\,b_<)]$ given, respectively, in (50) and (51).

For unpolarized photons, we average over the initial polarization states and sum over the final polarization states. Using the completeness relations in (56), then the following expression emerges for the probability that an unpolarized photon crosses the layer of metal

$$\text{Prob}\;=\sum_{\beta=1,2}\left(\frac{1}{2}\sum_{\alpha=1,2}|(\mathbf{k},\beta|\mathbf{k},\alpha)|^2\right)=\frac{1}{2}\left[|a_<|^2+\frac{\mathbf{k}^4}{q^4}\,|a_<+b_<|^2\right]. \tag{62}$$

For normal incidence, for example, $\mathbf{K}=\mathbf{0}$, and $(\mathbf{k}^2/q^2)(a_<+b_<)=-\,a_<$. Also, for optical properties, we may set $\mu=1$. The transition probability of a photon crossing a layer of metal of thickness D, then emerges from (61), (50) to be

$$\text{Prob}=\frac{\mathcal{N}}{\mathcal{D}}, \tag{63}$$

where

$$\mathcal{N}=[(A-1)^2+B^2]^2+[(A+1)^2+B^2]^2-2\,\Big([A^2-1+B^2]^2-4B^2\Big), \tag{64}$$

$$\begin{aligned}\mathcal{D}=&\,[(A-1)^2+B^2]^2\,\mathrm{e}^{-2|q|BD}+[(A+1)^2+B^2]^2\,\mathrm{e}^{2|q|BD}\\&-2\,\Big([A^2-1+B^2]^2-4B^2\Big)\cos 2|q|AD+8\,B[A^2-1+B^2]\sin 2|q|AD,\end{aligned} \tag{65}$$

and

$$Q=|q|\sqrt{\varepsilon+\mathrm{i}\frac{4\pi\sigma}{|q|}}\equiv|q|\,\big(A+\mathrm{i}B\big), \quad n_{\mathrm{c}}=\sqrt{\varepsilon+\mathrm{i}\frac{4\pi\sigma}{|q|}}, \tag{66}$$

with n_{c} defining the complex index of refraction,

$$\varepsilon\equiv\varepsilon_{\mathrm{R}}, \qquad \frac{4\pi\sigma}{|q|}\equiv\varepsilon_{\mathrm{I}}, \tag{67}$$

defining, respectively, the real and imaginary parts of the complex dielectric function. Moreover, A and B are defined through (66) for the given pair $(\varepsilon_{\mathrm{R}}, \varepsilon_{\mathrm{I}})$. It is impossible not to be tempted to investigate the fate of, say, a red and blue photon in the spirit of Feynman. For concreteness, consider silver, whose electromagnetic properties, in particular, have been well investigated experimentally in [5], with thickness $D = 50\,\mathrm{nm}$. For a red photon of wavelength $\lambda_{\mathrm{red}} = 625\,\mathrm{nm}$, at the lower end of the red part of the spectrum, [5]: $\varepsilon_{\mathrm{R}} = -18.18$, $\varepsilon_{\mathrm{I}} = 0.43$. These give $A = .0504$, $B = 4.264$ leading to a probability equal to .011 of such a photon crossing this thin silver film. For a blue photon of wavelength $\lambda_{\mathrm{blue}} = 450\,\mathrm{nm}$, corresponding to a blue photon at the lower end of the blue part of the spectrum, [5]: $A = .0523$, $B = 2.676$ leading to probability equal to .040 of such a photon crossing this thin silver film.

An important application of the exact expression of the probability of transmission of light through a metal obtained in (63)–(65), is in assessing the accuracy of the phenomenological expression of transmission with a simple exponential damping. The latter probability may be obtained directly from our exact expression by considering the condition $D \gg 1/2|q|B$. Hence, by conveniently denoting the probability in (63) by $\mathrm{Prob}_{\mathrm{exact}}$, we obtain

$$\mathrm{Prob}_{\mathrm{exact}} \rightarrow \left[1 + \frac{[(A-1)^2 + B^2]^2}{[(A+1)^2 + B^2]^2} - 2\,\frac{\left([A^2 - 1 + B^2]^2 - 4B^2\right)}{[(A+1)^2 + B^2]^2}\right] \mathrm{e}^{-2|q|BD} \equiv \mathrm{Prob}_{\infty}, \qquad (68)$$

where now Prob_{∞} is the limiting expression with a simple exponential damping often associated with transmissions through metals. To assess the accuracy of this phenomenological expression involving a simple exponential damping, we may, for any given thickness D of the metal, define the absolute relative error $E(D)$ by the following expression:

$$E(D) = \left|\frac{\mathrm{Prob}_{\mathrm{exact}} - \mathrm{Prob}_{\infty}}{\mathrm{Prob}_{\mathrm{exact}}}\right|. \qquad (69)$$

From Equations (63)–(65) and the right-hand side of Equation (68), $E(D)$ is readily worked out to be

$$\begin{aligned} E(D) = \Bigg| & 2\,\frac{\left([A^2 - 1 + B^2]^2 - 4B^2\right)}{[(A+1)^2 + B^2]^2}\,\mathrm{e}^{-2|q|BD} \cos 2|q|AD \\ & - 8B\,\frac{[A^2 - 1 + B^2]}{[(A+1)^2 + B^2]^2}\,\mathrm{e}^{-2|q|BD} \sin 2|q|AD \\ & - \frac{[(A-1)^2 + B^2]^2}{[(A+1)^2 + B^2]^2}\,\mathrm{e}^{-4|q|BD} \Bigg|. \end{aligned} \qquad (70)$$

The absolute (see Figure 1) relative error in (70) is plotted above for red and blue light for various thicknesses D of the metal in nm. In particular, we note that the simple expression with exponential damping sets in faster for blue light than for red light. The absolute relative errors for $D = 50\,\mathrm{nm}$, for example, are 1.5 % for red light, while only 0.26 % for blue light. The graphs also indicate that for very thin metal films the expression with simple exponential damping is not reliable and one has to recourse to the exact expression in (63)–(65). Needless to say, the general expression of the propagator given in (52) and (53), together with the ones in (50) and (51), are quite general and apply for materials of different thickness, not just thin films, and for a wide range of the spectrum

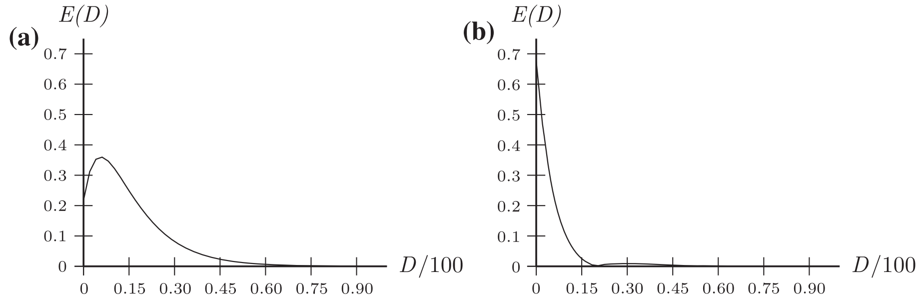

Figure 1. Absolute relative errors for red and blue lights (a) and (b), respectively.

of light. Similar applications may be carried for other metals such as aluminum, whose electromagnetic properties are well established experimentally [4], and for other metals as well [1,2]. Hopefully, this work will be also of interest to practitioners working on different aspects, and with different approaches in the applications of electromagnetic waves and realize the importance of the propagator approach for electromagnetic wave propagation, emphasizing, in turn, the concept of a photon, which goes beyond just the analysis of Maxwell's equations, and would be applicable when non-linearities are present in a theory, such as in quantum electrodynamics. This is not to mention the physically appealing approach using causal propagators as describing propagation between any two points, not necessarily in the same medium, as a time evolution process. This is in contrast, for example, to an oscillating electric field at a given one point in a medium.

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