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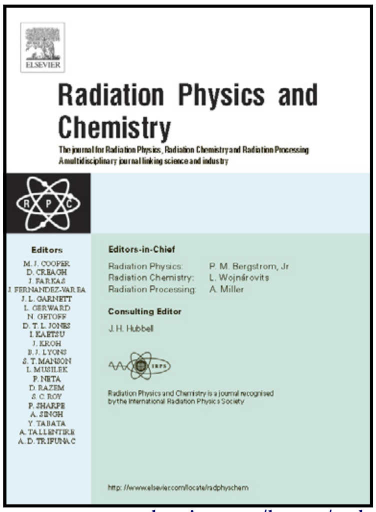

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Vacuum-to-Vacuum Transition Probability and the Classic Radiation Theory

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Abstract

Using the fact that the vacuum-to-vacuum transition probability for the interaction of the Maxwell field $A^\mu(x)$ with a given current $J_\mu(x)$ represents the probability of no photons emitted by the current of a Poisson distribution, the average number of photons emitted of given energies for the underlying distribution is readily derived. From this the classical power of radiation of Schwinger of a relativistic charged particle follows.

Key Words: electromagnetic radiation; quantum viewpoint of Maxwell's theory; classical and quantum probability; classic radiation theory

1 Introduction

The Maxwell Lagrangian density for the interaction of the vector potential $A_\mu(x)$ with an external current $J^\mu(x)$ is given by

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + J^\mu(x)A_\mu(x), \qquad F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{1}$$

Prior to switching on of the current, as a source of photon production, one is dealing with a vacuum state, denoted by $|0_-\rangle$, involving no photons. After switching on of the current, the state of the system may evolve to one involving any number of photons, or it may just stay in the vacuum state, involving no photons, with the latter state now denoted by $|0_+\rangle$. Quantum theory tells us that the vacuum-to-vacuum transition probability satisfies the inequality $|\langle 0_+|0_-\rangle|^2 < 1$, due to conservation of probability, allowing the possibility that the system evolves to other states as well involving an arbitrary number of photons that may be created by the current source. A very interesting property of this system, is that the probability distribution of the photon number N created by the current [1] is given by the Poisson distribution [2]. That is

$$\text{Prob}[N = n] \,=\, \frac{(\lambda)^n}{n!}\,\mathrm{e}^{-\lambda}, \quad n = 0, 1, ..., \qquad \lambda =< N >, \tag{2}$$

where $\lambda =< N >$ denotes the average number of photons created by the current source, and

$$\exp[- < N >] \,=\, |\langle 0_+|0_-\rangle|^2, \tag{3}$$

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denotes the probability that no photons are created by the current source, i.e., it represents the vacuum-to-vacuum transition probability $|\langle 0_+|0_-\rangle|^2$ as just stated.

The purpose of this communication is by using the expression of the vacuum-to-vacuum transition amplitude, derive the exact expression of the average number of photons, of a given frequency, produced by a given general current distribution, from which the classic radiation theory of radiation emitted from a relativistic charge particle may be readily obtained for the power of radiation as well as for the power of radiation emitted along a given direction.

Quantum viewpoint analysis, as discussed above, of electromagnetic phenomena and electromagnetic radiation, e.g., [9], and of related applications [4-9,11] turns out to be quite useful in applications and certainly in simplifying, to a large extent, derivations in this field as we will witness below. In particular, we note that due to the generality of the expressions leading to the total energy of radiation emitted, derived below in Eqn(18), and being valid for arbitrary current distributions, the analysis is expected to have further applications in the domain of synchrotron radiation as well as in considering quantum corrections to general physical problems in radiation theory. We are especially interested in generalizing the present analysis to radiation in the presence of obstacles as well in different media than just the vacuum and will be attempted in a future report.

2 Average Number of Photons Emitted of a Given Frequency

The vacuum-to-vacuum transition of the theory described by the the Lagrangian density in (1) is given by [10]

$$\langle 0_+|0_-\rangle = \exp\Big[\frac{\mathrm{i}}{2\hbar\,\mathrm{c}^3}\int(\mathrm{d}x')\,(\mathrm{d}x)J^\mu(x')\eta_{\mu\nu}D(x',x)J^\nu(x)\Big], \quad (\mathrm{d}x)=\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \tag{4}$$

$[\,\eta_{\mu\nu}\,]=\mathrm{diag}[-1,1,1,1]$, the photon propagator is given by

$$D(x',x)=\int\frac{(\mathrm{d}Q)}{(2\,\pi)^4}\,\frac{\mathrm{e}^{\mathrm{i}Q(x'-x)}}{Q^2-\mathrm{i}\epsilon},\quad Q^2=\mathbf{Q}^2-{Q^0}^2,\quad \epsilon\to+0, \tag{5}$$

and $J^\mu(x)$ is the conserved four-current $\partial_\mu J^\mu(x)=0$, $(J^\mu(x))^*=J^\mu(x)$.

The vacuum-to-vacuum transition probability is then

$$|\langle 0_+|0_-\rangle|^2=\exp\Big[-\frac{1}{\hbar\,\mathrm{c}^3}\int(\mathrm{d}x')\,(\mathrm{d}x)J^\mu(x')\eta_{\mu\nu}\big(\,\mathrm{Im}D(x',x)\,\big)J^\nu(x)\Big], \tag{6}$$

$$\mathrm{Im}D(x',x)=\pi\int\frac{(\mathrm{d}Q)}{(2\,\pi)^4}\,\delta(Q^2)\,\mathrm{e}^{\mathrm{i}Q(x'-x)}. \tag{7}$$

This gives from (6),

$$|\langle 0_+|0_-\rangle|^2=\exp\Big[-\frac{\pi}{\hbar\,\mathrm{c}^3}\int(\mathrm{d}x')\,(\mathrm{d}x)\,J^\mu(x')J_\mu(x)\int\frac{(\mathrm{d}Q)}{(2\,\pi)^4}\,\delta(Q^2)\,\mathrm{e}^{\mathrm{i}Q(x'-x)}\Big]. \tag{8}$$

We introduce the resolution of the identity expressing the equality of the energy of a photon $\hbar\,\omega=\hbar\,|\mathbf{Q}|\mathrm{c}$:

$$1=\int_0^\infty\mathrm{d}\omega\,\delta(\omega-|\mathbf{Q}|\mathrm{c}), \tag{9}$$

as well as the the average of the photon number density $N(\omega)$ of energy $\hbar\omega$:

$$< N > = \int_0^\infty \mathrm{d}\omega\, N(\omega). \tag{10}$$

Upon inserting the identity given in (9) in the Q-integral in the exponential in (8), we obtain the explicit expression for the photon number density to be given by

$$N(\omega) = \frac{\pi}{\hbar\, \mathrm{c}^4} \int (\mathrm{d}x')\,(\mathrm{d}x)\, J^\mu(x') J_\mu(x) \int \frac{(\mathrm{d}Q)}{(2\,\pi)^4}\, \delta\Big(|\mathbf{Q}| - \frac{\omega}{\mathrm{c}}\Big) \delta(Q^2)\, \mathrm{e}^{\mathrm{i}Q(x'-x)}, \tag{11}$$

where we have used the fact that $\delta(\omega - |\mathbf{Q}|\mathrm{c}) = \delta\big(|\mathbf{Q}| - (\omega/\mathrm{c})\big)/\mathrm{c}$.

Upon using the relation

$$\delta(Q^2) = \frac{\delta\big(Q^0 - |\mathbf{Q}|\big) + \delta\big(Q^0 + |\mathbf{Q}|\big)}{2\,|\mathbf{Q}|}, \tag{12}$$

in (11), we obtain

$$N(\omega) = \frac{1}{2\,\hbar\, \mathrm{c}^4} \int (\mathrm{d}x')\,(\mathrm{d}x)\, J^\mu(x') J_\mu(x) \int \frac{\mathrm{d}^3\mathbf{Q}}{2\,|\mathbf{Q}|\,(2\,\pi)^3}\, \delta\Big(|\mathbf{Q}| - \frac{\omega}{\mathrm{c}}\Big)$$
$$\times \Big[\mathrm{e}^{\mathrm{i}\mathbf{Q}.(\mathbf{x}'-\mathbf{x})} \mathrm{e}^{-\mathrm{i}|\mathbf{Q}|(x'^0 - x^0)} + \mathrm{e}^{\mathrm{i}\mathbf{Q}.(\mathbf{x}'-\mathbf{x})} \mathrm{e}^{\mathrm{i}|\mathbf{Q}|(x'^0 - x^0)}\Big]. \tag{13}$$

By making a simultaneous transformation $(x', \mathbf{Q}) \leftrightarrow (x, -\mathbf{Q})$ in the pair of the second exponentials within the square brackets, and taking into consideration of the symmetry of the product $J^\mu(x')J_\mu(x)$ under the transformation $x' \leftrightarrow x$, the above expression simplifies to

$$N(\omega) = \frac{1}{\hbar\, \mathrm{c}^4} \int (\mathrm{d}x')\,(\mathrm{d}x)\, J^\mu(x') J_\mu(x) \int \frac{\mathrm{d}^3\mathbf{Q}}{2\,|\mathbf{Q}|\,(2\,\pi)^3}\, \mathrm{e}^{\mathrm{i}Q(x'-x)}\, \delta\Big(|\mathbf{Q}| - \frac{\omega}{\mathrm{c}}\Big), \quad Q^0 = |\mathbf{Q}|. \tag{14}$$

We introduce the unit vector $\mathbf{n}$, via the equation

$$\mathbf{Q} = |\mathbf{Q}|\, \mathbf{n}, \tag{15}$$

and use the fact that

$$\frac{\mathrm{d}^3\mathbf{Q}}{|\mathbf{Q}|} = \mathrm{d}\Omega\, |\mathbf{Q}|\, \mathrm{d}|\mathbf{Q}|, \tag{16}$$

to carry out the $|\mathbf{Q}|$-integral in (14), using the property of the delta function $\delta\big(|\mathbf{Q}| - \omega/\mathrm{c}\big)$, giving the exact expression

$$N(\omega) = \frac{\omega}{16\,\pi^3\,\hbar\,\mathrm{c}^5} \int (\mathrm{d}x')\,(\mathrm{d}x)\, J^\mu(x') J_\mu(x) \int \mathrm{d}\Omega\, \mathrm{e}^{\mathrm{i}\,\omega(\mathbf{x}'-\mathbf{x}).\mathbf{n}/\mathrm{c}}\, \mathrm{e}^{-\mathrm{i}\,\omega(x'^0 - x^0)/\mathrm{c}}. \tag{17}$$

To obtain the total energy of radiation $E(\omega)$ per unit angular frequency about ω, we simply have to multiply the above expression by $\hbar\,\omega$, leading to

$$E(\omega) = \frac{\omega^2}{16\,\pi^3\,\mathrm{c}^5} \int (\mathrm{d}x')\,(\mathrm{d}x)\, J^\mu(x') J_\mu(x) \int \mathrm{d}\Omega\, \mathrm{e}^{\mathrm{i}\,\omega(\mathbf{x}'-\mathbf{x}).\mathbf{n}/\mathrm{c}}\, \mathrm{e}^{-\mathrm{i}\,\omega(x'^0 - x^0)/\mathrm{c}}, \tag{18}$$

and is independent of $\hbar$ as expected. The total energy is then given by

$$E = \int_0^\infty \mathrm{d}\omega\, E(\omega) = \frac{1}{2} \int_{-\infty}^\infty \mathrm{d}\omega E(\omega), \tag{19}$$

where in writing the last equality, we have used the reality condition on $E(\omega)$ implying that the latter is an even function of ω.

3 The Classic Radiation Theory

The current of a relativistic charged particle of charge e is given by the well known expressions

$$\mathbf{J}(x) \;=\; \mathrm{e}\,\dot{\mathbf{R}}\,\delta^{(3)}\big(\mathbf{x}-\mathbf{R}(t)\big), \quad \dot{\mathbf{R}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{R}, \quad x^0 = \mathrm{c}\,t, \tag{20}$$

$$J^0(x) \;=\; \mathrm{e}\,\mathrm{c}\,\delta^{(3)}\big(\mathbf{x}-\mathbf{R}(t)\big). \tag{21}$$

This gives for E

$$E = \frac{\mathrm{e}^2}{32\,\pi^3\,\mathrm{c}}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\omega^2\int\mathrm{d}\Omega\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}t'\left[\frac{\dot{\mathbf{R}}(t').\,\dot{\mathbf{R}}(t)}{\mathrm{c}^2}-1\right]\mathrm{e}^{\mathrm{i}\omega[\mathbf{R}(t')-\mathbf{R}(t)].\mathbf{n}]/\mathrm{c}}\mathrm{e}^{-\mathrm{i}\omega(t'-t)}. \tag{22}$$

For completeness and convenience of the reader we spell out the details in integrating this expression. To this end, we follow Schwinger and set

$$t'-t \;=\; \tau, \qquad \tau\left(1-\frac{[\mathbf{R}(t+\tau)-\mathbf{R}(t)]}{\tau\,\mathrm{c}}\right) \;=\; \gamma, \tag{23}$$

and note that

$$\left|\frac{[\mathbf{R}(t+\tau)-\mathbf{R}(t)]}{\tau}\right| < \mathrm{c}, \quad \tau\to\pm\infty \Rightarrow \gamma\to\pm\infty, \tag{24}$$

to obtain

$$E = \frac{\mathrm{e}^2}{32\,\pi^3\,\mathrm{c}}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\omega^2\int\mathrm{d}\Omega\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}\gamma\,\mathrm{e}^{-\mathrm{i}\omega\gamma}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\gamma}\right)\left[\frac{\dot{\mathbf{R}}(t+\tau).\,\dot{\mathbf{R}}(t)}{\mathrm{c}^2}-1\right]. \tag{25}$$

Using the fact that

$$\int_{-\infty}^{\infty}\mathrm{d}\omega\,\omega^2\,\mathrm{e}^{-\mathrm{i}\omega\gamma} = -(2\pi)\left(\frac{\mathrm{d}}{\mathrm{d}\gamma}\right)^2\delta(\gamma). \tag{26}$$

gives

$$E = -\frac{\mathrm{e}^2}{16\,\pi^2\,\mathrm{c}}\int\mathrm{d}\Omega\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}\gamma\,\delta(\gamma)\left(\frac{\mathrm{d}}{\mathrm{d}\gamma}\right)^2\left(\frac{\mathrm{d}\tau}{\mathrm{d}\gamma}\Big[\hat{\mathbf{V}}(t+\tau).\,\hat{\mathbf{V}}(t)-1\Big]\right), \quad \hat{\mathbf{V}} = \frac{\dot{\mathbf{R}}}{\mathrm{c}}. \tag{27}$$

as a consequence of a property of the delta function.

The chain rule

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} = \frac{\mathrm{d}\tau}{\mathrm{d}\gamma}\,\frac{\mathrm{d}}{\mathrm{d}\tau}, \qquad \frac{\mathrm{d}\tau}{\mathrm{d}\gamma} = \frac{1}{[1-\hat{\mathbf{V}}(t+\tau).\mathbf{n}]} \equiv f(t+\tau), \tag{28}$$

and the fact that $\gamma = 0$ gives $\tau = 0$, allow one to write

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}\gamma\,\delta(\omega)\frac{\mathrm{d}\tau}{\mathrm{d}\gamma}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\frac{\mathrm{d}\tau}{\mathrm{d}\gamma}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\gamma}\Big[\hat{\mathbf{V}}(t+\tau).\,\hat{\mathbf{V}}(t)-1\Big]\right)\\ &= \hat{\mathbf{V}}(t).\,f(t)\,\frac{\mathrm{d}}{\mathrm{d}\tau}f(t)\,\frac{\mathrm{d}}{\mathrm{d}t}f(t)\,\hat{\mathbf{V}}(t) \;-\; f(t)\,\frac{\mathrm{d}}{\mathrm{d}t}f(t)\,\frac{\mathrm{d}}{\mathrm{d}t}f(t)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}[F((t)] - \dot{\hat{\mathbf{V}}}.f(t)^2\frac{\mathrm{d}}{\mathrm{d}t}\big(f(t)\hat{\mathbf{V}}\big) - \hat{\mathbf{V}}\dot{f(t)}f(t)\frac{\mathrm{d}}{\mathrm{d}t}\big(f(t)\hat{\mathbf{V}}\big) + \dot{f(t)}^2 f(t),\end{aligned} \tag{29}$$

where $F(t)$ is a surface term given by

$$F(t) = f(t)^2\Big(\hat{\mathbf{V}}.\frac{\mathrm{d}}{\mathrm{d}t}\big(f(t)\hat{\mathbf{V}}\big) - \dot{f}(t)\Big)$$
$$= \dot{\hat{\mathbf{V}}}.\bigg(\frac{\hat{\mathbf{V}}}{[1-\hat{\mathbf{V}}.\mathbf{n}]^3} - \frac{\mathbf{n}(1-\hat{\mathbf{V}}^2)}{[1-\hat{\mathbf{V}}.\mathbf{n}]^4}\bigg). \tag{30}$$

and is proportional to the acceleration for $t \to \pm\infty$, while the remaining terms on the extreme right-hand of (29) are given by

$$\frac{\dot{\hat{\mathbf{V}}}^2}{[1-\hat{\mathbf{V}}.\mathbf{n}]^3} + 2\frac{\mathbf{n}.\dot{\hat{\mathbf{V}}}\,\hat{\mathbf{V}}.\dot{\hat{\mathbf{V}}}}{[1-\hat{\mathbf{V}}.\mathbf{n}]^4} - \frac{(1-\hat{\mathbf{V}}^2)(\mathbf{n}.\dot{\hat{\mathbf{V}}})^2}{[1-\hat{\mathbf{V}}.\mathbf{n}]^5}. \tag{31}$$

The power of radiation is independent of the surface term, and along the unit vector the Schwinger [11] expression for it follows

$$P(\mathbf{n},t) = \frac{\mathrm{e}^2}{16\,\pi^2\,\mathrm{c}}\bigg\{\frac{\dot{\hat{\mathbf{V}}}^2}{[1-\hat{\mathbf{V}}.\mathbf{n}]^3} + 2\frac{\mathbf{n}.\dot{\hat{\mathbf{V}}}\;\hat{\mathbf{V}}.\dot{\hat{\mathbf{V}}}}{[1-\hat{\mathbf{V}}.\mathbf{n}]^4} - \frac{(1-\hat{\mathbf{V}}^2)\,(\mathbf{n}.\dot{\hat{\mathbf{V}}})^2}{[1-\hat{\mathbf{V}}.\mathbf{n}]^5}\bigg\}. \tag{32}$$

Using the set of integrals

$$\int \frac{\mathrm{d}\Omega}{[1-\hat{\mathbf{V}}.\mathbf{n}]^3} = \frac{4\pi}{[1-\hat{\mathbf{V}}^2]^2}, \tag{33}$$

$$\int \frac{n^i\,\mathrm{d}\Omega}{[1-\hat{\mathbf{V}}.\mathbf{n}]^4} = \frac{16\pi}{3}\,\frac{\hat{V}^i}{[1-\hat{\mathbf{V}}^2]^3}, \tag{34}$$

$$\int \frac{n^i\,n^j\,\mathrm{d}\Omega}{[1-\hat{\mathbf{V}}.\mathbf{n}]^5} = \frac{4\,\pi}{[1-\hat{\mathbf{V}}^2]^3}\bigg[\frac{\delta^{ij}}{3} + 2\,\frac{\hat{V}^i\,\hat{V}^j}{[1-\hat{\mathbf{V}}^2]}\bigg], \tag{35}$$

(36)

gives the power of radiation

$$P(t) = \int \mathrm{d}\Omega\,P(\mathbf{n},t) = \frac{2}{3\,\mathrm{c}}\,\frac{\mathrm{e}^2}{4\pi}\,\frac{1}{[1-\hat{\mathbf{V}}^2]^2}\bigg[\dot{\hat{\mathbf{V}}}^2 + \frac{(\hat{\mathbf{V}}.\dot{\hat{\mathbf{V}}})^2}{[1-\hat{\mathbf{V}}^2]}\bigg]. \tag{37}$$

The simplicity of the derivations given above should be noticed. The results derived in § 2 are quite general and are valid for arbitrary current distributions and are expected to be applicable in other problems as well. Some of such applications and further generalizations were mentioned at the end of § 1, and will be attempted in a future report.

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