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RESEARCH ARTICLE

Quantum viewpoint of quadratic $f(R)$ gravity, constraints, vacuum-to-vacuum transition amplitude and particle content

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Abstract We investigate, from a quantum viewpoint, the nature of the linearized quadratic $f(R) = (R + aR^2)$-gravity. To derive the explicit expression of the underlying propagator of the theory, the field is coupled to an external energy–momentum $T^{\mu\nu}$, and an auxiliary field is introduced to deal with gauge constraints as done in gauge theories. In particular for a conserved $T^{\mu\nu}$, we establish the gauge invariance of the vacuum-to-vacuum transition amplitude $\langle 0_+|0_-\rangle$, and prove the necessary positivity condition $|\langle 0_+|0_-\rangle|^2 < 1$, for $a > 0$, required by the quantum theory aspect of the treatment. An exact expression is then derived of the number of particles emitted, at a given energy, by a circularly oscillating Nambu string from which we may compare the relative number of the spin 0 massive particles emitted to the graviton number which turns up to give a clear cut departure from the general relativity prediction.

Keywords Modified theories of gravitation · Constrained dynamics · Quantum viewpoint · Propagator theory · Vacuum-to-vacuum transition amplitude · Particles emissions

1 Introduction

There has been much interest over the years in extended theories of gravitation, e.g. [1–30], as generalizations of general relativity, from both the classical and the quantum view points. With the hope of compatibility with experimental data, correct Newtonian and post-Newtonian limits, and other consistency requirements, classically (e.g., [1–13]), the interest has been, for example, in developing alternatives to dark energy

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models in explaining the cosmic acceleration [9]. Quantum mechanically, the non-renormalizable aspect of perturbative general relativity [14–18] has led, in turn, to introduce counterterms to the Einstein–Hilbert action involving higher-order derivatives, thus necessarily modifying Einstein's theory. Some of such modified theories involving, a priori, higher-order derivatives in the Lagrangian density turned out to be renormalizable [19–23] (see also [24]) with the main reason for this was that an underlying propagator, carrying a momentum k would vanish like k^{-4}, instead of k^{-2}, for $k^2 \to \infty$, making the degree of divergence of subgraphs not to increase with the number of loops, in contradistinction to the case in general relativity theory where a degree of divergence increases with the number of loops with no limit. Unfortunately, this was achieved by paying a high price for violating, in a perturbative setting, the underlying quantum theory positivity requirement. The modifications to the Einstein–Hilbert action, were a byproduct of the severe test of renormalizability. It is important to realize that a consistent quantum gravity is essential to cope with the big-bang theory and near black-hole singularities. In applying general relativity at high energies, or at small distances with scales comparable to the Planck one, one seems to be pushing it beyond its limit of validity.

The above considerations, lead us to infer that from both the classical and quantum mechanical points of view, extended theories of gravitation certainly deserve further analyses. The simplest modifications to the Einstein–Hilbert action is the replacement of the Ricci scalar R by a general function $f(R)$ of it. Such theories involve (massive) scalar fields and it is of no surprise that their connections to the classic Brans-Dicke theory have been made [9–11,25–28]. In the present paper, we are interested in the linearized theory, e.g., [28–30], with a quadratic function $f(R) = R + aR^2$ from the quantum viewpoint [31–33], where the quantum-particle aspect, associated with the gravitational field, emerges by considering the small fluctuation of the metric about the Minkowskian one, as the limit of the full metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the gravitational field becomes *weaker and the underlying particles become identified*. For the application at hand, we need to derive the explicit expression of the propagator of the underlying theory as well as obtain the expression of the vacuum-to-vacuum transition amplitude $\langle 0_+|0_-\rangle$. To do this, we couple the field to an external energy momentum tensor, and to deal with gauge constraints, we introduce an auxiliary field to fix the gauges as normally done in gauge theories [34–38]. In view of an application to the particle aspect of the theory, once the vacuum-to-vacuum transition amplitude is obtained, we impose a conservation law $\partial_\mu T^{\mu\nu} = 0$ on $T^{\mu\nu}$. This allows us to establish the gauge invariance of the theory, and most importantly, we prove the quantum *positivity* requirement of the vacuum-to-vacuum transition probability $|\langle 0_+|0_-\rangle|^2 < 1$ (§ 4) for $a > 0$. We apply the expression for $|\langle 0_+|0_-\rangle|^2$ to obtain the *exact* expression for the number of particles emitted from a circularly oscillating Nambu string [39–43] of a given total energy, in order to compare the relative number of massive spin 0 particles to the number of massless spin 2 (the graviton) particles emitted (Sect. 5), which turns out to give a clear cut departure [42] from that of general relativity. It is worth mentioning that cylindrically symmetric *metrics* have been considered, e.g., [44–47] from which the energy momentum-tensors are then extracted from Einstein's equation having relatively simple structures. The corresponding emerging string structures are not cylindrically symmetric, however, and in [47], the string is infinitely long and straight. Cylindrical symmetric solutions for Nambu strings have been also studied by other authors, e.g., [48–50], which deal, respectively, with gravitational radiation in flat spacetime, in investigating string instabilities, and finally in investigating the field produced by a collapsing cosmic string, with the latter two in linearized general relativity. None of these, however, are directly relevant to the present study in this extended gravitational theory. The non-trivial limit $a \to 0$, in a quantum mechanical setting, will be finally discussed. The Minkowski metric in this work is defined by diag$[-1, 1, 1, 1]$.

2 Gauge constraints and modified propagator

For the subsequent analysis, we first record the following expressions arising from a small deviation of the metric from the Minkowski one:

$$\sqrt{-g} = 1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \mathcal{O}(h^3), \tag{1}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}\, h_\lambda{}^\nu + \mathcal{O}(h^3), \tag{2}$$

$$\Gamma_{\mu\nu}{}^\rho = \frac{1}{2}(\partial_\mu h_\nu{}^\rho + \partial_\nu h_\mu{}^\rho - \partial^\rho h_{\mu\nu}) + \mathcal{O}(h^2), \tag{3}$$

$$R_{\mu\nu} = R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \mathcal{O}(h^3), \qquad \partial_\mu \Gamma_{\nu\rho}{}^\rho = \frac{1}{2}\partial_\mu\partial_\nu h + \mathrm{O}(h^2), \tag{4}$$

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\Big(\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h\Big), \; R^{(1)} = \big(-\partial^2 h + \partial_\alpha\partial_\beta h^{\alpha\beta}\big), \tag{5}$$

$$\eta^{\mu\nu} R^{(2)}_{\mu\nu} = \frac{1}{2}\partial_\lambda h\, \partial_\rho h^{\rho\lambda} - \frac{1}{4}\partial_\rho h \partial^\rho h - \frac{1}{2}\partial_\rho h_{\sigma\lambda}\partial^\sigma h^{\rho\lambda} + \frac{1}{4}\partial^\rho h_{\sigma\lambda}\partial_\rho h^{\sigma\lambda}, \tag{6}$$

up to total derivatives. Applying the above to $\sqrt{-g}\big(R + aR^2\big)$, one is led to consider the following Lagrangian density quadratic in $h^{\mu\nu}$

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\,\partial_\sigma h^{\mu\nu}\partial^\sigma h_{\mu\nu} + \partial_\sigma h^\sigma{}_\nu\, \partial_\mu h^{\mu\nu} - \partial_\nu h\, \partial_\mu h^{\mu\nu} + \frac{1}{2}\,\partial_\mu h\, \partial^\mu h \\ &-2\,a\,\Big(2\,\partial_\sigma\partial_\mu h\, \partial^\sigma\partial_\nu h^{\mu\nu} - \partial_\alpha\partial_\beta h^{\alpha\beta}\, \partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h\, \partial^2 h\Big).\end{aligned} \tag{7}$$

We consider the celebrated de Donder gauge

$$\partial_\mu\Big(h^{\mu\nu} - \frac{\eta^{\mu\nu}}{2}h - 2\,a\,\eta^{\mu\nu}R^{(1)}\Big) = 0, \tag{8}$$

where $R^{(1)}$ is given in (5). The gauge constraints may be directly derived by modifying the Lagrangian density as done in gauge theories. To this end, one may generalize the de Donder gauge further, by introducing an auxiliary vector field χ^ν, and choose

$$\partial_\mu\Big(h^{\mu\nu} - \frac{\eta^{\mu\nu}}{2}h - 2\,a\,\eta^{\mu\nu}R^{(1)}\Big) = \lambda\,\chi^\nu, \tag{9}$$

where λ is an arbitrary parameter. The constraints in (9) may be now derived from a corresponding Lagrangian density. A new Lagrangian density may be now defined as follows:

$$\begin{aligned}\tilde{\mathcal{L}} = &-\frac{1}{2}\,\partial_\sigma h^{\mu\nu}\,\partial^\sigma h_{\mu\nu} + \partial_\sigma h^\sigma{}_\nu\,\partial_\mu h^{\mu\nu} - \partial_\nu h\,\partial_\mu h^{\mu\nu}\\ &+\frac{1}{2}\partial_\mu h\,\partial^\mu h - 4\,a\,\partial_\sigma\partial_\mu h\,\partial^\sigma\partial_\nu h^{\mu\nu}\\ &+2\,a\;\partial_\alpha\partial_\beta h^{\alpha\beta}\,\partial_\mu\partial_\nu h^{\mu\nu} + 2\,a\,\partial^2 h\,\partial^2 h\\ &-2\chi_\nu\partial_\mu\Big(h^{\mu\nu} - \frac{\eta^{\mu\nu}}{2}h - 2\,a\,\eta^{\mu\nu}R^{(1)}\Big) + \lambda\,\chi^\mu\chi_\mu + T^{\mu\nu}h_{\mu\nu},\end{aligned} \tag{10}$$

where we have also introduced a coupling to an external energy–momentum tensor $T^{\mu\nu}$, a priori not conserved, to the field $h_{\mu\nu}$. We will introduce the Newton gravitational constant G, self consistently, later as done earlier by Schwinger [31].

By varying χ^ν in the Lagrangian density $\tilde{\mathcal{L}}$ in (10), the gauge constraints in (9), now as derived conditions, immediately follow. On the other the variation of the field components $h^{\mu\nu}$ lead to

$$\begin{aligned}&-\partial^2 h_{\mu\nu} + \partial_\mu\partial_\sigma h^\sigma{}_\nu + \partial_\nu\partial_\sigma h^\sigma{}_\mu - \partial_\mu\partial_\nu h - \eta_{\mu\nu}R^{(1)} + 4\,a\,(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)R^{(1)}\\ &\quad+\big(\eta_{\mu\nu}\partial_\sigma\chi^\sigma - \partial_\mu\chi_\nu - \partial_\nu\chi_\mu\big) - 4\,a\,\big(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big)\partial_\sigma\chi^\sigma = T_{\mu\nu},\end{aligned} \tag{11}$$

with $R^{(1)}$ given in (5). Equations (9) and (11) allow us to eliminate the auxiliary field and finally obtain the following key equation for the field $h_{\mu\nu}$

$$\begin{aligned}-\partial^2 h_{\mu\nu} = &\;T_{\mu\nu} - \frac{\eta_{\mu\nu}}{2}T - \eta_{\mu\nu}\frac{a\,\partial^2}{[1-6\,a\,\partial^2]}T + 4\,a\,\partial_\mu\partial_\nu\frac{\partial_\alpha\partial_\beta}{\partial^2}T_{\alpha\beta}\\ &+(\lambda-1)\frac{\partial_\mu\partial^\sigma}{\partial^2}T_{\sigma\nu} + (\lambda-1)\frac{\partial_\nu\partial^\sigma}{\partial^2}T_{\sigma\mu},\end{aligned} \tag{12}$$

with $T = T_\mu{}^\mu$.

Upon taking the vacuum matrix element $\langle 0_+|\,\cdot\,|0_-\rangle$ of this equation we obtain

$$\begin{aligned}&\langle\, 0_+|h^{\mu\nu}(x)|0_-\rangle = \int (\mathrm{d}x')\,\triangle_+^{\mu\nu;\alpha\beta}(x-x')\,T_{\alpha\beta}(x')\,\langle\, 0_+|0_-\rangle,\\ &(\mathrm{d}x) = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3,\end{aligned} \tag{13}$$

where the following explicit expression for the propagator $\triangle_+^{\mu\nu;\alpha\beta}(x-x')$ emerges

$$\triangle_+^{\mu\nu;\alpha\beta}(x-x') = \int \frac{(\mathrm{d}k)}{(2\pi)^4}\,\triangle_+^{\mu\nu;\alpha\beta}(k)\,\mathrm{e}^{\mathrm{i}k(x-x')}, \tag{14}$$

$$\Delta_+^{\mu\nu;\alpha\beta}(k) = \frac{\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)}{2\,[k^2]} + \frac{\eta^{\mu\nu}\eta^{\alpha\beta}}{6\,[k^2 + (1/6a)]} + (\lambda - 1)$$
$$\times \frac{\left[k^\mu k^\beta\,\eta^{\nu\alpha} + k^\mu k^\alpha\,\eta^{\nu\beta} + k^\nu k^\beta\,\eta^{\mu\alpha} + k^\nu k^\alpha\,\eta^{\mu\beta}\right]}{2\,[k^2]^2} - 4\,a\,\frac{k^\mu k^\nu k^\alpha k^\beta}{[k^2]^2}. \tag{15}$$

The formal replacement $k^2 \to k^2 - \mathrm{i}\,\epsilon$, $\epsilon \to +0$, in the denominators is understood. We learn, in particular, that no value of the gauge parameter λ may be taken to gauge away the massive spin 0 particle. That is the latter is an observable particle, provided for consistency and for the absence of a ghost, the parameter a must be non-negative. Also note that one cannot find values for the gauge parameter λ to make the $(k^2)^2$ singularities in the propagator, in Minkowski spacetime, disappear.

3 Vacuum-to-vacuum transition amplitude and positivity

Since, a priori, no constraint, as a conservation law, was imposed on $T^{\mu\nu}$, we may vary all of its components independently, and use the functional equation

$$(-\mathrm{i})\frac{\delta}{\delta T^{\mu\nu}(x)}\langle 0_+|0_-\rangle = \langle\, 0_+|h_{\mu\nu}(x)|0_-\rangle, \tag{16}$$

to functionally integrate (16) by using, in the process, (13) to obtain

$$\langle 0_+|0_-\rangle = \exp\Big[8\pi\mathrm{G}\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')T_{\mu\nu}(x)\Delta_+^{\mu\nu;\alpha\beta}(x-x')T_{\alpha\beta}(x')\Big]. \tag{17}$$

Here we note now that the expression for $\langle 0_+|0_-\rangle$ is derived, we may impose the conservation law $\partial_\mu T^{\mu\nu}(x) = 0$. Also for $a \to 0$, the contribution of the spin 0 particle disappears. Finally note that in order that the potential energy of two widely separated particles coincides with the Newtonian one, for $a \to 0$, as shown by Schwinger [31], a scaling $T_{\mu\nu} \to \sqrt{8\pi\mathrm{G}}\,T_{\mu\nu}$ is carried out.

Hence with the conservation law $\partial_\mu T^{\mu\nu}(x) = 0$ imposed, we may write

$$\langle 0_+|0_-\rangle = \exp\left[8\pi\mathrm{G}\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')T^{\mu\nu}(x)\left(\eta_{\mu\alpha}\eta_{\nu\beta} - \frac{1}{2}\,\eta_{\mu\nu}\eta_{\alpha\beta}\right)D_+(x-x')\,T^{\alpha\beta}(x')\right]$$
$$\times \exp\left[8\pi\mathrm{G}\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')\,\frac{T(x)}{\sqrt{6}}\,\Delta_+\Big(x-x',\frac{1}{6a}\Big)\,\frac{T(x')}{\sqrt{6}}\right], \tag{18}$$

where $T = T^\mu{}_\mu$,

$$D_+(x-x') = \int\frac{(\mathrm{d}k)}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2 - \mathrm{i}\epsilon}, \quad \epsilon \to +0, \tag{19}$$

$$\Delta_+\Big(x-x',\frac{1}{6a}\Big) = \int\frac{(\mathrm{d}k)}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{k^2 + (1/6a) - \mathrm{i}\epsilon}, \quad \epsilon \to +0. \tag{20}$$

No further functional differentiations of the expression in (18), with respect to $T^{\mu\nu}$, may be carried out. In order to be able to do that, on has to use the rather general expression for $\langle 0_+|0_-\rangle$ given in (17). The independence of $\langle 0_+|0_-\rangle$ in (18), with a conserved $T^{\mu\nu}$, of any gauge parameter is evident.

Using the identity

$$\frac{1}{k^2+m^2-\mathrm{i}\epsilon}-\frac{1}{k^2+m^2+\mathrm{i}\epsilon}=2\,\mathrm{i}\,\pi\,\delta(k^2+m^2), \tag{21}$$

for m^2 real and non-negative, we obtain

$$\begin{aligned}|\langle 0_+|0_-\rangle|^2 = \exp\Bigg[-8\pi\,\mathrm{G}\int\frac{(\mathrm{d}k)}{(2\pi)^3 2k^0}\,\delta(k^0-|\mathbf{k}|)\,T^{\mu\nu *}(k)\\ \left(\eta_{\mu\alpha}\eta_{\nu\beta}-\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\right)\,T^{\alpha\beta}(k)\Bigg],\\ \times\exp\left[-8\pi\,\mathrm{G}\int\frac{(\mathrm{d}k)}{(2\pi)^3 2k^0}\,\delta(k^0-\sqrt{|\mathbf{k}|^2+(1/6a)}\,)\,\frac{|T(k)|^2}{6}\right].\end{aligned} \tag{22}$$

Upon introducing two ortho-normalized real vector fields $\mathrm{e}_\lambda=(\mathrm{e}^\mu_\lambda)$, $\lambda=1,2$, such that

$$k_\mu \mathrm{e}^\mu_\lambda=0, \qquad \lambda=1,2, \tag{23}$$

and carrying out the completeness relation

$$\eta^{\mu\nu}=\frac{k^\mu\bar{k}^\nu+k^\nu\bar{k}^\mu}{k\bar{k}}+\sum_{\lambda=1,2}\mathrm{e}^\mu_\lambda\mathrm{e}^\nu_\lambda, \tag{24}$$

where $\bar{k}=(k^0=|\mathbf{k}|,-\mathbf{k})$, $k=(k^0=|\mathbf{k}|,\mathbf{k})$, we may, now due to the conservation law $k_\mu T^{\mu\nu}(k)=0$, effectively carry out the replacement

$$\left(\eta_{\mu\alpha}\eta_{\nu\beta}-\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\right)\to\sum_{\lambda,\lambda'=1,2}\mathrm{e}^{\mu\nu}_{\lambda\lambda'}\,\mathrm{e}^{\alpha\beta}_{\lambda\lambda'}, \tag{25}$$

in (22), where

$$\mathrm{e}^{\mu\nu}_{\lambda\lambda'}=\frac{1}{2}\left[\mathrm{e}^\mu_\lambda\,\mathrm{e}^\nu_{\lambda'}+\mathrm{e}^\mu_{\lambda'}\,\mathrm{e}^\nu_\lambda-\delta_{\lambda\lambda'}\sum_{\gamma=1,2}\mathrm{e}^\mu_\gamma\,\mathrm{e}^\nu_\gamma\right], \tag{26}$$

to obtain

$$\sum_{\lambda,\lambda'=1,2}\mathrm{e}^{\mu\nu}_{\lambda\lambda'}\mathrm{e}^{\alpha\beta}_{\lambda\lambda'}=\sum_{\xi=1,2}\varepsilon^{\mu\nu}_\xi\,\varepsilon^{\alpha\beta}_\xi, \tag{27}$$

with

$$\varepsilon^{\mu\nu}_1=-\sqrt{2}\,\mathrm{e}^{\mu\nu}_{22}, \qquad \varepsilon^{\mu\nu}_2=\sqrt{2}\,\mathrm{e}^{\mu\nu}_{12}, \tag{28}$$

leading to

$$|\langle 0_+|0_-\rangle|^2 = \exp\left[-8\pi\,\mathrm{G}\int \frac{(\mathrm{d}k)}{(2\pi)^3 2k^0}\,\delta(k^0-|\mathbf{k}|)\sum_{\xi=1,2}\left|\varepsilon^{\mu\nu}_{\xi}\,T_{\mu\nu}(k)\right|^2\right]$$
$$\times\exp\left[-8\pi\,\mathrm{G}\int \frac{(\mathrm{d}k)}{(2\pi)^3 2k^0}\,\delta(k^0-\sqrt{|\mathbf{k}|^2+(1/6a)}\,)\,\frac{\left|T(k)\right|^2}{6}\right] < 1, \tag{29}$$

which is the self-consistency positivity requirement.

4 Average number of particles of a given total energy

A very interesting property of particle emission by the external source $T^{\mu\nu}$ is that it is given via the Poisson distribution [51–54]. That is, in particular,

$$\text{Prob[no particles emitted]} = |\langle 0_+|0_-\rangle|^2 \equiv \exp[-<N>], \tag{30}$$

where $<N>$ denotes the average number of particles emitted. Hence from (29), the following expression for $<N>$ emerges:

$$<N> = 8\pi\,\mathrm{G}\int_0^\infty \mathrm{d}\omega\int\frac{(\mathrm{d}^3\mathbf{k})}{(2\pi)^3 2\omega}\Big[\,\delta(\omega-|\mathbf{k}|)T^{\mu\nu *}(\omega,\mathbf{k})\left(\eta_{\mu\alpha}\eta_{\nu\beta}-\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\right)$$
$$\times\, T^{\alpha\beta}(\omega,\mathbf{k})+\delta(\omega-\sqrt{|\mathbf{k}|^2+(1/6a)}\,)\,\frac{|T(\omega,\mathbf{k})|^2}{6}\Big], \tag{31}$$

from which the average number of particle with total energy ω is given by

$$<N(\omega)> = 8\pi\,\mathrm{G}\int\frac{(\mathrm{d}^3\mathbf{k})}{(2\pi)^3 2\omega}\Big[\,\delta(\omega-|\mathbf{k}|)T^{\mu\nu *}(\omega,\mathbf{k})\left(\eta_{\mu\alpha}\eta_{\nu\beta}-\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\right)$$
$$T^{\alpha\beta}(\omega,\mathbf{k})+\,\delta(\omega-\sqrt{|\mathbf{k}|^2+(1/6a)}\,)\,\frac{|T(\omega,\mathbf{k})|^2}{6}\Big]. \tag{32}$$

5 A Nambu string and relative number of particles emitted

The trajectory of a string is described by a vector field $\mathbf{R}(\sigma,t)$, where σ parametrizes the string. The equation of motion of the closed string is taken to be [39–43]

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial\sigma^2}\right)\mathbf{R}=0, \tag{33}$$

with constraints $\partial_t\mathbf{R}\cdot\partial_\sigma\mathbf{R}=0,\ \left(\partial_t\mathbf{R}\right)^2+\left(\partial_\sigma\mathbf{R}\right)^2=1,\ \mathbf{R}(\sigma+(2\pi/m),t)=\mathbf{R}(\sigma,t)$, where m, so far, is an arbitrary mass scale. The general solution to the above equation is

$$\mathbf{R}(\sigma, t) = \frac{1}{2}\big[\Phi(\sigma - t) + \Psi(\sigma + t)\big], \tag{34}$$

where $\Phi,\ \Psi$ satisfy, in particular, the normalization conditions $(\partial_\sigma \Phi)^2 = (\partial_\sigma \Psi)^2 = 1$. For the above system, we consider a solution of the form [42,43]

$$\mathbf{R}(\sigma, t) = \frac{1}{m}\big(\cos m\sigma, \sin m\sigma\big) \sin mt. \tag{35}$$

The general expression of the energy–momentum tensor of the string is given by [40,41,55]

$$T^{\mu\nu}(t, \mathbf{r}, z) = \frac{m^2}{2\pi}\int_0^{2\pi/m} \mathrm{d}\sigma \,\big(\partial_t R^\mu\, \partial_t R^\nu - \partial_\sigma R^\mu\, \partial_\sigma R^\nu\big)\, \delta^2\big(\mathbf{r} - \mathbf{R}(\sigma, t)\big)\delta(z), \tag{36}$$

where $R^0 = t,\ R^3 = 0$, and $\mathbf{r}$ lies in the plane of the string. The evaluation of $T^{\mu\nu}$ is quite tedious and is spelled out in [56]. In particular, with $\mathbf{r} = r(\cos\phi, \sin\phi)$,

$$T^{00} = \frac{m}{2\pi r}\,\delta\Big(r - \frac{|\sin mt|}{m}\Big)\,\delta(z), \tag{37}$$

$$(T^{01}, T^{02}) = \frac{m(\cos\phi, \sin\phi)}{r}\,\delta\Big(r - \frac{|\sin mt|}{m}\Big)\delta(z)\,\cos(mt)\,\mathrm{sgn}(\sin mt), \tag{38}$$

$$T^{11} = \frac{m}{2\pi r}\,\delta\Big(r - \frac{|\sin mt|}{m}\Big)\delta(z)\,[\cos^2 mt - \sin^2\phi], \tag{39}$$

$$T^{12} = \frac{m}{2\pi r}\,\delta\Big(r - \frac{|\sin mt|}{m}\Big)\,\delta(z)\,\frac{\sin 2\phi}{2}, \tag{40}$$

$$T^{22} = \frac{m}{2\pi r}\,\delta\Big(r - \frac{|\sin mt|}{m}\Big)\delta(z)\,[\cos^2 mt - \cos^2\phi], \tag{41}$$

$$T^{\mu 3} = 0. \tag{42}$$

$$\int \mathrm{d}^3\mathbf{x}\, T^{00} = m. \tag{43}$$

The Fourier transform of $T^{\mu\nu}$ at hand reads [56]

$$B^{\mu\nu}(\mathbf{p}, n) = \frac{m}{2\pi}\int_{-\pi/m}^{\pi/m} \mathrm{d}t\, \mathrm{e}^{2\mathrm{i}nmt}\int \mathrm{d}^2\mathbf{r}\int_{-\infty}^{\infty} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\mathbf{p}.\mathbf{r}}\, \mathrm{e}^{-\mathrm{i}qz}\, T^{\mu\nu}(t, \mathbf{r}, z), \tag{44}$$

with

$$B^{00} = \beta_n\, J_n^2(x), \quad B^{0a} = \beta_n \frac{p^0\, p^a}{\mathbf{p}^2}\, J_n^2(x), \quad a = 1, 2, \tag{45}$$

$$B^{ab} = \beta_n \Big[A_n\, \delta^{ab} + E_n\, \frac{p^a\, p^b}{\mathbf{p}^2}\Big], \quad a, b = 1, 2, \quad p^0 \equiv \omega = 2nm, \tag{46}$$

$$B^{\mu 3} = 0, \tag{47}$$

where

$$A_n = \frac{1}{4}\big[J_{n+1}(x) - J_{n-1}(x)\big]^2, \quad E_n = J_{n+1}(x) J_{n-1}(x), \tag{48}$$

$$x = |\mathbf{p}|/2m, \qquad \beta_n = m(-1)^n \cos(n\pi), \tag{49}$$

and J_n is a Bessel function of the first kind of integral order n.

With $p^0 = \omega$, we then have from (32) for the average number of particles with total energy in the interval $(\omega, \omega+\mathrm{d}\omega)$ in a period $2\pi/m$ of oscillation of a string, the explicit expressions

$$< N(\omega) > | =< N_T(\omega) > | + < N_S(\omega) > |, \quad \omega = 2nm, \tag{50}$$

$$< N_T(\omega) > | = \frac{4\mathrm{G}m^2\pi}{\hbar \mathrm{c}} U_n, \tag{51}$$

$$< N_S(\omega) > | = \frac{4\mathrm{G}m^2\pi}{\hbar \mathrm{c}} V_n, \tag{52}$$

where we have re-inserted the fundamental constants $\hbar$ and c, and where for the spin 2 (tensor) particles, and spin 0 (scalar) particles, we have, respectively, the exact expressions

$$U_n = n \int_0^\pi \sin\theta \mathrm{d}\theta \left[J_n^2(n\sin\theta) - J_{n+1}(n\sin\theta)\, J_{n-1}(n\sin\theta)\right]^2, \tag{53}$$

$$V_n = \frac{1}{12}\bar{n} \int_0^\pi \sin\theta \mathrm{d}\theta \left[J_{n+1}^2(\bar{n}\sin\theta) + J_{n-1}^2(\bar{n}\sin\theta) - 2\, J_n^2(\bar{n}\sin\theta)\right]^2, \tag{54}$$

$$\bar{n} = \sqrt{n^2 - \frac{1}{24m^2a}}. \tag{55}$$

Here we have used the fact that with $\mathbf{k} = (\mathbf{p}, q)$, $|\mathbf{p}| = |\mathbf{k}| \sin\theta$, in spherical coordinates. Also recall that the mass squared of the spin 0 particle is $= 1/(6a)$.

For numerical estimates, we may conveniently set $m^2 = 1/24a$, corresponding to a period of oscillation of a string equal to $\pi\sqrt{96a}$. Some numerical values are

$$\begin{aligned}
&U_1 = 1.25 \times 10^{-2},\ V_1 = 0, && U_2 = 4.52 \times 10^{-3},\ V_2 = 9.48 \times 10^{-3},\\
&U_3 = 2.31 \times 10^{-3},\ V_3 = 3.78 \times 10^{-3}, && U_4 = 1.41 \times 10^{-3},\ V_4 = 2.02 \times 10^{-3},\\
&U_5 = 9.41 \times 10^{-4},\ V_5 = 12.61 \times 10^{-4}, && U_6 = 6.81 \times 10^{-4},\ V_6 = 8.62 \times 10^{-4},\\
&U_7 = 5.13 \times 10^{-4},\ V_7 = 6.26 \times 10^{-4}, && U_8 = 4.01 \times 10^{-4},\ V_8 = 4.76 \times 10^{-4},\\
&U_9 = 3.22 \times 10^{-4},\ V_9 = 3.74 \times 10^{-4}, && U_{10} = 2.65 \times 10^{-4},\ V_{10} = 3.02 \times 10^{-4},\\
&U_{11} = 2.21 \times 10^{-4},\ V_{11} = 2.49 \times 10^{-4}, && U_{12} = 1.88 \times 10^{-4},\ V_{12} = 2.09 \times 10^{-4},\\
&U_{13} = 1.61 \times 10^{-4},\ V_{13} = 1.77 \times 10^{-4}, && U_{14} = 1.40 \times 10^{-4},\ V_{14} = 1.53 \times 10^{-4}.
\end{aligned}$$

Needless to say, for a large number of oscillating strings, the corresponding number of particles emitted may be significant. It is interesting to note that the number of spin

0 particles are, in general, greater than the number of spin 2, indicating a significant departure from the general relativity prediction [42].

6 Conclusions

In particular we have found that no gauge parameter λ may be chosen to gauge away the scalar field, and thus the latter is a real observable particle provided $a > 0$, to ensure that its mass is real and positive. For a conserved external energy–momentum tensor, we have proved the gauge invariance, and the fundamental positivity condition of the vacuum-to-vacuum transition probability: $|\langle 0_+|0_-\rangle|^2 < 1$, as an important consistency condition set up, particularly, by Schwinger [51] in his studies. The Nambu string considered above leads to an exact expression for the average number of spin 0 particles emitted relative to that of the gravitons at a given energy. The number of spin 0 emissions compares quite well with the spin 2 one, indicating a significant difference from that of general relativity which could formally provide a test on the validity of the modified theory. Although taking the limit $a \to 0$ in a classical context is uncomplicated, this becomes much more involved in a quantum setting. In the latter one has to invoke the decoupling theorem [57–59], to infer that for energies much less than $\sqrt{1/6a}$, the contribution of the massive scalar particle may be omitted altogether from the theory. This is reminiscent of omitting the contribution of all those quarks with masses much greater than the energy in question at hand in quantum chromodynamics.

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References

1. Buchdahl, H.A.: Mon. Not. R. Astron. Soc. **150**, 1 (1970)
2. Stelle, K.S.: Gen. Relativ. Gravit. **9**, 353 (1978)
3. Starobinsky, A.A.: Phys. Lett. B **91**, 99 (1980)
4. Zhang, Y.Z., Guan, K.Y.: Gen. Relativ. Gravit. **16**, 835 (1984)
5. Rippl, S., van Elst, H., Tavakol, R.: Gen. Relativ. Gravit. **28**, 193 (1996)
6. Kleinert, H., Schmidt, H.-J.: Gen. Relativ. Gravit. **34**, 1295 (2002)
7. Carroll, S.M., Duvvuri, V., Trodden, M., Turner, M.S.: Phys. Rev. D **70**, 043528 (2004)
8. Nojiri, S., Odinstov, S.D.: Int. J. Geom. Methods Phys. **4**, 115 (2007)
9. Faraoni, V.: Presented at the 18th Congress of General Relativity and Gravitation, pp. 22–25. Cosenza (2008)
10. Sotiriou, T.P.: Rev. Mod. Phys. **82**, 451 (2010)
11. Capozziello, S., De Laurentis, M.: Phys. Rep. **509**, 167 (2011)
12. Lambiase, G., Mohanty, S., Pizza, L.: Gen. Relativ. Gravit. **45**, 1771 (2013)
13. Tsokaros, A.: Class. Quantum Gravity **31**, 025021 (2014)
14. Utimaya, R., DeWitt, B.S.: J. Math. Phys. **3**, 608 (1962)
15. 't Hooft, G., Veltman, M.: Ann. Inst. Henri Poincaré **20**, 69–94 (1974)
16. Goroff, M.H., Sagnotti, A.: Nucl. Phys. B **266**, 709 (1986)
17. van de Ven, A.E.M.: Nucl. Phys. B **378**, 309 (1992)
18. Deser, S.: Ann. Phys. **9**, 299 (2000)
19. Stelle, K.S.: Phys. Rev. D **16**, 953 (1977)
20. Julve, J., Tonin, M.: Nuovo Cimento **B46**, 137 (1978)
21. Fradkin, E.S., Tseytlin, A.A.: Nucl. Phys. B **201**, 469 (1983)
22. Strominger, A.: Phys. Rev. D **30**, 2257 (1984)

23. Vilkovsky, G.: Class. Quantum Gravity **9**, 985 (1992)
24. Schmidt, H.-J.: Int. J. Geom. Methods Phys. **4**, 209 (2007)
25. Flanagan, E.E.: Class. Quantum Gravity **21**, 3817 (2004)
26. Olmo, G.J.: Phys. Rev. Lett. **95**, 261102 (2005)
27. Chiba, T.: Phys. Lett. B **575**, 1 (2005)
28. Barraco, D.E., Hamity, V.H., Vucetich, : Gen. Relativ. Gravit. **34**, 533 (2002)
29. Jaekel, M.-T., Reynaud, S.: Class. Quantum Gravity **22**, 2135 (2005)
30. Berry, C.P.L., Gair, J.R.: Phys. Rev. D **83**, 104022 (2011)
31. Schwinger, J.: Gen. Relativ. Gravit. **7**, 251 (1976)
32. Manoukian, E.B.: Gen. Relativ. Gravit. **22**, 501 (1989)
33. Schwinger, J., Tsai, W.-Y.: Ann. Phys. (N.Y) **96**, 303 (1975)
34. Manoukian, E.B.: Phys. Rev. D **34**, 3739 (1986)
35. Manoukian, E.B.: Int. J. Theor. Phys. **27**, 401 (1988)
36. DeWitt, B.S.: Phys. Rev. Lett. **12**, 742 (1964)
37. DeWitt, B.S.: Phys. Rev. **162**, 1195 (1967)
38. Faddeev, L.D., Popov, V.N.: Phys. Lett. B **25**, 29 (1967)
39. Goddard, P., Goldstone, J., Rebbi, C., Thorne, C.B.: Nucl. Phys. B **36**, 109 (1973)
40. Kibble, T.W.B., Turok, N.: Phys. Lett. B **116**, 141 (1982)
41. Albrecht, A., Turok, N.: Phys. Rev. D **40**, 973 (1989)
42. Manoukian, E.B.: Gen. Relativ. Gravit. **29**, 705 (1997)
43. Manoukian, E.B.: Nuovo Cimento **A104**, 1459 (1991)
44. Vilenkin, A.: Phys. Rev. D **23**, 852 (1981)
45. Hiscock, W.A.: Phys. Rev. D **31**, 3288 (1985)
46. Gott, J.: Astrophys. J. **288**, 422 (1985)
47. Shaver, E., Lake, K.: Phys. Rev. D **40**, 3287 (1989)
48. Allen, B., Casper, P.: Phys. Rev. D **50**, 2496 (1994)
49. Larsen, A.L.: Phys. Rev. D **50**, 2623 (1994)
50. Garfunkle, D., Duncan, G.C.: Phys. Rev. D **49**, 2752 (1994)
51. Schwinger, J.: Particles, Sources, and Fields. Addison-Wesley. Reading, Massahusetts, pp. 67–68 (1970)
52. Manoukian, E.B.: Radiat. Phys. Chem. **106**, 268 (2015)
53. Manoukian, E.B.: Radiat. Phys. Chem. **112**, 104 (2015)
54. Manoukian, E.B.: Modern Concepts and Theorems of Mathematical Statistics, Paperback edn. Springer, New York (2011)
55. Sakellariadou, M.: Phys. Rev. D **42**, 354 (1990)
56. Manoukian, E.B., Ungkitchanukit, A., Eab, C.H.: Hadron. J. **18**, 15 (1995)
57. Manoukian, E.B.: J. Math. Phys. **25**, 1519 (1984)
58. Manoukian, E.B.: J. Math. Phys. **27**, 1879 (1986)
59. Manoukian, E.B.: Renormalization. Academic Press, New York (1983)