MASSIVE GRAVITY IN HIGHER DIMENSIONS

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Ratchaphat Nakarachinda

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ABSTRACT

We are interested in explaining the dynamics of the universe using the modification of General Relativity. Many models of modified gravity theories have been constructed for many decades. In this work, we focus on one of them which is the massive gravity theory. The famous massive gravity model proposed by de Rham, Gabadadze and Tolley provided the self-accelerated expansion of the universe. However, the de Rham-Gabadadze-Tolley (dRGT) massive gravity theory encounters the problem about the lack of the number of propagating degrees of freedom. One of solutions is to introduce an additional field to the theory. In this work, we will add a field resulting from the dimensional reduction of the higher-dimensional massive gravity. We obtain an effective four-dimensional massive gravity theory with a scalar field. Moreover, the resulting theory corresponds to a combined description of two extensions of the dRGT theory namely the mass-varying and quasi-dilaton models. By using a dynamical system approach, we found regions of model parameters for which the late-time expansion of the universe is a stable fixed point.

CHAPTER I

INTRODUCTION

Nowadays, the universe is expanding with an acceleration. Many astronomical observations confirm this phenomena, for example, type Ia supernovae (SN-Ia) [1, 2], the cosmic microwave background radiation [3] and large scale structure [4, 5]. It has motivated physicists to construct the theoretical models which can explain this expansion of the universe. Since gravity is the major cause of the formulation of stars, galaxies and the other astronomical objects in the universe, it is possible to apply the theory of gravitation to explain the universe. That is why the study about the universe is based on General Relativity (GR). Unfortunately, GR with ordinary matter cannot be used to describe the accelerated expansion of the universe as we will see in Chapter II. We have to put unknown matter into GR in order to predict this phenomena. This matter is called dark energy (see [6, 7] for reviews). However, the theories of dark energy are still not good enough to explain the observations. On the other hand, the modification of GR called modified gravity theory is another possible way to solve the problem. This way is more elegant because it is not necessary to introduce any strange matter in order to predict the dynamics of the universe. There are many theories of modified gravity (see [8, 9, 10] for reviews) which are studied and developed.

The alternative theoretical models have been intensively constructed in order to describe the accelerated expansion of the universe. One possibility to construct this theoretical model is to modify GR at large scale, driving the expansion at cosmological scale while recovering GR at local gravity scale. One of the simple models of modified gravity is GR with a constant called cosmological constant. This constant is introduced to drive the accelerated expansion. Although the model can predict most of the phenomena satisfying the observations, the value of the cosmological constant needs to be fine-tuned. This does not provide a proper model in the theoretical point of view. Many models of modified gravity has been proposed for the last couple of decades. In this work, we focus on the other modification of GR by considering the spin-2 graviton with non-zero mass (while the graviton is massless in GR). The field theory for massive graviton was first proposed in 1939, but it encountered many problems. We will review this topic in Chapter III. In 2010, the consistent theory at a classical level for the theory of massive graviton was proposed by de Rham, Gabadadze and Tolley [11, 12] namely dRGT massive gravity theory. The dRGT theory is the modification of GR by adding suitable mass term into Einstein-Hilbert action. To explain the dynamics of the universe, it was found that the dRGT theory does not admit flat and closed Friedmann-Laîrmatre-Robertson-Walker (FLRW) solutions [13] when the fiducial metric is taken as the Minkowski one. Two of the possible ways to obtain all kinds of FLRW solutions suggested in [14] are applied in this work. The first way is taking other forms of fiducial metric e.g. FLRW or de-Sitter [15, 16, 17, 18]. The second one is adding more degrees of freedom into the original dRGT theory. Although the dRGT theory with FLRW fiducial metric is able to predict the accelerated expansion of the universe, it was found that the number of the propagating degrees of freedom is two which is not correct (it should be five) [15]. To add the external degrees of freedom may be a solution for solving the lack of the number of propagating degrees of freedom (e.g. adding a scalar field to the theory, it is possible to find that there are six propagating degrees of freedom). Since the additional degrees of freedom, in this work, is chosen to interpret as the effect of the extra dimensions, we devote Chapter IV to review the higher-dimensional gravity theories. Actually, we consider the dRGT theory in higher dimensions. By a mechanism called the Kaluza-Klein dimensional reduction, a scalar field is able to emerge in the four-dimensional effective theory from the higher-dimensional one as we will see in Chapter V. The cosmological solutions of this effective theory with the additional scalar field are analyzed in this chapter.

Since we usually consider the arbitrary dimensions of spacetime, we use three types of alphabets referring to each type of considered spacetime. The ordinary four-dimensional spacetime is described by the Greek indices ($\mu, \nu, \ldots$ run over $0, 1, 2, 3$). The extra dimensional spacetime is described by the small

Latin ones ($a, b, \ldots$ run over $5, 6, \ldots, n$ where n is the total number of dimensions of the spacetime). We skip the use of the number 4 in order to avoid some confusion. The higher dimensional spacetime is described by the capital Latin indices ($A, B, \ldots$ run over $0, \ldots, 3, 5, \ldots, n$). The variable with tilde such as $\tilde{v}$ refers to the n-dimensional quantities. We use the mostly plus Lorentz signature for the metric, $(-, +, +, \ldots)$. The symmetrized and anti-symmetrized tensor are respectively defined as $A_{(\mu_1 \ldots \mu_n)} = \frac{1}{n!} \left(A_{\mu_1 \ldots \mu_n} + \text{all permutation of } A_{\mu_1 \ldots \mu_n} \right)$ and $A_{[\mu_1 \ldots \mu_n]} = \frac{1}{n!} \left\{ A_{\mu_1 \ldots \mu_n} + \sum_i (-1)^i (\text{permutation}\, i \text{ times of } A_{\mu_1 \ldots \mu_n}) \right\}$. The natural units are used in the thesis such as the speed of light and the Planck constant setting to be the unity, $c = \hbar \equiv 1$. This means that the unit of length is the same as the unit of time, inverse of energy and inverse of mass, $[L] = [T] = 1/[E] = 1/[M]$.

CHAPTER II

GENERAL RELATIVITY AND COSMOLOGY

Gravitational force is one of the fundamental forces of nature. There were many attempts to explain it for a long time. Newton proposed the theory of gravitational force between two objects. His theory is very useful in predicting how the object moves under the gravitational force. However, the theory faces a problem such that the shift of the Mercury's orbit around the sun. Although including effect of the planets neighbouring our solar system, this phenomenon is perfectly unpredictable by the Newton's theory. Fortunately, this problem was solved by using the General Relativity. GR is used to describe the gravity in new aspect. Gravity is not a force as in the Newton's theory but a curvature of spacetime. In this chapter, we will review GR as well as its consequence in the context of cosmology.

2.1 General Relativity

The recent description for gravity was proposed by the famous physicist Einstein in 1915 [19, 20]. His theory has become the one of pillars in the modern physics. It is still the best theory for gravitation. Einstein's gravity theory is covered in many standard books e.g. [21, 22, 23]. The theory is the subject of the differential geometry. To understand the motion of the object in spacetime from GR point of view, we will start by discussing about the extremum interval between two events in spacetime which is called a geodesic. Mathematically, the solution $x^\mu(\lambda)$ is a geodesic if it satisfies the geodesic equation,

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\lambda^2} + \Gamma^\rho_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0, \tag{2.1}$$

where λ is an affine parameter which parametrizes the curve on spacetime and $\Gamma^\rho_{\mu\nu}$ is the Christoffel connection which describes how two events in spacetime are related together. It can be written in terms of the metric as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \tag{2.2}$$

This geodesic equation tells us how a particle moves in spacetime.

Another main point of GR is the beautiful relation between the curvature of spacetime and matter. The curvature implies the existence of the matter and vice versa. We can say that the matter dictates how spacetime curves and curved spacetime provides how the matter moves. This relation is described by the Einstein equation,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.3}$$

where $G_{\mu\nu}$ is the Einstein tensor which describes the curvature of spacetime, G is Newton's gravitational constant (the constant $8\pi G$ is just a constant in which GR can be reduced to Newtonian theory) and $T_{\mu\nu}$ is the energy-momentum tensor which describes the matter in spacetime. This matter also obeys the conservation law, $\nabla_\mu T^{\mu\nu} = 0$ where ∇_μ is the covariant derivative with respect to the coordinate x^μ defined by

$$\begin{aligned}\nabla_\rho A^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m} &= \partial_\rho A^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m} + \sum_{i=1}^{n} \left(\Gamma^{\mu_i}_{\rho\sigma} A^{\mu_1 \ldots \mu_{i-1}\sigma\mu_{i+1} \ldots \mu_n}{}_{\nu_1 \ldots \nu_m}\right) \\ &\quad - \sum_{i=1}^{m} \left(\Gamma^{\sigma}_{\rho\nu_i} A^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_{i-1}\sigma\nu_{i+1} \ldots \nu_m}\right),\end{aligned} \tag{2.4}$$

where $A^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m}$ is an arbitrary (n, m)-tensor. It is consistent with conservation of the Einstein tensor, $G_{\mu\nu}$ as $\nabla_\mu G^{\mu\nu} = 0$. Fortunately, the (curvature) rank-2 tensorial quantity which satisfies this condition was founded by Einstein, it is defined as $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, where R and $R_{\mu\nu}$ are the Ricci scalar and Ricci tensor respectively defined as

$$R_{\mu\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\mu \Gamma^{\rho}_{\rho\nu} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}, \tag{2.5}$$

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}. \tag{2.6}$$

Note that the field equations (2.3) is the nonlinear differential equations (second order of $g_{\mu\nu}$). For example, we have to put many symmetries in order to find a solution. If we have less symmetries, it is more difficult to solve the field equations.

In field theory, Einstein equation (2.3) can be derived from the variational principle of the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_{\mathrm{Pl}}^2}{2}R + \mathcal{L}_{\mathrm{m}}\right), \tag{2.7}$$

where M_{Pl} is the Planck mass ($M_{\mathrm{Pl}} \equiv 1/\sqrt{8\pi G}$). The action $S_{\mathrm{EH}} = \int \mathrm{d}^4x\sqrt{-g}R$ is called Einstein-Hilbert (EH) action yielding the left hand side of (2.3) by varying the action with respect to the metric tensor $g^{\mu\nu}$, $\mathcal{L}_{\mathrm{m}}$ is the Lagrangian density of matter which corresponds to the right hand side of (2.3).

Before we move on to discuss how GR explains the universe at large scale, it is worthwhile to note that there is an important symmetry in GR called general coordinate invariance. This means that the laws of physics are the same in any coordinates of consideration. It is freely to consider a system in many choices of coordinate. We will see how this symmetry is important to construct the nonlinear massive theory in the next chapter.

2.2 Cosmology

There are many experimental evidences confirming predictions from GR, for example, perihelion precession of the orbit of Mercury, light bending, gravitational redshift, the existence of gravitational waves, etc. These phenomena cannot be explained by Newton's gravity theory. This is why Einstein's theory has been the one of pillars in the modern physics. To study our universe using GR, we have to start with the cosmological principle which states that the universe are invariant under spatial translation (homogeneous) and spatial rotation (isotropic) in macroscopic scale. The metric which is consistent with these two symmetries and non-static is called Friedmann-Laîrmatre-Robertson-Walker (FLRW) metric,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right), \tag{2.8}$$

where $a(t)$ is a scale factor providing how the universe evolves and k is a constant which describes the geometry of the universe (flat, closed and open FLRW universes correspond to $k = 0, 1, -1$ respectively). Let's turn our attention to the matter sector, the simplest matter which is consistent with the above symmetries is called the perfect fluid, the fluid with no heat transfer and viscosity. The energy-momentum tensor of the perfect fluid can be written as

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}, \tag{2.9}$$

where ρ and p are the energy density and pressure of the fluid respectively and u^μ is a four velocity, $u^\mu = (-1, 0, 0, 0)$. We have both curvature and matter sector of (2.3) completely. Then plugging (2.8) and (2.9) into Einstein equation (2.3), we can derive the acceleration equation as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1 + 3w)\rho. \tag{2.10}$$

To obtain (2.10), we have to apply the relation between the energy density and pressure of matter which is called the equation of state, $p = w\rho$ where w is an equation of state parameter. Note that the fluid which corresponds to the accelerated expansion of the universe must satisfies the condition, $w < -1/3$. This implies the fluid has negative pressure, it is impossible to be a usual matter. Indeed, the equation of state parameter of the known matter are $w_{\mathrm{m}} = 0$ for non-relativistic matter or dust and $w_{\mathrm{r}} = 1/3$ for radiation. Moreover, the observations suggest that the effective equation of state parameter $w_{\mathrm{eff}} \sim -0.7$ for the accelerated expansions nowadays (the unknown matter called dark energy with $w_{\mathrm{DE}} \sim -1$ is required to exist in our universe [24]). It is found that we cannot use GR with the ordinary matter to explain the dynamics of the universe at late time. It makes a big challenge to seek for the theory which can explain the accelerated expansion. In this work, we choose to solve the problem using one of the modified gravity theories called massive gravity theory. We will discuss in detail of this gravity theory in the next chapter.

CHAPTER III

MASSIVE GRAVITY THEORY

Massive gravity theory is one of modified gravity theories. This theory corresponds to a non-zero mass of spin-2 graviton while GR is the theory corresponding a massless one. We devote this section to review the construction of this gravity theory. The first attempt was proposed as the linear mass theory (it obeys the linear gauge symmetry). There are problems arose when we take the massless limit of the massive theory. The linear massive theory in the massless limit is not only different with the massless one in the theoretical point of view, but also in the observational point of view. Thus the linear massive theory is ruled out by many observations at the solar system scale. The solution for the existence of such a theory is to add the nonlinear correction to the linear massive theory. After facing instability problems for almost forty years, the stable nonlinear massive theory is successfully constructed. We also discuss about some consequences of the nonlinear massive theory below.

3.1 Massless theory

We pay our attention to a symmetric spin-2 field, $h_{\mu\nu}$ which obeys the Lorentz symmetry. To construct a theory of this massless spin-2 field, the Lorentz invariant and local Lagrangian density has only a kinetic term. All of the possible contributions of kinetic terms in Lagrangian density for this spin-2 field (in which each contribution is not equivalent to others up to a boundary) can be written as

$$\mathcal{L}_{\text{kin}} = \partial^\rho h^{\mu\nu} \left[a_1 \partial_\rho h_{\mu\nu} + a_2 \partial_{(\mu} h_{\nu)\rho} + a_3 \eta_{\mu\nu} \partial_\rho h + a_4 \eta_{\rho(\mu} \partial_{\nu)} h \right], \tag{3.1}$$

where the indices are raised and lowered with respect to the Minkowski metric, $\eta_{\mu\nu}$ and h is the trace of $h_{\mu\nu}$, $h = h^\mu_\mu$. The coefficients, a_1, a_2, a_3 and a_4 are constants. These constants are not arbitrary. The constants are chosen later in order to avoid the existence of ghost term. To determine the values of these constants, we split the spin-2 field into a longitudinal part, $h^{(\mathrm{L})}_{\mu}$ and a transverse part, $h^{(\mathrm{T})}_{\mu\nu}$ as

$$h_{\mu\nu} = h^{(\mathrm{T})}_{\mu\nu} + 2\partial_{(\mu} h^{(\mathrm{L})}_{\nu)}, \tag{3.2}$$

where the transverse part satisfies $\partial^{\mu} h^{(\mathrm{T})}_{\mu\nu} = 0$. Substituting these decomposition to the kinetic term (3.1), it becomes

$$\begin{aligned}
\mathcal{L}_{\mathrm{kin}} &= \partial^{\rho} h^{(\mathrm{T})\,\mu\nu} \left[a_1 \partial_{\rho} h^{(\mathrm{T})}_{\mu\nu} + a_2 \partial_{(\mu} h^{(\mathrm{T})}_{\nu)\rho} + a_3 \eta_{\mu\nu} \partial_{\rho} h^{(\mathrm{T})} + a_4 \eta_{\rho(\mu} \partial_{\nu)} h^{(\mathrm{T})}\right] \\
&\quad + 2\partial^{\rho} \partial^{(\mu} h^{(\mathrm{L})\,\nu)} \left[\begin{array}{l} a_1 \left(2\partial_{\rho} \partial_{(\mu} h^{(\mathrm{L})}_{\nu)}\right) + \frac{a_2}{2} \left(2\partial_{\mu} \partial_{(\nu} h^{(\mathrm{L})}_{\rho)} + 2\partial_{\nu} \partial_{(\mu} h^{(\mathrm{L})}_{\rho)}\right) \\ + a_3 \left(2\eta_{\mu\nu} \partial_{\rho} \partial^{\sigma} h^{(\mathrm{L})}_{\sigma}\right) + \frac{a_4}{2} \left(2\eta_{\rho\mu} \partial_{\nu} \partial^{\sigma} h^{(\mathrm{L})}_{\sigma} + 2\eta_{\rho\nu} \partial_{\mu} \partial^{\sigma} h^{(\mathrm{L})}_{\sigma}\right) \end{array}\right] \\
&\quad + 2\partial^{\rho} \partial^{(\mu} h^{(\mathrm{L})\,\nu)} \left[a_1 \partial_{\rho} h^{(\mathrm{T})}_{\mu\nu} + a_2 \partial_{(\mu} h^{(\mathrm{T})}_{\nu)\rho} + a_3 \eta_{\mu\nu} \partial_{\rho} h^{(\mathrm{T})} + a_4 \eta_{\rho(\mu} \partial_{\nu)} h^{(\mathrm{T})}\right] \\
&\quad + \partial^{\rho} h^{(\mathrm{T})\,\mu\nu} \left[\begin{array}{l} a_1 \left(2\partial_{\rho} \partial_{(\mu} h^{(\mathrm{L})}_{\nu)}\right) + \frac{a_2}{2} \left(2\partial_{\mu} \partial_{(\nu} h^{(\mathrm{L})}_{\rho)} + 2\partial_{\nu} \partial_{(\mu} h^{(\mathrm{L})}_{\rho)}\right) \\ + a_3 \left(2\eta_{\mu\nu} \partial_{\rho} \partial^{\sigma} h^{(\mathrm{L})}_{\sigma}\right) + \frac{a_4}{2} \left(2\eta_{\rho\mu} \partial_{\nu} \partial^{\sigma} h^{(\mathrm{L})}_{\sigma} + 2\eta_{\rho\nu} \partial_{\mu} \partial^{\sigma} h^{(\mathrm{L})}_{\sigma}\right) \end{array}\right].
\end{aligned} \tag{3.3}$$

The first line is the contribution of the transverse part, the second line is the contribution of the longitudinal one and last two lines are the mixed contributions of both parts. To avoid the ghost instability, which emerges from the higher (than second order) derivative terms, we consider the last three lines of (3.3) (the first line contains only the second order derivative of $h^{(\mathrm{T})}_{\mu\nu}$ after integrating by parts) as follows

$$\begin{aligned}
\mathcal{L}_{\text{kin, 2nd line of (3.3)}} &= (2a_1 + a_2) h^{(\mathrm{L})\,\nu} \partial^2 \partial^2 h^{(\mathrm{L})}_{\nu} \\
&\quad + (2a_1 + 3a_2 + 4a_3 + 4a_4) h^{(\mathrm{L})\,\nu} \partial^2 \partial_{\nu} \partial^{\sigma} h^{(\mathrm{L})}_{\sigma},
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\mathcal{L}_{\text{kin, 3rd line of (3.3)}} &= -(2a_1 + a_2) h^{(\mathrm{T})}_{\mu\nu} \partial^2 \partial^{\mu} h^{(\mathrm{L})\,\nu} - a_2 h^{(\mathrm{T})}_{\nu\rho} \partial^{\nu} \partial^{\rho} \partial_{\mu} h^{(\mathrm{L})\,\mu} \\
&\quad - 2(a_3 + a_4) h^{(\mathrm{T})} \partial^2 \partial^{\nu} h^{(\mathrm{L})}_{\nu},
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\mathcal{L}_{\text{kin, 4th line of (3.3)}} &= -h^{(\mathrm{T})\,\mu\nu} \left[(2a_1 + a_2) \partial^2 \partial_{\mu} h^{(\mathrm{L})}_{\nu} + (a_2 + 2a_4) \partial_{\mu} \partial_{\nu} \partial^{\sigma} h^{(\mathrm{L})}_{\sigma}\right] \\
&\quad - 2a_3 h^{(\mathrm{T})} \partial^2 \partial^{\sigma} h^{(\mathrm{L})}_{\sigma},
\end{aligned} \tag{3.6}$$

where $\partial^2 = \eta^{\mu\nu} \partial_{\mu} \partial_{\nu}$. Thus the higher order derivative part in the kinetic term in (3.1) can be written as

$$\begin{aligned}
\mathcal{L}^{\text{higher der}}_{\text{kin}} &= (2a_1 + a_2) h^{(\mathrm{L})\,\nu} \partial^2 \partial^2 h^{(\mathrm{L})}_{\nu} + (2a_1 + 3a_2 + 4a_3 + 4a_4) h^{(\mathrm{L})\,\nu} \partial^2 \partial_{\nu} \partial^{\sigma} h^{(\mathrm{L})}_{\sigma} \\
&\quad - 2(2a_1 + a_2) h^{(\mathrm{T})}_{\mu\nu} \partial^2 \partial^{\mu} h^{(\mathrm{L})\,\nu} - 2(a_2 + a_4) h^{(\mathrm{T})}_{\mu\nu} \partial^{\mu} \partial^{\nu} \partial^{\rho} h^{(\mathrm{L})}_{\rho} \\
&\quad - 2(2a_3 + a_4) h^{(\mathrm{T})} \partial^2 \partial^{\rho} h^{(\mathrm{L})}_{\rho}.
\end{aligned} \tag{3.7}$$

In order to avoid ghost instability, all terms in the above equation must be eliminated. As a result, a_1, a_2, a_3 and a_4 must satisfy

$$2a_1 = -a_2 = -2a_3 = a_4. \tag{3.8}$$

The parameter a_1 will be set as $a_1 = -1/8$ for obtaining standard convention. Eventually, the healthy kinetic term for massless spin-2 field, $h_{\mu\nu}$ is

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}h^{(\text{T})\,\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h^{(\text{T})}_{\rho\sigma} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma}, \tag{3.9}$$

where $\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} = -\frac{1}{2}\left[\partial^2 h_{\mu\nu} - 2\partial^\rho\partial_{(\mu}h_{\nu)\rho} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}\left(\partial_\rho\partial_\sigma h^{\rho\sigma} - \partial^2 h\right)\right]$, and the operator $\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}$ is called Lichnerowicz operator. Not only the transverse mode $h^{(\text{T})}_{\mu\nu}$ but the whole field $h_{\mu\nu}$ also satisfy this fact. We also found that the kinetic term is invariant under the linear gauge transformation,

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}, \tag{3.10}$$

where ξ_μ is an arbitrary vector field. This massless theory propagates two tensor mode (or helicity-2) degrees of freedom for four-dimensional spacetime. In general, we can count the number of degrees of freedom for any field theories by using Hamiltonian formalism (see [25] for background knowledge about Hamiltonian formalism). For an arbitrary n-dimensional spacetime, the number of propagating degrees of freedom is $n(n-3)/2$ for $n > 2$.

Since the kinetic term in (3.9) is free from the ghost instability, it is convenient to use this term as the kinetic term. We also need to find the appropriated mass term, if we want to construct the massive theory as we will discuss in the next subsection. The massive theory thus consists of the kinetic term (3.9) and the additional mass term. However, the gauge symmetry in (3.10) no longer exists in the massive theory. Indeed, it is broken by the mass term which is constructed for the massive theory.

Moreover, this massless spin-2 theory is consistent with the linearization of GR. We consider the metric which can be perturbed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll |\eta_{\mu\nu}|, \tag{3.11}$$

where $h_{\mu\nu}$ is a symmetric tensor. The first order perturbation of $h_{\mu\nu}$ for the Christoffel symbol, Ricci tensor, Ricci scalar and Einstein tensor can be evaluated respectively as

$$\begin{aligned}
\Gamma^{(1)\,\rho}_{\ \ \mu\nu} &= \frac{1}{2}\left(\partial_\mu h^\rho_\nu + \partial_\nu h^\rho_\mu - \partial^\rho h_{\mu\nu}\right), \\
R^{(1)}_{\mu\nu} &= \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h\right), \\
R^{(1)} &= \partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h, \\
G^{(1)}_{\mu\nu} &= \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h\right) - \frac{1}{2}\eta_{\mu\nu}\left(\partial_\rho\partial_\sigma h^{\rho\sigma} - \partial^2 h\right).
\end{aligned} \tag{3.12}$$

Thus the field equations in vacuum for the linearized GR, $G^{(1)}_{\mu\nu} = 0$ reads

$$\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h + \eta_{\mu\nu}\left(\partial^2 h - \partial_\rho\partial_\sigma h^{\rho\sigma}\right) = 0. \tag{3.13}$$

We also notice that these field equations are obtained by varying the action (3.9) ($S = \int \mathrm{d}^4x\mathcal{L}_{\mathrm{kin}}$) with respect to $h^{\mu\nu}$. Thus, (linearized) GR is a theory of massless spin-2 graviton in the aspect of field theory.

In addition, the $h_{\mu\nu}$ field is able to be decomposed as a scalar mode h_{00} with one degree of freedom, vector mode h_{0i} with three degrees of freedom and tensor mode h_{ij} with six degrees of freedom. After substituting this decomposition to the field equations (3.13), it is found that both scalar and vector modes do not propagate (there are no time derivative acting in them). Only the tensor mode is the propagating degrees of freedom in the massless theory. From the linear gauge symmetry (3.10), we can fixed the gauge parameters. For example, in the transverse and traceless gauge, the tensor mode is able to be fixed as $h^i_i = 0$ and $\partial^i h_{ij} = 0$ in which eliminates one and three degrees of freedom respectively. We can conclude that there are two (transverse-traceless) tensor degrees of freedom which propagate in the massless theory as we have mentioned before. Next, we will move our consideration to a massive theory for spin-2 graviton which is our choice for modifying GR.

3.2 Linear massive theory

The kinetic term in (3.9) is still useful for both massless and massive theory. Since we try to construct the massive theory, an interaction term due to non-zero mass has to be introduced. So, the object of this section (and next section) is to construct the mass term which does not contain any problems.

3.2.1 Fierz-Pauli mass term

The simplest contributions for this mass term are constructed from quadratic contributions for the field, $h_{\mu\nu}$ as $h_{\mu\nu}h^{\mu\nu}$ and h^2. The general form of a quadratic mass term can be written as

$$\mathcal{L}_{\text{mass}} = m_g^2 \left(b_1 h_{\mu\nu}h^{\mu\nu} + b_2 h^2 \right), \tag{3.14}$$

where m_g is a constant interpreted as the graviton mass and b_1, b_2 are also constants. As the same strategy in the kinetic term, we apply the decomposition (3.2) to the above mass terms in order to find the unhealthy higher derivative terms. As a result, we obtain

$$\mathcal{L}_{\text{mass}} = m_g^2 \left[\begin{array}{l} b_1 \left(\begin{array}{l} h^{(\mathrm{T})}_{\mu\nu} h^{(\mathrm{T})\,\mu\nu} + 2h^{(\mathrm{T})}_{\mu\nu}\partial^\mu h^{(\mathrm{L})\,\nu} + 2h^{(\mathrm{T})}_{\mu\nu}\partial^\nu h^{(\mathrm{L})\,\mu} \\ +2\partial^\mu h^{(\mathrm{L})\,\nu}\partial_\mu h^{(\mathrm{L})}_\nu + 2\partial^\mu h^{(\mathrm{L})\,\nu}\partial_\nu h^{(\mathrm{L})}_\mu \end{array} \right) \\ +b_2 \left(h^{(\mathrm{T})\,2} + 4h^{(\mathrm{T})}\partial_\rho h^{(\mathrm{L})\,\rho} + 4\partial_\rho h^{(\mathrm{L})\,\rho}\partial_\sigma h^{(\mathrm{L})\,\sigma} \right) \end{array} \right]. \tag{3.15}$$

As we have seen, these mass terms do not contain any higher derivative terms for a tensor field $h^{(\mathrm{T})}_{\mu\nu}$ and a vector field $h^{(\mathrm{L})}_\mu$. However, there exists more degrees of freedom hiding in the vector field $h^{(\mathrm{L})}_\mu$. To see these modes, we choose to decompose it as

$$h^{(\mathrm{L})}_\mu = l^\perp_\mu + \partial_\mu l^\parallel, \tag{3.16}$$

where $l^\perp_\mu$ and $l^\parallel$ are a vector mode (or helicity-1) and a scalar mode (or helicity-0) of the vector field $h^{(\mathrm{L})}_\mu$ respectively. Then, applying this decomposition to the mass term (3.15), the unhealthy higher order derivative for the scalar mode $l^\parallel$ in this mass term can be shown as

$$\mathcal{L}^{\text{higher der}}_{\text{mass}} = m_g^2 4(b_1 + b_2) l^\parallel \partial^2\partial^2 l^\parallel. \tag{3.17}$$

Thus we obtain a condition which eliminates ghost instability from higher order derivative term as

$$b_1 = -b_2. \tag{3.18}$$

We also set $b_1 = -1/8$ for obtaining the standard convention. This healthy mass term can be written as

$$\mathcal{L}_{\mathrm{FP\,mass}} = -\frac{1}{8} m_g^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right). \tag{3.19}$$

This mass term was proposed by Fierz and Pauli in 1939 [26]. It is called Fierz-Pauli (FP) mass term. It is found that this mass term is not invariant under the linear gauge transformation (3.10). Therefore, the whole massive theory for spin-2 field has no gauge symmetry. However, there is a procedure to restore the gauge symmetry to the theory which is discussed in the next subsection.

Before moving on to the next subsection, we discuss the number of propagating degrees of freedom for the FP massive theory. The Lagrangian density can be written as

$$\mathcal{L}_{\mathrm{FP}} = -\frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu} h_{\rho\sigma} - \frac{1}{8} m_g^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right). \tag{3.20}$$

By varying this action with respect to $h^{\mu\nu}$, we obtain

$$\left[\begin{array}{l} \partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu - \partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu h \\ + \eta_{\mu\nu} \left(\partial^2 h - \partial_\rho \partial_\sigma h^{\rho\sigma} \right) + m_g^2 \left(h_{\mu\nu} - \eta_{\mu\nu} h \right) \end{array} \right] = 0. \tag{3.21}$$

Operating this fields equations with ∂^μ, we obtain constraints

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = 0, \tag{3.22}$$

for $m_g^2 \neq 0$. Then applying these constraints to the field equations (3.21), they become

$$\partial_\mu \partial_\nu h - \partial^2 h_{\mu\nu} + m_g^2 \left(h_{\mu\nu} - \eta_{\mu\nu} h \right) = 0. \tag{3.23}$$

Taking the trace of them, we obtain $h = 0$. From the constraints (3.22), we also obtain $\partial^\mu h_{\mu\nu} = 0$. Substituting constraints $h = 0$ and $\partial^\mu h_{\mu\nu} = 0$ into (3.23), the field equations become

$$\left(\partial^2 - m_g^2 \right) h_{\mu\nu} = 0. \tag{3.24}$$

Notice that these equations with constraints $\partial^\mu h_{\mu\nu} = 0$ and $h = 0$ are just other form of the field equations (3.21). They make it easy to count the number of degrees of freedom for the massive theory. As we have known that the symmetric tensor field, $h_{\mu\nu}$, in four dimensions contains ten independent components with five constraints: $\partial^\mu h_{\mu\nu} = 0$ and $h = 0$ (the equations (3.24) are just the consequences of applying these constraints to (3.21)). Finally, it gives us that the linear massive spin-2 theory propagates five degrees of freedom in four dimensions. In n-dimensional spacetime, there are $\frac{n(n-1)}{2}-1$ propagating degrees of freedom. This number is the same with the counting via Hamiltonian approach [27].

3.2.2 Stückelberg trick

The differences between massless and massive theories are the existence of the (linear) gauge symmetry and the number of propagating degrees of freedom. Is it possible to construct the massive theory which is invariant under the gauge transformation? In 1938, Stückelberg proposed a procedure which is used to restore a gauge symmetry to any field theories [28, 29, 30] (see [31] for review). This process is called Stückelberg trick. By taking the massless limit, $m_g \to 0$, the FP theory faces the many problems about the discontinuity. The Stückelberg trick is also used to see what the proper problem is. To see the power of this trick [27], it is worthy to consider the FP theory with matter or source. The matter term due to the coupling between the spin-2 field, $h_{\mu\nu}$ and the energy momentum of matter field, $T^{\mu\nu}$ can be written as

$$\mathcal{L}_{\text{matter}} = \frac{1}{2M_{\text{Pl}}} h_{\mu\nu} T^{\mu\nu}, \tag{3.25}$$

where a coefficient $1/2M_{\text{Pl}}$ is the constant corresponding to Newton's gravitational force. Note that the matter is, in general, not necessary conserved. Thus the full Lagrangian density for the FP theory with matter is

$$\mathcal{L} = -\frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu} h_{\rho\sigma} - \frac{1}{8} m_g^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) + \frac{1}{2M_{\text{Pl}}} h_{\mu\nu} T^{\mu\nu}. \tag{3.26}$$

To restore the linear gauge symmetry, we will introduce a new vector field called Stückelberg field, χ_μ via $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)}$. The above Lagrangian density becomes

$$\begin{aligned}
\mathcal{L} \;\; = \;\; & -\frac{1}{4} h^{\mu\nu} \hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu} h_{\rho\sigma} - \frac{1}{8} m_g^2 \left[\left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) + \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + 4 \left(h_{\mu\nu}\partial^\mu \chi^\nu - h \partial_\rho \chi^\rho \right) \right] \\
& + \frac{1}{2M_{\text{Pl}}} \left(h_{\mu\nu} T^{\mu\nu} - 2\chi_\mu \partial_\nu T^{\mu\nu} \right),
\end{aligned} \tag{3.27}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu \chi_\nu - \partial_\nu \chi_\mu$ is taken in the same form as Maxwell stress tensor. We can see that the term $m_g^2 \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ is taken in the form of the kinetic term for spin-1 field or Maxwell's kinetic term. In order to obtain a canonical form of Maxwell's kinetic term, the Stückelberg field will be rescaled as $\chi_\mu \to \chi_\mu / m_g$. Then, the Lagrangian density is expressed as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}m_g^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) - \frac{1}{8}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m_g\left(h_{\mu\nu}\partial^\mu\chi^\nu - h\partial_\rho\chi^\rho\right) \\ &\quad + \frac{1}{2M_{\mathrm{Pl}}}\left(h_{\mu\nu}T^{\mu\nu} - \frac{2}{m_g}\chi_\mu\partial_\nu T^{\mu\nu}\right). \end{aligned} \tag{3.28}$$

It is found that this Lagrangian density is invariant under the transformations:

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}, \qquad \chi_\mu \to \chi_\mu - m_g\xi_\mu. \tag{3.29}$$

Let's check whether this gauge invariant massive theory has any problems at massless limit or not. Taking $m_g \to 0$, the Stückelberg field will be strongly coupled to the divergence of matter. The problem at the massless limit arises if we consider the non-conserved matter. It is useful to move our attention to the conserved matter. Moreover, we see that the FP theory in the massless limit propagates only two tensor degrees of freedom (represented by $h_{\mu\nu}$) and two vector degrees of freedom (represented by χ_μ). The total number of the propagating degrees of freedom is four, there exist the unsmoothness of this number in the FP massive theory at the massless limit.

Going to another step by introducing a new Stückelberg scalar field π, we plug this field in via $\chi_\mu \to \chi_\mu + \partial_\mu\pi$. So, the Lagrangian density (3.27) becomes

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}m_g^2\left[\begin{array}{l}(h_{\mu\nu}h^{\mu\nu} - h^2) + \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \\ +4\left(h_{\mu\nu}\partial^\mu\chi^\nu + h_{\mu\nu}\partial^\mu\partial^\nu\pi - h\partial_\rho\chi^\rho - h\partial^2\pi\right)\end{array}\right] \\ &\quad + \frac{1}{2M_{\mathrm{Pl}}}\left(h_{\mu\nu}T^{\mu\nu} - 2\chi_\mu\partial_\nu T^{\mu\nu} + 2\pi\partial_\mu\partial_\nu T^{\mu\nu}\right), \end{aligned} \tag{3.30}$$

which is invariant under transformations:

$$\begin{aligned} h_{\mu\nu} &\to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}, & \chi_\mu &\to \chi_\mu - m_g\xi_\mu, \\ \chi_\mu &\to \chi_\mu + \partial_\mu\theta, & \pi &\to \pi - m_g\theta, \end{aligned} \tag{3.31}$$

where θ is an arbitrary scalar field. Similarly, we rescale both Stückelberg vector and scalar fields as $\chi_\mu \to \chi_\mu/m_g$ and $\pi \to \pi/m_g^2$ respectively. The Lagrangian density (3.30) reads

$$\mathcal{L} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}m_g^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) - \frac{1}{8}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}m_g\left(h_{\mu\nu}\partial^\mu\chi^\nu - h\partial_\rho\chi^\rho\right)$$
$$-\frac{1}{2}\left(h_{\mu\nu}\partial^\mu\partial^\nu\pi - h\partial^2\pi\right) + \frac{1}{2M_{\mathrm{Pl}}}h_{\mu\nu}T^{\mu\nu}. \quad (3.32)$$

We choose to consider the conserved; once again, the $\partial_\mu T^{\mu\nu}$ terms vanish. Moreover, the strong couplings for scalar and vector modes with matter at $m_g \to 0$ limit are avoided. To count the number of propagating degrees of freedom, we should simplify it into the suitable form. As we have seen in the massless limit, this form of Lagrangian density is expressed as mixing of the kinetic term of scalar mode and tensor mode. To see the unmix Stückelberg scalar field, π (with tensor field, $h_{\mu\nu}$), we will consider the Lagrangian density (3.30) in the other suitable frame,

$$h'_{\mu\nu} = h_{\mu\nu} - \pi\eta_{\mu\nu}, \qquad \chi'_\mu = \chi_\mu, \qquad \pi' = \pi. \quad (3.33)$$

We can obtain the gauge invariant massive theory in this new frame by transforming from the unprime field to prime one. Eventually, this theory at the massless limit can be written as

$$\mathcal{L} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \frac{1}{8}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{3}{4}\partial_\mu\pi\partial^\mu\pi + \frac{1}{2M_{\mathrm{Pl}}}h_{\mu\nu}T^{\mu\nu} + \frac{1}{2M_{\mathrm{Pl}}}\pi T. \quad (3.34)$$

We have removed the prime. This Lagrangian density explicitly propagates two tensor, two vector and one scalar degrees of freedom. We have already eliminated the unsmoothness of the number of propagating degrees of freedom at the massless limit and completely constructed the gauge invariant massive theory for spin-2 field as written in (3.30). Besides the gauge symmetry and the number of the propagating degrees of freedom, another important feature of the FP massive theory at the massless limit is the coupling between the scalar degree of freedom and matter in the last term of (3.34). The problem immediately emerges because this feature does not exist in massless theory as we will discuss in the next subsection.

3.2.3 van Dam-Veltman-Zakharov discontinuity

As we have realized that the FP massive spin-2 theory has many aspects about discontinuity at the massless limit such as the jump to the number of propagating degrees of freedom (before restore the gauge symmetries via Stückelberg trick) which does not make any sense why it is different from linearized GR at this limit. To show more clearly, if we are interested in the motion of a test particle such as a photon in the spin-2 field, we are able to calculate the trajectories from both massive (at the massless limit) and massless theories. It is found that we obtain the different light bending angles. The angle for massive theory at the massless limit is 3/4 times of the result in massless theory. In other words, the graviton in the massive theory taking $m_g \to 0$ gravitates weaker interaction than the massless one by the scaling 3/4 (see [27] for more detail of the calculation).

There is an important aspect about this discontinuity when we consider the matter at the massless limit. According to (3.34), we can see that there are not only the tensor modes which are coupled with matter, but the scalar mode is also coupled with matter (trace of matter field $T^{\mu\nu}$) as

$$\mathcal{L}^{m_g \to 0}_{\text{matter}} = \frac{1}{2M_{\text{Pl}}} h_{\mu\nu} T^{\mu\nu} + \frac{1}{2M_{\text{Pl}}} \pi T. \tag{3.35}$$

Unfortunately, a problem arises again since the effect of this coupling is different to that of linearized GR. For example, the exchange amplitudes between two conserved sources which are predicted from the massive at the massless limit and massless theories are not the same (see [32]). As we have known that the massless theory (linearized GR) is correct in the solar system scale, thus our massive theory (at the massless limit) could not contradict this fact. This failure of the FP linear massive theory at the massless limit was pointed out by van Dam, Veltman and Zakharov in 1970 [33, 34]. It is called van Dam-Veltman-Zakharov (vDVZ) discontinuity.

The way to solve the vDVZ discontinuity had beed proposed by Vainshtein in 1972 [35]. He proposed a mechanism which states that the linear massive theory (FP theory) is able to be used at the large enough distance away from the heavy source, the nonlinear effect is required if we want to consider the short distance behavior. In this mechanism, called Vainshtein mechanism, the nonlinearity is used to screen the effect of the scalar mode within the short distance consideration. The regimes for these two theories are separated by a radius from the source called Vainshtein radius. In the next section, we will discuss how the nonlinear theory is constructed.

3.3 Nonlinear massive theory

As we have mentioned in the previous section, the massive theory which explains within the small distance from source should be nonlinear. To construct this theory, we have to promote any structures in the linear theory to the nonlinear one.

3.3.1 From linear to nonlinear theory

Let's discuss the kinetic term in the massive theory first. We have known that the kinetic term of the linear massive theory is the linearization of GR. It is worthwhile to promote the kinetic term (3.9) to a nonlinear version as

$$\mathcal{L}_{\text{kin}}^{\text{nonlinear}} = \frac{M_{\text{Pl}}^2}{2} R[g], \tag{3.36}$$

which is invariant under the general coordinate transformations

$$g_{\mu\nu}(x) \to \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} g_{\rho\sigma}\left(y(x)\right). \tag{3.37}$$

For $y^\mu = x^\mu + \xi^\mu(x)$, these transformations can be written in the infinitesimal version [27] as

$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\sigma g_{\mu\sigma} + L_\xi g_{\mu\nu} \tag{3.38}$$

where ξ^μ is a gauge parameter and L_ξ is the Lie derivative with respect to ξ^μ. In order to obtain the theory for the spin-2 field, the metric tensor, $g_{\mu\nu}$ is expanded about the Minkowski metric as in (3.11). Then the above transformations read

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} + \ldots, \tag{3.39}$$

which are the nonlinear version of the gauge transformation for the spin-2 field. Next, we will construct the mass term for this nonlinear theory.

We have already known that the fundamentally dynamical field for this nonlinear theory is the metric. Unfortunately, any contributions which are constructed from $g_{\mu\nu}$ are just constants. Such contributions are not different to the cosmological constant. One of the solutions for constructing the mass term is to introduce a new field. In this consideration, we choose to introduce the new field as the non-dynamical reference metric, $f_{\mu\nu}$ (we also call the fiducial metric) in which the field $h_{\mu\nu}$ represents perturbation about it. The cost of introducing this reference metric is the loss of the interpretation of massive spin-2 field. Thus, the metric $g_{\mu\nu}$ called the physical metric can be expanded with the perturbed field $H_{\mu\nu}$ about the reference metric, $f_{\mu\nu}$ as

$$g_{\mu\nu} = f_{\mu\nu} + \frac{1}{M_{\rm Pl}} H_{\mu\nu}. \tag{3.40}$$

Then, the FP mass term (3.19) is promoted to be

$$\begin{aligned} \mathcal{L}^{\rm nonlinear}_{\rm FP\ mass} &= -\frac{1}{2} m_g^2 \left(H_{\mu\nu} H^{\mu\nu} - H^2 \right), \\ &= -\frac{1}{2} m_g^2 M_{\rm Pl}^2 \left([(\mathbb{I} - \mathbb{X})^2] - [\mathbb{I} - \mathbb{X}]^2 \right), \end{aligned} \tag{3.41}$$

where in the second line we write the equation in a matrix form where $\mathbb{I}$ is the identity matrix and $\mathbb{X}^{\mu}_{\ \nu}$ is a new matrix defined by $\mathbb{X}^{\mu}_{\ \nu} = g^{\mu\rho} f_{\mu\nu}$. $[\mathbb{A}]$ is the trace of the matrix $\mathbb{A}$. The graviton mass is also rescaled by $m_g \to 2m_g$.

The same Stückelberg trick is used again in order to restore the general coordinate invariance. We replace $f_{\mu\nu}$ by

$$f_{\mu\nu} \to \hat{f}_{\mu\nu} = \partial_\mu \psi^{\bar{\rho}} \partial_\nu \psi^{\bar{\sigma}} f_{\bar{\rho}\bar{\sigma}}, \tag{3.42}$$

where the bar index runs over four dimensional spacetime but does not depend on the unbar index. Note that the Stückelberg fields, $\psi^{\bar{\mu}}$ transform as the scalar field $\psi^{\bar{\mu}}(x) \to \psi^{\bar{\mu}}(y(x))$ in order to obtain the covariant mass term. The mass term (3.41) with the replacement (3.42) will be invariant under the general coordinate transformation. There is another way to restore the symmetry under general coordinate transformation by replacing in the physical metric [27]. By setting $\psi^{\bar{\mu}} = x^{\bar{\mu}}$, we will obtain $\hat{f}_{\mu\nu} = f_{\mu\nu}$. It is the unitary gauge in the nonlinear theory. We may be confused which the metric ($g_{\mu\nu}$ or $\hat{f}_{\mu\nu}$) is used for raising and lowering the indices of any tensorial quantities. The answer is that we always use the dynamical metric which is the physical metric $g_{\mu\nu}$ for moving the indices of any quantities except moving ones of another metric $\hat{f}_{\mu\nu}$. Thus we define $f^{\mu\nu}$ as the inverse of $\hat{f}_{\mu\nu}$ not $g^{\mu\rho} g^{\nu\sigma} \hat{f}_{\rho\sigma}$.

Finally, we have already constructed the nonlinear FP massive gravity. Note that this is not the nonlinear theory for a massive spin-2 field. This nonlinear gravity theory can be written in the form of action as

$$S_{\mathrm{FP}}^{\mathrm{nonlinear}} = \int \mathrm{d}^4x\sqrt{-g}\frac{M_{\mathrm{Pl}}^2}{2}\left[R + m_g^2\left([\mathbb{I}-\hat{\mathbb{X}}]^2 - [(\mathbb{I}-\hat{\mathbb{X}})^2]\right)\right], \tag{3.43}$$

where $\hat{\mathbb{X}}^{\mu}_{\ \nu} = g^{\mu\rho}\hat{f}_{\mu\nu}$. However, this nonlinear theory still has a problem due to the higher derivative terms which will be discussed later.

3.3.2 Vainshtein mechanism

Before discussing the problem of the nonlinear massive theory, it is worthwhile to see how the nonlinear theory is possible to recover GR at the short distance (with respect to the Vainshtein radius) following the Vainshtein mechanism. Let's start by considering any nonlinear massive theory (not only (3.43), but the theory with a more general mass term) in the frame that each mode is separated. In this frame, the kinetic terms are the same as in (3.34). However, there exists an additional interaction term which we will see below. Since the vector modes do not influence the vDVZ discontinuity (these modes do not couple to matter), we will ignore them for convenience. Consequently, the action can be written as

$$S = \int \mathrm{d}^4x \left[\begin{array}{l} -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} + \frac{3}{4}\pi\partial^2\pi + \frac{1}{2}M_{\mathrm{Pl}}^2 m_g^2\Phi(\pi,\partial\pi,...) \\ +\frac{1}{2M_{\mathrm{Pl}}}\left(T_{\mu\nu}h^{\mu\nu} - T\pi\right) \end{array}\right], \tag{3.44}$$

where $h_{\mu\nu}$ and π are the tensor and scalar modes respectively. The function Φ is an arbitrary function of the scalar mode and its derivatives. The field equations associated with the above action are separated into the equations for tensor and scalar modes respectively,

$$\hat{\mathcal{E}}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} = \frac{1}{M_{\mathrm{Pl}}}T_{\mu\nu}, \tag{3.45}$$

$$3\partial^2\pi + M_{\mathrm{Pl}}^2 m_g^2\frac{\delta\Phi}{\delta\pi} = \frac{1}{M_{\mathrm{Pl}}}T. \tag{3.46}$$

We notice that the left hand side of the equation for π contains both linear and nonlinear terms. If the linear term is dominant, the solution of π is in the order of h. This means that there exists the scalar mode. On the other hand, if the nonlinear term is dominant, the effect of π almost vanishes comparing to one of h. Therefore, the nonlinear theories are able to recover GR. Moreover, [36] shows that the spherically symmetric solution in unitary gauge at linear order exhibits the vDVZ discontinuity. If the nonlinear correction terms are added, the discontinuity will be eliminated. Unfortunately, the nonlinear theory contains a ghost instability as we will discuss in the next subsection.

3.3.3 Boulware-Deser ghost

To see the higher derivative terms explicitly, we will expand the spin-2 field, $\hat{H}_{\mu\nu}$ (the symmetry under general coordinate transformation are restored i.e. $\hat{H}_{\mu\nu} = M_{\rm Pl}(g_{\mu\nu} - \partial_\mu \psi^{\bar{\rho}} \partial_\nu \psi^{\bar{\sigma}} f_{\bar{\rho}\bar{\sigma}})$). The Stückelberg field can be expanded as

$$\psi^{\bar{\mu}} = x^{\bar{\mu}} - \frac{1}{M_{\rm Pl}} \varphi^{\bar{\mu}}, \tag{3.47}$$

where $x^{\bar{\mu}}$ and $\varphi^{\bar{\mu}}$ are the coordinates and infinitesimal scalar fields respectively. Using Taylor's expansion about $x^{\bar{\mu}}$, the fiducial metric becomes

$$f_{\bar{\rho}\bar{\sigma}}(\psi^{\bar{\mu}}) = f_{\bar{\rho}\bar{\sigma}}(x^{\bar{\mu}}) - \frac{1}{2! M_{\rm Pl}} \frac{\partial f_{\bar{\rho}\bar{\sigma}}}{\partial \psi^{\bar{\nu}}}(x^{\bar{\mu}}) \varphi^{\bar{\nu}} + \frac{1}{3! M_{\rm Pl}^2} \frac{\partial^2 f_{\bar{\rho}\bar{\sigma}}}{\partial \psi^{\bar{\nu}\, 2}}(x^{\bar{\mu}}) \varphi^{\bar{\nu}\, 2} + \ldots. \tag{3.48}$$

Thus the expansion of spin-2 field reads

$$\hat{H}_{\mu\nu} = H_{\mu\nu} + \partial_\mu \varphi^{\bar{\rho}} f_{\bar{\rho}\nu} + \partial_\nu \varphi^{\bar{\sigma}} f_{\mu\bar{\sigma}} - \frac{1}{M_{\rm Pl}} \partial_\mu \varphi^{\bar{\rho}} \partial_\nu \varphi^{\bar{\sigma}} f_{\bar{\rho}\bar{\sigma}} + \ldots, \tag{3.49}$$

where the derivative of $f_{\bar{\rho}\bar{\sigma}}$ terms are not written down explicitly (they vanish when we consider the flat fiducial metric, $f_{\bar{\rho}\bar{\sigma}} = \eta_{\bar{\rho}\bar{\sigma}}$). Under the infinitesimal general coordinate transformation, $x^\mu \to x^\mu - \xi^\mu / M_{\rm Pl}$, the fields $H_{\mu\nu}$ and $\varphi^{\bar{\mu}}$ must respectively be transformed as

$$\begin{aligned} H_{\mu\nu} &\to H_{\mu\nu} + \nabla^{(f)}_\mu \xi_\nu + \nabla^{(f)}_\nu \xi_\mu + L_\xi H_{\mu\nu}, && (3.50) \\ \varphi^{\bar{\mu}} &\to \varphi^{\bar{\mu}} - \xi^{\bar{\mu}} + \xi^{\bar{\nu}} \partial_{\bar{\nu}} \varphi^{\bar{\mu}}, && (3.51) \end{aligned}$$

where $\nabla^{(f)}_\mu$ denotes the covariant derivative with respect to the fiducial metric. ξ_μ is the gauge parameter.

In the case of $f_{\bar{\rho}\bar{\sigma}} = \eta_{\bar{\rho}\bar{\sigma}}$, the expansion (3.49) is reduced as

$$\hat{H}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \varphi_\nu + \partial_\nu \varphi_\mu - \frac{1}{M_{\rm Pl}} \partial_\mu \varphi^\rho \partial_\nu \varphi_\rho. \tag{3.52}$$

Then splitting the field φ_μ into the transverse mode, χ_μ and longitudinal mode, π as follows

$$\varphi_\mu = \frac{1}{m_g} \chi_\mu + \frac{1}{m_g^2} \partial_\mu \pi. \tag{3.53}$$

We have

$$\hat{H}_{\mu\nu} = h_{\mu\nu} + \frac{2}{m_g}\partial_{(\mu}\chi_{\nu)} + \frac{2}{m_g^2}\partial_\mu\partial_\nu\pi - \frac{1}{M_{\rm Pl}m_g^2}\partial_\mu\chi^\rho\partial_\nu\chi_\rho - \frac{1}{M_{\rm Pl}m_g^3}\partial_\mu\partial^\rho\pi\partial_\nu\chi_\rho$$
$$-\frac{1}{M_{\rm Pl}m_g^3}\partial_\mu\chi^\rho\partial_\nu\partial_\rho\pi - \frac{1}{M_{\rm Pl}m_g^4}\partial_\mu\partial^\rho\pi\partial_\nu\partial_\rho\pi. \quad (3.54)$$

In order to see the higher derivative terms, we will focus only on the scalar mode π (ignore $h_{\mu\nu}$ and χ_μ). The matrix $\hat{\mathbb{X}}$ can be expressed in terms of π as

$$\hat{\mathbb{X}}^\mu{}_\nu = \delta^\mu{}_\nu - \frac{2}{M_{\rm Pl}m_g^2}\partial^\mu\partial_\nu\pi + \frac{1}{M_{\rm Pl}^2 m_g^4}\partial^\mu\partial_\rho\pi\partial^\rho\partial_\nu\pi. \quad (3.55)$$

Substituting it back into the mass term (3.41) and keeping only the behavior of π, this reads

$$\mathcal{L}^{\rm nonlinear}_{\text{FP mass}, \pi} = -\frac{2}{m_g^2}\left([\Pi^2] - [\Pi]^2\right) + \frac{2}{M_{\rm Pl}m_g^4}\left([\Pi^3] - [\Pi][\Pi^2]\right)$$
$$+\frac{1}{2M_{\rm Pl}^2 m_g^6}\left([\Pi^4] - [\Pi^2]^2\right), \quad (3.56)$$

where $\Pi^\mu_\nu \equiv \partial^\mu\partial_\nu\pi$. It is found that the first term which contains the fourth order derivative of π is just a boundary term (it can be written in the form of the total derivative). However, the second and third terms do not vanish. Therefore, these terms are the real higher derivative terms of π in which the second and third terms contain the sixth and eighth order derivative terms respectively. They also lead to an appearance of the sixth scalar degree of freedom which is a ghost interpreted as the wrong sign kinetic energy. This ghost was found by Boulware and Deser in 1972 [37]. It is called Boulware-Deser (BD) ghost. Not only the massive theory with the mass term (3.41) contains the BD ghost, but the theory with various mass terms (e.g. a function of nonlinear FP mass term [37]) is also proven that there exists this ghost degree of freedom. This obstacle makes the nonlinear massive theory is unpopular to study. In 2010, de Rham, Gabadadze and Tolley succeed to construct the appropriate form of the nonlinear mass term which eliminates BD ghost [11].

3.3.4 dRGT massive gravity theory

The general form of the healthy nonlinear FP massive gravity theory is called the de Rham-Gabadadze-Tolley (dRGT) massive gravity theory [11, 12].

The action can be written as

$$S_{\rm dRGT} \;=\; \int {\rm d}^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}\left[R+m_g^2\,\mathcal{U}(g,\hat{f}\,)\right]. \tag{3.57}$$

The potential $\mathcal{U}$ is taken in a specific combination of functions of $g_{\mu\nu}$ and $\hat{f}_{\mu\nu}$ in order to eliminate the BD ghost. It reads

$$\begin{aligned}
\mathcal{U} &= \mathcal{U}_2+\alpha_3\,\mathcal{U}_3+\alpha_4\,\mathcal{U}_4,\\
\mathcal{U}_2 &\equiv [\mathcal{K}]^2-[\mathcal{K}^2],\\
\mathcal{U}_3 &\equiv [\mathcal{K}]^3-3[\mathcal{K}][\mathcal{K}^2]+2[\mathcal{K}^3],\\
\mathcal{U}_4 &\equiv [\mathcal{K}]^4-6[\mathcal{K}]^2[\mathcal{K}^2]+3[\mathcal{K}^2]^2+8[\mathcal{K}][\mathcal{K}^3]-6[\mathcal{K}^4],
\end{aligned} \tag{3.58}$$

with

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - (\sqrt{g^{-1}\hat{f}}\,)^\mu_\nu, \quad \hat{f}_{\mu\nu} = \partial_\mu\psi^{\bar{\rho}}\partial_\nu\psi^{\bar{\sigma}} f_{\bar{\rho}\bar{\sigma}}. \tag{3.59}$$

The square root matrix $\mathbb{M}^\mu{}_\nu = (\sqrt{g^{-1}\hat{f}}\,)^\mu{}_\nu$ is defined from $\mathbb{M}^\mu{}_\rho\mathbb{M}^\rho{}_\nu = g^{\mu\rho}\hat{f}_{\rho\nu}$. The coefficient parameters α_3 and α_4 are arbitrary constants. We also see that the nonlinear gauge symmetries are restored. The fiducial metric, $\hat{f}_{\mu\nu}$ will play an important role in the theory because the solution depends on the form of this metric as we will see later.

To see how the massive theory with the mass term in (3.58) is free from the BD ghost, we first consider the martix $\hat{\mathbb{X}}$ in terms of $\mathcal{K}^\mu{}_\nu$ as follows

$$\begin{aligned}
\hat{\mathbb{X}}^\mu{}_\nu &= g^{\mu\rho}\hat{f}_{\rho\nu} = (\sqrt{g^{-1}\hat{f}}\,)^\mu{}_\rho(\sqrt{g^{-1}\hat{f}}\,)^\rho{}_\nu,\\
&= \delta^\mu{}_\nu - 2\mathcal{K}^\mu{}_\nu + \mathcal{K}^\mu{}_\rho\mathcal{K}^\rho{}_\nu.
\end{aligned} \tag{3.60}$$

By comparing to (3.55), we can see that $\mathcal{K}^\mu{}_\nu$ is proportional to the scalar mode ($\mathcal{K}^\mu{}_\nu = \Pi^\mu{}_\nu/M_{\rm Pl}m_g^2$). Therefore, it is useful to construct the mass term in which the scalar mode being the total derivative e.g. the first term on the left hand side of (3.56). As in the Galileon theory [38], the ghost-free mass terms can be constructed in (3.58) for four dimensions.

One method to obtain the potential $\mathcal{U}$ is motivated from the fact that the number of the propagating degrees of freedom for five-dimensional GR is five. It is the same number with the massive gravity theory in four-dimensional spacetime. The process which we use in order to obtain the four-dimensional massive gravity from five-dimensional GR is called the deconstruction (see Appendix A).

3.3.5 Regimes of validity

Since we have constructed the mass term in the nonlinear massive theory by introducing the fiducial metric, this theory is just an effective field theory. Its validity is only at the classical regime. For the quantum regime, we need to use the other quantum theory. In other words, the screening mechanism (Vainshtein mechanism) is no longer valid at the very short length scale. The scalar degrees of freedom is strongly coupled to the other fields at scale called the strong coupling scale.

The massless theory (GR) also has this scale which is a very high energy scale called Planck scale ($\sim 10^{-35}$ m), but the strong coupling scale in the massive gravity is much lower than one in GR. To find the strong coupling scale, we will consider the quantum correction due to the interaction among the tensor mode $h_{\mu\nu}$, vector mode χ_μ and scalar mode π which is the higher derivative terms for these modes. These interaction terms provide us the limit of validity of the massive gravity theory. The general form of the interaction can be written as

$$\mathcal{L}_{\text{int}} = m_g^2 M_{\text{Pl}}^2 \left(h\right)^{n_h} \left(\partial\chi\right)^{n_\chi} \left(\partial^2\pi\right)^{n_\pi}, \tag{3.61}$$

where n_h, n_χ and n_π are the power of $h_{\mu\nu}, \chi_\mu$ and π respectively. The derivative in $\partial\chi$ and $\partial^2\pi$ exist because of the decomposition of $H_{\mu\nu}$ as we discussed before (it is symbolically different, but the same idea). Then we normalize these fields canonically as

$$h'_{\mu\nu} = M_{\text{Pl}} h_{\mu\nu}, \qquad \chi'_\mu = m_g M_{\text{Pl}} \chi_\mu, \qquad \pi' = m_g^2 M_{\text{Pl}} \pi. \tag{3.62}$$

The interaction becomes

$$\begin{aligned} \mathcal{L}_{\text{int}} &= m_g^2 M_{\text{Pl}}^2 \left(h\right)^{n_h} \left(\partial\chi\right)^{n_\chi} \left(\partial^2\pi\right)^{n_\pi}, \\ &= \Lambda_\lambda^{4-n_h-2n_\chi-3n_\pi} \left(h'\right)^{n_h} \left(\partial\chi'\right)^{n_\chi} \left(\partial^2\pi'\right)^{n_\pi}, \end{aligned} \tag{3.63}$$

where the scale is

$$\Lambda_\lambda = \left(M_{\text{Pl}} m_g^{\lambda-1}\right)^{1/\lambda}, \qquad \lambda = \frac{4 - n_h - 2n_\chi - 3n_\pi}{2 - n_h - n_\chi - n_\pi}. \tag{3.64}$$

Note that $n_h + n_\chi + n_\pi > 2$ because we consider only the interaction which is not the normal mass term (containing second order). The contribution from scalar degree of freedom is the lowest in energy scale and one of tensor is the highest.

For the arbitrary nonlinear massive gravity theories besides the dRGT theory, the interaction corresponding the lowest scale is the case of $n_h = n_\chi = 0$ and $n_\pi = 3$. Thus this lowest scale is $\Lambda_5 = (M_{\rm Pl} m_g^4)^{1/5}$. This is the cutoff for such nonlinear theories. The scalar degree of freedom is active at the scale Λ_5. We also called the Λ_5 theory as a quantum theory using to explain in the quantum regime for the (arbitrary) nonlinear massive gravity theory. [27] also shows that the nonlinearity is important (the linearity breaks down) at the Vainshtein radius $r_{\rm V} \sim \frac{1}{\Lambda_5} (M/M_{\rm Pl})^{1/5}$ from the heavy source with mass M. Then coming back to consider quantum corrections, it is found that these corrections become relevent at distance $r_{\rm Quant} \sim \frac{1}{\Lambda_5} (M/M_{\rm Pl})^{1/3}$. We can see that the distance $r_{\rm Quant}$ in which the classical theory cannot be trusted is larger than $r_{\rm V}$. These results are very strange ($r_{\rm V} \sim 10^{21}$ m and $r_{\rm Quant} \sim 10^{27}$ m by substituting $M \sim M_{\rm sun}$ and $m_g \sim H_0 \sim 10^{-33}$ eV), this is another problem of the arbitrary nonlinear theories before the construction of the dRGT theory.

In the dRGT theory, the interaction for the quantum theory is inactive at the scale Λ_5 [12]. We then move to consider the higher scale. It is found the next higher scale is the $\Lambda_4 = (M_{\rm Pl} m_g^3)^{1/4}$, but the specific form of mass term (3.58) eliminate interactions at this scale (they can be written in the form of the total derivative, so it vanishes up to boundary surface). Eventually, we obtain the strong coupling scale for the dRGT theory which is $\Lambda_3 = (M_{\rm Pl} m_g^2)^{1/3}$. With the similar analysis, the Vainshtein radius and the distance for quantum regime can be evaluated as $r_{\rm V} \sim \frac{1}{\Lambda_3} (M/M_{\rm Pl})^{1/3} \sim 10^{19}$ m and $r_{\rm Quant} \sim (\Lambda_{(3)})^{-1} \sim 10^6$ m respectively. It is reasonable to study such a nonlinear theory.

We have seen that the dRGT massive gravity theory gorgeously solve both problems about BD ghost and the regimes of validity. Although the linear theory (3.20) can be used at large scale ($r > r_{\rm V}$), we will analyze the nonlinear (dRGT) theory in the cosmological aspect.

3.4 Cosmological implication

Let's move our attention to analyze the consequence of this theory. The unitary gauge are chosen for our convenience. From the action (3.57), we can find the field equation by varying this action with respect to the physical metric, $g^{\mu\nu}$ which is

$$G_{\mu\nu} + m_g^2 X_{\mu\nu} = 0, \tag{3.65}$$

with

$$\begin{aligned} X_{\mu\nu} &= \mathcal{K}_{\mu\nu} - [\mathcal{K}]g_{\mu\nu} - (3\alpha_3 + 1)\left(\mathcal{K}^2_{\mu\nu} - [\mathcal{K}]\mathcal{K}_{\mu\nu} + \frac{1}{2}\mathcal{U}_2 g_{\mu\nu}\right) \\ &\quad + 3(\alpha_3 + 4\alpha_4)\left(\mathcal{K}^3_{\mu\nu} - [\mathcal{K}]\mathcal{K}^2_{\mu\nu} + \frac{1}{2}\mathcal{U}_2\mathcal{K}_{\mu\nu} - \frac{1}{6}\mathcal{U}_3 g_{\mu\nu}\right), \end{aligned} \tag{3.66}$$

where $\mathcal{K}^2_{\mu\nu} \equiv \mathcal{K}^\rho_\mu \mathcal{K}_{\rho\nu}$ and $\mathcal{K}^3_{\mu\nu} \equiv \mathcal{K}^\rho_\mu \mathcal{K}^\sigma_\rho \mathcal{K}_{\sigma\nu}$. Since we have the constraint $\nabla_\mu G^{\mu\nu} = 0$ for the Einstein tensor, $G_{\mu\nu}$, it implies that we also have another constraint for the tensor $X_{\mu\nu}$ which is $\nabla_\mu X^{\mu\nu} = 0$. Note that the constraint $\nabla_\mu X^{\mu\nu} = 0$ can be derived by varying the action (3.57) with respect to the fiducial metric, $f_{\mu\nu}$. In general, we work without fixing the gauge, we have the physical metric and the Stückelberg scalar fields as the dynamical fields. The constraints $\nabla_\mu X^{\mu\nu} = 0$ are also obtained by varying the action with respect to the Stückelberg fields (see [39] for the detail). When we compare (3.65) to (2.3), the tensor $X_{\mu\nu}$ looks like the energy-momentum tensor, $T_{\mu\nu}$. So, we call $X_{\mu\nu}$ the effective energy-momentum tensor.

Next, we try to explain the universe by using the dRGT theory. We have to find the solution of the field equation (3.65) which corresponds to the homogeneity and isotropy of the universe. Without a doubt, the physical metric will be considered the flat FLRW metric, $g_{\mu\nu} = \mathrm{diag}\left(-1, a^2(t), a^2(t), a^2(t)\right)$. Our work is choosing the form of the fiducial metric. For our convenience, the unitary gauge, $\psi^{\bar{\mu}} = x^{\bar{\mu}}$ is chosen. Firstly, we will discuss the simple metric which is the Minkowski metric, $f_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}\left(-1, 1, 1, 1\right)$. From these two metrics, we can compute the important quantities as

$$\mathbb{M}^{\mu}_{\nu} = \mathrm{diag}\left(-1, a^{-1}, a^{-1}, a^{-1}\right), \tag{3.67}$$

$$(\mathcal{K}^l)^{\mu}_{\nu} = \left(\frac{a-1}{a}\right)^l \mathrm{diag}\,(0,1,1,1), \tag{3.68}$$

for $l = 1, 2, \ldots$. Substituting the matrix $(\mathcal{K}^l)_{\mu\nu}$ and its trace into (3.66), the non-zero components of the effective energy-momentum tensor, X^{μ}_{ν} are

$$X^0_0 = -3\left(\frac{a-1}{a}\right) - 3\alpha\left(\frac{a-1}{a}\right)^2 - 3\beta\left(\frac{a-1}{a}\right)^3, \tag{3.69}$$

$$X^i_j = \left[-2\left(\frac{a-1}{a}\right) - \alpha\left(\frac{a-1}{a}\right)^2\right]\delta^i_j, \tag{3.70}$$

where $\alpha \equiv 3\alpha_3 + 1$ and $\beta \equiv \alpha_3 + 4\alpha_4$. From the constraint $\nabla_\mu X^{\mu\nu} = 0$, let's consider the non zero component which is $\nu = 0$. Finally, its solution are $H = 0$ or $1 + 2\alpha\left(\frac{a-1}{a}\right) + 3\beta\left(\frac{a-1}{a}\right)^2 = 0$. Both of them give us the scale factor, $a(t)$ is a constant. We can conclude that the case of Minkowski fiducial metric predict only the static universe. Note that this type of the fiducial metric can predict the accelerated expansion if the geometry of the universe is only open space [14].

Since we can choose the form of the fiducial metric arbitrarily, it is useful to consider the more interesting case. This is the case of the (flat) FLRW-like fiducial metric, $f_{\mu\nu} = \mathrm{diag}\,(-1, b^2(t), b^2(t), b^2(t))$, where $b(t)$ is the function of time. This function plays the role the scale factor for this fiducial metric. Then we can repeat the process to find the cosmological solution. Firstly, we can find the tensors $\mathbb{M}^{\mu}_{\nu}$ and $(\mathcal{K}^l)^{\mu}_{\nu}$ as

$$\mathbb{M}^{\mu}_{\nu} = \mathrm{diag}\left(1, \frac{b}{a}, \frac{b}{a}, \frac{b}{a}\right) \tag{3.71}$$

$$(\mathcal{K}^l)^{\mu}_{\nu} = \left(\frac{a-b}{a}\right)^l \mathrm{diag}\,(0,1,1,1). \tag{3.72}$$

Then, the two useful forms of the effective energy-momentum tensor, X^{μ}_{ν} are

$$\begin{aligned} X^{\mu}_{\nu} &= -3\left(A + \alpha A^2 + \beta A^3\right)\mathrm{diag}(1,0,0,0) - \left(2A + \alpha A^2\right)\mathrm{diag}(0,1,1,1), \\ &= -\left(A + 2\alpha A^2 + 3\beta A^3\right)\mathrm{diag}(1,0,0,0) - \left(2A + \alpha A^2\right)\mathrm{diag}(1,1,1,1), \end{aligned} \tag{3.73}$$

where $A \equiv (a-b)/a$. From the constraint $\nabla_\mu X^{\mu}_{\nu} = 0$, we obtain the condition

$$0 = 3\left(\dot{A} + AH\right)\left(1 + 2\alpha A + 3\beta A^2\right). \tag{3.74}$$

It gives us the two possible solutions. The first one is $1 + 2\alpha A + 3\beta A^2 = 0$. From (3.73), the effective energy density and pressure can be defined as $\rho_X = 2A + \alpha A^2 = -P_X$. It corresponds to $w_X = -1$. This result implies that the universe is expanding with acceleration with the rate which corresponds to the observation (In the observation, the rate of the expansion the universe due to dark energy provides us through the parameter, $w_{\mathrm{DE}} \approx -1$ [21, 22]). This case is called the self-accelerating branch. The second solution is $\dot{A} + AH = 0$. We obtain $a(t) - b(t) = C$, where C is a integration constant. We also define the effective energy density and effective pressure as $\rho_X = -3\left[\left(\frac{C}{a}\right) + \alpha\left(\frac{C}{a}\right)^2 + \beta\left(\frac{C}{a}\right)^3\right]$ and $P_X = -\left[2\left(\frac{C}{a}\right) + \alpha\left(\frac{C}{a}\right)^2\right]$ respectively. In some condition, it gives us the solution that corresponds to the accelerated expansion of the universe. This case is called the normal branch.

According to the above analysis, the late time universe is predictable from this theory. However, in cosmological perturbations, this model gives us the wrong number of propagating degrees of freedom [15]. It predicts only two propagating degrees of freedom while it should be five for the four-dimensional massive gravity. As we have seen in the above calculation, the solution depends on the fiducial metric form. Therefore, one way to find the better model is choosing the proper fiducial metric. Moreover, we have had the problem about the lack of the number of degrees of freedom, one of the possible ways to extend the dRGT theory is adding the external field, e.g. mass-varying model and quasi-dilaton model. For the mass-varying model, the graviton mass which is a constant in the dRGT theory will be promoted to be a function of the external scalar field ϕ, $m_g \to m_g(\phi)$, which has its own dynamics [40, 41, 42, 43, 44, 45]. It is possible to find the range of parameters such that the model is stable. However, the graviton mass will decays and then vanishes at the accelerating phase of the universe [46]. The quasi-dilaton model is the model which extends the dRGT theory by adding the scaling symmetry, the fiducial metric which is invariant under the transformation $f_{\mu\nu} \to \partial_\mu \psi^{\hat{\rho}} \partial_\nu \psi^{\hat{\sigma}} f_{\hat{\rho}\hat{\sigma}}$ in the dRGT theory will be promoted to be a function of the quasi-dilaton scalar field ψ, $f_{\mu\nu} \to e^{2\phi/M_{\mathrm{Pl}}} \partial_\mu \psi^{\hat{\rho}} \partial_\nu \psi^{\hat{\sigma}} f_{\hat{\rho}\hat{\sigma}}$ [47, 48]. Unfortunately, we found that the quasi-dilaton model is unstable. The more general model which is stable was successfully constructed in [49, 50]. It is called the generalized quasi-dilaton theory. In addition, decreasing some symmetry of the physical metric is the possible way to solve this problem, the cosmological solution which is homogeneous but anisotropic was found in [51, 52, 53]. The other possible way is promoting the fiducial metric as the dynamical field, called the massive bi-gravity [54]. These are the topics in the active area of massive gravity.

Before finishing this chapter, we comment on the matter coupling in the dRGT theory. The simple contribution is constructed by coupling the physical metric covariantly with the matter field. [54] proved that this simple coupling is free from BD ghost. Since there are two metrics in the dRGT theory, the question is how to construct the matter term with both metrics. One of the attempts to construct the matter term is assumed that we have two independent matter fields, we are able to construct the healthy couplings term in which each matter couples to only one metric. The problems arise when we choose these two matter fields to be the same field. There exists the ghosts in both classical and quantum levels of consideration [55]. Fortunately, the theory with particular form of the coupling terms, which is free from ghosts, is proposed in [55]. The matter field is not directly coupled to both physical and fiducial metrics, but the effective metric which is

$$g_{\text{eff}\,\mu\nu} = \gamma_1^2 g_{\mu\nu} + \gamma_2^2 \hat{f}_{\mu\nu} + 2\gamma_1\gamma_2 g_{\mu\rho}(\sqrt{g^{-1}\hat{f}}\,)^\rho_\nu, \tag{3.75}$$

where γ_1 and γ_2 are arbitrary parameters. The coupling between the massive matter field ψ with the mass m_ψ and the effective metric can be written as

$$\mathcal{L}_{\text{matter}} = -\frac{1}{2}\sqrt{-g_{\text{eff}}}\left(g^{\mu\nu}_{\text{eff}}\partial_\mu\psi\partial_\nu\psi + m_\psi^2\psi^2\right). \tag{3.76}$$

The square root of the determinant of the effective metric can be written in the form of one of the physical metric as $\sqrt{-g_{\text{eff}}} = \sqrt{-g}\,\det\left(\gamma_1 + \gamma_2(\sqrt{g^{-1}\hat{f}}\,)^\mu_\nu\right)$. It is seen that this is the covariant coupling. We will finish our discussion about the massive gravity here, and move our attention to the other modified theory which is consideration of gravity theories in higher dimensions.

CHAPTER IV

GRAVITY THEORIES WITH EXTRA DIMENSION(S)

One of the wonderful and surprising ideas in General Relativity (GR) is that the gravity cannot be described by the theory in three-dimensional spatial coordinates but in four-dimensional spacetime coordinates including time. The additional time plays the role of the coordinate similar to other spatial coordinates (there is no absolute time but we have only the relative one) in the theory of relativity. It is not only the theoretical imagination but also confirmed by many experiments such as the time dilation and the length contraction [56, 57].

A possible way to solve some theoretical problems is to extend a four dimensional spacetime to one in higher dimensions. For simplicity, we consider only the case of the extra spatial dimensions because the extra temporal dimensions may lead us to the problem of causality. We do not want to deal with the complicated case right now. The spacetime with an extra temporal dimension is an active area of study (see e.g., [58]). The evidence of existence of the extra dimensions is tightly constrained by the experiments. For example, in the stability of the moon's orbit around the earth, the Newton potential behaves like $1/r$ not $1/r^l$ (for $l \neq 1$). In general, this potential is proportional to $1/r^{n-3}$ in n-dimensional spacetime. If the spacetime had five dimensions or more, it should not appear in the solar system (must be hidden from the experiments). In order to distinguish between GR in four dimensions and one in higher dimensions, the effect of the higher dimensions should be taken as a correction to GR at very small and very large scale. One of the possible ways to explain the dynamics of the universe is using the higher-dimensional gravity theory. In this chapter, we review the higher-dimensional gravity theories which are constructed for various objectives.

4.1 Einstein-Gauss-Bonnet gravity theory

In this section, we pay our attention to the theory of gravitation which is constructed from the metric tensor as one fundamental field. As we have known, in Lagrangian formalism, the Einstein field equations can be derived from the EH action. It will be shown that this action is not only one choice constructed from the metric tensor that gives the Einstein field equations. The difference between the field equations obtained from EH action and ones from other action cannot be seen in four dimensions, but it will be arisen when we consider in higher-dimensional field equations as we will see below.

4.1.1 Lovelock's theorem

The most general Lagrangian density constructed from the metric $g_{\mu\nu}$ contains $g_{\mu\nu}$ and its infinite order derivatives. We start by considering the four-dimensional action,

$$S = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\rho\partial_\sigma g_{\mu\nu}, \ldots). \tag{4.1}$$

The Euler-Lagrange field equations obtained by extremizing this action with respect to the metric tensor can be expressed as

$$E^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} - \partial_\rho\frac{\partial\mathcal{L}}{\partial(\partial_\rho g_{\mu\nu})} + \partial_\sigma\partial_\rho\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\partial_\rho g_{\mu\nu})} + \ldots = 0. \tag{4.2}$$

The theorem called Lovelock's theorem [59] states that the second order derivatives Euler-Lagrange field equations obtainable from the action (4.1) are

$$E^{\mu\nu} = a_1\sqrt{-g}G^{\mu\nu} + a_2\sqrt{-g}g^{\mu\nu} = 0. \tag{4.3}$$

These second order field equations correspond to the linear combination between the Einstein field equations and cosmological constant where a_1 and a_2 are constants. However, the EH action is not the only action that yields the Einstein equations (with the cosmological constant). The general Lagrangian density, in fact, should be

$$\begin{aligned}\mathcal{L} &= a_1\sqrt{-g}R + 2a_2\sqrt{-g} + a_3\epsilon^{\mu\nu\rho\sigma}R^{\alpha\beta}{}_{\mu\nu}R_{\alpha\beta\rho\sigma} \\ &\quad + a_4\sqrt{-g}\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\right),\end{aligned} \tag{4.4}$$

where a_3 and a_4 are also constants. These last two terms of this Lagrangian density correspond to field equations as follows

$$E^{\mu\nu}\epsilon^{\gamma\lambda\rho\sigma}R^{\alpha\beta}{}_{\gamma\lambda}R_{\alpha\beta\rho\sigma} = 0, \tag{4.5}$$

$$E^{\mu\nu}\sqrt{-g}\left(R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma}\right) = 0. \tag{4.6}$$

These field equations (4.5) and (4.6) are trivial up to the field equations (4.3), they can be thought of as the boundary surface terms. Moreover, the field equation (4.5) is valid in any number of dimensions but the field equations (4.6) is valid in four or less dimensions.

As we have seen, the above consideration satisfies only in four-dimensional spacetime. Now, we extend our attention to the theory in n-dimensional spacetime without considering the third term in Lagrangian density (4.4). The general form of the Lagrangian density is

$$\mathcal{L} = \sqrt{-\tilde{g}}\sum_{i=0}^{m} b_i\mathcal{R}^i, \quad \mathcal{R}^i = \frac{i!}{2^i}\delta^{\mu_1}_{[\rho_1}...\delta^{\mu_i}_{\rho_i}\delta^{\nu_1}_{\sigma_1}...\delta^{\nu_i}_{\sigma_i]}\prod_{j=1}^{i}\tilde{R}^{\rho_j\sigma_j}{}_{\mu_j\nu_j}, \tag{4.7}$$

or it can be explicitly expanded as

$$\mathcal{L} = \sqrt{-\tilde{g}}\left[\begin{array}{l} b_0 + b_1\tilde{R} + b_2\left(\tilde{R}^2 - 4\tilde{R}^{AB}\tilde{R}_{AB} + \tilde{R}^{ABCD}\tilde{R}_{ABCD}\right) \\ +b_3\mathcal{O}(\tilde{R}^3) + \ldots \end{array}\right], \tag{4.8}$$

where quantities with tilde refer to quantities in n dimensions and each of b_i is a constant. If we compare the above Lagrangian density to the one in (4.4), we get $b_0 = 2a_2$, $b_1 = a_1$ and $b_2 = a_4$. $\mathcal{R}^i$ contributes to the field equations only in $n > 2i$ dimensions. Thus, m is taken to be $(n-1)/2$ for odd dimensions and to be $(n-2)/2$ for even dimensions. Note that the field equations obtained from the above generalized Lagrangian density contain only the second order derivatives. In the next subsection, we will see the application of this theorem in the construction of higher-dimensional gravity theories which is distinguishable from GR in higher dimensions.

4.1.2 Einstein-Gauss-Bonnet gravity theory

From the Lovelock's theorem, one notices that the boundary surface terms, e.g. the second order of the curvature quantities term, $\sqrt{-g}(R^2 - 4R^{\alpha\beta}R_{\alpha\beta} + R^{\alpha\beta\rho\sigma}R_{\alpha\beta\rho\sigma})$ called Gauss-Bonnet term, does not contribute to the field equations in four or less dimensional spacetime, but it can contribute to the field equations in five or more dimensional spacetime. Thus we can see what difference between the GR (with a cosmological constant) and generalized theory followed from Lovelock's theorem by considering these theories in higher dimensions. For simplicity, we will study this generalized theory called Einstein-Gauss-Bonnet (EGB) theory in five or six dimensions. The action for the (five or six)-dimensional EGB theory can be written as

$$S^{(5\mathrm{D},6\mathrm{D})} = \int \mathrm{d}^n X \left[\begin{array}{l} \sqrt{-\tilde{g}} \frac{M_{(n)}^{n-2}}{2} \left\{ \begin{array}{l} \tilde{R} - 2\Lambda \\ +a \left(\tilde{R}^2 - 4\tilde{R}^{AB}\tilde{R}_{AB} + \tilde{R}^{ABCD}\tilde{R}_{ABCD} \right) \end{array} \right\} \\ +\mathcal{L}_{\mathrm{m}}(\tilde{g}_{AB}, \psi) \end{array} \right], \tag{4.9}$$

where a is a constant which is related to the previous consideration as $b_2 = aM_{(n)}^{n-2}/2$, $M_{(n)}$ is n-dimensional Planck mass and X^A are the n-dimensional coordinates. Moreover, $\mathcal{L}_{\mathrm{m}}$ is the Lagrangian density of the n-dimensional matter with an arbitrary matter field, ψ. The field equations for this theory are

$$\tilde{G}_{AB} + \Lambda\tilde{g}_{AB} + a\tilde{H}_{AB} = M_{(n)}^{2-n}\tilde{T}_{AB}, \tag{4.10}$$

where the tensor $\tilde{H}_{AB}$ is obtained by varying the action in part of Gauss-Bonnet terms,

$$\begin{aligned} \tilde{H}_{AB} &= 2\tilde{R}\tilde{R}_{AB} - 4\tilde{R}_{AC}\tilde{R}_B^C - 4\tilde{R}_{ACBD}\tilde{R}^{CD} + 2\tilde{R}_{ACDE}\tilde{R}_B{}^{CDE} \\ &\quad -\frac{1}{2}\tilde{g}_{AB}\left(\tilde{R}^2 - 4\tilde{R}^{CD}\tilde{R}_{CD} + \tilde{R}^{CDEF}\tilde{R}_{CDEF}\right), \end{aligned} \tag{4.11}$$

and the tensor, $\tilde{T}_{AB} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta}{\delta\tilde{g}^{AB}}\mathcal{L}_{\mathrm{m}}$ is the energy-momentum tensor for matter. $\tilde{H}_{AB}$ in (4.10) called the Lovelock tensor. It is the additional term to the field equations in GR.

To see more detail about some consistency of this theory, we will consider the maximally symmetric vacuum solutions. Let the metric which describes the maximally symmetric vacuum geometry be $\tilde{g}_{AB}^{(0)}$. Then the Riemannian tensor, Ricci tensor and Ricci scalar become $\tilde{R}_{ABCD} = \frac{2\kappa}{(n-1)(n-2)}\left(\tilde{g}_{AC}^{(0)}\tilde{g}_{BD}^{(0)} - \tilde{g}_{AD}^{(0)}\tilde{g}_{BC}^{(0)}\right)$, $\tilde{R}_{AB} = \frac{2\kappa}{n-2}\tilde{g}_{AB}^{(0)}$ and $\tilde{R} = \frac{2n\kappa}{n-2}$ respectively where κ is a constant which describes the curvature of spacetime. Applying these quantities to the field equations (4.10) as well as taking $\tilde{T}^{(0)}_{AB} = 0$, the solutions exist only for two values of κ as

$$\kappa_{\pm} = \bar{\Lambda}\left(1 \pm \sqrt{1 - \frac{2\Lambda}{\bar{\Lambda}}}\right), \quad \bar{\Lambda} = -\frac{1}{4a}\frac{(n-1)(n-2)}{(n-3)(n-4)}. \tag{4.12}$$

If we consider a limit in which the theory should be reduced to higher-dimensional GR, i.e., the limit $a \to 0$ (or $\bar{\Lambda} \to -\infty$), the constants κ_+ and κ_- approach to $\bar{\Lambda}$ and Λ respectively. This means that the solution with κ_- will reduce smoothly to one in the higher-dimensional GR (with the curvature Λ for considering in the maximally symmetric spacetime), this branch is called the Einstein branch. However, taking $a \to 0$ does not reduce to the higher-dimensional GR in the case of κ_+. It is called the Gauss-Bonnet branch which identifies the new feature of the EGB theory compared to GR. Moreover, these two vacua will be the same at $\kappa_- = \kappa_+ = \bar{\Lambda} = 2\Lambda$ and we found the relations,

$$\frac{\kappa_-}{\bar{\Lambda}} \leq 1 \leq \frac{\kappa_+}{\bar{\Lambda}}. \tag{4.13}$$

Now, we show some problem of the solutions in Gauss-Bonnet branch by considering the perturbation about these vacuum solutions, $\tilde{g}_{AB} = \tilde{g}^{(0)}_{AB} + \tilde{g}^{(1)}_{AB}$. Substituting this expression into the field equations (4.10) and keeping only the first order perturbation, we obtain the linearized field equation as

$$\tilde{G}^{(1)}_{AB} + \kappa\tilde{g}^{(1)}_{AB} = M^{2-n}_{(n),\text{eff}}\tilde{T}^{(1)}_{AB}, \quad M^{2-n}_{(n),\text{eff}} = M^{2-n}_{(n)}\bigg/\left(1 - \frac{\kappa}{\bar{\Lambda}}\right), \tag{4.14}$$

where $\tilde{T}^{(1)}_{AB}$ is the first order perturbation of energy-momentum part and $M_{(n),\text{eff}}$ is the effective Planck mass for the linearized field equations. As we have known, the Planck mass, $M_{(n)}$ can be written in the form of the Newton constant, $G_{(n)}$ in n dimensions as $M^{n-2}_{(n)} = 1/8\pi G_{(n)}$. Thus the relation between the ordinary Newton constant and the effective one can be written as

$$G_{(n),\text{eff}} = G_{(n)}\bigg/\left(1 - \frac{\kappa}{\bar{\Lambda}}\right). \tag{4.15}$$

For the solution with κ_+ in (4.12), the sign of the effective Newton constant, $G_{(n),\text{eff}}$ is opposite to the ordinary one (for $G_{(n)}$ is positive, we obtain the negative $G_{(n),\text{eff}}$). This implies that the Gauss-Bonnet branch contains a perturbative ghost [60]. Moreover, the spherically symmetric solutions were studied in [61]. It is found that there is an event horizon covered the singularity at $r = 0$ for the Einstein branch while the singularity is naked for the Gauss-Bonnet branch. The cosmological aspect of the EGB theory was studied in [62], it is found that there exists the stable cosmological solutions with exponentially time dependent scale factor. [8] also discusses the mechanism for obtaining the effective gravity theories in four dimensions from the higher-dimensional EGB theory. In the next two sections, we will introduce the mechanisms which can explain how the (spatial) extra dimensions exists in the real world without being detected by observation.

4.2 Kaluza-Klein theory

This theory was proposed in order to unify two known fundamental interactions in 1921 [63, 64] which are electromagnetism and gravity by considering GR in five dimensions (one more extra spatial dimension). Kaluza had succeed in constructing a four-dimensional theory of electromagnetism and gravitation from the five-dimensional GR. However, he faced a problem of why we cannot see the fifth dimension. Fortunately, Klein solved this problem by proposing a mechanism in which this fifth dimension should be compacted as a tiny circle.

The Kaluza's ansatz corresponding to this unification theory is to interpret components of the five-dimensional metric $\tilde{g}_{AB}$ as four-dimensional metric $g_{\mu\nu}$, vector field A_μ called gauge field and scalar field ϕ called dilaton field. Thus the five-dimensional metric can be written as

$$\tilde{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi^2 \end{pmatrix}. \tag{4.16}$$

Kaluza believed that the five-dimensional universe contains only the pure gravity. The four-dimensional fields can be interpreted from the five-dimensional geometry. Therefore, the action contains only the curvature sector as

$$S^{(5)} = \frac{M_{(5)}^3}{2} \int \mathrm{d}^5 X \sqrt{-\tilde{g}} \tilde{R}. \tag{4.17}$$

Applying the decomposition (4.16) to this empty five-dimensional GR, we obtain the effective gravity theory in four-dimensional spacetime as

$$\begin{aligned} S_{\text{eff}}^{(4)} &= \frac{M_{(5)}^3}{2} \int \mathrm{d}^4 x \mathrm{d}y \phi \sqrt{-g} \left(R - \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} + \frac{2\partial_\mu\phi\partial^\mu\phi}{3\phi^2} \right) \\ &= \int \mathrm{d}^4 x \sqrt{-g}\phi \left(\frac{M_{\text{Pl}}^2}{2} R - \frac{M_{\text{Pl}}^2}{8} F_{\mu\nu}F^{\mu\nu} + M_{\text{Pl}}^2 \frac{\partial_\mu\phi\partial^\mu\phi}{3\phi^2} \right) \end{aligned} \tag{4.18}$$

where the Planck mass in four dimensions, $M_{\rm Pl}$ can be defined in terms of mass scale in five dimensions, $M_{(5)}$ and volume of the fifth dimension as $M_{\rm Pl}^2 = \int {\rm d}y M_{(5)}^3$ and $F_{\mu\nu}$ is the electromagnetic field strength tensor defined as $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. To obtain GR with electromagnetic matter, we can set $\phi = 1$ and also redefine the gauge field, A_μ as $M_{\rm Pl} A_\mu/\sqrt{2} \to A_\mu$. Finally, the unified theory can be obtained as

$$S_{\rm eff} = \int {\rm d}^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]. \tag{4.19}$$

This is a success for constructing the unified theory of the GR and electromagnetism in four-dimensional spacetime. In addition, we will obtain another kind of gravity theory if we set the gauge field to be zero, $A_\mu = 0$ (without setting ϕ to be unity). This effective theory is the scalar-tensor theory called Brans-Dicke theory. We also note that the process in which we obtain the four-dimensional theory from the higher one, for example, from (4.17) to (4.18) in this discussion, is called the Kaluza-Klein (KK) dimensional reduction.

A new question may have been arisen which states that if the Kaluza's theory is true, why we cannot observe the fifth dimension. Fortunately, Klein already proposed a mechanism to solve this problem in 1926 [65]. His mechanism told us that there exists the fifth extra dimension in nature, but it is compacted as a circle $\mathbb{S}^1$ with a tiny radius L (possibly the Planck length scale). This compactification is performed by the topology of spacetime as $\mathbb{M}^4 \times \mathbb{S}^1$. Consequently, fields in five-dimensional spacetime can be expanded by Fourier expansion as

$$\tilde{\phi}(x,y) = \sum_{n=-\infty}^{\infty} \tilde{\phi}^{(n)}(x) e^{iny/L}, \tag{4.20}$$

$$\tilde{A}_B(x,y) = \sum_{n=-\infty}^{\infty} \tilde{A}_B^{(n)}(x) e^{iny/L}, \tag{4.21}$$

$$\tilde{g}_{AB}(x,y) = \sum_{n=-\infty}^{\infty} \tilde{g}_{AB}^{(n)}(x) e^{iny/L}, \tag{4.22}$$

where the integer n is the Fourier mode for this expansion. By considering a five-dimensional matter field, the matter Lagrangian density can be expanded following the above expansion. It is found that the matter field can be interpreted as a particle [67]. The n-th mode of this matter field will contain a quantized charge associated with the Fourier mode, n ($q_n \propto n/L$) and mass associated with the Fourier mode, n ($m_n \propto |n|/L$). Moreover, the mass states for any fields are constructed as a stack called Kaluza-Klein tower. Unfortunately, a particle predicted from KK theory cannot be interpreted as an elementary particle. The KK theory is thus ruled out by this reason.

Although, the main objective of this theory is not successful and then ruled out by experiments because it cannot provide the elementary particles, the process called the dimensional reduction leads us the very useful idea to construct a fundamental theory which requires the higher-dimensional spacetime as we can see in many studies. In the next section, we will discuss about an another idea to hide the extra dimensions. It is called theory of large extra dimensions.

4.3 Braneworld scenario

Besides the theory of extremely small extra dimensions, there are other theories which contain the idea of large extra dimensions. Some of them does not require a small size of extra dimensions but it can be large as 0.1 mm (its maximum limit depends on the experiment test [66]). Some theories introduce the warped extra dimensions in which we impose the specific geometry for bulk spacetime. There are also the theories of an infinite size of extra dimensions. These theories lead us to the new aspect of the existence of the extra dimensions in nature. Our universe is confined to live on a four-dimensional spacetime called 3-brane which is embedded in a higher-dimensional spacetime called bulk while the gravity is able to propagate through out this whole bulk. For the next model, the extra dimensions in the braneworld scenario is also compactified as in the KK theory in order to solve the problem about the huge gap between gravity interaction and other fundamental interactions in nature. This problem is called the hierarchy problem. Moreover, we also review a theory constructed in order to explain the dynamics of the universe.

4.3.1 Arkani-Hamed-Dimopoulos-Dvali model

First consider a model proposed by Arkani-Hamed, Dimopoulos and Dvali in 1998 [68], called ADD model. We have already mentioned about the habitations for gravity and the standard model (SM) fields. The action which describes this scenario (in n-dimensional bulk) can be written as

$$S = S_{\text{bulk}} + S_{\text{brane}}, \tag{4.23}$$

with each action reads

$$S_{\text{bulk}} = \frac{M_{(n)}^{n-2}}{2} \int \mathrm{d}^n X \sqrt{-\tilde{g}} \tilde{R}, \tag{4.24}$$

$$S_{\text{brane}} = \int \mathrm{d}^4 x \sqrt{-g_{\text{br}}} \mathcal{L}_{\text{matter}}, \tag{4.25}$$

where $\tilde{g}_{AB}$ and $g_{\text{br}\,\mu\nu}$ are the metric of the bulk spacetime and the induced metric on the brane respectively. The Lagrangian density $\mathcal{L}_{\text{matter}}$ describes matter fields on the brane. To see how this model is a candidate for solving the hierarchy problem, it is worthwhile to first consider only the bulk sector (4.24). The four-dimensional theory obtained by the dimensional reduction from the theory in higher dimensions has a mass scale which is proportional to the size (finite volume) of the extra space as

$$M_{\text{eff}}^2 = M_{(n)}^{n-2} \int \mathrm{d}^{n-4} y = M_{(n)}^{n-2} V_{(n-4)}, \tag{4.26}$$

where $V_{(n-4)}$ is the volume of $(n-4)$-dimensional extra space. It implies that the four-dimensional theory is the effective theory of the more fundamental theory which is the theory in higher-dimensional bulk. To explain how to solve the hierarchy problem between the electroweak scale and the gravity scale (Planck scale), we assume that the more fundamental mass scale is the electroweak scale ($M_{(n)} = M_{\text{EW}} \sim$ TeV) and the effective mass scale is set to be the Planck scale ($M_{\text{eff}} = M_{\text{Pl}} \sim 10^{16}$ TeV). From calculation, we found that, for $n = 5$, the radius of the extra space is too large ($\sim 10^{11}$ m) which is the observable scale. The ADD model will not be ruled out by the short distance gravity tests if we consider the case of $n \geq 6$ [8].

We can see that the hierarchy problem is solved by using the mass scale $M_{(n)}$ which can explain both gravity and electroweak theories. The Planck mass is just the effective mass scale of the more fundamental theory. Unfortunately, another question arises. Why are the size of the extra dimensions and the length scale of the fundamental theory much different? In other words, the ADD model does not actually solve the hierarchy problem, it just introduces the new hierarchy between the size of the extra dimensions and the scale of the fundamental theory in n dimensions. For example, for $n = 2$, the inverse of size of the extra dimensions, $L_{\rm extra}^{-1} \sim 10^{-4}$ eV and $M_{\rm EW} \sim 10^{12}$ eV. The problem is actually not solved. The other models provide in the next subsection are candidates for solving the hierarchy problem.

Moreover, the ADD model has the problems in cosmology. The KK modes for gravitons in this model is very light ($\gtrsim 10^{-4}$ eV) and the number of the KK gravitons is numerous ($\lesssim 10^{32}$) [8]. It is possible to obtain over-production of KK modes for graviton at high temperature and then the standard Big Bang Nucleosynthesis may be destroyed [69].

4.3.2 Randall-Sundrum models

As we have seen in the ADD model, the hierarchy problem is not solved completely. Another model was proposed by Randall and Sundrum, called the Randall-Sundrum (RS) model, in 1999 [70]. The five-dimensional bulk spacetime is non-flat. Such a spacetime is helpful for compactification without requiring the large extra dimensions as in the ADD model. Its setup requires two 3-branes. The first brane is the SM brane where we live on and the second one is the Planck brane. A function W is introduced in order to describe the curvature of bulk. The action associated with the RS model can be written as

$$S = S_{\rm bulk} + S_{\rm brane1} + S_{\rm brane2}. \tag{4.27}$$

The two branes are located at $y = 0$ and $y = \bar{y}$. The $\mathbb{Z}_2$ symmetry is imposed so that $y = -y$ in this model. It is not necessary to consider the whole compactified extra dimension from $y = 0$ to $y = 2\pi L$ where L is its radius. We thus choose to consider a half of this size which corresponds to set $\bar{y} = \pi L$ so that we consider the compactified interval from $y = 0$ to $y = \pi L$. By considering the anti-de Sitter (AdS) bulk spacetime, there exists the cosmological constant Λ. The brane tensions for two branes $\Lambda_{\rm (I)}$ and $\Lambda_{\rm (II)}$ are also introduced against Λ. Each sector of (4.27) becomes

$$S_{\rm bulk} = \int \mathrm{d}^4x \int_0^{\pi L} \mathrm{d}y \sqrt{-\tilde{g}} \left(\frac{M_{(5)}^3}{2} \tilde{R} - 2\Lambda \right), \tag{4.28}$$

$$S_{\rm brane1} = \int_{y=0} \mathrm{d}^4x \sqrt{-g_{(\mathrm{I})}} \left(\mathcal{L}_{\mathrm{matter}\,(\mathrm{I})} + \Lambda_{(\mathrm{I})} \right), \tag{4.29}$$

$$S_{\rm brane2} = \int_{y=\pi L} \mathrm{d}^4x \sqrt{-g_{(\mathrm{II})}} \left(\mathcal{L}_{\mathrm{matter}\,(\mathrm{II})} + \Lambda_{(\mathrm{II})} \right), \tag{4.30}$$

where $\mathcal{L}_{\mathrm{matter}\,(\mathrm{I})}$ and $\mathcal{L}_{\mathrm{matter}\,(\mathrm{II})}$ are the Lagrangian density for any matter fields on branes which are located on $y = 0$ and $y = \pi L$ respectively. The geometry of the brane1 and brane2 are described by the induced metrics $g_{(\mathrm{I})\,\mu\nu}$ and $g_{(\mathrm{II})\,\mu\nu}$ respectively, so we have $g_{(\mathrm{I})\,\mu\nu}(x) = \tilde{g}_{\mu\nu}(x, y = 0)$and $g_{(\mathrm{II})\,\mu\nu}(x) = \tilde{g}_{\mu\nu}(x, y = \pi L)$. According to the AdS bulk, the metric with the warp factor can be written as

$$\mathrm{d}s^2 = e^{W(y)} \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu - \mathrm{d}y^2. \tag{4.31}$$

where the function W is assume to be depended on only the extra coordinate y. Then applying this metric to the field equations corresponding to the action (4.27), we find the solution of $W(y)$ depending on brane we live on. If we live on brane1, we obtain $W(y) = ky$ where $k = \sqrt{\Lambda/6M_{(5)}^3}$. On the other hand, we obtain $W(y) = -ky$ for brane2. These solutions also require that the brane tensions must satisfy $\Lambda_{(\mathrm{I})} = -\Lambda_{(\mathrm{II})} = \Lambda/k$. The effective mass scales, which will be interpreted as the Planck masses in this model, can be defined as

$$M_{\mathrm{Pl}\,(\pm)}^2 = M_{(5)}^3 \int_0^{\pi L} \mathrm{d}y\, e^{\mp 2ky} = \pm \frac{M_{(5)}^3}{2k} \left(1 - e^{\mp 2k\pi L} \right), \tag{4.32}$$

where $M_{\mathrm{Pl}\,(+)}$ and $M_{\mathrm{Pl}\,(-)}$ are the Planck masses on the brane1 (we live on brane2) and brane2 (we live on brane1) respectively. We suppose to live on the brane2 which located at $y = \pi L$. We can interpret that the more fundamental mass scale $M_{(5)}$ is enlarged by a factor $\exp(2k\pi L)$ and becomes $M_{\mathrm{Pl}\,(-)}$ on brane2. The factor $\exp(W)$ is called the warp factor. In the other words, the warped factor makes the gravitational interaction much weaker than the other interactions on the SM brane. By calculation, the size of the compactified extra space of the RS model is slightly larger than the Planck length [71]. It is found that the hierarchy problem between the Planck scale and the energy scale in the other interactions is solved without introducing another hierarchy problem. In addition, for the case that we live on brane1, it is impossible to solve the hierarchy problem as in the above case

[70].

Unfortunately, the model (4.27) called RS1 model still have a problem because the effective theory in four-dimensional spacetime is not GR, but a scalar-tensor theory called the Brans-Dicke theory [8]. The additional scalar field comes from fluctuations between branes. Another model called RS2 was proposed in the same year [72]. In RS2, one brane is taking to be located at infinity along the direction of extra dimension. So, this model is considered as the infinitely large extra dimension. The effective theory becomes GR for the RS2 model. However, the other problem is arisen as in the ADD model. There is the new hierarchy problem between the Planck scale and the curvature scale so that the hierarchy problem is not completely solved. There are many attempts to solve the hierarchy problem. For example, the other mechanism for the warped extra dimensions is also studied [73]. There is a mechanism called the relaxion mechanism which was proposed to solve the hierarchy problem [74]. Next, we will move our attention to discuss the other model in the braneworld scenario which is proposed in order to explain some aspect in cosmology.

4.3.3 Dvali-Gabadadze-Porrati model

This is another model in the infinite extra dimension scenario which was proposed by Dvali, Gabadadze and Porrati in 2000 [75]. In the Dvali-Gabadadze-Porrati (DGP) model, the four-dimensional brane is embedded in the empty flat five-dimensional bulk which has infinite size ($-\infty < y < \infty$). It also imposes the $\mathbb{Z}_2$ symmetry as in the RS model. It is possible to choose where the brane is located in the bulk. For simplicity, its location is at $y = 0$ and it is enough to consider only one side which is $y > 0$. The action can be splitted into the empty bulk sector and the brane sector as

$$S = S_{\text{bulk}} + S_{\text{brane}}, \tag{4.33}$$

with

$$\begin{aligned} S_{\text{bulk}} &= \frac{M_{(5)}^3}{4}\int \mathrm{d}^4x\mathrm{d}y\sqrt{-\tilde{g}}\tilde{R}, && (4.34)\\ S_{\text{brane}} &= \frac{M_{\text{Pl}}^2}{2}\int \mathrm{d}^4x\sqrt{-g_{\text{br}}}\left(R + \mathcal{L}_{\text{matter}}\right), && (4.35)\end{aligned}$$

where the metric $\tilde{g}_{AB}(x,y)$ describes the geometry of the bulk and $g_{\mathrm{br}\,\mu\nu}(x) = \tilde{g}_{\mu\nu}(x, y = 0)$ is the induced metric on the brane. We use the factor 1/4 in (4.34) instead of 1/2 because we consider only a half of the bulk ($y > 0$). We also notice that there are two mass scales $M_{(5)}$ and M_{Pl} associated with the Planck mass in five and four dimensions respectively. The crossover (length) scale which tells us where the gravity changes its behavior can be determined by

$$r_0 \sim M_{\mathrm{Pl}}^2/M_{(5)}^3. \tag{4.36}$$

This means that if we consider distance scale $r \ll r_0$, gravity will have four-dimensional behavior and if we consider $r \gg r_0$, it has five-dimensional one. The brane tension and the cosmological constant in the bulk are not introduced here as in the RS model. The field equations are written as

$$M_{(5)}^3 \tilde{G}_{AB} + 2\delta(y)\left[\left(M_{\mathrm{Pl}}^2 G_{\mu\nu} - T_{\mu\nu}\right)\delta^\mu_A \delta^\nu_B\right] = 0 \tag{4.37}$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\mathcal{L}_{\mathrm{m}}$ is the energy momentum tensor for matter on brane. It is found that the DGP model admits two brunches of the solution. The first one is the AdS brane in the bulk, it is called the normal branch. The another solution is the dS brane with a Hubble radius $H \sim M_{(5)}^3/M_{\mathrm{Pl}}^2$, it is called the self-accelerating branch. The cosmological implication of this model is very interesting because the solution in the second branch can be used to explain the dynamics of the late-time universe without introducing the cosmological constant [76]. Unfortunately, this model faces the problem such that the solution in this branch is unstable [77]. In other words, there exists the ghost in this branch. Although the normal branch cannot predict the accelerated expansion of the universe, it has no such problem. It is more useful to analyze the normal branch because we do not want to deal with the unhealthy theory.

Although the DGP model is not successful in explaining the accelerated expansion, it is found that this model contains some feature of massive gravity theory [8, 32, 78]. For example, there exists the vDVZ discontinuity in the DGP model, which is solved by Vainshtein mechanism. It motivates theorists to revisit the problem of nonlinear massive gravity theory and leads to the construction of the dRGT theory. Notice that the DGP model is qualitatively similar to the massive gravity theory since it contains the infinite gravitons while the massive gravity is the theory of the single graviton. We finish the brief introduction and discussion about the extra dimensions in the gravity theories here. In this work, the KK mechanism as we have discussed in Section 4.2 will be applied in order to add the external scalar field to the dRGT theory. Thus this work is motivated from the two modified gravity theories which are the massive gravity theory and the KK theory.

CHAPTER V

HIGHER-DIMENSIONAL MASSIVE GRAVITY THEORY

The objective of this work is to explain the dynamics of the universe by using the effective massive gravity in four dimensions. We choose to extend the dRGT theory by introducing an external scalar field. Moreover, this scalar field is interpreted as the size of the extra spatial dimensions. The extra spatial dimension is also compactified as in the KK theory. Thus the effective theory in four dimensions can be obtained by using dimensional reduction as reviewed in Section 4.2. The application of this effective theory in the cosmological context will be analyzed below.

5.1 Four-dimensional effective theory

Let's start with considering the higher-dimensional dRGT massive gravity theory, the action for this theory in n dimensions can be written as

$$S_{\mathrm{dRGT}} = \int \mathrm{d}^n X \sqrt{-\tilde{g}} \frac{M_{(n)}^{n-2}}{2} \left(\tilde{R} + m_g^2 \tilde{\mathcal{U}} \right), \tag{5.1}$$

where

$$\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_2 + \alpha_3 \tilde{\mathcal{U}}_3 + \alpha_4 \tilde{\mathcal{U}}_4 + \alpha_5 \tilde{\mathcal{U}}_5 + \alpha_6 \tilde{\mathcal{U}}_6 + \dots, \tag{5.2}$$

with

$$\begin{aligned}
\tilde{\mathcal{U}}_2 &= [\tilde{\mathcal{K}}]^2 - [\tilde{\mathcal{K}}^2], \\
\tilde{\mathcal{U}}_3 &= [\tilde{\mathcal{K}}]^3 - 3[\tilde{\mathcal{K}}][\tilde{\mathcal{K}}^2] + 2[\tilde{\mathcal{K}}^3], \\
\tilde{\mathcal{U}}_4 &= [\tilde{\mathcal{K}}]^4 - 6[\tilde{\mathcal{K}}]^2[\tilde{\mathcal{K}}^2] + 3[\tilde{\mathcal{K}}^2]^2 + 8[\tilde{\mathcal{K}}][\tilde{\mathcal{K}}^3] - 6[\tilde{\mathcal{K}}^4], \\
\tilde{\mathcal{U}}_5 &= [\tilde{\mathcal{K}}]^5 - 10[\tilde{\mathcal{K}}]^3[\tilde{\mathcal{K}}^2] + 20[\tilde{\mathcal{K}}]^2[\tilde{\mathcal{K}}^3] - 20[\tilde{\mathcal{K}}^2][\tilde{\mathcal{K}}^3] + 15[\tilde{\mathcal{K}}][\tilde{\mathcal{K}}^2]^2 - 30[\tilde{\mathcal{K}}][\tilde{\mathcal{K}}^4] + 24[\tilde{\mathcal{K}}^5],
\end{aligned}$$

$$
\begin{aligned}
\tilde{\mathcal{U}}_6 &= [\tilde{\mathcal{K}}]^6 - 15[\tilde{\mathcal{K}}]^4[\tilde{\mathcal{K}}^2] + 40[\tilde{\mathcal{K}}]^3[\tilde{\mathcal{K}}^3] - 90[\tilde{\mathcal{K}}]^2[\tilde{\mathcal{K}}^4] + 45[\tilde{\mathcal{K}}]^2[\tilde{\mathcal{K}}^2]^2 - 15[\tilde{\mathcal{K}}^2]^3 + 40[\tilde{\mathcal{K}}^3]^2 \\
&\quad -120[\tilde{\mathcal{K}}^3][\tilde{\mathcal{K}}^2][\tilde{\mathcal{K}}] + 90[\tilde{\mathcal{K}}^4][\tilde{\mathcal{K}}^2] + 144[\tilde{\mathcal{K}}^5][\tilde{\mathcal{K}}] - 120[\tilde{\mathcal{K}}^6], \\
&\ \vdots
\end{aligned} \tag{5.3}
$$

Here $\tilde{\mathcal{K}}^A_B = (\sqrt{\tilde{g}^{-1}\hat{\tilde{f}}}\,)^A_B$ and $\hat{\tilde{f}}_{AB} = \partial_A\tilde{\psi}^{\bar{C}}\partial_B\tilde{\psi}^{\bar{D}}\tilde{f}_{\bar{C}\bar{D}}$.

In order to obtain the four-dimensional effective gravity theory, we assume that the d extra dimensions are compactified in small size. The ansatz for this compactification can be written as

$$
\tilde{g}_{AB} = \begin{pmatrix} t(\phi)^{-2}g_{\mu\nu}(x) & 0 \\ 0 & p(\phi)^2\gamma_{ab}(y) \end{pmatrix}, \tag{5.4}
$$

where we separate the higher-dimensional coordinate, X^A into the ordinary four-dimensional coordinate, x^μ and d extra dimensional coordinate, y^a as $X^A = (x^\mu, y^a)$. $g_{\mu\nu}$ is the physical metric in four-dimensional spacetime, γ_{ab} is the metric in the extra dimensions. Moreover, we consider the maximally symmetric extra dimensions, so the geometry of the extra spatial dimensions can be described by a constant. Two functions of a scalar field, ϕ are introduced. The first function is $p(\phi)$ playing a role of the radius of the extra dimensions and the second one is $t(\phi)$ being a conformal factor for the conformal transformation. This conformal transformation is the transformation which preserves an angle between two vectors on spacetime. We introduce the conformal factor in order to obtain the effective theory in four dimensions in a frame where ϕ is decoupled to the kinetic term of $g_{\mu\nu}$. The scalar field is also assumed to be a function of the ordinary spacetime coordinates, $\phi = \phi(x)$. In the same way, the ansatz for the fiducial metric is

$$
\hat{\tilde{f}}_{AB} = \begin{pmatrix} \hat{f}_{\mu\nu}(x) & 0 \\ 0 & q(\phi)^2\gamma_{ab}(y) \end{pmatrix}, \tag{5.5}
$$

where $\hat{f}_{\mu\nu}$ is the fiducial metric in four-dimensional spacetime and a function $q(\phi)$ playing a role of the radius of the extra dimensions for the fiducial sector. The higher dimensional Strückelberg fields are splitted into the ordinary four-spacetime part $\tilde{\psi}^{\bar{\mu}} = \psi^{\bar{\mu}}$ and d extra spatial part $\tilde{\psi}^{\bar{a}}$ satisfying the condition $\partial_a\tilde{\psi}^{\bar{c}}\partial_b\tilde{\psi}^{\bar{d}}\tilde{f}_{\bar{c}\bar{d}} = q^2\gamma_{ab}$.

From these two ansatz metrics (5.4) and (5.5), the curvature quantities are evaluated as follow. The non-zero components of the connection $\tilde{\Gamma}^A_{BC}$ are

$$\begin{aligned}
\tilde{\Gamma}^\rho_{\mu\nu} &= \Gamma^\rho_{\mu\nu}[g] + C^\rho_{\mu\nu}, \qquad \text{where } \; C^\rho_{\mu\nu} \equiv \frac{1}{t}\left(g_{\mu\nu}\nabla^\rho t - \delta^\rho_\mu \nabla_\nu t - \delta^\rho_\nu \nabla_\mu t\right), \\
\tilde{\Gamma}^\rho_{ab} &= -t^2 p \nabla^\rho p \gamma_{ab}, \\
\tilde{\Gamma}^a_{b\mu} &= \frac{1}{p}\nabla_\mu p \delta^a_b, \\
\tilde{\Gamma}^a_{bc} &= \frac{1}{2}\gamma^{ad}\left[\partial_b(\gamma_{cd}) + \partial_c(\gamma_{bd}) - \partial_d(\gamma_{bc})\right] = \Gamma^a_{bc}[\gamma].
\end{aligned} \tag{5.6}$$

The non-zero components of the higher-dimensional Ricci tensor, $\tilde{R}_{AB}$ are

$$\begin{aligned}
\tilde{R}_{\mu\nu} &= R_{\mu\nu} + \frac{1}{t}\left[g_{\mu\nu}\nabla^2 t + 2\nabla_\mu\nabla_\nu t\right] + \frac{1}{t^2}\left[-3g_{\mu\nu}(\nabla t)^2\right] - d\frac{1}{p}\nabla_\mu\nabla_\nu p \\
&\quad + \frac{1}{tp}\left[d g_{\mu\nu}\nabla_\rho t \nabla^\rho p - d\nabla_\mu t \nabla_\nu p - d\nabla_\nu t \nabla_\mu p\right], \\
\tilde{R}_{ab} &= R_{ab} + \gamma_{ab}\left[-t^2 p \nabla^2 p + 2tp\nabla_\rho t \nabla^\rho p + (1-d)\, t^2 (\nabla p)^2\right].
\end{aligned} \tag{5.7}$$

The higher-dimensional Ricci scalar is

$$\tilde{R} = t^2 \left[\begin{array}{c} R[g] + \frac{1}{t^2p^2}R[\gamma] + 6\nabla_\rho\left(\frac{1}{t}\nabla^\rho t\right) - (d+2)\nabla_\rho\left(\frac{1}{p}\nabla^\rho p\right) \\ -6\frac{1}{t^2}(\nabla t)^2 - 3d\frac{1}{p^2}(\nabla p)^2 + 2(d+2)\frac{1}{tp}\nabla_\rho t \nabla^\rho p \end{array}\right]. \tag{5.8}$$

Thus, the dimensional reduction of the curvature sector reads

$$\begin{aligned}
S_{\text{curv}} &= \int \mathrm{d}^{4+d}X\sqrt{-\tilde{g}}\frac{M^{2+d}_{(4+d)}}{2}\tilde{R}, \\
&= \int \mathrm{d}^4x \int \mathrm{d}^d y \sqrt{-g}\sqrt{\gamma}\frac{p^d}{t^4}\frac{M^{2+d}_{(4+d)}}{2}t^2 \\
&\qquad \left[\begin{array}{c} R[g] + \frac{1}{t^2p^2}R[\gamma] + 6\nabla_\rho\left(\frac{1}{t}\nabla^\rho t\right) - (d+2)\nabla_\rho\left(\frac{1}{p}\nabla^\rho p\right) \\ -6\frac{1}{t^2}(\nabla t)^2 - 3d\frac{1}{p^2}(\nabla p)^2 + 2(d+2)\frac{1}{tp}\nabla_\rho t \nabla^\rho p \end{array}\right], \\
&= \int \mathrm{d}^4x\sqrt{-g}\left(\int \mathrm{d}^d y\sqrt{\gamma}\frac{M^{2+d}_{(4+d)}}{2}\right)\frac{p^d}{t^2} \\
&\qquad \left[R[g] + \frac{1}{t^2p^2}R[\gamma] - 6\frac{1}{t^2}(\nabla t)^2 - 3m\frac{1}{p^2}(\nabla p)^2 + 2(m+2)\frac{1}{tp}\nabla_\rho t \nabla^\rho p\right].
\end{aligned} \tag{5.9}$$

Let's define the four-dimensional Planck mass as $M^2_{\text{Pl}} \equiv \int \mathrm{d}^d y \sqrt{\gamma} M^{2+d}_{(4+d)}$. Moreover, the conformal factor is set as $t^2 = p^d$ for obtaining the theory in the frame that ϕ and the kinetic term of $g_{\mu\nu}$ are decoupled as we mentioned before. The action of the curvature sector becomes

$$S_{\text{curv}} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M^2_{\text{Pl}}}{2}R[g] + \frac{M^2_{\text{Pl}}}{2p^{d+2}}R[\gamma] - \frac{M^2_{\text{Pl}}}{4}d(d+2)\frac{p^2_{,\phi}}{p^2}(\nabla\phi)^2\right]. \tag{5.10}$$

To obtain the canonical form of the kinetic term, we have to set $\frac{M_{\rm Pl}^2}{2}d\,(d+2)\,\frac{p_{,\phi}^2}{p^2}=1$, we can solve this condition to obtain the form of the function p as

$$p(\phi)=\exp\left[\sqrt{\frac{2}{d(d+2)}}\frac{\phi}{M_{\rm Pl}}\right]. \tag{5.11}$$

Since we consider the maximally symmetric extra space which is described by the metric γ_{ab}, the curvature tensor for the extra space can be written as

$$\begin{aligned} R_{abcd} &= \kappa\left(\gamma_{ac}\gamma_{bd}-\gamma_{ad}\gamma_{bc}\right), \\ R_{bd} &= \gamma^{ac}R_{abcd}=\kappa\,(d-1)\,\gamma_{bd}, \\ R[\gamma] &= d\,(d-1)\,\kappa, \end{aligned} \tag{5.12}$$

where κ is a constant which gives us the geometry of the extra space being hyperbolic, flat and spherical for $\kappa<0, \kappa=0$ and $\kappa>0$ respectively. Finally, the curvature sector from the dimensional reduction is

$$S_{\rm curv}=\int {\rm d}^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R[g]-\frac{1}{2}\left(\nabla\phi\right)^2+V(\phi)\right], \tag{5.13}$$

where the potential of the scalar field can be defined as $V(\phi)\equiv\frac{d(d-1)}{2}\frac{M_{\rm Pl}^2}{p^{d+2}}\kappa=\frac{1}{2}\frac{M_{\rm Pl}^2}{p^{d+2}}R[\gamma]$. We notice that the potential V will vanish when we consider the one-dimensional extra space ($d=1$). Next, we will move our attention to consider the interaction sector,

$$\begin{aligned} S_{\rm int} &= \int {\rm d}^{4+d}X\sqrt{-\tilde{g}}\frac{M_{(4+d)}^{2+d}}{2}\left[\,m_g^2\tilde{\mathcal{U}}\,\right], \\ &= \int {\rm d}^4x\sqrt{-g}\frac{M_{\rm Pl}^2}{2}\left[\,M_g^2\tilde{\mathcal{U}}\,\right], \end{aligned} \tag{5.14}$$

where the new graviton mass, $M_g(\phi)$ is reintroduced via $M_g^2\equiv p(\phi)^{-d}m_g^2$. In orther words, the graviton mass has been promoted to a function of ϕ. The matrix $\tilde{\mathcal{K}}^A_B$ can be written as

$$\tilde{\mathcal{K}}^A_B=\delta^A_B-\left(\sqrt{\tilde{g}^{-1}\hat{\tilde{f}}}\,\right)^A_B=\delta^A_B-\tilde{\mathbb{M}}^A_B, \tag{5.15}$$

where the matrix $\tilde{\mathbb{M}}^A_B$, using the ansatz (5.4) and (5.5), is

$$\tilde{\mathbb{M}}^A_B=\begin{pmatrix} p^{d/2}\left(\sqrt{g^{-1}\hat{f}}\,\right)^\mu_\nu & 0 \\ 0 & \frac{q}{p}\delta^a_b \end{pmatrix}. \tag{5.16}$$

Thus the non-zero components of the matrix $\tilde{\mathcal{K}}^A_B$ are

$$\tilde{\mathcal{K}}^\mu_\nu = \delta^\mu_\nu - p^{d/2}(\sqrt{g^{-1}\hat{f}}\,)^\mu_\nu, \qquad \tilde{\mathcal{K}}^a_b = \frac{q}{p}\delta^a_b. \tag{5.17}$$

The potential due to interaction, $\tilde{\mathcal{U}}$, becomes

$$\begin{aligned}\tilde{\mathcal{U}} &= \mathcal{U} + d\,r\mathcal{F}, \\ &= \mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4 + \alpha_5\mathcal{U}_5 + \alpha_6\mathcal{U}_6 + \ldots \\ &\quad + d\,r\left(\mathcal{F}_2 + \alpha_3\mathcal{F}_3 + \alpha_4\mathcal{F}_4 + \alpha_5\mathcal{F}_5 + \alpha_6\mathcal{F}_6 + \ldots\right),\end{aligned} \tag{5.18}$$

with $\mathcal{U}_i$ being the function of the trace $[\mathcal{K}^j]$ takes the same form as the potential $\tilde{\mathcal{U}}_i$ which is the function of the trace $[\tilde{\mathcal{K}}^j]$ expressed in (5.3) and the function $\mathcal{F}_i$ can be written as

$$\begin{aligned}\mathcal{F}_2 &= 2[\mathcal{K}] + r(d-1), \\ \mathcal{F}_3 &= 3\mathcal{U}_2 + 3r(d-1)[\mathcal{K}] + r^2(d-1)(d-2), \\ \mathcal{F}_4 &= 4\mathcal{U}_3 + 6r(d-1)\mathcal{U}_2 + 4r^2(d-1)(d-2)[\mathcal{K}] + r^3(d-1)(d-2)(d-3), \\ \mathcal{F}_5 &= 5\mathcal{U}_4 + 10r(d-1)\mathcal{U}_3 + 6r^2(d-1)(d-2)\mathcal{U}_2 + 5r^3(d-1)(d-2)(d-3)[\mathcal{K}] \\ &\quad + r^4(d-1)(d-2)(d-3)(d-4), \\ \mathcal{F}_6 &= 6\mathcal{U}_5 + 15r(d-1)\mathcal{U}_4 + 20r^2(d-1)(d-2)\mathcal{U}_3 + 15r^3(d-1)(d-2)(d-3)\mathcal{U}_2 \\ &\quad + 6r^4(d-1)(d-2)(d-3)(d-4)[\mathcal{K}] + r^5(d-1)(d-2)(d-3)(d-4)(d-5). \\ &\vdots\end{aligned} \tag{5.19}$$

where $r = 1 - q/p$. Eventually, the effective theory which is obtained from the higher dimensional dRGT theory (5.1) via the KK dimensional reduction is

$$S_{\text{eff}} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\text{Pl}}^2}{2}R[g] - \frac{1}{2}\left(\nabla\phi\right)^2 + V + \frac{M_{\text{Pl}}^2}{2}M_g^2\left(\mathcal{U} + d\,r\mathcal{F}\right)\right]. \tag{5.20}$$

This theory is the massive gravity with the scalar field introducing through the existence of the extra dimensions. Moreover, it contains two description of the extensions of dRGT theory since we have the graviton mass which depend on the scalar field ϕ and the potential $\mathcal{U}$ and $\mathcal{F}$ which are invariant under a global symmetry,

$$\phi \to \phi - \phi_0, \qquad \psi^{\bar{a}} \to \psi^{\bar{a}}\exp\left[-\sqrt{\frac{d}{2(d+2)}}\frac{\phi_0}{M_{PL}}\right], \tag{5.21}$$

which is the feature of the quasi-dilaton model. ϕ_0 is an arbitrary symmetry transformation parameter and is independent of the spacetime coordinates x^μ. The first feature is the one of the mass-varying model [41, 42, 43, 44, 45] and the second feature is one of the quasi-dilaton model [47, 48]. However, the graviton mass breaks this global symmetry. Therefore, the effective theory is somewhat a combination of these two extensions of dRGT theory.

The reason why we have shown the explicit form of the potentials up to $i = 6$ is the main idea for our consideration is the same for the case of six and higher than six dimensions. We would like to analyze whether the existence of the potential of the scalar field affects the prediction of the dynamics of the universe. It is enough to consider only five and six dimensions as we will see later. Then, we will discuss the cosmological implication for this effective theory.

5.2 Field equations

There are three dynamical fields in the effective massive gravity (5.20) which are $g_{\mu\nu}$, $\psi^{\bar{\mu}}$ and ϕ. We choose to work in the unitary gauge, $\psi^{\bar{\mu}} = x^{\bar{\mu}}$ or $\hat{f}_{\mu\nu} = f_{\mu\nu}$. Thus the dynamical fields become $g_{\mu\nu}$ and $f_{\mu\nu}$. The field equations for $g_{\mu\nu}$ are obtained by varying (5.20) with respect to $g^{\mu\nu}$,

$$G_{\mu\nu} = \frac{1}{M_{\rm Pl}^2}\left(T^{(X)}_{\mu\nu} + T^{(\phi)}_{\mu\nu}\right), \tag{5.22}$$

where

$$\begin{aligned} T^{(X)}_{\mu\nu} &= -M_{\rm Pl}^2 M_g^2\left(X_{\mu\nu} + d\,rY_{\mu\nu}\right), &(5.23)\\ T^{(\phi)}_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 - g_{\mu\nu}V, &(5.24)\end{aligned}$$

with

$$\begin{aligned} X_{\mu\nu} &= \frac{\delta\mathcal{U}}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{U} = X^{(2)}_{\mu\nu} + \alpha_3 X^{(3)}_{\mu\nu} + \alpha_4 X^{(4)}_{\mu\nu} + \alpha_5 X^{(5)}_{\mu\nu} + \alpha_6 X^{(6)}_{\mu\nu} + \ldots,\\ X^{(2)}_{\mu\nu} &= -\mathcal{K}^2_{\mu\nu} + ([\mathcal{K}]+1)\mathcal{K}_{\mu\nu} - \frac{1}{2}(\mathcal{U}_2 + 2[\mathcal{K}])g_{\mu\nu},\\ X^{(3)}_{\mu\nu} &= \frac{1}{2}\left\{6\mathcal{K}^3_{\mu\nu} - 6([\mathcal{K}]+1)\mathcal{K}^2_{\mu\nu} + 3(\mathcal{U}_2 + 2[\mathcal{K}])\mathcal{K}_{\mu\nu} - (\mathcal{U}_3 + 3\mathcal{U}_2)g_{\mu\nu}\right\},\\ X^{(4)}_{\mu\nu} &= 2\left\{-6\mathcal{K}^4_{\mu\nu} + 6([\mathcal{K}]+1)\mathcal{K}^3_{\mu\nu} - 3(\mathcal{U}_2 + 2[\mathcal{K}])\mathcal{K}^2_{\mu\nu} + (\mathcal{U}_3 + 3\mathcal{U}_2)\mathcal{K}_{\mu\nu}\right\}\\ &\quad -\frac{1}{2}(\mathcal{U}_4 + 4\mathcal{U}_3)g_{\mu\nu},\end{aligned}$$

$$
\begin{aligned}
X^{(5)}_{\mu\nu} &= \frac{1}{2}\left[5\left\{\begin{array}{l} 24\mathcal{K}^5_{\mu\nu} - 24([\mathcal{K}]+1)\mathcal{K}^4_{\mu\nu} + 12(\mathcal{U}_2 + 2[\mathcal{K}])\mathcal{K}^3_{\mu\nu} \\ -4(\mathcal{U}_3 + 3\mathcal{U}_2)\mathcal{K}^2_{\mu\nu} + (\mathcal{U}_4 + 4\mathcal{U}_3)\mathcal{K}_{\mu\nu} \end{array}\right\} - (\mathcal{U}_5 + 5\mathcal{U}_4)g_{\mu\nu}\right], \\
X^{(6)}_{\mu\nu} &= 3\left\{\begin{array}{l} -120\mathcal{K}^6_{\mu\nu} + 120([\mathcal{K}]+1)\mathcal{K}^5_{\mu\nu} - 60(\mathcal{U}_2 + 2[\mathcal{K}])\mathcal{K}^4_{\mu\nu} \\ +20(\mathcal{U}_3 + 3\mathcal{U}_2)\mathcal{K}^3_{\mu\nu} - 5(\mathcal{U}_4 + 4\mathcal{U}_3)\mathcal{K}^2_{\mu\nu} + (\mathcal{U}_5 + 5\mathcal{U}_4)\mathcal{K}_{\mu\nu} \end{array}\right\} \\
&\quad -\frac{1}{2}(\mathcal{U}_6 + 6\mathcal{U}_5)g_{\mu\nu}, \\
&\vdots \quad ,
\end{aligned} \tag{5.25}
$$

and

$$
\begin{aligned}
Y_{\mu\nu} &= \frac{\delta\mathcal{F}}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{F} = Y^{(2)}_{\mu\nu} + \alpha_3 Y^{(3)}_{\mu\nu} + \alpha_4 Y^{(4)}_{\mu\nu} + \alpha_5 Y^{(5)}_{\mu\nu} + \alpha_6 Y^{(6)}_{\mu\nu} + \ldots, \\
Y^{(2)}_{\mu\nu} &= \frac{1}{2}\left[2\left\{\mathcal{K}_{\mu\nu} - ([\mathcal{K}]+1)g_{\mu\nu}\right\} + r(d-1)g_{\mu\nu}\right], \\
Y^{(3)}_{\mu\nu} &= \frac{1}{2}\left[\begin{array}{l} -3\left\{2\mathcal{K}^2_{\mu\nu} - 2([\mathcal{K}]+1)\mathcal{K}_{\mu\nu} + (\mathcal{U}_2 + 2[\mathcal{K}])g_{\mu\nu}\right\} \\ +3r(d-1)\left\{\mathcal{K}_{\mu\nu} - g_{\mu\nu}([\mathcal{K}]+1)\right\} - r^2(d-1)(d-2)g_{\mu\nu} \end{array}\right], \\
Y^{(4)}_{\mu\nu} &= \frac{1}{2}\left[\begin{array}{l} 4\left\{6\mathcal{K}^3_{\mu\nu} - 6([\mathcal{K}]+1)\mathcal{K}^2_{\mu\nu} + 3(\mathcal{U}_2 + 2[\mathcal{K}])\mathcal{K}_{\mu\nu} - (\mathcal{U}_3 + 3\mathcal{U}_2)g_{\mu\nu}\right\} \\ -6r(d-1)\left\{2\mathcal{K}^2_{\mu\nu} - 2([\mathcal{K}]+1)\mathcal{K}_{\mu\nu} + (\mathcal{U}_2 + 2[\mathcal{K}])g_{\mu\nu}\right\} \\ +4r^2(d-1)(d-2)\left\{\mathcal{K}_{\mu\nu} - ([\mathcal{K}]+1)g_{\mu\nu}\right\} - r^3(d-1)(d-2)(d-3)g_{\mu\nu} \end{array}\right], \\
Y^{(5)}_{\mu\nu} &= \frac{1}{2}\left[\begin{array}{l} -5\left\{\begin{array}{l} 24\mathcal{K}^4_{\mu\nu} - 24([\mathcal{K}]+1)\mathcal{K}^3_{\mu\nu} + 12(\mathcal{U}_2 + 2[\mathcal{K}])\mathcal{K}^2_{\mu\nu} - 4(\mathcal{U}_3 + 3\mathcal{U}_2)\mathcal{K}_{\mu\nu} \\ +(\mathcal{U}_4 + 4\mathcal{U}_3)g_{\mu\nu} \end{array}\right\} \\ +10r(d-1)\left\{6\mathcal{K}^3_{\mu\nu} - 6([\mathcal{K}]+1)\mathcal{K}^2_{\mu\nu} + 3(\mathcal{U}_2 + 2[\mathcal{K}])\mathcal{K}_{\mu\nu} - (\mathcal{U}_3 + 3\mathcal{U}_2)g_{\mu\nu}\right\} \\ -10r^2(d-1)(d-2)\left\{2\mathcal{K}^2_{\mu\nu} - 2([\mathcal{K}]+1)\mathcal{K}_{\mu\nu} + (\mathcal{U}_2 + 2[\mathcal{K}])g_{\mu\nu}\right\} \\ +5r^3(d-1)(d-2)(d-3)\left\{\mathcal{K}_{\mu\nu} - ([\mathcal{K}]+1)g_{\mu\nu}\right\} \\ -r^4(d-1)(d-2)(d-3)(d-4)g_{\mu\nu} \end{array}\right], \\
Y^{(6)}_{\mu\nu} &= \frac{1}{2}\left[\begin{array}{l} 6\left\{\begin{array}{l} 120\mathcal{K}^5_{\mu\nu} - 120([\mathcal{K}]+1)\mathcal{K}^4_{\mu\nu} + 60(\mathcal{U}_2 + 2[\mathcal{K}])\mathcal{K}^3_{\mu\nu} \\ -20(\mathcal{U}_3 + 3\mathcal{U}_2)\mathcal{K}^2_{\mu\nu} + 5(\mathcal{U}_4 + 4\mathcal{U}_3)\mathcal{K}_{\mu\nu} - (\mathcal{U}_5 + 5\mathcal{U}_4)g_{\mu\nu} \end{array}\right\} \\ -15r(d-1)\left\{\begin{array}{l} 24\mathcal{K}^4_{\mu\nu} - 24([\mathcal{K}]+1)\mathcal{K}^3_{\mu\nu} + 12(\mathcal{U}_2 + 2[\mathcal{K}])\mathcal{K}^2_{\mu\nu} \\ -4(\mathcal{U}_3 + 3\mathcal{U}_2)\mathcal{K}_{\mu\nu} + (\mathcal{U}_4 + 4\mathcal{U}_3)g_{\mu\nu} \end{array}\right\} \\ +20r^2(d-1)(d-2)\left\{\begin{array}{l} 6\mathcal{K}^3_{\mu\nu} - 6([\mathcal{K}]+1)\mathcal{K}^2_{\mu\nu} + 3(\mathcal{U}_2 + 2[\mathcal{K}])\mathcal{K}_{\mu\nu} \\ -(\mathcal{U}_3 + 3\mathcal{U}_2)g_{\mu\nu} \end{array}\right\} \\ -15r^3(d-1)(d-2)(d-3)\left\{2\mathcal{K}^2_{\mu\nu} - 2([\mathcal{K}]+1)\mathcal{K}_{\mu\nu} + (\mathcal{U}_2 + 2[\mathcal{K}])g_{\mu\nu}\right\} \\ +6r^4(d-1)(d-2)(d-3)(d-4)\left\{\mathcal{K}_{\mu\nu} - ([\mathcal{K}]+1)g_{\mu\nu}\right\} \\ -r^5(d-1)(d-2)(d-3)(d-4)(d-5)g_{\mu\nu} \end{array}\right], \\
&\vdots \quad .
\end{aligned} \tag{5.26}
$$

By using the Bianchi identity, the conservation of the total energy momentum tensor is obtained as

$$
\nabla_\mu \left(T^{(X)\mu}{}_{\nu} + T^{(\phi)\mu}{}_{\nu}\right) = 0, \tag{5.27}
$$

which can also be derived by varying the action (5.20) with respect to $f_{\mu\nu}$ in the unitary gauge or varying the action (5.20) (without gauge fixing) with respect to $\psi^{\bar{\mu}}$. The field equation for ϕ is

$$\nabla^2\phi - V_{,\phi} + \sqrt{\frac{d}{2(d+2)}} M_{\rm Pl} M_g^2 \left(\Phi + d\, r\Psi\right) = 0, \tag{5.28}$$

where

$$\begin{aligned}
\Phi &= M_{\rm Pl}\sqrt{\frac{d+2}{2d}}\frac{\delta\mathcal{U}}{\delta\phi} - \mathcal{U} = \Phi_2 + \alpha_3\Phi_3 + \alpha_4\Phi_4 + \alpha_5\Phi_5 + \alpha_6\Phi_6 + \dots, \\
\Phi_2 &= -3[K], \qquad \Phi_3 = \frac{1}{2}\mathcal{U}_3 - 3\mathcal{U}_2, \qquad \Phi_4 = \mathcal{U}_4 - 2\mathcal{U}_3, \\
\Phi_5 &= \frac{3}{2}\mathcal{U}_5, \qquad \Phi_6 = 2\mathcal{U}_6 + 3\mathcal{U}_5, \qquad \dots,
\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}
\Psi &= M_{\rm Pl}\sqrt{\frac{d+2}{2d}}\frac{\delta\mathcal{F}}{\delta\phi} - \mathcal{F} = \Psi_2 + \alpha_3\Psi_3 + \alpha_4\Psi_4 + \alpha_5\Psi_5 + \alpha_6\Psi_6 + \dots, \\
\Psi_2 &= -([K]+4) - r(d-1), \\
\Psi_3 &= -9[K] - \frac{3}{2}r(d-1)([K]+4) - r^2(d-1)(d-2), \\
\Psi_4 &= 2(\mathcal{U}_3 - 6\mathcal{U}_2) - 18r(d-1)[K] - 2r^2(d-1)(d-2)([K]+4) \\
&\quad -r^3(d-1)(d-2)(d-3), \\
\Psi_5 &= 5(\mathcal{U}_4 - 2\mathcal{U}_3) + 5r(d-1)(\mathcal{U}_3 - 6\mathcal{U}_2) - 30r^2(d-1)(d-2)[K] \\
&\quad -\frac{5}{2}r^3(d-1)(d-2)(d-3)([K]+4) - r^4(d-1)(d-2)(d-3)(d-4), \\
\Psi_6 &= 9\mathcal{U}_5 + 15r(d-1)(\mathcal{U}_4 - 2\mathcal{U}_3) + 10r^2(d-1)(d-2)(\mathcal{U}_3 - 6\mathcal{U}_2) \\
&\quad -45r^3(d-1)(d-2)(d-3)[K] - 3r^4(d-1)(d-2)(d-3)(d-4)([K]+4) \\
&\quad -r^5(d-1)(d-2)(d-3)(d-4)(d-5), \\
&\ \vdots \quad .
\end{aligned} \tag{5.30}$$

Moreover, the field equations (5.22) and (5.28) can be derived by the dimensional reduction from the higher dimensional fields equations as show in Appendix C

5.3 Cosmological solutions

In this section, we attempt to use the effective massive gravity theory (5.20) to explain the dynamics of the universe at late-time. The original dRGT theory has no flat FLRW solution. As we proposed, one of the possible ways to solve this problem is to introduce the additional scalar field (we introduce it from the existence of the extra dimensions). Thus it is convenient to adopt the physical metric as the flat FLRW metric

$$g_{\mu\nu} = \mathrm{diag}\left(-1, a(t)^2, a(t)^2, a(t)^2\right), \tag{5.31}$$

where $a(t)$ is a scale factor determining the scale of the spatial distance. The gauge is fixed to the unitary gauge, $\psi^{\bar{\mu}} = x^{\bar{\mu}}$ for our convenience. As we have mentioned, the flat FLRW fiducial metric is considered in this work,

$$f_{\mu\nu} = \mathrm{diag}\left(-1, b(t)^2, b(t)^2, b(t)^2\right), \tag{5.32}$$

where $b(t)$ is a scale factor for this fiducial metric. To avoid a long calculation, it is enough to work in the simple case with $p = q$ or $r = 0$. For simplicity, we interpret the mass term and the potential of the scalar field in (5.20) as something driving the accelerated expansion of the late-time universe so that the radiation and matter are included in this consideration. The action of the model can be written as

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}\left(R + M_g^2(\phi)\mathcal{U}\right) - \frac{1}{2}\nabla_\rho\phi\nabla^\rho\phi - V + \mathcal{L}_{\mathrm{m}} + \mathcal{L}_{\mathrm{r}}\right], \tag{5.33}$$

where $\mathcal{L}_{\mathrm{m}}$ and $\mathcal{L}_{\mathrm{r}}$ are the Lagrangian density of the matter and radiation respectively. As a result, the field equations (5.22) become

$$G_{\mu\nu} = \frac{1}{M_{\mathrm{Pl}}}\left(T_{\mu\nu}^{(X)} + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\mathrm{m})} + T_{\mu\nu}^{(\mathrm{r})}\right), \tag{5.34}$$

where $T_{\mu\nu}^{(\mathrm{m})}$ and $T_{\mu\nu}^{(\mathrm{r})}$ are the energy momentum tensors associated with $\mathcal{L}_{\mathrm{m}}$ and $\mathcal{L}_{\mathrm{r}}$ respectively. From the ansatz (5.31) and (5.32), the non-zero components imply that

$$3M_{\mathrm{Pl}}^2H^2 = M_{\mathrm{Pl}}^2M_g^2A + \left(\frac{1}{2}\dot{\phi}^2 + V\right) + \rho_{\mathrm{m}} + \rho_{\mathrm{r}}, \tag{5.35}$$

$$M_{\mathrm{Pl}}^2(2\dot{H} + 3H^2) = M_{\mathrm{Pl}}^2M_g^2B - \left(\frac{1}{2}\dot{\phi}^2 - V\right) - p_{\mathrm{m}} - p_{\mathrm{r}}. \tag{5.36}$$

The first and second equations are respectively consequences of $(0,0)$ and (i,j) components of (5.34). $H = \dot{a}/a$ is the Hubble parameter, ρ_{m} and p_{m} are energy density and pressure of the matter respectively. ρ_{r} and p_{r} are ones of the radiation. The short-hand function A and B obtained from the tensor X^μ_ν can be expressed as

$$
\begin{aligned}
A &= X^0_0 = -3(y + \alpha y^2 + \beta y^3), \\
B\delta^i_j &= X^i_j = -\left[\left(\frac{y+s-1}{s}\right)(1 + 2\alpha y + 3\beta y^2) + y(2+\alpha y)\right]\delta^i_j, \\
y &= 1 - p^{d/2}s, \qquad s = \frac{b}{a}, \qquad \alpha = 1 + 3\alpha_3, \qquad \beta = \alpha_3 + 4\alpha_4. \qquad (5.37)
\end{aligned}
$$

For the scalar field, the equation of motion (5.28) in this consideration becomes

$$
\nabla_\rho \nabla^\rho \phi - V_{,\phi} + \sqrt{\frac{d}{2(d+2)}} M_{\rm Pl} M_g^2 \Phi = 0. \qquad (5.38)
$$

By applying the constraint equation (5.27), we obtain

$$
\left(\nabla_\rho \nabla^\rho \phi - V_{,\phi}\right) \nabla_\nu \phi = M_{\rm Pl}^2 \nabla_\rho \left(M_g^2 X^\rho_\nu\right), \qquad (5.39)
$$

As a result, the field equation (5.38) can be rewrite in a convenient form as

$$
\sqrt{\frac{d}{2(d+2)}} \frac{\dot{\phi}}{M_{\rm Pl} H} = \frac{3(1-s)(A-B)}{(2A - \Phi)(1-s) + 3s(A-B)}, \qquad (5.40)
$$

where dot denotes the derivative with respect to t. This is one of the constraints in our consideration. Note that the energy density and pressure contribution from graviton mass can be respectively written as

$$
\rho_g = A M_{\rm Pl}^2 M_g^2, \qquad p_g = -B M_{\rm Pl}^2 M_g^2. \qquad (5.41)
$$

We immediately see that an interesting branch of the solution is the case $A = B$. For this branch, called self-accelerating branch, the graviton mass will play the role of cosmological constant since its equation of state parameter equals to minus unity, $w_g = -1$. Moreover, it is possible to find the the non-trivial solution which is $\dot{\phi}/H = 0$. This also means that the graviton mass does not decay. Instead, it is a constant while the one in the mass-varying model always decrease in late-time universe [46].

Moreover, the equations of conservation (5.27) can be viewed as the coupling equations between a scalar field and the graviton mass. By rewriting the conservation equations, the coupling equations can be written as

$$
\begin{aligned}
\rho'_\phi + 3(1 + w_\phi)\rho_\phi &= \sqrt{\frac{d}{2(d+2)}} M_{\rm Pl} M_g^2 \Phi\, \phi', \qquad (5.42) \\
\rho'_g + 3(1 + w_g)\rho_g &= -\sqrt{\frac{d}{2(d+2)}} M_{\rm Pl} M_g^2 \Phi\, \phi', \qquad (5.43)
\end{aligned}
$$

where

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V, \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V, \qquad w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}. \tag{5.44}$$

and prime refers to the derivative with respect to $N = \ln a$. From these equations, it is found that the interaction term, which actually corresponds to the energy transfer between the two contents, vanishes when $\phi' = 0$. We will see the behaviors of each cosmic content more clearly when we analyze the model by the dynamical system approach and this is the main issue in the next section. The main issue in the next section is to find the stability of this cosmological model.

5.4 Dynamical system

We start this section by rewriting (5.35) in the appropriated form as

$$\begin{aligned} 1 &= -(y + \alpha y^2 + \beta y^3)\frac{M_g^2}{H^2} + z^2 + v + \Omega_\mathrm{m} + \Omega_\mathrm{r}, \\ &= -\frac{A}{3}x + z^2 + v + \Omega_\mathrm{m} + \Omega_\mathrm{r}, \end{aligned} \tag{5.45}$$

where

$$x = -\frac{M_g^2}{H^2}, \quad z^2 = \frac{\dot{\phi}^2}{6M_\mathrm{Pl}^2 H^2}, \quad v = \frac{V}{3M_\mathrm{Pl}^2 H^2}, \quad \Omega_\mathrm{m} = \frac{\rho_\mathrm{m}}{3M_\mathrm{Pl}^2 H^2}, \quad \Omega_\mathrm{r} = \frac{\rho_\mathrm{r}}{3M_\mathrm{Pl}^2 H^2}. \tag{5.46}$$

This is one of the constraints which contains six variables including y. The other one recalled from (5.40) is expressed in terms of the above dynamical variables as

$$\sqrt{\frac{3d}{(d+2)}}z = \frac{3(A-B)(1-s)}{(2A-\Phi)(1-s)+3s(A-B)}, \tag{5.47}$$

$$A - B = \frac{1-s}{s}(y-1)Y, \quad Y = 3\beta y^2 + 2\alpha y + 1. \tag{5.48}$$

We also note that the dynamics of the universe with the additional scalar field in this model is different from ones in the other usual scalar field models by virtue of the constraint (5.47) where $z \propto \phi'$. From this equation, the dynamic of scalar field is constrianed by its own equation without coupling to the other dynamical variables as found in the usual scalar field models. The last constraint is obtained form the existence of the potential for scalar field, $V(\phi)$ as

$$v = \gamma x \left(\frac{s}{1-y}\right)^{\frac{4}{d}}, \qquad \gamma = \frac{d(d-1)}{6}\frac{\kappa}{m_g^2}. \tag{5.49}$$

A new parameter γ is introduced in order to characterize an effect of the potential term compared to the graviton mass. Note that this is not the trace of the extra spatial metric in the ansatz (5.4) and (5.5). This parameter actually tells us how the curvature of the extra dimensions affects the dynamics of four-dimensional universe compared to the effect of the graviton mass. Moreover, the geometry of the extra space is characterized by this parameter as follows; the geometry of the hyperbolic, flat and spherical extra space for $\gamma < 0$, $\gamma = 0$ and $\gamma > 0$ respectively. We also see that the case $d = 1$, there is no extra curvature so that the potential of scalar field automatically vanishes.

Since we have six dynamical variables and three constraints. By choosing to eliminate three dynamical variables, z, v and $\Omega_{\rm m}$, three dynamical equations for the other variables can be written as

$$x' = -x\left(2\sqrt{\frac{3d}{d+2}}z + 2\frac{\dot{H}}{H^2}\right), \tag{5.50}$$

$$y' = \sqrt{\frac{3d}{d+2}}(y-1)z, \tag{5.51}$$

$$\Omega_{\rm r}' = -\Omega_{\rm r}\left(3(1+w_{\rm r}) + 2\frac{\dot{H}}{H^2}\right). \tag{5.52}$$

As a result of (5.35) and (5.36), the effective equation of state parameter can be expressed as

$$w_{\rm eff} = -1 - \frac{2\dot{H}}{3H^2}, \quad \frac{2\dot{H}}{3H^2} = \frac{x}{3}(A-B) - 2z^2 - (1+w_{\rm m})\Omega_{\rm m} - (1+w_{\rm r})\Omega_{\rm r}, \tag{5.53}$$

where $w_{\rm m}$ and $w_{\rm r}$ are the equation of state parameters for matter and radiation respectively ($w_{\rm m} = 0$ and $w_{\rm r} = 1/3$).

After substituting z from (5.47), this dynamical system is independent of the number of the extra dimensions, d. Even if the number of the extra dimensions is changed, the dynamics of contents in the universe still evolves following the same set of equations. However, the effect of more than one of extra dimensions is implicitly found in the constraint (5.45) where the potential V exists if $d > 1$.

Substituting z form (5.47) into (5.51), the non-trivial fixed points always exist at $A-B=0$. This equation is able to be solved for y as

$$y_{\pm}=\frac{-\alpha\pm\sqrt{\alpha^2-3\beta}}{3\beta}. \tag{5.54}$$

Note also that $x'=0$ in (5.50) is satisfied due to $\dot{H}=0$. Moreover, from (5.51) with (5.47), y' depends only on y so that one can separately examine the stability of the fixed points. As a result, the stability condition can be found by

$$(y-1)\partial_y z\Big|_{y=y_{\pm}}<0. \tag{5.55}$$

The left hand side is the quantities of three parameters which are α, β and s. The initial condition is able to use in order to eliminate one of them. We consider the contents at present as the initial condition, $\Omega_{\rm m0}\sim 0.25, \Omega_{\rm r0}\sim 0$ and $\Omega_{g0}=-Ax/3+v\sim 0.75$. As we discussed before, there is more parameter γ in the model with higher than five dimensions. It is useful to separate our analysis into two parts which are the five-dimensional case and higher than five-dimensional one (in this work, we consider in six dimensions).

5.4.1 Five-dimensional model

For five-dimensional model, the potential V (or variable v) disappears then the constraint (5.45) becomes $-Ax/3+\Omega_{\rm m0}+\Omega_{\rm r0}=1$. We choose to find the value of parameter s from the initial condition, $\Omega_{\rm m0}\sim 0.25, \Omega_{\rm r0}\sim 0$ and $\Omega_{g0}=-Ax/3\sim 0.75$. Thus we have only two free parameters α and β. By using a numerical method, the regions satisfying the stability condition for y_+ and y_- can be illustrated as in Fig. 1. In this calculation, we substitute the original graviton mass as the scale of Hubble radius at present, $m_g\sim H_0$. It is important to note that the stability condition (5.55) can be rewritten explicitly by

$$\frac{1}{2A-\Phi}\partial_y Y\Big|_{y=y_{\pm}}<0. \tag{5.56}$$

Since solutions $y_{\pm}$ are solved from $Y=0$, $\partial_y Y|_{y_+}$ is positive and $\partial_y Y|_{y_-}$ is negative. The stability condition can thus be inferred from the sign of $2A-\Phi$. In other words, the stable fixed points are obtained when $2A-\Phi<0$ for y_+ and $2A-\Phi>0$ for y_-. The regions illustrated in Fig. 1 does not include only these conditions but also conditions such that y and s are real numbers.

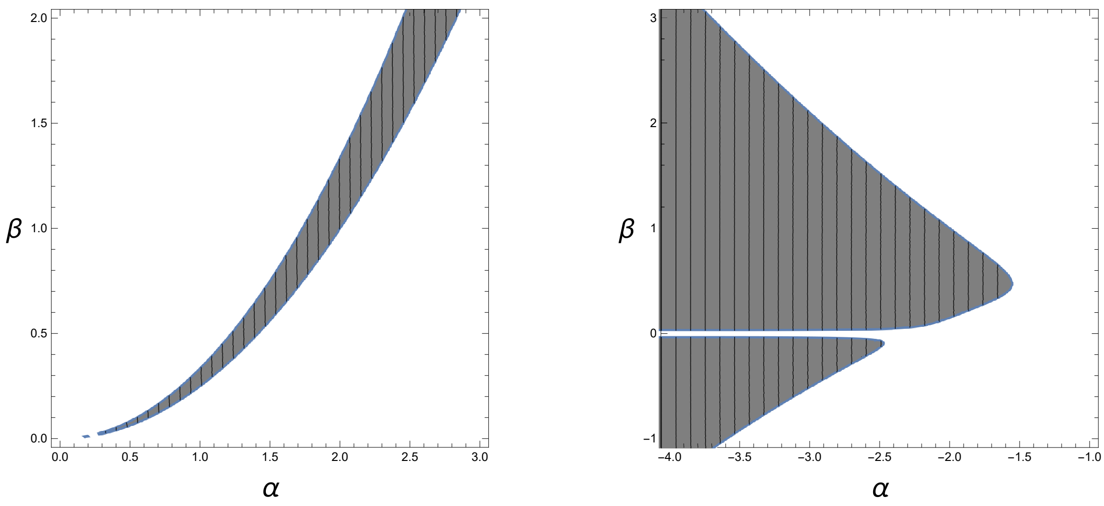

Figure 1 The left panel and the right panel show stability region for y_- solution and y_+ solution respectively.

There is a special case which is $\partial_y Y|_{y_\pm} = 0$. The eigenvalue is zero and the stability cannot be inferred from linear analysis [79]. Such a fixed point corresponding to the minimum of Y guarantees the stability of the nonlinear analysis. This case gives us the relation between two free parameter as $3\beta = \alpha^2$ which also turns out the merging of two solution $y_+ = y_-$. Moreover, it implies that $z = 0$ and $z' = 0$. This provides us that the scalar field is fixed all the time of the evolution of the universe. By choosing $\alpha = 2$, the evolution of cosmic contents ($\Omega_g, \Omega_\mathrm{m}$ and Ω_r) in the universe and the w_eff can be numerically evaluated as shown in Fig. 2. From this figure, one can see that all contents evolve as in standard evolution such that there exist the radiation, matter and dark energy dominated periods.

Now we analyze the case of $\partial_y Y|_{y_\pm} \neq 0$ which is expected that the dynamics of the scalar field will affect the predicted evolution of the universe. Since we have $\partial_y z \neq 0$, it leads $z' \neq 0$.

Let's discuss the y_- solution first, the stability region is below the line $\alpha^2 = 3\beta$ where $\alpha > 0$ in the left panel of Fig 1. To avoid the much effect due to the kinetic term of scalar field, it is found that choosing the parameters closed to the line $\alpha^2 = 3\beta$ does not affect the standard evolution too much. As shown in Fig. 3, we choose $\alpha = 2$ and $\beta = 1.32$ and then obtain the very small peak of z ($\sim 10^{-3}$).

For y_+ solution, the stability region is larger than one of y_- solution (see

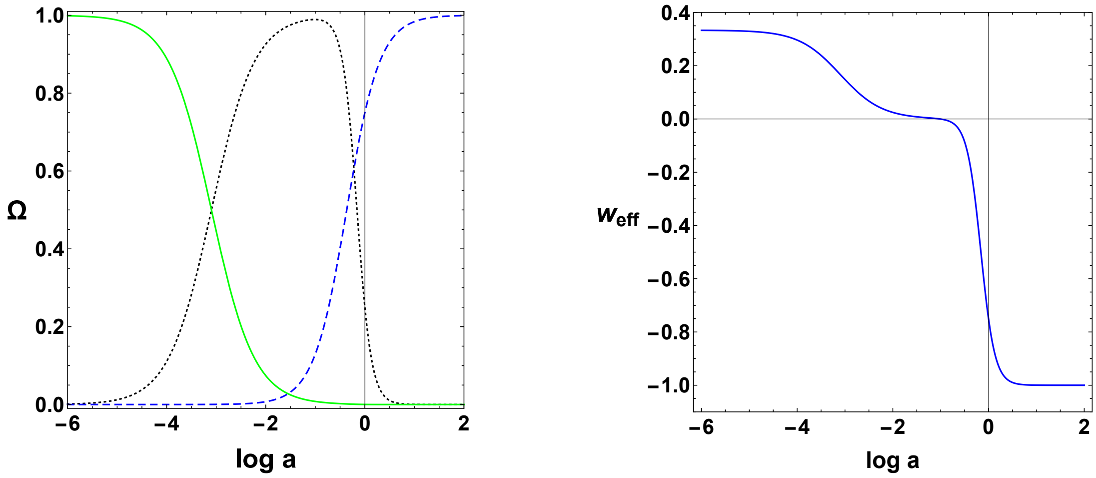

Figure 2 The left panel shows the evolution of the density parameters of Ω_g (dashed-blue line), $\Omega_{\rm m}$ (dotted-black line) and $\Omega_{\rm r}$ (solid-green line). The right panel shows the evolution of the $w_{\rm eff}$.

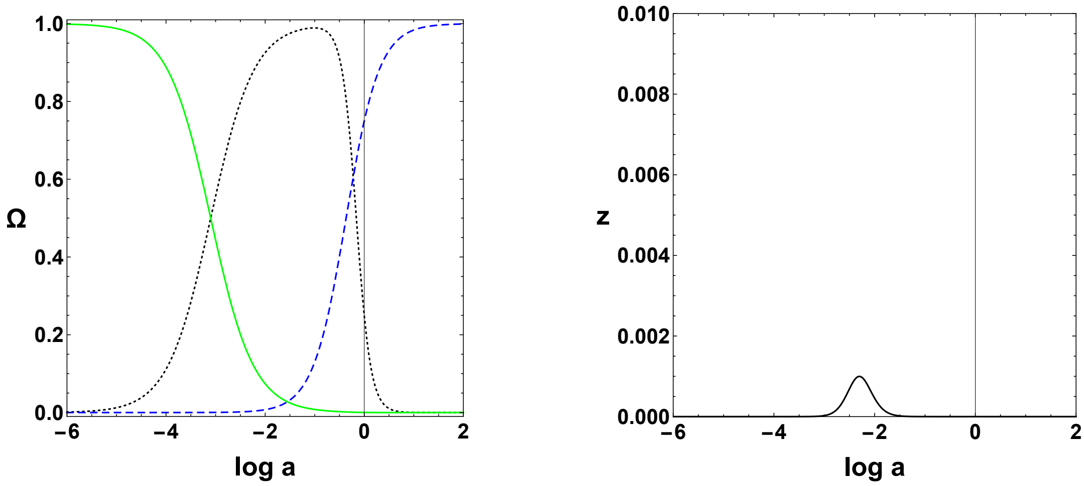

Figure 3 The left panel shows the evolution of the density parameters of Ω_g (dashed-blue line), $\Omega_{\rm m}$ (dotted-black line) and $\Omega_{\rm r}$ (solid-green line). The right panel shows the evolution of z.

Fig 1). We found that it is possible to obtain the standard evolution without significant effect from z, for example, choosing $\alpha = -2.0$ and $\beta = 0.5$. We do not show the evolutions of cosmic contents because they are similar to case of y_-.

Moreover, this analysis has another strong condition for prediction the evolution of this model in which the effect of z must be not seen at very early universe. If we break this condition, the Big Bang Nucleosynthesis will be affected. Fortunately, the parameters discussed above are also tuned in order to avoid the domination of z at early time.

5.4.2 Six-dimensional model

We move to consider the case of the model including the potential term of scalar field. The parameter γ introduced in (5.49) characterizes the effect of this potential term. Note that the parameter γ represents the curvature of extra space, it vanishes when we consider the flat extra dimensions (we have the same set of variables and parameters as one in the consideration of five dimensions in the previous subsection). The main purpose in this subsection is to study how the potential term affect the dynamics of the universe. It is found that the variable v does not evolve freely since it is constrained by (5.49). The effect of the potential of scalar field thus relates to the stability regions (v can be written in the term of y).

As we have known, there exists the special case of the solution which is the case $y_+ = y_-$ (or $3\beta = \alpha^2$ in form of parameters). It is convenient to study how the potential term affects the dynamics of the universe in this model. From the initial condition, $\Omega_{\rm m0} \sim 0.25, \Omega_{\rm r0} \sim 0$ and $\Omega_{g0} = -Ax/3 + v \sim 0.75$, we can determine the parameter s as

$$s^2 = \frac{6(1+\alpha)^2(1-\Omega_{\rm m0})}{\alpha(1 \pm \sqrt{1-36\alpha^2\gamma(1-\Omega_{\rm m0})})}. \tag{5.57}$$

For obtaining real value of s, we also have the condition for γ (also taking $m_g \sim H_0$) as

$$\gamma \leq \frac{1}{36\alpha^2(1-\Omega_{\rm m0})}. \tag{5.58}$$

We can notice that the parameter γ can be a very large negative while it is in the order of 10^{-2} where α is the order of unity. The stability regions for y_- solution substituting $\gamma = -0.1$ and $\gamma = 0.01$ are illustrated in Fig. 4. This figure shows that the stability regions for the negative γ is larger but the stability regions of the positive γ is less than ones of $\gamma = 0$. Moreover, it is found that the behavior of the stability regions for y_+ solution is similar to the case of y_- solution. Thus it is enough to analyze the case of y_- solution.

The different feature of six-dimensional consideration is that both graviton

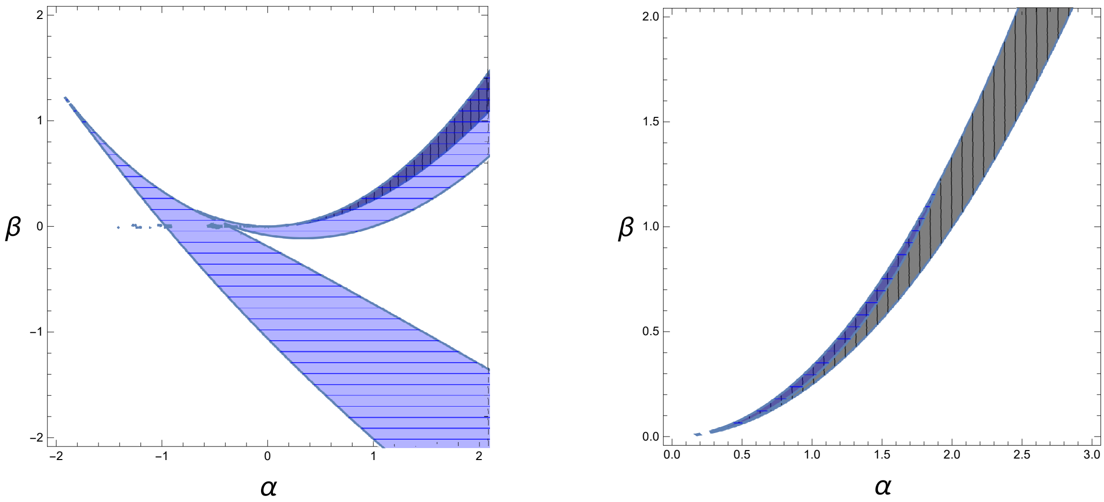

Figure 4 The left panel and the right panel show stability region for y_{-} solution with $\gamma = -0.1$ and 0.01 respectively. The horizontal-blue-shaded region corresponds to the stability region in six-dimensional model while the vertical-black shaded region corresponds to the stability region in five-dimensional model.

mass and scalar potential are contribution to drive the accelerated expansion while there is only the contribution from graviton mass driving this phenomenon in five dimensions. As shown in Fig. 5, this is the illustration of the cosmic contents including the contribution from scalar potential, v (dashed-red line). The evolution of Ω_{g} (dashed-blue line) is caused by contribution from the graviton mass and scalar potential (associated to extra-spatial curvature) in the six-dimensional model.

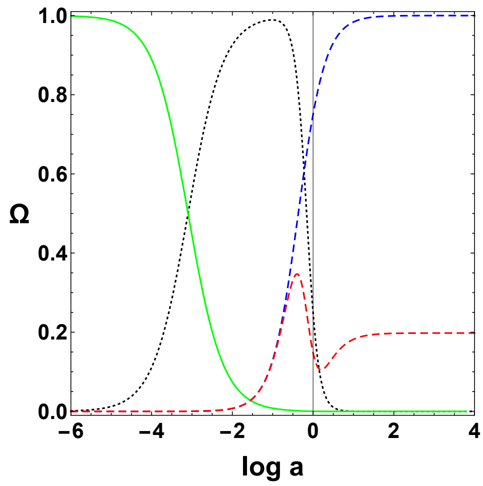

Figure 5 This figure shows evolution of the density parameters of Ω_g (dashed-blue line), v (dashed-red line), Ω_{m} (dotted-black line) and Ω_{r} (solid-green line) for y_{+} solution with $\gamma = -2.0, \alpha = -2, \beta = 0.5$.

Note that for the case $s = 1$, y will be a non-dynamical variable then the scalar field is just a constant in the dynamical system. Therefore, this case gives us the same prediction as GR with the cosmological constant.

We can conclude that the effective massive gravity theory with additional scalar field (5.20) is able to predict the standard evolution of the universe. Unfortunately, the strong coupling scale of the higher-dimensional dRGT theory (see in Appendix D) always larger than the radius of compactified extra dimensions. This means that the Kaluza-Klein mechanism can not be trusted. This notice becomes the main problem of the model.

CHAPTER VI

CONCLUSIONS

General Relativity (GR) is the description for gravitation nowadays. At large scale, it encounters the problem since it cannot predict the accelerated expansion of the universe without exotic matter. Many modifications of GR have been proposed in order to explain this phenomenon. One of them is a theory in which the graviton has non-zero mass while GR is the theory of massless spin-2 graviton. It is called the massive gravity theory. We are interested to investigate the model in massive gravity theory which can predict the dynamics of universe at late time.

The construction of the massive gravity theory started by introducing the Fierz-Pauli mass term into linearized GR. This linear massive gravity theory has the van Dam-Veltman-Zakharov discontinuity at the massless limit. This means that the linear massive theory cannot be properly reduced to the linearized GR by taking $m_g \to 0$. Vainshtein proposed a mechanism to eliminate the discontinuity by including nonlinear correction. The kinetic term for the nonlinear massive theory is the Einstein-Hilbert action. For the mass term, it is necessary to introduce the non-dynamical fiducial metric in order to construct the mass term. Introducing such a metric may be paid the price of losing the description for spin-2 field since the notion of spin following the Poincaré symmetry. The theory with other forms of fiducial metric besides the Minkowski form does not contain this symmetry [32]. We have seen that the nonlinear theory with various mass terms contains the scalar ghost degree of freedom called Boulware-Deser (BD) ghost. Until last eight years ago, the ghost-free nonlinear theory called de Rham-Gabadadze-Tolley (dRGT) massive gravity theory was proposed. The dRGT theory is able to eliminate the BD ghost and present the reasonable the regime of validity.

In the cosmological context, the dRGT theory with the Minkowski fiducial metric cannot predict the dynamics of the universe at late time, there is only the open Friedmann-Laîrmatre-Robertson-Walker (FLRW) solution is admitted. By using the flat FLRW fiducial metric, all kind of the FLRW solutions are obtained. However, there is only two propagating degrees of freedom while it should be five. In order to obtain the correct number of degrees of freedom, we choose to add the external degrees of freedom. The external degrees of freedom are obtained from the Kaluza-Klein dimensional reduction of the higher-dimensional dRGT theory. The extra dimensions are assumed to be compactified and maximally symmetric. By this ansatz, an external scalar field interpreted as the radius of extra dimensions is introduced. We obtain the resulting theory containing two features of the mass-varying and quasi-dilaton models so that it is a new kind of extension of dRGT theory.

We investigate the cosmological solutions by adopting flat FLRW form for both physical and fiducial metrics. The matter and radiation are also included in order to analyze the dynamics of cosmological contents more clearly. It is found that there exists a fixed point corresponding to the accelerated expansion which is the case $A = B$ (A and B are defined (5.37)) where A and B are associated with the energy density and (minus) pressure due to the term contributed from the graviton mass. It is found that the graviton mass is a constant at this fixed point. This is an advantage of this model comparing to the original mass-varying model where the graviton mass sink to zero at late time universe [46].

The cosmological model contains six dynamical variables, $x, y, z, v, \Omega_{\rm m}, \Omega_{\rm r}$ and four free parameters, α, β, s and γ. The dynamical variable y is defined in (5.37) and others are defined in (5.46). α, β are the appropriated parameters in the cosmological analysis, s is the ratio between the scale factors of physical and fiducial metrics defined in (5.37) and γ is the parameter characterized the effect of scalar potential defined in (5.49). The existence of the last parameter (or the variable v) associated the curvature of extra dimensions depends on the number of (higher) dimensions. Only in the consideration of five dimensions, there are three parameters (γ vanishes in this case). In this work, we thus consider only two cases of five and six dimensions because the six and higher than six dimensions are not different in the aspect of containing the extra-dimensional curvature parameter.

It is found that we can find the regions of model parameters α and β which correspond to the stable fixed point while s is set by the initial condition and γ is chosen by hand. Eventually, the standard evolution of the universe is obtained by choosing the appropriated parameters as discussed in Section 5.4. Moreover, the difference between five and six dimensions is that both graviton mass and scalar potential are the cosmic contents driving the accelerated expansion in six dimensions while only the graviton mass drives in five dimensions.

We found that the radius of extra dimension is always smaller than the strong coupling scale. This means that the geometry in the scale of compactification cannot be trusted. In other words, the radius of extra dimensions is out of the validity regime of the higher-dimensional dRGT theory. One of the possible ways to avoid this problem is considering the higher-dimensional dRGT theory in the braneworld scenario (see some of simple models in Chapter IV). The external field(s) introduced for solving the problem in dRGT theory may be interpreted from the behavior of 3-brane. After introducing the external field, it is important to check the propagating degrees of freedom as a further study. The higher-dimensional dRGT theory is not proven that it is free from the BD ghost or other ghosts.

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APPENDICES

APPENDIX A DIMENSIONAL DECONSTRUCTION FOR MASSIVE GRAVITY

As we have mentioned in Section 3.3.4, we start by considering the five-dimensional GR,

$$S^{(5)} = \frac{M_{(5)}^3}{2} \int \mathrm{d}^5X \sqrt{-\tilde{g}} \tilde{R}. \tag{A.1}$$

We will analyze this theory by decomposition into the four-ordinary spacetime part and one-extra spatial part. Therefore, the line element can be written as

$$\mathrm{d}s^2 = \tilde{g}_{AB} \mathrm{d}X^A \mathrm{d}X^B = g_{\mu\nu}(x, y) \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2. \tag{A.2}$$

This ansatz corresponds to the setting the lapse $N \equiv 1/\sqrt{g^{yy}}$ is unity and the shift $N_\mu \equiv g_{\mu y}$ vanishes. By applying the above ansatz, the action (A.1) becomes

$$S^{(5)} = \frac{M_{(5)}^3}{2} \int \mathrm{d}^4x \mathrm{d}y \sqrt{-g} \left(R + [K]^2 - [K^2] \right), \tag{A.3}$$

where $K_{\mu\nu} = \frac{1}{2} \partial_y g_{\mu\nu}$ is the extrinsic curvature along the extra dimension corresponding to our ansatz (A.2). Then we suppose that the extra space described by coordinate y is the two points y_1 and y_2. In other words, we make the continuous coordinate y to be the discrete one y_i. This process is called discretization. The metric at y_1 is chosen to be dynamical as the physical metric $g_{\mu\nu}(x, y_1) = g_{\mu\nu}(x)$ and the metric at y_2 be non-dynamical as the fiducial metric $g_{\mu\nu}(x, y_2) = f_{\mu\nu}(x)$. The external curvature, $K_{\mu\nu}$ can be evaluated from the subtraction between $g_{\mu\nu}$ and $f_{\mu\nu}$ since this curvature is defined by the derivative of the metric with respect to y. It is found that our discretizaton does not give us a structure of dRGT mass term which contains the square of the matrix $\mathbb{M}^\mu_\nu = g^{\mu\rho} f_{\rho\nu}$ (we have already discussed in Section 3.3). This structure is the feature for the ghost-free massive gravity theory. The discretization in the metric formalism always introduce the BD ghost into the theory [80]. Then we will move our consideration from the metric formalism to the vielbein one. Now we introduce the vielbein which satisfies the condition,

$$\tilde{g}_{AB} = \tilde{e}_A^{\hat{C}} \tilde{e}_B^{\hat{D}} \tilde{\eta}_{\hat{C}\hat{D}}, \tag{A.4}$$

where the indices with hat $\hat{A}, \hat{\mu}, \hat{a}, \ldots$ denote ones in the vielbein frame. The spin connection and Riemann curvature 2-form are defined as

$$\tilde{\omega}_A^{\hat{B}\hat{C}} = \frac{1}{2}\tilde{e}_A^{\hat{D}}\left(\tilde{O}^{\hat{B}\hat{C}}{}_{\hat{D}} - \tilde{O}_{\hat{D}}{}^{\hat{B}\hat{C}} - \tilde{O}^{\hat{B}}{}_{\hat{D}}{}^{\hat{C}}\right), \tag{A.5}$$

$$\tilde{\mathcal{R}}^{\hat{A}\hat{B}} = \tilde{d}\tilde{\omega}^{\hat{A}\hat{B}} + \tilde{\omega}^{\hat{A}}{}_{\hat{C}}\tilde{\omega}^{\hat{C}\hat{B}}, \tag{A.6}$$

where $\tilde{O}^{\hat{A}\hat{B}}{}_{\hat{E}} \equiv \tilde{e}^{\hat{A}C}\tilde{e}^{\hat{B}D}\left(\partial_C\tilde{e}_{D\hat{E}} - \partial_D\tilde{e}_{C\hat{E}}\right)$. $\tilde{d}$ is the directional exterior derivative in five dimensions (we also use d refering the four-dimensional one as we will see below). The torsion-free condition in the vielbein form is $\tilde{d}\tilde{e}^{\hat{A}} + \tilde{\omega}^{\hat{A}}{}_{\hat{B}} \wedge \tilde{e}^{\hat{B}} = 0$. The action (A.1) can be written in vielbein formalism as

$$S^{(5)}_{\mathrm{vielb}} = \frac{1}{3!}\frac{M^3_{(5)}}{2}\int \epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}}\tilde{\mathcal{R}}^{\hat{A}\hat{B}} \wedge \tilde{e}^{\hat{C}} \wedge \tilde{e}^{\hat{D}} \wedge \tilde{e}^{\hat{E}}, \tag{A.7}$$

where $\epsilon_{\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}}$ is the five-dimensional Levi-Civita tensor. The spactime can be decomposed into the ordinarily four-dimensional spacetime part and one-extra spatial part and then the ansatz which is equivalent to (A.2) can be written as

$$\tilde{e}^{\hat{\mu}} = e^{\hat{\mu}}, \qquad \tilde{e}^{\hat{y}} = \mathrm{d}y. \tag{A.8}$$

This ansatz also correspond to setting the lapse, N ($N \equiv 1/\sqrt{\tilde{g}^{yy}}$) is unity and the shift, N_μ ($N_\mu \equiv \tilde{g}_{\mu y}$) vanishes in the metric formalism. Each component of the spin connection becomes

$$\tilde{\omega}^{\hat{\mu}\hat{\nu}} = \left(\omega_\rho^{\hat{\mu}\hat{\nu}}\mathrm{d}x^\rho, \tilde{\omega}_y^{\hat{\mu}\hat{\nu}}\mathrm{d}y\right), \qquad \tilde{\omega}_y^{\hat{\mu}\hat{\nu}} = \frac{1}{2}\left(e^{\rho\hat{\mu}}\partial_y e_\rho^{\hat{\nu}} - e^{\rho\hat{\nu}}\partial_y e_\rho^{\hat{\mu}}\right), \tag{A.9}$$

$$\tilde{\omega}^{\hat{y}\hat{\mu}} = K^{\hat{\mu}} = \frac{1}{2}\left(e^{\rho\hat{\nu}}\partial_y e_\rho^{\hat{\mu}} + e^{\rho\hat{\mu}}\partial_y e_\rho^{\hat{\nu}}\right)e_{\sigma\hat{\nu}}\mathrm{d}x^\sigma, \tag{A.10}$$

$$\tilde{\omega}^{\hat{y}\hat{y}} = 0. \tag{A.11}$$

The tensor $K^{\hat{\mu}} = K_\nu^{\hat{\mu}}\mathrm{d}x^\nu$ corresponds to the above extrinsic curvature in the metric formalism, $K_{\mu\nu} = \frac{1}{2}\partial_y g_{\mu\nu}$ by

$$K_\nu^{\hat{\mu}} = e^{\rho\hat{\mu}}K_{\nu\rho}. \tag{A.12}$$

Without the ansatz (A.2) (the decomposition into the general form of lapse and shift), the extrinsic curvature in the metric formalism can be calculated as $K_{\mu\nu} = \frac{1}{2N}\left(\partial_y g_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu\right)$. We have seen that there are ten independent components for the metric in the ansatz (A.2), but, in the vielbein formalism, we have sixteen components. It is convenient to fix the gauge,

$$\tilde{\omega}_y^{\hat{\mu}\hat{\nu}} = e^{\rho[\hat{\mu}}\partial_y e_\rho^{\hat{\nu}]} = 0, \tag{A.13}$$

which eliminates additional six components. By applying our ansatz, each component of Riemann curvature 2-form is

$$\begin{aligned}
\tilde{\mathcal{R}}^{\hat{\mu}\hat{\nu}} &= \mathcal{R}^{\hat{\mu}\hat{\nu}} - K^{\hat{\mu}} \wedge K^{\hat{\nu}} - \partial_y \omega^{\hat{\mu}\hat{\nu}} \wedge \mathrm{d}y, & \text{(A.14)}\\
\tilde{\mathcal{R}}^{\hat{y}\hat{\mu}} &= dK^{\hat{\mu}} + \omega^{\hat{\mu}}{}_{\hat{\nu}} K^{\hat{\nu}} - \partial_y K^{\hat{\mu}} \wedge \mathrm{d}y, & \text{(A.15)}\\
\tilde{\mathcal{R}}^{\hat{y}\hat{y}} &= 0. & \text{(A.16)}
\end{aligned}$$

The action (A.7) is expressed as

$$\begin{aligned}
S_{\text{vielb}}^{(4)} &= \frac{1}{2}\frac{M_{(5)}^3}{2}\int \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \begin{pmatrix} \mathcal{R}^{\hat{\mu}\hat{\nu}} \wedge e^{\hat{\rho}} \wedge e^{\hat{\sigma}} - K^{\hat{\mu}} \wedge K^{\hat{\nu}} \wedge e^{\hat{\rho}} \wedge e^{\hat{\sigma}} \\ +2K^{\hat{\mu}} \wedge \partial_y e^{\hat{\nu}} \wedge e^{\hat{\rho}} \wedge e^{\hat{\sigma}} \end{pmatrix} \wedge \mathrm{d}y, \\
&= \frac{M_{(5)}^3}{4}\int \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \left(\mathcal{R}^{\hat{\mu}\hat{\nu}} \wedge e^{\hat{\rho}} \wedge e^{\hat{\sigma}} + \partial_y e^{\hat{\mu}} \wedge \partial_y e^{\hat{\nu}} \wedge e^{\hat{\rho}} \wedge e^{\hat{\sigma}}\right) \wedge \mathrm{d}y. \quad \text{(A.17)}
\end{aligned}$$

The second equality is obtained by using the gauge (A.13). It implies $K^{\hat{\mu}} = \partial_y e^{\hat{\mu}}$.

Now, we perform the discretization of the coordinate y by

$$y \to y_i. \tag{A.18}$$

Since the massive gravity theory contains two metrics ($g_{\mu\nu}$ and $f_{\mu\nu}$), we consider the case that i runs over 1 and 2 called the discretization with two sites. It is explicitly to construct a multi-gravity theory by extending to consider the case of many sites. The vielbein on both sites are defined as

$$e^{\hat{\mu}}(x, y_1) \to e^{(1)\,\hat{\mu}}(x), \qquad e^{\hat{\mu}}(x, y_2) \to e^{(2)\,\hat{\mu}}(x), \tag{A.19}$$

and their derivative with respect to the coordinate y become

$$\begin{aligned}
\partial_y e^{\hat{\mu}}(x, y_1) &\to m_g \left(e^{(2)\,\hat{\mu}}(x) - e^{(1)\,\hat{\mu}}(x)\right), & \text{(A.20)}\\
\partial_y e^{\hat{\mu}}(x, y_2) &\to m_g \left(e^{(1)\,\hat{\mu}}(x) - e^{(2)\,\hat{\mu}}(x)\right). & \text{(A.21)}
\end{aligned}$$

We also see that we choose the distance between two points y_1 and y_2 is $1/m_g$ and $e^{(3)\,\hat{\mu}}(x) = e^{(1)\,\hat{\mu}}(x)$ following the two sites consideration. By using this setting, (A.13) becomes

$$e^{(1)\,\rho[\hat{\mu}} e^{(2)}{}_{\rho}^{\hat{\nu}]} = 0. \tag{A.22}$$

To see more explicitly, looking at the extrinsic curvature in the metric formalism which are taken in the form of vielbein as

$$K_{\mu\nu} = \frac{1}{2}\partial_y g_{\mu\nu} = \frac{1}{2}\left(e^{\hat{\rho}}_{\mu}\partial_y e^{\hat{\sigma}}_{\nu} + \partial_y e^{\hat{\rho}}_{\mu} e^{\hat{\sigma}}_{\nu}\right)\eta_{\hat{\rho}\hat{\sigma}}. \tag{A.23}$$

This curvature can be discretized as

$$K_{\mu\nu} \to \mathcal{K}_{\mu\nu} = -m_g\left(g_{\mu\nu} - \frac{1}{2}(e^{(1)}{}_{\mu}^{\hat{\rho}} e^{(2)}{}_{\nu}^{\hat{\sigma}} + e^{(2)}{}_{\mu}^{\hat{\sigma}} e^{(1)}{}_{\nu}^{\hat{\rho}})\eta_{\hat{\rho}\hat{\sigma}}\right), \tag{A.24}$$

where $g_{\mu\nu} \equiv e^{(1)\,\hat{\rho}}_{\mu} e^{(1)\,\hat{\sigma}}_{\nu}\eta_{\hat{\rho}\hat{\sigma}}$. Then applying the gauge (A.22), we obtain

$$\mathcal{K}_{\mu\nu} = -m_g\left(g_{\mu\nu} - e^{(1)}{}_{\mu}^{\hat{\rho}} e^{(2)}{}_{\nu}^{\hat{\sigma}}\eta_{\hat{\rho}\hat{\sigma}}\right) = -m_g\left(g_{\mu\nu} - g_{\mu\rho} e^{(1)}{}^{\rho}_{\hat{\sigma}} e^{(2)}{}^{\hat{\sigma}}_{\nu}\right). \tag{A.25}$$

The fiducial metric also be defined as $f_{\mu\nu} \equiv e^{(2)}{}_{\mu}^{\hat{\rho}} e^{(2)}{}_{\nu}^{\hat{\sigma}}\eta_{\hat{\rho}\hat{\sigma}} = e^{(2)}{}_{\mu}^{\hat{\rho}} e^{(2)}{}_{\nu\hat{\rho}}$. As a result, we have relations

$$e^{(1)}{}^{\mu}_{\hat{\sigma}} e^{(2)}{}^{\hat{\sigma}}_{\nu} = (\sqrt{g^{-1}f}\,)^{\mu}_{\nu}, \tag{A.26}$$

or

$$\mathcal{K}^{\mu}_{\nu} = -m_g\left(\delta^{\nu}_{\nu} - (\sqrt{g^{-1}f}\,)^{\mu}_{\nu}\right). \tag{A.27}$$

We have seen that the structure of the square root is obtained from the discretization in vielbein formalism. Applying the above discretization, the action (A.17) becomes

$$S^{(4)}_{\text{discr, vielb}} = \frac{M^2_{\text{Pl}}}{4}\int \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\left[\begin{array}{l}\mathcal{R}^{\hat{\mu}\hat{\nu}} \wedge e^{\hat{\rho}} \wedge e^{\hat{\sigma}} \\ +m_g^2\left(\begin{array}{l}(e^{(2)\,\hat{\mu}} - e^{(1)\,\hat{\mu}}) \wedge (e^{(2)\,\hat{\nu}} - e^{(1)\,\hat{\nu}}) \\ \wedge e^{(1)\,\hat{\rho}} \wedge e^{(1)\,\hat{\sigma}}\end{array}\right)\end{array}\right], \tag{A.28}$$

$$S^{(4)}_{\text{discr, metric}} = \frac{M^2_{\text{Pl}}}{2}\int \mathrm{d}^4x\sqrt{-g}\left[R + m_g^2\left([\mathcal{K}]^2 - [\mathcal{K}^2]\right)\right], \tag{A.29}$$

where we defined the four-dimensional Planck mass as $M^2_{\text{Pl}} = M^3_{(5)}/m_g$. The volume of the extra dimension is $1/m_g$ ($\int \mathrm{d}y = 1/m_g$). The above results are just the simple form of nonlinear theory of massive gravity theory in vielbein and metric formalism respectively. Moreover, we can construct the more general form by choosing the more general discretization. We consider the one-site discretrization with the new discretizatied vielbein. These are chosen to be the average value with a weight r. The more general discretization for vielbein can be expressed as

$$e^{\hat{\mu}}(x, y_i) \to e^{(i+1)\,\hat{\mu}} - r\left(e^{(i+1)\,\hat{\mu}} - e^{(i)\,\hat{\mu}}\right). \tag{A.30}$$

Although we consider on one site e.g. $i = 1$, it is able to generate the vielbein corresponding to two metrics which are labeled by i and $i+1$. The discretized derivatives of vielbein with respect to y are fixed as the difference between two vielbein with the different labels,

$$\partial_y e^{\hat{\mu}} \to m_g \left(e^{(i+1)\,\hat{\mu}} - e^{(i)\,\hat{\mu}}\right). \tag{A.31}$$

Substituting $i = 1$, the action (A.17) with the more general discretization in vielbein formalism becomes

$$S^{(4)}_{\text{discr2, vielb}} = \frac{M_{\text{Pl}}^2}{4} \int \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \left[\begin{array}{l} \mathcal{R}^{\hat{\mu}\hat{\nu}} \wedge e^{\hat{\rho}} \wedge e^{\hat{\sigma}} \\ +m_g^2 \left(\begin{array}{l} (e^{(2)\,\hat{\mu}} - e^{(1)\,\hat{\mu}}) \wedge (e^{(2)\,\hat{\nu}} - e^{(1)\,\hat{\nu}}) \\ \wedge(e^{(2)\,\hat{\mu}} - r\left\{e^{(2)\,\hat{\mu}} - e^{(1)\,\hat{\mu}}\right\}) \\ \wedge(e^{(2)\,\hat{\mu}} - s\left\{e^{(2)\,\hat{\mu}} - e^{(1)\,\hat{\mu}}\right\}) \end{array} \right) \end{array} \right], \tag{A.32}$$

where s is another weight parameter. The above resulting theory thus contains two free parameters. This number of the free parameter equals to the number for the dRGT theory in Section 3.3. By transforming this action to the metric formalism, we obtain the familiar theory which is

$$S^{(4)}_{\text{discr2, metric}} = \frac{M_{\text{Pl}}^2}{2} \int \mathrm{d}^4x \sqrt{-g} \left[R + m_g^2 \left(\frac{1}{4}\mathcal{U}_2 + \frac{r+s}{2}\mathcal{U}_3 + \frac{rs}{2}\mathcal{U}_4 \right) \right], \tag{A.33}$$

where the potential $\mathcal{U}_i$ is the same as the definition (3.58). We also obtain the four-dimensional dRGT massive gravity theory (3.57) by redefining the parameters in the above action as $m_g \to 2m_g$, $2(r+s) \equiv \alpha_3$ and $2rs \equiv \alpha_4$. Moreover, it is clearly that we can construct the general multi-gravity theory by extension the number of sites.

APPENDIX B VARIATION OF $[\mathcal{K}^j]$

B.1 Detail of varying quantities with respect to $g^{\mu\nu}$

From $\mathbb{M}^\mu_\rho \mathbb{M}^\rho_\nu = g^{\mu\rho} f_{\rho\nu}$, we can write $f_{\rho\nu} = \mathbb{M}_{\rho\sigma}\mathbb{M}^\sigma_\nu$. Then varying of $\mathbb{M}^\mu_\nu$ are

$$\begin{aligned} \delta_g \mathbb{M}^\mu_\rho \mathbb{M}^\rho_\nu + \mathbb{M}^\mu_\rho \delta_g \mathbb{M}^\rho_\nu &= \delta g^{\mu\rho} f_{\rho\nu} \\ 2\delta_g \mathbb{M}^\mu_\rho \mathbb{M}^\rho_\nu &= \delta g^{\mu\rho} \mathbb{M}_{\rho\sigma}\mathbb{M}^\sigma_\nu, \end{aligned}$$

then

$$\delta_g \mathbb{M}^\mu_\alpha = \frac{1}{2}\delta g^{\mu\rho} g_{\rho\beta} \mathbb{M}^\beta_\alpha. \tag{B.1}$$

From $\mathcal{K}^\mu_\nu = \delta^\mu_\nu - p^{d/2}\mathbb{M}^\mu_\nu$, their varying with respect to $g^{\mu\nu}$ are

$$\delta_g \mathcal{K}^\mu_\nu = -p^{d/2}\delta_g \mathbb{M}^\mu_\nu = \frac{1}{2} g_{\rho\beta}(-p^{d/2}\mathbb{M}^\beta_\nu)\delta g^{\mu\rho} = \frac{1}{2}(\mathcal{K}_{\rho\nu} - g_{\rho\nu})\delta g^{\mu\rho}. \tag{B.2}$$

Finally, the varying of the trace are

$$\begin{aligned} \delta_g [K] &= \frac{1}{2}(K_{\mu\nu} - g_{\mu\nu})\delta g^{\mu\nu}, & \text{(B.3)} \\ \delta_g [K^j] &= \frac{j}{2}(K^j_{\mu\nu} - K^{j-1}_{\mu\nu})\delta g^{\mu\nu}, \qquad j > 1. & \text{(B.4)} \end{aligned}$$

B.2 Detail of varying quantities with respect to $f_{\mu\nu}$

From $\mathbb{M}^\mu_\rho \mathbb{M}^\rho_\nu = g^{\mu\rho} f_{\rho\nu}$, we can write $g^{\mu\sigma} = \mathbb{M}^\mu_\rho \mathbb{M}^\rho_\nu f^{\nu\sigma}$. Then varying $\mathbb{M}^\mu_\nu$ are

$$\begin{aligned} \delta_f \mathbb{M}^\mu_\rho \mathbb{M}^\rho_\nu + \mathbb{M}^\mu_\rho \delta_f \mathbb{M}^\rho_\nu &= g^{\mu\rho}\delta f_{\rho\nu}, \\ 2\mathbb{M}^\mu_\rho \delta_f \mathbb{M}^\rho_\nu &= \mathbb{M}^\mu_\alpha \mathbb{M}^\alpha_\sigma f^{\sigma\rho}\delta f_{\rho\nu}, \end{aligned}$$

then

$$\delta_f \mathbb{M}^\gamma_\nu = \frac{1}{2}\mathbb{M}^\gamma_\sigma f^{\sigma\rho}\delta f_{\rho\nu}. \tag{B.5}$$

From $\mathcal{K}^\mu_\nu = \delta^\mu_\nu - p^{d/2}\mathbb{M}^\mu_\nu$, their varying with respect to $f_{\mu\nu}$ are

$$\delta_f \mathcal{K}^\mu_\nu = -p^{d/2}\delta_f \mathbb{M}^\mu_\nu = \frac{1}{2}(-p^{d/2}\mathbb{M}^\mu_\sigma) f^{\sigma\rho}\delta f_{\rho\nu} = \frac{1}{2}(\mathcal{K}^\mu_\sigma - \delta^\mu_\sigma) f^{\sigma\rho}\delta f_{\rho\nu}. \tag{B.6}$$

Finally, the varying of the trace are

$$\delta_f[\mathcal{K}] = \frac{1}{2}(\mathcal{K}^\nu_\sigma - \delta^\nu_\sigma) f^{\sigma\mu}\delta f_{\mu\nu}, \tag{B.7}$$

$$\delta_f[\mathcal{K}^j] = \frac{j}{2}(\mathcal{K}^{j\,\nu}{}_\sigma - \mathcal{K}^{j-1\,\nu}{}_\sigma) f^{\sigma\mu}\delta f_{\mu\nu}, \qquad j > 1. \tag{B.8}$$

B.3 Detail of varying quantities with respect to ϕ

From $\mathcal{K}^\mu_\nu = \delta^\mu_\nu - p^{d/2}\mathbb{M}^\mu_\nu$, their varying with respect to ϕ are

$$\delta_\phi \mathcal{K}^\mu_\nu = \frac{d}{2}\frac{p_{,\phi}}{p}(-p^d \mathbb{M}^\mu_\nu) = \frac{1}{M_{\rm Pl}}\sqrt{\frac{d}{2(d+2)}}(\mathcal{K}^\mu_\nu - \delta^\mu_\nu)\delta\phi. \tag{B.9}$$

Finally, the varying of the trace are

$$\delta_\phi[\mathcal{K}] = \frac{1}{M_{\rm Pl}}\sqrt{\frac{d}{2(d+2)}}([\mathcal{K}] - 4)\delta\phi, \tag{B.10}$$

$$\delta_\phi[\mathcal{K}^j] = \frac{1}{M_{\rm Pl}}\sqrt{\frac{d}{2(d+2)}}([\mathcal{K}^j] - [\mathcal{K}^{j-1}])\delta\phi, \qquad j > 1. \tag{B.11}$$

APPENDIX C DIMENSIONAL REDUCTION AT THE FIELD EQUATIONS

We choose to show in the unitary gauge, $\psi^{\bar{\mu}} = x^{\bar{\mu}}$. To derive the field equations (5.22) and (5.28) in another way, we firstly find the higher dimensional field equations by varying the action (5.1) with respect to $\tilde{g}^{AB}$ which are

$$\tilde{G}_{AB} + m_g^2 \left(\tilde{U}_{AB} - \frac{1}{2}\tilde{g}_{AB}\tilde{\mathcal{U}} \right) = 0, \tag{C.1}$$

where

$$\tilde{U}_{AB} = \frac{\delta \tilde{\mathcal{U}}}{\delta \tilde{g}^{AB}}. \tag{C.2}$$

The constraints which are obtained by varying the same action with respect to f_{AB} are

$$\frac{\delta \tilde{\mathcal{U}}}{\delta \tilde{f}_{AB}} = 0, \tag{C.3}$$

Since we have the relations (see Appendix B),

$$\frac{\delta[\tilde{\mathcal{K}}^l]}{\delta \tilde{f}_{CB}} = \frac{\delta[\tilde{\mathcal{K}}^l]}{\delta \tilde{g}^{AB}}\, \tilde{g}^{CA}\, \tilde{f}^{BD}, \qquad \text{for } l = 1, 2, \ldots. \tag{C.4}$$

The constraints (C.3) become

$$\frac{\delta \tilde{\mathcal{U}}}{\delta \tilde{f}^{AB}} = \frac{\delta \tilde{\mathcal{U}}}{\delta \tilde{g}^{CD}}\, \tilde{g}^{AC}\, \tilde{f}^{DB} = \tilde{U}_{CD}\tilde{g}^{AC}\, \tilde{f}^{DB} = 0. \tag{C.5}$$

For non-vanishing physical and fiducial metrics, we obtain

$$\tilde{U}_{AB} = 0. \tag{C.6}$$

They also imply one more constraint which is obtained by taking the trace,

$$\tilde{U}^A_A = \tilde{U} = \tilde{\mathcal{U}} + \tilde{W} = 0, \tag{C.7}$$

where

$$\begin{aligned}
\tilde{W} &= \tilde{W}_2+\alpha_3\tilde{W}_3+\alpha_4\tilde{W}_4+\alpha_5\tilde{W}_5+\alpha_6\tilde{W}_6+\dots,\\
\tilde{W}_2 &= -(d+3)[\tilde{\mathcal{K}}],\\
\tilde{W}_3 &= \frac{1}{2}\left\{-3(d+2)[\tilde{\mathcal{K}}]^2+3(d+2)[\tilde{\mathcal{K}}^2]+[\tilde{\mathcal{K}}]^3-3[\tilde{\mathcal{K}}][\tilde{\mathcal{K}}^2]+2[\tilde{\mathcal{K}}^3]\right\},\\
\tilde{W}_4 &= -2(d+1)[\tilde{\mathcal{K}}]^3+[\tilde{\mathcal{K}}]\left\{6(d+1)[\tilde{\mathcal{K}}^2]+8[\tilde{\mathcal{K}}^3]\right\}-4d[\tilde{\mathcal{K}}^3]+[\tilde{\mathcal{K}}]^4\\
&\quad -6[\tilde{\mathcal{K}}]^2[\tilde{\mathcal{K}}^2]+3[\tilde{\mathcal{K}}^2]^2-4[\tilde{\mathcal{K}}^3]-6[\tilde{\mathcal{K}}^4],\\
\tilde{W}_5 &= \frac{1}{2}\left[\begin{array}{l} 3\left\{\begin{array}{l}[\tilde{\mathcal{K}}]^5-10[\tilde{\mathcal{K}}]^3[\tilde{\mathcal{K}}^2]+20[\tilde{\mathcal{K}}]^2[\tilde{\mathcal{K}}^3]+15[\tilde{\mathcal{K}}]([\tilde{\mathcal{K}}^2]^2-2[\tilde{\mathcal{K}}^4])\\ -20[\tilde{\mathcal{K}}^2][\tilde{\mathcal{K}}^3]+24[\tilde{\mathcal{K}}^5]\end{array}\right\}\\ -5d([\tilde{\mathcal{K}}]^4-6[\tilde{\mathcal{K}}]^2[\tilde{\mathcal{K}}^2]+8[\tilde{\mathcal{K}}][\tilde{\mathcal{K}}^3]+3[\tilde{\mathcal{K}}^2]^2-6[\tilde{\mathcal{K}}^4])\end{array}\right],\\
\tilde{W}_6 &= -3(d-1)[\tilde{\mathcal{K}}]^5+10[\tilde{\mathcal{K}}]^3\left\{3(d-1)[\tilde{\mathcal{K}}^2]+8[\tilde{\mathcal{K}}^3]\right\}\\
&\quad +30[\tilde{\mathcal{K}}]^2(-2d[\tilde{\mathcal{K}}^3]+3[\tilde{\mathcal{K}}^2]^2+2[\tilde{\mathcal{K}}^3]-6[\tilde{\mathcal{K}}^4])\\
&\quad -3[\tilde{\mathcal{K}}]\left(15(d-1)[\tilde{\mathcal{K}}^2]^2-6\left\{5(d-1)[\tilde{\mathcal{K}}^4]+16[\tilde{\mathcal{K}}^5]\right\}+80[\tilde{\mathcal{K}}^2][\tilde{\mathcal{K}}^3]\right)\\
&\quad -30[\tilde{\mathcal{K}}]^4[\tilde{\mathcal{K}}^2]-30[\tilde{\mathcal{K}}^2]^3+60[\tilde{\mathcal{K}}^2]((d-1)[\tilde{\mathcal{K}}^3]+3[\tilde{\mathcal{K}}^4])\\
&\quad +8(-9d[\tilde{\mathcal{K}}^5]+10[\tilde{\mathcal{K}}^3]^2+9[\tilde{\mathcal{K}}^5]-30[\tilde{\mathcal{K}}^6])+2[\tilde{\mathcal{K}}]^6,\\
&\vdots\quad .
\end{aligned} \tag{C.8}$$

Applying the constraints (C.6) and (C.7) to the field equations (C.1), so the field equations can be simplified as

$$\tilde{G}_{AB}+\frac{1}{2}m_g^2\tilde{g}_{AB}\tilde{W}=0. \tag{C.9}$$

The next step is applying the ansatz of the metrics (5.4) and (5.5) to these field equations, we can write the higher-dimensional quantities in the form of the four-dimensional one. The non-zero components of the Einstein tensor $\tilde{G}_{AB}$ are

$$\tilde{G}_{\mu\nu} = G_{\mu\nu}-\frac{p^{-(d+2)}}{2}g_{\mu\nu}R[\gamma]+\frac{1}{2M_{\rm Pl}^2}g_{\mu\nu}\left(\nabla\phi\right)^2-\frac{1}{M_{\rm Pl}^2}\nabla_\mu\phi\nabla_\nu\phi, \tag{C.10}$$

$$\tilde{G}_{ab} = G_{ab}+p^{d+2}\gamma_{ab}\left(-\frac{1}{2}R[g]-\frac{1}{M_{\rm Pl}}\sqrt{\frac{d+2}{2d}}\nabla^2\phi+\frac{1}{2M_{\rm Pl}^2}\left(\nabla\phi\right)^2\right). \tag{C.11}$$

As a result, the matrix $\tilde{\mathbb{M}}^A_B$ also be expressed as

$$\tilde{\mathbb{M}}^A_B=\begin{pmatrix} p^{d/2}(\sqrt{g^{-1}f}\,)^\mu_\nu & 0\\ 0 & q/p\,\delta^a_b\end{pmatrix}. \tag{C.12}$$

Then the matrix $\tilde{\mathcal{K}}^{l\,A}_{\ \ B}$ and their trace are

$$\tilde{\mathcal{K}}^{l\,A}_{\ \ B}=\begin{pmatrix} \mathcal{K}^{l\,\mu}_{\ \ \nu} & 0\\ 0 & r^l\delta^a_b\end{pmatrix},\qquad [\tilde{K}^l]=[\mathcal{K}^l]+d\,r^l,\qquad \text{for } l=1,2,\dots, \tag{C.13}$$

where

$$\mathcal{K}^{\mu}_{\nu} = \delta^{\mu}_{\nu} - p^{d/2}(\sqrt{g^{-1}f}\,)^{\mu}_{\nu}. \tag{C.14}$$

Let's consider each component of (C.9). Firstly, the $(\mu\nu)$-components are

$$\begin{aligned} G_{\mu\nu} - \frac{p^{-(d+2)}}{2} g_{\mu\nu} R[\gamma] + \frac{1}{2M_{\rm Pl}^2}\left(\nabla\phi\right)^2 - \frac{1}{M_{\rm Pl}^2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}M_g^2 g_{\mu\nu}\tilde{W} = 0, \\ G_{\mu\nu} + \frac{1}{M_{\rm Pl}^2} g_{\mu\nu} V + \frac{1}{2M_{\rm Pl}^2} g_{\mu\nu}(\nabla\phi)^2 - \frac{1}{M_{\rm Pl}^2}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}M_g^2 g_{\mu\nu}\tilde{W} = 0. \end{aligned} \tag{C.15}$$

Their trace gives us

$$R[g] = -2p^{-(d+2)}R[\gamma] + \frac{1}{M_{\rm Pl}^2}\left(\nabla\phi\right)^2 + 2M_g^2\tilde{W}. \tag{C.16}$$

The other non-vanishing part of (C.9) are the (ab)-components,

$$G_{ab} + p^{d+2}\gamma_{ab}\left\{-\frac{1}{2}R[g] - \frac{1}{M_{\rm Pl}}\sqrt{\frac{d+2}{2d}}\nabla^2\phi + \frac{1}{2M_{\rm Pl}^2}\left(\nabla\phi\right)^2\right\} + \frac{1}{2}m_g^2 p^2\gamma_{ab}\tilde{W} = 0. \tag{C.17}$$

Substituting $\tilde{G}_{ab} = -\left(\frac{d-2}{2d}\right)\gamma_{ab}R[\gamma]$ (from (5.12)) and $R[g]$ with (C.16),

$$\left[\frac{d+2}{2d}R[\gamma] + p^{d+2}\left\{-\frac{1}{M_{\rm Pl}}\sqrt{\frac{d+2}{2d}}\nabla^2\phi - \frac{1}{2}M_g^2\tilde{W}\right\}\right]\gamma_{ab} = 0. \tag{C.18}$$

Finally, we obtain

$$\begin{aligned} \frac{1}{M_{\rm Pl}}\nabla^2\phi - \sqrt{\frac{d+2}{2d}}p^{-(d+2)}R[\gamma] + \sqrt{\frac{d}{2(d+2)}}M_g^2\tilde{W} &= 0, \\ \nabla^2\phi - V_{,\phi} + \sqrt{\frac{d}{2(d+2)}}M_{\rm Pl}M_g^2\tilde{W} &= 0. \end{aligned} \tag{C.19}$$

These are two field equations (C.15) and (C.19) which are the same as the field equations for $g^{\mu\nu}$ in (5.22) and for ϕ in (5.28) respectively in Section 5.2 since it is found that $\tilde{W} = \Phi + d\,r\Psi$ and $\frac{1}{2}g_{\mu\nu}\tilde{W} = X_{\mu\nu} + d\,rY_{\mu\nu}$.

APPENDIX D STRONG COUPLING SCALE FOR THE HIGHER-DIMENSIONAL dRGT THEORY

To find the strong coupling scale for the n-dimensional massive gravity theory, we will consider the interaction terms which are possible to emerge in the quantum regime as

$$\mathcal{L}_{\rm int}^{(n)} = m_g^2 M_{(n)}^{n-2} (h)^{n_h} (\partial\chi)^{n_\chi} (\partial^2\pi)^{n_\pi}, \tag{D.1}$$

where n_h, n_χ and n_π are the power of h_μ, χ_μ and π respectively. The normalized fields are

$$h'_{\mu\nu} = M_{(n)}^{(n-2)/2} h_{\mu\nu}, \qquad \chi'_\mu = m_g M_{(n)}^{(n-2)/2} \chi_\mu, \qquad \pi' = m_g^2 M_{(n)}^{(n-2)/2} \pi. \tag{D.2}$$

The interaction becomes

$$\begin{aligned} \mathcal{L}_{\rm int}^{(n)} &= m_g^2 M_{(n)}^{n-2} (h)^{n_h} (\partial\chi)^{n_\chi} (\partial^2\pi)^{n_\pi}, \\ &= (\Lambda_\lambda^{(n)})^{n-n_h-2n_\chi-3n_\pi} (h')^{n_h} (\partial\chi')^{n_\chi} (\partial^2\pi')^{n_\pi}, \end{aligned} \tag{D.3}$$

where the strong coupling scale is

$$\Lambda_\lambda^{(n)} = \left(M_{(n)} m_g^{\lambda-1}\right)^{1/\lambda}, \qquad \lambda^{(n)} = \frac{2n-(n-2)n_h-(n)n_\chi-(n+2)n_\pi}{(n-2)(2-n_h-n_\chi-n_\pi)}, \tag{D.4}$$

for $n_h + n_\chi + n_\pi > 2$. From the KK reduction, we have the relation between two mass scale which is

$$M_{\rm Pl}^2 = V_{(n-4)} M_{(n)}^{n-2} \propto r^{n-4} M_{(n)}^{n-2} \rightarrow \; M_{(n)} \sim \left(\frac{M_{\rm Pl}^2}{r^{n-4}}\right)^{1/(n-2)} \tag{D.5}$$

where $V_{(n-4)}$ and r are the volume and radius of the extra dimensional space respectively. We get the strong coupling scale in the unit of length as

$$(\Lambda_\lambda^{(n)})^{-1} = \left(\frac{r^{(n-4)/(n-2)}}{M_{\rm Pl}^{2/(n-2)} m_g^{\lambda-1}}\right)^{1/\lambda}. \tag{D.6}$$

Consider the case of $\lambda = 3$, we get

$$(\Lambda_3^{(n)})^{-1} = \left(\frac{r^{(n-4)/(n-2)}}{M_{\rm Pl}^{2/(n-2)} m_g^{2}}\right)^{1/3}. \tag{D.7}$$

By substituting $m_g \sim 10^{-33}$ eV and $M_{\rm Pl} \sim 10^{29}$ eV, we can conclude that the radius of $(n-4)$-dimensional compacted extra space is always less than the strong coupling scale for the arbitrary n-dimensional dRGT massive gravity theory, $r < (\Lambda_3^{(n)})^{-1} = r^{\frac{n-4}{3(n-2)}} 10^{\frac{24}{2-n}+17.3}$. We also see that the idea of using the KK reduction in this work cannot be trusted because the size of the extra dimensions is out of the regime of validity for the higher-dimensional dRGT theory. However, the effective massive gravity theory with the scalar field (5.20) is somehow possible to propose as an extension of the original dRGT theory (3.57).

BIOGRAPHY

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