PRESENT ACCELERATION OF THE UNIVERSE FROM DARK COSMIC INGREDIENTS

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ABSTRACT

Two Palatini versions of modified gravity theories, EiBI and NMDC-Palatini, are studied in this work. In the scenario of NMDC-Palatini, the derivative of scalar field of non-minimal coupling with Palatini Einstein tensor is studied in details with allowance only a single NMDC coupling constant. Conformal factor is found to be a function of time derivative of scalar field in FLRW universe. NMDC coupling constant is limited in a range of $-2/\dot{\phi}^2 < \kappa \leq \infty$ to preserve Lorentz signature of the conformal metric. The coupling constant is allowed to take large value in the slow-roll regime. We have derived cosmological field equations and considered the equations in the slow-roll regime in which the acceleration condition is modified to $w_{\text{eff}} \simeq -(1/3)(1 + 2\kappa\dot{\phi}^2)$, resulting that the acceleration could occur even w_{eff} is less than $-1/3$. Effective gravitational coupling strength and modification of the entropy of blackhole's apparent horizon and inflationary stage of this theory are also investigated.

The stability of three fixed points at late time evolution of EiBI cosmology are completely investigated by three different methods since the linear stability method is insufficient to indicate the (in)stability two fixed points which have zero eigenvalue. With helping of Kosambi-Cartan-Chern (KCC) theory and Lyapunov functions, the prediction of the (in)stability of the leftover ones are possible. The dark matter dominated is an unstable fixed point by the linear stability method. Specifying the stability with the KCC method ,the new discovering fixed point so-called ΛEiBI, i.e. $(0, \frac{1}{2}, \frac{1}{2})$, is indeed an unstable point. The vacuum dominated is confirmed to be an unstable fixed point by Lyapunov functions method.

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Conventions, Units and Notations

In this work, we use the mostly plus metric signature, i.e. (- ,+,+,+). ∇_σ is the covariant derivative defined with the Levi-Civita connection $\{^\lambda{}_{\mu\nu}\}$, while ∇^Γ_σ denotes the covariant derivative defined the independent connection $\Gamma^\lambda{}_{\mu\nu}$.

Units with $c = \hbar = 1$

[Length] = [time] = [L]

[mass] = [energy] = [M]

$[\mathrm{energydensity} : \rho] = [\mathrm{pressure : p}] = \frac{\mathrm{M}}{\mathrm{L}^3} = \frac{1}{[\mathrm{L}^4]}$

$L = T = M^{-1} = E^{-1}$

$[g_{\mu\nu}] = [\mathrm{dimensionless}]$

$[\Gamma^\mu_{\nu\kappa}] = [L^{-1}]$

$[R^\mu{}_{\nu\kappa\sigma}] = [R_{\mu\nu}] = [R] = [L^{-2}]$

$[\Lambda] = [L^{-2}]$

Units with $c = k_\mathrm{B} = \hbar = 1$ and $[G_\mathrm{N}] = [M_\mathrm{pl}^{-2}]$

$\ell_\mathrm{pl} = \frac{1}{M_\mathrm{pl}}$

$[\kappa^2] = [8\pi G_\mathrm{N}] = [\frac{8\pi}{m^2_\mathrm{pl}}] = [M_\mathrm{pl}^{-2}] = [\ell^2_\mathrm{pl}]$

The Planckian units or natural units $G_\mathrm{N} = \hbar = c = k_\mathrm{B} = 1$

Planck length: $\ell_\mathrm{pl} \equiv (\frac{G_\mathrm{N}\hbar}{c^3})^{1/2} = 1.616 \times 10^{-33}$ cm,

Planck time: $t_\mathrm{pl} \equiv \frac{\ell_\mathrm{pl}}{c} = 5.391 \times 10^{-44}$ s,

Planck mass: $m_\mathrm{pl} \equiv (\frac{\hbar c}{G_\mathrm{N}})^{1/2} = 2.177 \times 10^{-5}$ g,

Planck temperature: $T_\mathrm{pl} \equiv \frac{m_\mathrm{pl} c^2}{k_\mathrm{B}} = 1.416 \times 10^{32}$ K,

Planck energy: $E_\mathrm{pl} \equiv \sqrt{\frac{\hbar c^5}{G_\mathrm{N}}} = 1.221 \times 10^{19}$ GeV.

Notations

G_N : Newton's gravitational constant ($G_\mathrm{N} = 6.67 \times 10^{-8} \mathrm{cm}^3 \mathrm{g}^{-1} \mathrm{sec}^{-2}$)

G_eff : Effective gravitational coupling strength

M_pl : Reduced Planck mass ($M_\mathrm{pl} = \frac{1}{\sqrt{8\pi G_\mathrm{N}}} = 2.4357 \times 10^{18}\,\mathrm{GeV}$)

a : Scale factor of the universe (Traditionally, we use $a_0 = 1$ at present day)

t : The cosmic time

N : Number of e-foldings, i.e. $N = \ln a$

($\cdot$) : Derivative with respect to t

($'$) : Derivative with respect to $N = \ln a$

H : Hubble parameter, i.e. $H = \frac{\dot{a}}{a}$

ρ : Energy density

p : Pressure

w : Equation of state parameter (EoS), i.e. $w = \frac{p}{\rho}$

$w_{\rm eff}$: Effective equation of state parameter

Λ : The cosmological constant or the curverture associated to vacuum

$\rho_\Lambda = \frac{\Lambda}{8\pi G_{\rm N}}$: Energy density of vacuum energy

$\rho_{\rm c} \equiv 3H^2/8\pi G_{\rm N}$: The critical density

R : Ricci scalar

Ω : Density parameter

$\Omega_\Lambda = \frac{\Lambda}{3H^2}$: Density parameter of the vacuum energy

k : The spatially intrinsic curvature of the space geometry(the curvature of the universe, the spatial curvature constant), i.e. $k = -1, 0, 1$ are correspond to an open, flat and closed universe respectively.

$\Omega_k = \frac{-k}{3a^2H^2}$: Density parameter of curvature of the space geometry

$\rho_k = -\frac{3k}{8\pi G_{\rm N}}\frac{1}{a^2}$: Energy density of the curvature of space geometry

EoM : Equation of motion

GR : General Relativity or Einstein's General Relativity Theory

EiBI : Eddington inspired Born-Infeld

EBI : Eddington Born-Infeld

T : Temperature

S : Action

$S_{\rm g}$: Gravitational action

S_{m}: Matter action

Ψ : The collecting of matter fields

$g_{\mu\nu}$: Metric tensor

$G_{\mu\nu}$: Einstein tensor in metric formalism

$G_{\mu\nu}(\Gamma)$: Einstein tensor in Palatini formalism

$T_{\mu\nu}$: Energy- momentum tensor

$\tilde{T}_{\mu\nu}$: Energy- momentum tensor in conformal frame

$T^{\mathrm{vac}}_{\mu\nu} = \frac{\Lambda}{8\pi G_{\mathrm{N}}} g_{\mu\nu} = \frac{\Lambda}{\kappa^2} g_{\mu\nu}$: Energy-momentum tensor for vacuum energy

ϕ : Scalar field

$V(\phi)$: Scalar-field potential

$\mathcal{L} = \sqrt{-g}L$: Lagrange density

$\sqrt{-g} = \sqrt{|g_{\mu\nu}|}$: Square root of the absolute determinant of metric tensor

(x_c, y_c) : critical point or fixed point

Dimensions analysis

$[\rho] = [\frac{\mathrm{energy}}{\mathrm{volume}}] = \frac{[L^{-1}]}{[L^3]} = [L^{-4}] = [M^4]$

$[\rho_\Lambda] = \frac{[\Lambda]}{[8\pi G_{\mathrm{N}}]} = [M^4_{\mathrm{Pl}}](10^{18}\mathrm{GeV})^4 \sim 10^{112}$ erg $\cdot$ cm^{-3}

$\kappa^2\rho \equiv \kappa^2(\rho_{\mathrm{m}} + \rho_\Lambda) = \kappa^2\rho_{\mathrm{m}} + \frac{\kappa^2\Lambda}{\kappa^2} = \kappa^2\rho_{\mathrm{m}} + \Lambda$

$S_{\mathrm{AH}} \equiv \frac{k_{\mathrm{B}}c^3}{\hbar G}\frac{A_{\mathrm{AH}}}{4}$: Entropy-area law

$S_{\mathrm{AH}} \equiv \frac{A_{\mathrm{AH}}}{4G_{\mathrm{N}}}$: Entropy-area law is expressed in natural unit

For example: The scalar field Lagrange density $\mathcal{L} = -\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$,

$[V(\phi)] = M^4, [\phi] = M^1, [\partial_\mu] = [\frac{\partial}{\partial X^\mu}] = M, [\partial_\mu\phi] = M^2$

Operations

$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha$

$\nabla_\mu B^\nu = \partial_\mu B^\nu + \Gamma^\alpha_{\mu\alpha} B_\nu$

$R_{\alpha\beta\gamma\delta} \equiv g_{\alpha\rho} R^\rho{}_{\beta\gamma\delta}$

$R_{\mu\nu} \equiv g^{\alpha\gamma} R_{\alpha\mu\gamma\nu} \equiv R^\gamma{}_{\mu\gamma\nu}$

CHAPTER I

INTRODUCTION

"Einstein would be one of the theoretical physicists of all times even if he had not written a single on relativity." *Max Born*

1.1 Background and motivation

The current data implies that the accelerated universe may be caused by the effect of dark energy whereas the successful formation of large scale structures of the universe has to include the outcome of the non-relativistic dark matter [1, 2, 3]. One of the reliable models of describing universe at present is ΛCDM, this model, however, faces with several problems, e.g. fine-tuning problem, coincidence problem (see Section 3.6). This includes the old problem (before discovering that the universe is in an accelerating phase) of an inconsistency of the cosmological constant (the vacuum energy density) in quantum field theory point of view and precise cosmological observation [4, 5]. In addition, it appears an existence of singularity in classical GR, e.g. black hole singularity and the cosmological singularity or the Big Bang. It is suggested that the Einstein-Hilbert action may be the low-energy approximation of Grand Unified theory whereas the higher order scalar terms, e.g. R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}$ should be included in the gravitational action to explain the high energy phenomenon even if described infrared behaviours of spacetime [6, 7]. Modified gravity theories beyond classical GR are expected to get rid of those concerning problems mentioned before. Of course we cannot get to the scale factor $a(t)$ for the entire evolution of the universe, but we can get the present day scale factor represented by $a(t_0)$ and its time derivative $\dot{a}(t_0)$ which is encoded in the Hubble parameter $H(t_0)$ and an effective EoS parameter $w_0 \simeq -1$ is also consistent with present observation [8]. If $w < -1$ this will violate the null energy condition by phantom matter or exotic matter that leads to a big rip. However, the observation today is not good enough to clearly distinguish around this value and may include the possibility among three cases of space geometry $k = \pm 1, 0$ [8]. Additionally, the present equation of state parameter (EoS) does not tell anything about the changing ratio of dark energy and dark matter for the near future [8].

Motivated by the Palatini formalism, the metric tensor and the connection are independent objects on the same manifold. It is believed that connection field, i.e. $\Gamma^{\lambda}_{\mu\nu}$, be leftover from the symmetry breaking of the maximal symmetry as the result of the first order phase transitions which took place at very high temperature near the Big bang [9, 10]. The dynamical metric tensor $g_{\mu\nu}$ entered simultaneously at the beginning of the cosmic time (Weyl hypothesis) of the universe. After the creation of the metric field, it is expected that the coupling of two independent field $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ will stimulate a new dynamical field $q_{\mu\nu}$ which is mathematically expressed as $q_{\mu\nu} = g_{\mu\nu} + R_{\mu\nu}(\Gamma)$ (see section 5.1-5.3 for more details).

Bañados examined the theoretical structure of the pre-metric and the left-over connection field in [11]. In this theory, so long as the metric disappear and the connection fields does not vanish, the Einstein's field equation can be written as

$$G_{\mu\nu}[g_{\mu\nu}(x) = 0, \Gamma_0(x)] = \lambda(x) R_{\mu\nu}\left[\Gamma_0(x)\right], \tag{1.1}$$

where $\lambda(x)$ is a dimensionless parameter depending on the spacetime trajectories.

Bañados also suggested Vienbein formulation to explain the pre-metric formalism. The Vienbian $e^I = e^I{}_{\mu} dx^{\mu}$ is the gauge field for translations and the spin connection $\omega_{IJ} = \omega_{IJ\mu} dx^{\mu}$ is the gauge field for rotational with the antisymmetric property $\omega_{IJ} = -\omega_{JI}$ in which the Latin indices $\{I, J, ..\}$ denote for the Minkowski metric component and the Greek indices $\{\mu, \nu, ...\}$ reserve for the Lorentzian indices [12, 13]. Both independent fields obey the relation as follows [14]:

$$de^I + \omega^I{}_J \wedge e^J = 0, \tag{1.2}$$

$$R_{IJ}\wedge e^{J}=0. \tag{1.3}$$

The trivial solution of Eq.(1.2) and Eq.(1.3) is $e^I = (e^0, e^i) = 0$ where the upper script 0 refers to time component and i refers to spatial components. Hence, the metric tensor vanishes, i.e. $g_{\mu\nu} = \eta_{IJ} {e^I}_\mu {e^J}_\nu = 0$ where η_{IJ} is a Minkowski metric. ω_{IJ} in the second term of Eq.(1.2), however, does not need to be zero. It is suggested that the solution for $g_{\mu\nu} = 0$ is valid at high temperature and high curvature where the general (diffeomorphism) invariance is broken and the metrical volume element $dV = \sqrt{-g}d^4x$ is no definite there, yet it may replace by a manifold volume $dV = 4!d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 \wedge d\varphi_4 \equiv \Phi d^4x$ where φ_a is the scalar field ($a = 1, 2, 3, 4$) and $\Phi = \epsilon_{abcd}\epsilon^{\mu\nu\lambda\sigma}(\partial_\mu\varphi_a)(\partial_\nu\varphi_b)(\partial_\lambda\varphi_c)(\partial_\sigma\varphi_d)$ [15]. We note that the pre-metric hypothesis in previous paragraph can be compared to the occurrence of changing from paramagnetism system to ferromagnetism system. This phenomenon occurs by applying external magnetic fields (H_0) and increasing temperature (T) to the paramagnetism system.

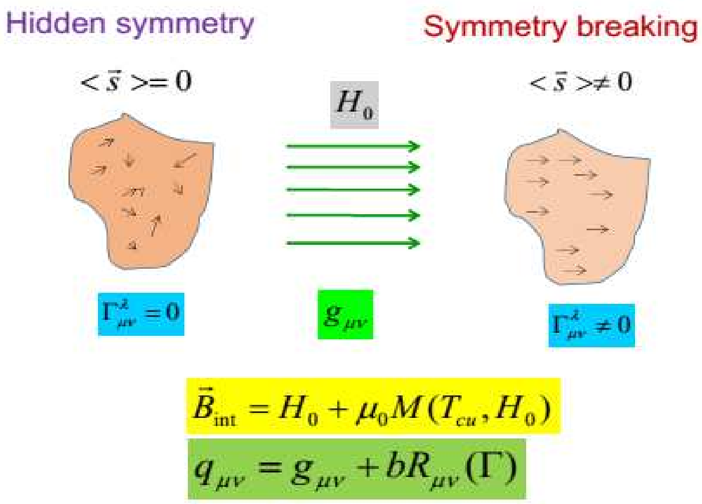

Figure 1.1: The analogy between EiBI gravity and ferromagnetic system

In paramagnetism system, there exists the collection of magnetic dipoles or domains. If the ambient temperature exceeds a critical value ($T > T_c$) and an external magnetic field, i.e. H_0, which is analogous to the non-vanishing metric tensor $g_{\mu\nu} \neq 0$, is applied and suddenly turned it off. Domains, which here is analogous to the independent connection fields or the spin connection, will distribute randomly in ferromagnetism system within the scope of the relaxation time. In this case the spin averaging of all domains approaches nearly zero, i.e. $< s >= 0$. Pre-metric hypothesis tells us that $\Gamma^{\lambda}_{\mu\nu}$ does exist before the Big Bang and it waits to couple with the metric tensor $g_{\mu\nu}$ to construct the auxiliary metric $q_{\nu\nu}$. Set $T < T_c$ and sudden remove the external magnetic field , activated domains still orientate in the direction of magnetic field which we applied before, i.e.$< s >\neq 0$(see Figure 2.1). To be precise, the process above is called the broken of directional symmetry [7][17]. Mathematically, we can express the analogy between ferromagnetism system and EiBI gravity as follows:

$$\text{Ferromagnetic system}: \; B_{\text{int}} = \mu_0 \Big[H_0 + M(T, H_0, < \vec{s} >) \Big] \, , \tag{1.4}$$

$$\text{EBI and EiBI gravity}: \; q_{\mu\nu} = g_{\mu\nu} + bR_{\mu\nu}(\Gamma), \tag{1.5}$$

where $M(T, H_0) \propto \frac{H_0}{T}$ is the magnetization per unit volume which is non-vanishing for $T < T_c$. The comparative picture of the domains' orientation after applying the external magnetic field represents by the the Eddington's inspited term $\sqrt{|g_{\mu\nu} + R_{\mu\nu}(\Gamma)|}$. This shows the coupling between two independent fields and it also reduces to pure affinity as $\sqrt{|R_{\mu\nu}(\Gamma)|}$ by removing the metric tensor. We will see later that the coupling term under square root operation may have originate from classical mechanics (see section 5.1).

1.2 Objectives

In this work , it is a good opportunity to work out in two gravity models that are different interests, i.e. modified gravity and dark energy point of view. The first one is the Eddington inspired Born-Infeld theory (EiBI). This models is affiliated with modified geometry part of Einstein's field equation and it is the prototype of the non-linear coupling between matter and gravity. The second one is Non-Minimally Derivative Coupling -Palatini theory (NMDC-Palatini). It is dark energy model of which the scalar field and its derivative are included in the gravitation action. Both gravity models share the same manner by working on Palatini formulation and have one parameter, b for EiBI gravity and κ for NMDC-Palatini. In chapter III, we review basic ideas of GR and the standard cosmological model (ΛCDM). In chapter IV, some topics about physics of scalar field, bouncing effect, and turn around point are examined under the regime of cosmology. In chapter V and VI, applying variational methods to both gravitational actions, we obtained modified Einstein's field equations and also examined how much do they deviate from GR. We apply two gravity models to the spatially flat FLRW universe, and some implications at early time and late time evolution of FLRW universe are reported. In chapter VII, late time evolution of EiBI gravity are investigated by using three different methods: Linear stability , Kosambi-Cartan-Chern (KCC) theory; Lyapunov functions method. Finally, conclusions, discussions and future perspectives are presented in chapter VIII.

CHAPTER II

FOUNDATIONS OF GRAVITATIONAL THEORY AND COSMOLOGY

In this chapter, we review basic knowledge about variational methods to derive Einstein field equation from the Einstein-Hilbert action in both metric and Palatini formulations . We describe physical meaning and express mathematical form of the energy - momentum tensor of perfect fluid and how to derive the continuity equation from the covariant conservation of energy-momentum tensor. The definition of energy-momentum tensor of matter field and the quantity in metric affine formalism which is so-called the hypermomentum are shortly reviews.

2.1 Variational principle in Palatini formalism

In this section we propose the mathematical tools using throughout this work. The Riemann tensor which is an antisymmetric in the last two indices can be expressed as

$$R^{\mu}{}_{\nu\sigma\lambda} = \partial_\sigma \Gamma^{\mu}{}_{\nu\lambda} - \partial_\lambda \Gamma^{\mu}{}_{\nu\sigma} + \Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\alpha\lambda}\Gamma^{\alpha}{}_{\nu\sigma}. \tag{2.1}$$

The Ricci tensor can be determined by contracting one of indices of the Riemann tensor,

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\lambda}. \tag{2.2}$$

By taking trace of the Ricci tensor, it allows us to write the Ricci scalar as

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.3}$$

In Palatini formulation, we take a variation of the action with respect to the metric and the connection independently. In addition, the connection field does not enter the matter action in this formulation. In case of the connection field is allowed to enter the matter action, the algebraic derivation is performed under the metric-affine formalism [19]. The variation of the Ricci tensor in metric and Palatini can be expressed as (see derivation in references [19, 20]).

$$\delta R_{\alpha\beta}(g,\partial g) = \nabla^g_\lambda \,\delta\Gamma^\lambda{}_{\alpha\beta} - \nabla^g_\beta \,\delta\Gamma^\lambda{}_{\alpha\lambda}\,, \tag{2.4}$$

$$\delta R_{\alpha\beta}(\Gamma,\partial\Gamma) = \nabla^\Gamma_\lambda \,\delta\Gamma^\lambda{}_{\alpha\beta} - \nabla^\Gamma_\beta \,\delta\Gamma^\lambda{}_{\alpha\lambda}\,, \tag{2.5}$$

respectively.[1] For simplicity, we will use $R_{\mu\nu}(g)$ and $R(g)$ to signify that our calculation is performed under metric formalism. On the one hand, we prefer to use shorthand notations $R_{\mu\nu}(\Gamma)$ and $R(\Gamma)$ to represent that our derivation is performed under Palatini formulation. With allowing torsion, i.e. $\Gamma^\lambda_{\mu\nu} \neq \Gamma^\lambda_{\nu\mu}$, the variation of the Ricci tensor, however, yields the following relation

$$\delta R_{\alpha\beta}\left(\Gamma^\lambda{}_{[\alpha\beta]}\right) = \nabla^\Gamma_\lambda \,\delta\Gamma^\lambda{}_{\alpha\beta} - \nabla^\Gamma_\beta \,\delta\Gamma^\lambda{}_{\alpha\lambda} + 2\Gamma^\sigma{}_{[\beta\lambda]}\,\delta\Gamma^\lambda_{\alpha\sigma}, \tag{2.6}$$

where the appearing of the last term is due to a non-vanishing torsion.

The variation of the gravitation action $\delta S_{\rm g}$ in Palatini formalism can be performed as

$$\begin{aligned} \delta S_{\rm g} \;=\; & \int d^4x\,\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm g})}{\delta g^{\mu\nu}}\delta g^{\mu\nu} + \int d^4x\,\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm g})}{\delta\Gamma^\lambda{}_{\mu\nu}}\,\delta\Gamma^\lambda{}_{\mu\nu} + \\ & \int d^4x\,\frac{\delta(\sqrt{-g}\mathcal{L}_{\rm g})}{\delta\phi}\,\delta\phi = 0, \end{aligned} \tag{2.7}$$

where the last two terms are the variation with respect to the scalar field and its derivative respectively.

2.2 The energy-momentum tensor

The energy-momentum tensor $T_{\mu\nu}$ is a symmetric tensor of non-geometric matter fields which is defined locally at each point of spacetime [21]. The important

[1]The operation of covariant derivative ∇_λ depends on the Christoffel symbos which can be constructed from the metric $g_{\mu\nu}$, whereas ∇^Γ_λ depends on the independent connection which cannot be constructed from to the metric tenser $g_{\mu\nu}$.

properties of the energy-momentum tensor are listed as follows [22]:

1. Locality: $T_{\mu\nu}$ can be constructed from the collection of the matter fields $\Psi(x^\mu)$ and its derivative $[\nabla_\lambda \Psi(x^\mu)]$ at the spacetime point on a manifold.

2. Diffeomorphic covariant: $T_{\mu\nu}$ transforms as a tensor under the diffeomorphism of manifolds.

3. The covariant divergence: The expression, $\nabla_\nu T^{\mu\nu} = 0$ tells us that there are ten covariantly conservative quantities for 4 dimensional spacetime.

$$\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\mu_{\nu\sigma} T^{\sigma\nu} + \Gamma^\nu_{\sigma\nu} T^{\sigma\mu} = 0. \tag{2.8}$$

The appearing of terms like $\Gamma^\mu_{\nu\sigma} T^{\sigma\nu}$ and $\Gamma^\nu_{\sigma\nu} T^{\sigma\mu}$ means that there is an allowance to transfer of energy between the matter fields and the gravitational fields [23]. This sources the difficulty to designate a local energy density of the gravitational field[1]. The significant properties of a perfect fluid can be listed as the follows [24]:

1. Each mass element carries a 4-velocity (u^μ) or 4-momentum (p^μ) to move through spacetime.

2. Each fluid element is surrounded by a mass-energy density ρ and an isotropic pressure in the fluid's rest frame.

3. Shear stress, anisotropic pressure and viscosity do not appear because there are no interactions between different components and then the exchanging of energy and momentum do not occur [25].

---

[1]The expansion of universe makes the metric $g_{\mu\nu}$ changing with time and there is no isometry in time direction, so the locally gravitational energy does not conserve[23].

We can show that for spatially flat FLRW universe [23](p.118-p.119)

$$\begin{aligned}
\nabla_\nu T^{0\nu} &= \frac{\partial}{\partial x^\nu} T^{0\nu} + \Gamma^{\nu}_{\nu\lambda} T^{\lambda 0} + \Gamma^{0}_{\nu\lambda} T^{\nu\lambda} \,, \\
&= \frac{\partial}{\partial t} T^{00} + \Gamma^{\nu}_{\nu 0} T^{00} + \Gamma^{0}_{\nu\lambda} T^{\nu\lambda} \,, \\
&= \frac{\partial \rho}{\partial t} + (\Gamma^{1}_{10} + \Gamma^{2}_{20} + \Gamma^{3}_{30})\rho + \Gamma^{0}_{11} T^{11} + \Gamma^{0}_{22} T^{22} + \Gamma^{0}_{33} T^{33} \,, \\
&= \dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3a^2 \frac{\dot{a}}{a}\frac{p}{a^2} \\
&= \dot{\rho} + 3H(\rho + p) \qquad (2.9)
\end{aligned}$$

is equivalence to the continuity equation,

$$\begin{aligned}
\dot{\rho} + 3H(\rho + p) &= 0, \\
\dot{\rho} + 3H\rho(1 + w) &= 0, \qquad (2.10)
\end{aligned}$$

where the equation of state parameter for perfect fluid is defined as

$$w = \frac{p}{\rho} \,. \qquad (2.11)$$

Mathematically, the energy-momentum tensor for a perfect fluid is constructed from a metric tensor ($g_{\mu\nu}$), the 4-velocity (u^μ), and the rest frame total energy density (ρ) and the total pressure (p). This is

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \qquad (2.12)$$

where ρ is the rest frame total energy density that may originate from rest mass energy, compressed energy, nuclear binding energy and all other sources of mass-energy density [24].

We define the energy-momentum tensor by a variation of matter action with respect to a variation of the mutual distances of the events of spacetime ($\delta g_{\mu\nu}$).

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}(\Psi, g_{\mu\nu})}{\delta g^{\mu\nu}} \qquad (2.13)$$

or

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_{\rm m}(\Psi, g_{\mu\nu})}{\delta g_{\mu\nu}}, \tag{2.14}$$

where the last one we use the relation $g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}$. Accordingly, this variation is a deformation under a translational type [26].

In addition, the connection field can be included into the matter action in the framework of the metric-affine formalism, in this case the hypermomentum tensor related to the intrinsic spin of matter is introduced as [27]

$$\Delta_\lambda{}^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\rm m}[\Psi, g, \Gamma]}{\delta \Gamma^\lambda_{\mu\nu}}. \tag{2.15}$$

2.3 The equivalence principle

The equivalence principle is the local principle which is different from the global version of Mach's principle which depends on distribution of matter in the universe [28]. It is important to distinguish three types of the definitions of the equivalence principle.

The first one is the weak equivalent principle (WEP) which hypothesis is set out that [29]:

1. The laws of physics reduced to special relativity (SR) in small regions of space-time [23].
2. The world line of the free falling body is independent of its mass, internal structure, and composition.
3. A test body does not effect and modify the gravitational field created by other (non-test) bodies.

The second one is the Einstein's equivalence principle (EEP) which is WEP plus the condition that "Any locally physical experiments are independent of the apparatus' velocity, when, and where the experiments are performed."

The third one is the strong equivalence principle (SEP) is introduced by adding the condition to the WEP that

" The test-body is self-gravitating and local test experiments are allowed to probe its gravitational effects."

It is indicated that alternative gravity theories which contain additional fields, e.g. Scalar-tensor theory, Non-minimal coupling theory (NMC), Non- minimal derivative coupling theory (NMDC), predict the violation of the weak, Einstein, and strong equivalence principle at some levels [29].

2.4 Derivation of Einstein field equations from the Einstein-Hilbert action in metric formalism

The Einstein-Hilbert action with metric field can be written as

$$S_{\rm EH}(g) = \frac{c^4}{16\pi G_{\rm N}}\int d^4x\,\sqrt{-g}\,R(g) + \int d^4x\,\sqrt{-g}\,\mathcal{L}_{\rm m}(g_{\mu\nu},\psi)\,. \qquad (2.16)$$

Varying of Einstein Hilbert action with respect to the metric $g_{\mu\nu}$, we have

$$\begin{aligned}\delta S_{\rm EH}(g) &= \frac{c^4}{16\pi G_{\rm N}}\int d^4x\,\delta\Big[\sqrt{-g}\,R(g)\Big] + \int d^4x\,\delta\Big[\sqrt{-g}\,\mathcal{L}_m\left(g_{\mu\nu},\psi\right)\Big]\,,\\ &= \frac{c^4}{16\pi G_{\rm N}}\int d^4x\,\Big[\delta(\sqrt{-g})\,R(g)\delta g^{\mu\nu} + \sqrt{-g}\,\delta R(g)\Big] \qquad (2.17)\\ &\quad + \int d^4x\,\Big[-\frac{\sqrt{-g}}{2}\,T^{\rm (m)}_{\mu\nu}\delta g^{\mu\nu}\Big]\,.\end{aligned}$$

By using the relations (see appendix A for derivation)

$$\begin{aligned}\delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta g_{\mu\nu},\\ \delta R(g) &= \left[\,R_{\mu\nu} + \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\,\right]\delta g^{\mu\nu}, \qquad (2.18)\end{aligned}$$

and the definition of energy-momentum tensor expressed in Eq.(2.13) to Eq.(2.19) , we obtain

$$\begin{aligned}\delta S_{\rm EH}(g) &= \frac{c^4}{16\pi G_{\rm N}}\int d^4x\Bigg[-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}R(g)\,\delta g^{\mu\nu} + \sqrt{-g}\Big[\,R_{\mu\nu}(g)\delta g^{\mu\nu}\\ &\quad + \cancel{\nabla_\mu\nabla_\nu(\delta g^{\mu\nu})} - \cancel{g_{\mu\nu}\Box(\delta g^{\mu\nu})}\Big]\Bigg] + \Big[-\frac{1}{2}\sqrt{-g}\,T^{\rm (m)}_{\mu\nu}\,\delta g^{\mu\nu}\Big]\,, \qquad (2.19)\end{aligned}$$

where terms $\int d^4x\sqrt{-g}\Big[\nabla_\mu\nabla_\nu(\delta g^{\mu\nu})\Big]$ and $\int d^4x\sqrt{-g}\Big[\Box(\delta g^{\mu\nu})\Big]$ represent for the surface terms. We then write

$$\delta S_{\rm EH}(g) \;=\; \int d^4x\sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}R(g)+R_{\mu\nu}(g)-\frac{16\pi G_{\rm N}}{2c^4}\,T^{(\rm m)}_{\mu\nu}\right]\delta g^{\mu\nu}.$$

Since $\delta g^{\mu\nu}$ does not vanish, we then obtain

$$R_{\mu\nu}(g)-\frac{1}{2}g_{\mu\nu}R(g) \;=\; \frac{8\pi G_{\rm N}}{c^4}T^{(\rm m)}_{\mu\nu}\,, \tag{2.20}$$

$$G_{\mu\nu}(g) \;=\; \frac{8\pi G_{\rm N}}{c^4}T^{(\rm m)}_{\mu\nu}\,. \tag{2.21}$$

Now, we completely derive Einstein field equation from Einstein-Hilbert action in the metric formalism.

2.5 Derivation of Einstein field equations from the Einstein-Hilbert action in Palatini formalism

The Einstein Hibert action with matter field in Palatini formalism can be written as

$$S_{\rm EH}(g,\Gamma)=\frac{c^4}{16\pi G_{\rm N}}\int d^4x\,\sqrt{-g}\,R(\Gamma)+\int d^4x\,\sqrt{-g}\;\mathcal{L}_{\rm m}\left(g_{\mu\nu},\psi\right). \tag{2.22}$$

In this approach, the variation of the Einstein-Hilbert action depends on two independent objects, i.e. the metric $g_{\mu\nu}$ and the connection $\Gamma^\lambda{}_{\mu\nu}$. Hence we separate the variation of this action into two parts,

$$\delta S_{\rm EH}(g,\Gamma)=\frac{\delta S_{\rm EH}(g,\Gamma)}{\delta g^{\mu\nu}}\;\delta g^{\mu\nu}+\frac{\delta S_{\rm EH}(g,\Gamma)}{\delta\Gamma^\lambda{}_{\mu\nu}}\delta\Gamma^\lambda{}_{\mu\nu}\,. \tag{2.23}$$

The first part is to perform variation of the Einstein-Hilbert action with respect to the metric $g_{\mu\nu}$. This shows

$$\begin{aligned}
\delta_g S = \frac{\delta S(g,\Gamma)}{\delta g^{\mu\nu}}\delta g^{\mu\nu} &= \frac{c^4}{16\pi G_{\rm N}}\int d^4x\,\delta\left[\sqrt{-g}\,R(\Gamma)\right] + \int d^4x\left[-\frac{\sqrt{-g}}{2}\,T^{\rm (m)}_{\mu\nu}\,\delta g^{\mu\nu}\right], \\
&= \frac{c^4}{16\pi G_{\rm N}}\int d^4x\left[\delta\sqrt{-g}\,R(\Gamma) + \sqrt{-g}\,\delta(g^{\mu\nu}\,R_{\mu\nu}(\Gamma))\right] \\
&\quad + \int d^4x\left[-\frac{\sqrt{-g}}{2}\,T^{\rm (m)}_{\mu\nu}\,\delta g^{\mu\nu}\right], \qquad (2.24) \\
&= \frac{c^4}{16\pi G_{\rm N}}\int d^4x\left[-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}R(\Gamma) + \cancel{\sqrt{-g}\,g^{\mu\nu}\frac{\delta R_{\mu\nu}(\Gamma)}{\delta g_{\alpha\beta}}}\delta g_{\alpha\beta}\right. \\
&\quad \left. + \sqrt{-g}R_{\mu\nu}(\Gamma)\delta g^{\mu\nu}\right] + \int d^4x\left[-\frac{\sqrt{-g}}{2}T^{\rm (m)}_{\mu\nu}\delta g^{\mu\nu}\right], \\
&= \frac{c^4}{16\pi G_{\rm N}}\int d^4x\,\sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}\,R(\Gamma) + R_{\mu\nu}(\Gamma)\right]\delta g^{\mu\nu} \qquad (2.25) \\
&\quad + \int d^4x\sqrt{-g}\left[-\frac{T^{\rm (m)}_{\mu\nu}}{2}\right]\delta g^{\mu\nu}.
\end{aligned}$$

Hence, what we obtain is the Einstein field equations on which the Ricci tensor and Ricci scalar depend solely the connection field.

$$R_{\mu\nu}(\Gamma) - \frac{1}{2}g_{\mu\nu}R(\Gamma) = \frac{8\pi G_{\rm N}}{c^4}T^{\rm (m)}_{\mu\nu}. \qquad (2.26)$$

The story is not end due to the existence of the second part of variation. The variation of the Einstein-Hilbert action with respect to ${\Gamma^\lambda}_{\mu\nu}$ can be expressed as

$$\begin{aligned}
\delta_\Gamma S &= \frac{\delta S_{\rm EH}(g,\Gamma)}{\delta {\Gamma^\lambda}_{\mu\nu}}\,\delta{\Gamma^\lambda}_{\mu\nu} = \frac{c^4}{16\pi G_{\rm N}}\int d^4x\,\sqrt{-g}\left[\delta_\Gamma\,R(\Gamma)\right], \\
&= \frac{c^4}{16\pi G_{\rm N}}\int d^4x\,\sqrt{-g}\left[\delta_\Gamma\,(g^{\mu\nu}\,R_{\mu\nu}(\Gamma))\right], \\
&= \frac{c^4}{16\pi G_{\rm N}}\int d^4x\,\sqrt{-g}\left[\cancel{\delta_\Gamma(g^{\mu\nu})\,R_{\mu\nu}(\Gamma)} + g^{\mu\nu}\,\delta_\Gamma R_{\mu\nu}(\Gamma)\right], \\
&= \frac{c^4}{16\pi G_{\rm N}}\int d^4x\,\sqrt{-g}\left[g^{\mu\nu}\,\delta_\Gamma R_{\mu\nu}(\Gamma)\right]. \qquad (2.27)
\end{aligned}$$

Applying the variation of the Palatini Ricci tensor,

$$\delta_\Gamma R_{\mu\nu}(\Gamma) = \nabla^\Gamma_\lambda\delta{\Gamma^\lambda}_{\mu\nu} - \nabla^\Gamma_\nu\delta{\Gamma^\lambda}_{\mu\lambda}, \qquad (2.28)$$

into Eq.(2.27), this gives

$$\delta S_\Gamma = \frac{c^4}{16\pi G_{\rm N}}\int d^4x\sqrt{-g}\,g^{\mu\nu}\left[\nabla^\Gamma_\lambda\delta{\Gamma^\lambda}_{\mu\nu} - \nabla^\Gamma_\nu\delta{\Gamma^\lambda}_{\mu\lambda}\right]. \qquad (2.29)$$

Having defined $\sqrt{-g}g^{\mu\nu} = \tilde{g}^{\mu\nu}$, we can rewrite Eq.(2.29) as

$$\delta S_\Gamma = \frac{c^4}{16\pi G_N}\int d^4x\left[\tilde{g}^{\mu\nu}\nabla^\Gamma_\lambda\delta\Gamma^\lambda_{\mu\nu} - \tilde{g}^{\mu\nu}\nabla^\Gamma_\nu\delta\Gamma^\lambda_{\mu\lambda}\right]. \tag{2.30}$$

Using by part integration and omitting the boundary terms, we have

$$\begin{aligned}\delta_\Gamma S_{EH} &= \frac{c^4}{16\pi G_N}\int d^4x\Big[\cancel{\nabla^\Gamma_\lambda(\tilde{g}^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu})} - \nabla^\Gamma_\lambda(\tilde{g}^{\mu\nu})\delta\Gamma^\lambda_{\mu\nu} - \cancel{\nabla^\Gamma_\nu(\tilde{g}^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda})} + \nabla^\Gamma_\nu(\tilde{g}^{\mu\nu})\delta\Gamma^\lambda_{\mu\lambda}\Big], \\ &= \frac{c^4}{16\pi G_N}\int d^4x\Big[-\nabla^\Gamma_\lambda(\tilde{g}^{\mu\nu}) + \delta^\nu_\lambda\nabla^\Gamma_\alpha(\tilde{g}^{\mu\alpha})\Big]\delta\Gamma^\lambda_{\mu\nu} = 0.\end{aligned} \tag{2.31}$$

If we demand that action is stationary, in order that $\delta_\Gamma S_{EH} = 0$, under the arbitrary variations $\delta\Gamma^\lambda_{\mu\nu}$, so we require that

$$-\nabla^\Gamma_\lambda\tilde{g}^{\mu\nu} + \delta^\nu_\lambda\nabla^\Gamma_\alpha\tilde{g}^{\mu\alpha} = 0. \tag{2.32}$$

After setting $\lambda = \nu$, Eq.(2.32) becomes

$$-\nabla^\Gamma_\nu\tilde{g}^{\mu\nu} + 4\nabla^\Gamma_\nu\tilde{g}^{\mu\nu} = 3\nabla^\Gamma_\alpha\tilde{g}^{\mu\alpha} = 0. \tag{2.33}$$

We can conclude that

$$\nabla^\Gamma_\alpha\tilde{g}^{\mu\alpha} = 0. \tag{2.34}$$

Next, we substitute the result back to Eq.(2.32). Hence it is easy to see that

$$\nabla^\Gamma_\lambda\tilde{g}^{\mu\nu} = \nabla_\lambda(\sqrt{-g}g^{\mu\nu}) = 0. \tag{2.35}$$

The solution of Eq.(2.35) is therefore

$$\begin{aligned}\nabla^\Gamma_\lambda(\sqrt{-g}g^{\mu\nu}) &= (\nabla^\Gamma_\lambda\sqrt{-g})g^{\mu\nu} + (\nabla^\Gamma_\lambda g^{\mu\nu})\sqrt{-g}, \\ &= (\partial_\lambda\sqrt{-g} - \Gamma^\rho_{\lambda\rho}\sqrt{-g})g^{\mu\nu} + (\nabla^\Gamma_\lambda g^{\mu\nu})\sqrt{-g}, \\ &= \cancel{(\partial_\lambda\sqrt{-g} - \frac{\sqrt{-g}}{\sqrt{-g}}\partial_\lambda\sqrt{-g})g^{\mu\nu}} + (\nabla^\Gamma_\lambda g^{\mu\nu})\sqrt{-g}, \\ &= (\nabla^\Gamma_\lambda g^{\mu\nu})\sqrt{-g} = 0,\end{aligned} \tag{2.36}$$

where $\Gamma^\rho_{\lambda\rho} = \frac{1}{\sqrt{-g}}\partial_\lambda\sqrt{-g}$ is used to derived the second line of Eq.(2.36). Because $\sqrt{-g}$ is non-degenerate, we have to set

$$\nabla^\Gamma_\lambda g^{\mu\nu} = 0. \tag{2.37}$$

This condition is nothing but metric the compatibility relation for the metric $g_{\mu\nu}$, so we can use it to construct the Levi-Civita connection $\Gamma^{\lambda}_{\mu\nu}$ by the following expression (see appendix C)

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \Big(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu} \Big). \tag{2.38}$$

The above relation shows the equivalence between the metric and Palatini approach for derivation of Einstein field equations from Einstein-Hilbert action.

2.6 The standard model of Cosmology

The full Einstein equations with including a cosmological constant can be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\pi G_{\rm N} T^{(\rm m)}_{\mu\nu}. \tag{2.39}$$

When we move the cosmological term to the right-hand side of Eq.(2.39), it represents for the vacuum energy density which has negative pressure to accelerate the universe today. The energy-momentum tensor of vacuum energy density is (see also Eq.(4.74))

$$T^{\mu(\Lambda)}_{\nu} = \begin{pmatrix} -\rho_\Lambda & 0 & 0 & 0 \\ 0 & \rho_\Lambda & 0 & 0 \\ 0 & 0 & \rho_\Lambda & 0 \\ 0 & 0 & 0 & \rho_\Lambda \end{pmatrix}, \quad T^{(\Lambda)}_{\mu\nu} = \begin{pmatrix} \rho_\Lambda & 0 & 0 & 0 \\ 0 & -\rho_\Lambda a^2 & 0 & 0 \\ 0 & 0 & -\rho_\Lambda a^2 & 0 \\ 0 & 0 & 0 & -\rho_\Lambda a^2 \end{pmatrix}, \tag{2.40}$$

where $\rho_\Lambda = \Lambda/8\pi G_{\rm N}$ and $p_\Lambda = -\rho_\Lambda$.

The line element of FLRW universe is

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right). \tag{2.41}$$

The time-time component of Eq.(2.39) is called the Friedmann equation

$$(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{8\pi G_{\rm N}}{3}\rho, \tag{2.42}$$

where the total energy density $\rho = \rho_{\rm m} + \rho_{\rm rad} + \rho_\Lambda$. The space-space components of Eq.(2.39) becomes

$$\frac{2\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = -8\pi G_{\rm N} p, \tag{2.43}$$

where p denotes the total pressure $p = p_{\rm m} + p_{\rm rad} + p_\Lambda$. Substituting Eq.(2.42) into Eq.(2.43) , this give another Friedmann equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_{\rm N}}{3}(\rho + 3p). \tag{2.44}$$

With the definition of equation of state parameter , i.e., $w = \frac{p}{\rho}$ and the result from $\nabla_\nu T^{\mu\nu} = 0$ (which gives $\frac{d}{dt}(\rho a^3) = -p\frac{d}{dt}a^3$), we derive

$$\rho = \text{const} \cdot a^{-3(1+w)}. \tag{2.45}$$

Hence, for a radiation dominated universe with $w = \frac{1}{3}$,

$$\rho_{\text{rad}} \sim \frac{1}{a^4}, \tag{2.46}$$

for a matter dominated universe with $w = 0$,

$$\rho_{\text{m}} \sim \frac{1}{a^3}, \tag{2.47}$$

and for vacuum dominated universe $w = -1$,

$$\rho_\Lambda \sim \text{const}. \tag{2.48}$$

Substituting $\rho = \text{const} \cdot \text{a}^{-3(1+\text{w})}$ into Eq.(2.42) and performing integration, then we get

$$a(t) \sim t^{2/(3(1+w))}. \tag{2.49}$$

Therefore the scale factor for radiation, matter are

$$a(t) \sim t^{1/2}, \quad \text{(radiation dominated)} \tag{2.50}$$

$$a(t) \sim t^{2/3}, \quad \text{(matter dominated)} \tag{2.51}$$

respectively. The scale factor for vacuum dominated universe

$$a(t) \sim e^{Ht} = e^{\sqrt{\frac{\Lambda}{3}}\,t}. \quad \text{(vacuum dominated).} \tag{2.52}$$

The definition of the deceleration parameter is

$$q(t) = -\frac{\ddot{a}}{aH^2}. \tag{2.53}$$

2.7 Some problems of standard model of cosmology

2.7.1 The horizon problem

It is known that the scale factor $a(t)$,the Hubble parameter $H(t)$, the deceleration parameter $q(t)$ depend on time only. These parameters are expected to be the same everywhere for the homogeneity and isotropy universe [30]. These cosmological parameters describe past, present and future universe. Let us go back to the high energy regime for radiation dominated epoch. The scale factor at that time became $a(t) \propto t^{1/2}$, hence the particle horizon during the radiation dominated universe is [31],

$$\begin{aligned} \Upsilon_{\rm PH} &= a(t)\int_0^t \frac{cdt}{a(t)}, \\ &= t^{1/2} 2ct^{1/2}, \\ &= 2ct\,, \end{aligned} \quad (2.54)$$

where we set $t = 0$ at the starting time of the universe. Using the Friedmann equation, the energy density of radiation $\rho_{\rm rad} \propto T^4$,and assuming the spatial curvature constant $k = 0$ at radiation dominated universe, we get[31]

$$t = (\frac{3c^2}{16\pi G_{\rm N} a_{\rm B}})^{1/2} g^{-1/2} T^{-2}, \quad (2.55)$$

where the radiation constant $a_{\rm B} = \frac{8\pi^5 k_{\rm B}^4}{15c^3h^3} = 4.7211 \times 10^{-9}{\rm MeV} \cdot {\rm cm}^{-3} \cdot {\rm K}^{-4}$ and the total g-factor comes from $g = g_{\rm b} + \frac{7}{8} g_{\rm f}$ which g_b and $g_{\rm f}$ denote for boson and fermion respectively. We use the unit conversions as follows

$$1\,{\rm K} \sim 8.617 \times 10^{-11}{\rm MeV} = 8.617 \times 10^{-14}{\rm GeV}, \quad (2.56)$$

to express the present averaging temperature of cosmic microwave background (T_0) in terms of GeV. Hence Eq.(2.55) can be written as

$$t_{\rm second} = 2.4 g^{-1/2} T_{\rm MeV}^{-2} = 2.4 \times 10^{-6} g^{-1/2} T_{\rm GeV}^{-2}\,. \quad (2.57)$$

By using the unit converter of temperature from Kelvin (K) to GeV, i.e.

$$T_0(\mathrm{GeV}) = 2.3 \times 10^{-13}(\frac{\mathrm{T}_0}{2.7\mathrm{K}}), \tag{2.58}$$

the present limit of particle horizon of the universe can be expressed in terms of temperature as

$$\begin{aligned} \Upsilon_p(t_0) &= 2ct & (2.59) \\ &= 6.2 \times 10^{17} T_{\mathrm{GeV}}^{-1} g^{-1/2} (\frac{2.7K}{T_0}) \text{ cm.} & (2.60) \end{aligned}$$

After substituting $T_{\mathrm{GeV}} \simeq 10^{15}, g \simeq 100$, and $T_0 \approx 2.7K$, we get $\Upsilon_p(t_0) \sim 62$ cm. It can be interpreted that the homogeneity of the universe on a scale larger than this value cannot be observed today. Of course, it is impossible by the existence of the nearly uniform temperature of CMB on the scale of $\sim 10^{28}$ cm. The conflict between two different distances is called the horizon problem. Without causal contact and overlapped of the past light cones, most-spots in the CMB at the decoupling time had the same temperature. There have to suggest some processes to allow exchanging of information before the space expanded beyond the speed of light.

2.7.2 The flatness problem

Let us start from the Friedmann equation

$$\begin{aligned} H^2 + \frac{kc^2}{a^2} &= \frac{8\pi G_{\mathrm{N}}}{3}\rho, \\ &= \frac{8\pi G_{\mathrm{N}}}{3c^2}\rho c^2, & (2.61) \end{aligned}$$

where the radiation energy density is

$$\rho c^2 = \frac{1}{2} g a_{\mathrm{B}}^2 T^4 = \frac{1}{2}(g_{\mathrm{b}} + \frac{7}{8} g_{\mathrm{f}}) a_{\mathrm{B}}^2 T^4. \tag{2.62}$$

By using $\Omega = \frac{\rho}{\rho_c}$ and $a(t) \propto t^{1/2}$ for the radiation dominated universe, Eq.(2.61) becomes

$$\begin{aligned} \frac{k\,c^2}{a^2} &= (\Omega - 1)\frac{\dot{a}^2}{a^2}, & (2.63) \\ &= \frac{\Omega - 1}{4t^2}, & (2.64) \end{aligned}$$

where the Hubble parameter for radiation dominated universe, $H^2 = \frac{\dot{a}^2}{a^2} = \frac{1}{2t}$. For the present epoch, we have

$$\frac{k\,c^2}{a_0^2} = (\Omega_0 - 1)H_0^2\,. \tag{2.65}$$

Dividing Eq.(2.63) by Eq.(2.65), using the relation $a \propto \frac{1}{T}$ for radiation dominated universe, and setting $k = \pm 1$, we get

$$\Omega - 1 = 4H_0^2 t^2 \frac{T^2}{T_0^2}(\Omega_0 - 1). \tag{2.66}$$

Replacing the relations expressed in Eq.(2.57) and Eq.(2.58) into Eq.(2.66), we obtain [31]

$$\Omega - 1 \simeq 4.3 h_0^2 g^{-1} \times 10^{-21} T_{\rm GeV}^{-2} (\frac{2.7K}{T_0})^2 (\Omega_0 - 1)\,. \tag{2.67}$$

Substituting $T_{\rm GeV} = 10^{15}$, $g \sim 100$ at the GUT epoch, and $T_0 \approx 2.7K$, we get

$$\Omega - 1 \simeq 4.3 h_0^2 \times 10^{-53}(\Omega_0 - 1). \tag{2.68}$$

The present observation shows that $\mid \Omega_0 - 1 \mid \sim \mathcal{O}(1)$. During the present epoch, this compels us to believe that the spatially flat FLRW universe must be flat since 10^{-35}sec. Hence, the flatness problem is how does the universe know that it should converge to $\Omega = 1$ with going backward on time[31].

2.7.3 The conflict between observational and theory of vacuum energy

By the present observation, the value of the cosmological constant getting from the Friedmann equation is

$$\Lambda \simeq H_0^2 \simeq (2.1332 h \times 10^{-42} \text{GeV})^2. \tag{2.69}$$

This equals to the energy density of vacuum energy

$$\rho_\Lambda \simeq \frac{\Lambda m_{\rm pl}^2}{8\pi} \simeq 10^{-47}\text{GeV}^4 \simeq 10^{-123}\text{m}_{\rm pl}^4, \tag{2.70}$$

where we use $h \simeq 0.7$ and $m_{\rm pl} \simeq 10^{19}$GeV [32].

Theoretically, the vacuum energy can be explained by the zero-point energy (in natural unit)

$$E = \frac{\omega}{2} = \frac{1}{2}\sqrt{k^2 + m^2} \tag{2.71}$$

where k, and ω denote the momentum and frequency of some field which has rest-mass m respectively. The vacuum energy of all normal modes are summed up to k_{max}

$$\begin{aligned} \rho_{\text{vac}} &= \int_0^{k_{\text{max}}} \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2}\,, \\ &= \int_0^{k_{\text{max}}} \frac{4\pi k^2 dk}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2}\,, \quad \text{where } k \gg m \\ &\approx \frac{k_{\text{max}}^4}{16\pi^2}\,. \end{aligned} \tag{2.72}$$

It is possible to replace k_{max} with m_{pl} near Planck regime. The vacuum energy density then becomes

$$\rho_{\text{vac}} \simeq 10^{74}\text{GeV}^4. \tag{2.73}$$

It is larger than the observational vacuum energy density with 121 orders, i.e.

$$\frac{\rho_{\text{theory}}}{\rho_{\text{observation}}} \simeq \frac{10^{74}\,\text{GeV}^4}{10^{-47}\,\text{GeV}^4} \sim 10^{121}. \tag{2.74}$$

2.7.4 The entropy problem

The entropy in a co-moving volume remains constant in an expansion of the universe under adiabatic process. To proof this, we will start with the first law of thermodynamics as follows [33]

$$\begin{aligned} TdS(V,T) &= d\Big[\rho(T)V\Big] + p(T)dV\,, \\ &= d\Big[(\rho(T)+p(T))V\Big] - Vdp(T)\,. \end{aligned} \tag{2.75}$$

where $\rho(T)$ and p(T) are the equilibrium energy density and pressure respectively. The relation between energy density and pressure in equilibrium state is

$$T\frac{dp(T)}{dT} = \rho(T) + p(T) \tag{2.76}$$

or

$$dp(T) \;=\; \frac{\rho(T)+p(T)}{T}dT. \tag{2.77}$$

Substituting Eq.(2.77) into Eq.(2.75) and using Eq.(2.75), we find that

$$\begin{aligned} dS(V,T) &= \frac{1}{T}d\Big[\Big(\rho(T)+p(T)\Big)V\Big] - \Big[\rho(T)+p(T)\Big]V\frac{dT}{T^2}\,, \\ &= d\Big[\frac{(\rho(T)+p(T))V}{T} + \text{const}\Big]\,. \end{aligned} \tag{2.78}$$

Hence, the entropy per co-moving volume can be defined as $s = \frac{S}{V}$. This is

$$s(T) = \frac{[\rho(T)+p(T)]}{T}\,. \tag{2.79}$$

The entropy of photons in the an observable universe within the Hubble radius,i.e. $\Upsilon_0 \approx h_0^{-1}\times 10^{28}$ cm [31],

$$S_\gamma = \frac{4\pi}{3k_B}\frac{a_B}{k_B}T_0^3\Upsilon_0^3 \approx 4.4h_0^{-1}\times 10^{87}(\frac{T_0}{2.7\,K})^3\,. \tag{2.80}$$

The entropy is conserved through the evolution of the universe under adiabatic process. At high-energy regime $ST = \text{const}$ is violate the constant entropy. Therefore one way to solve this problem is to relax the restriction of adiabatic expansion at some stage to generate the huge amount of entropy to the present observation.

CHAPTER III

SOME VIEWS ON DARK ENERGY MODELS

"The greatness of Einstein lies in his tremendous imagination, in the unbelievable obstinacy with which he pursues his problem."

Leopold Infeld

3.1 The different between dark energy and modified gravity

Dark energy models are proposed by adding the scalar fields and also tensors rank n ($n = 1, 2, 3, ...$), cosmic fluids, etc. Those fields represent for effective the energy-momentum tensors $T^{(\text{eff})}_{\mu\nu}$ on the right-hand side of Einstein field equations, whereas modified gravity models are suggested some extended forms of the Einstein tensor on the left-hand side of the Einstein field equations. In some modified gravity theories, e.g. f(R) gravity, scalar tensor theory, we can move the extended terms beyond Einstein gravity to the right-hand side and redefine those terms to be the effective energy-momentum tensor. But some gravity models, e.g. EBI and EiBI gravity , the modified forms of field equations on the geometrical side cannot rearrange to move easily to be dark energy sources on the right-hand side of the field equations except for rewriting the modified Einstein tensor term in an expanded form by allowance some conditions (see section 5.2).

3.2 Physics of scalar fields

Scalar field is associated with spin-0 particles which keep invariant under coordinate transformations and does not violate Lorentz invariance. The well-known scalar fields are Higgs field that gives mass to the standard model particles and inflaton field which generated inflation at very early universe. It is also suspected that a new kind of yet to be discovered (very light) scalar field may play the role of the dark energy which creates the tremendous negative pressure to drive an accelerated phase of universe. Assuming that scalar field is the source of dark energy, we have to accept that some sources of scalar fields have very small mass which is around $m_\phi \sim H_0 \sim 10^{-33}$ eV. It does not know for sure that whether it can interact minimally or non-minimally to another standard model matters. Hence Higgs boson which mass $\sim 10^{11}$ eV seems to be hopelessly responsible to be a source of the expansion [34]. Matter fields weakly interact with scalar fields via gravity in matter action, i.e. $L_m(\varphi, \Omega(\phi) g_{\mu\nu})$, where $\Omega(\phi)$ is the conformal factor depended on scalar field, i.e. $\phi(x^\mu)$, which depends on spacetime point. This shows that matters have to follow geodesic of the Jordan frame metric $\tilde{g}_{\mu\nu} = \Omega(\phi) g_{\mu\nu}$ [35]. The action of scalar field can be written as

$$S_\phi = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] , \tag{3.1}$$

where the first term in square bracket is the kinetic term and the second term is the potential term. Calling kinetic term and potential term of scalar field in Eq.(3.1) is due to these forms of scalar field are analogous with the Lagrangian of a single particle moving in one-dimension in classical mechanics. The dimension analysis of scalar field and relevant terms [36] are expressed as $[\phi] = [L^{-1}] , [\dot{\phi}] = [L^{-2}]$, $[\dot{\phi}^2] = [L^{-4}]$, and $[V(\phi)] = [L^{-4}]$.

Varying the action in Eq.(3.1) with respect to the metric $g_{\mu\nu}$,

$$\delta S_\phi = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right) - \frac{1}{2} g^{\alpha\mu} g^{\beta\nu} \partial_\alpha \phi \partial_\beta \phi \right] \delta g_{\mu\nu}. \tag{3.2}$$

The energy momentum tensor of the canonical scalar fields can be obtained directly by comparing Eq.(3.2) with

$$\begin{aligned} \delta S_\phi &= \int d^4x \frac{\delta \mathcal{L}_\phi}{\delta g^{\mu\nu}} \delta g^{\mu\nu}, \\ &= \frac{1}{2} \int d^4x \sqrt{-g}\, T^{(\phi)}_{\mu\nu}\, \delta g^{\mu\nu} = -\frac{1}{2} \int d^4x \sqrt{-g}\, T^{\mu\nu}_{(\phi)}\, \delta g_{\mu\nu} \, . \end{aligned} \tag{3.3}$$

The energy momentum for the canonical scalar field is immediately found that

$$T^{\mu\nu}_{(\phi)} = -g^{\mu\nu} \left(\frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + V(\phi) \right) + \partial^\mu \phi \partial^\nu \phi \, . \tag{3.4}$$

According to the vanishing of the spatial derivatives because of the homogeneity of the universe, it is rational to write the energy-momentum tensor for scalar field as

$$T^{\mu\nu}_{(\phi)} = \dot{\phi}^2\delta^\mu_0\delta^\nu_0 + g^{\mu\nu}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right), \tag{3.5}$$

where dot (.) denotes a time derivative of scalar field.

Comparing with the energy - momentum tensor of a perfect fluid,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{3.6}$$

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{3.7}$$

$$\tag{3.8}$$

where u^μ is the four-velocity which null spatial part of $u^\mu = (1, 0, 0, 0)\,, u_\mu = (-1, 0, 0, 0)$ and $g_{\mu\nu}u^\mu u^\nu = -1$ (in natural unit c = 1). For example

$$T^{00} = (\rho + p)u^0u^0 + pg^{00} = \rho + p - p = \rho. \tag{3.9}$$

The energy density and pressure of homogeneous scalar field can be defined as

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{3.10}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{3.11}$$

respectively. The equation of state parameter of scalar field is

$$w_\phi = \frac{p_\phi}{\rho_\phi}\,. \tag{3.12}$$

The range of the EoS parameter of scalar field is $-1 \le w_\phi \le 1$. To describe the dark energy effect the EoS of scalar field must be in the renge $-1 \le w_\phi \le -\frac{1}{3}$ to generate the huge negative pressure.

Next, we will derive the Euler - Lagrange equation by varying Eq.(3.1) with respect to ϕ and its derivative[1], then we get

$$\delta S_\phi = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right]. \tag{3.13}$$

[1]It can be noticed that only the Lagrange density $\mathcal{L}$ (not for Lagrangian $L = \sqrt{-g}\,\mathcal{L}$) is used in derivation the Euler-Lagrange equation of scalar field [23].

Integrating the second term of Eq.(3.13) in the squared bracket by part, we get

$$\delta S_\phi = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] \right\} \delta\phi + \int d^4x \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right], \qquad (3.14)$$

where $\delta\phi$ can be set arbitrarily and the last term is vanishing for it is the boundary term. We thus obtain the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] = 0. \qquad (3.15)$$

The equation of motion for cosmological scalar field can be derived directly from the Euler-Lagrange equations,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \qquad (3.16)$$

where the second term represents the effect of gravity on scalar field. It should be noted that the covariant derivative of the tensor rank 1

$$\begin{aligned} \nabla_\mu \nabla^\nu \phi \equiv \nabla_\mu(\nabla^\nu \phi) &= \partial_\mu(\nabla^\nu \phi) + \Gamma^\sigma_{\mu\sigma}(\nabla^\nu \phi), \\ &= \partial_\mu(\partial^\nu \phi) + \Gamma^\sigma_{\mu\sigma}(\partial^\nu \phi) \end{aligned} \qquad (3.17)$$

was used between the step of derivation Eq.(3.16).

One of the scalar - tensor gravity theory that we work out in details is NMDC-Palatini gravity. Although there are no contribution of scalar field in the original form of EiBI theory, it is presumably that all kinds of matters in the universe may convert to scalar field at high energy regime near the Big Bounce which predicted by the existence of the critical density in EiBI gravity itself.

3.3 Physics at bouncing and turning around point

The bouncing effect is revealed by the existence of the critical density at high energy regime in some modified gravity theories [37]. When the universe reaches the maximum expansion point is called the turnaround point, it begins to collapse and reaches the bounce and stating expansion phase of the Universe again by the some unknown quantum gravity mechanism. If the bouncing effect appears in some modified gravity it is unclear that the physics at turn around point and at the bouncing point are the same or not. The quantum bounce have been found in EiBI gravity as well as in Loop quantum gravity. Both gravity models show a significant role of their free parameters which is encoded in the forms of critical density. The modified Friedmann equation is a starting point to examine the critical density in modified gravity models [37]. Physics at bounce requires the smallest scale factor and then $H(a_{\rm min}) = \frac{\dot{a}_{\rm min}}{a_{\rm min}} = 0$ where $\dot{a}_{\rm min} = 0$ and $\ddot{a}_{\rm min} > 0$. The condition at the turnaround point requires a local maximum of scale factor $a_{\rm max}$ then $\dot{a}_{\rm max} = 0$. The Hubble parameter at turn around point is $H(a_{\rm max}) = \frac{\dot{a}_{\rm max}}{a_{\rm max}} = 0$ and $\ddot{a}_{\rm max} < 0$ for deceleration phase. Hence, the condition that $H = 0$ covers both the bouncing point and the turnaround point. In case of the loitering neither a local maximum $a_{\rm max}$ nor minimum of $a_{\rm min}$ appears hence we cannot set $\dot{a}$ and $\ddot{a}$ to be zero at this point [38].

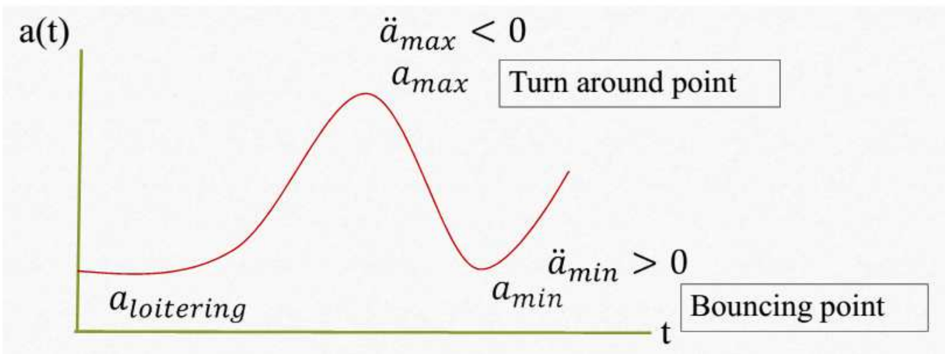

Figure 3.1: Loitering, bouncing and turn around points

CHAPTER IV

EIBI GRAVITY MODEL AND COSMOLOGY

"Einstein gave his wife the greatest care and sympathy. But in this atmosphere of coming death, Einstein remained serene and worked constantly."

Leopold Infeld

In this chapter, we will examine the physics of the coupling form of independent gravitational objects and scalar fields, e.g. $g_{\mu\nu}$, $R_{\mu\nu}(\Gamma)$, $\nabla_\mu\phi\nabla_\nu\phi$ under square root operation. For example, In EBI and EiBI gravity models, two independent geometrical objects $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ are treated to couple under square root operation, i.e. $\sqrt{g_{\mu\nu} + bR_{\mu\nu}(\Gamma)}$.

4.1 Historical of Born-Infeld types theory

The Einstein general relativity based on an affine connection was first noticed by Hermann Weyl in 1922 in his famous book " Space- Time - Matter" [39].

Two years later, Eddington proposed an alternative action for gravity without the contribution of matter fields [40]. That purely affine action with the invariant volume element is written as

$$S = \frac{2}{\Lambda}\int d^4x\sqrt{|R_{\mu\nu}(\Gamma)|}\,, \tag{4.1}$$

where Λ is the cosmological constant. By taking variation this action with respect to the connection, it can lead to the field equations $\nabla_\lambda(\frac{1}{\Lambda}\sqrt{|R|}R^{\mu\nu}) = 0$.

In 1934, Max Born and Leopole Infeld [41] inspected a new form of Lagrangian to unify theory of gravitation and electro-magnetics field in order to get rid of the divergence from the electron self energy. This action is here given by

$$S = \frac{1}{8\pi G_{\rm N} b}\int d^4x\left[\sqrt{|g_{\mu\nu} + bF_{\mu\nu}|} - \sqrt{|g_{\mu\nu}|}\right], \tag{4.2}$$

where the EiBI parameter b has dimension $[b] = [M_{\rm Pl}^{-2}]$, $G_{\rm N}$ is the universal gravitational constant, and $F_{\mu\nu}$ is the antisymmetric field strength tensor.

In 1947, Erwin Schrödinger developed the Eddington theory by proposing the anti-symmetric affine connection that showed in his famous book " Space-time structure" [42].

In 1998, Deser and Gibbon suggested Eddington-Born-Infeld (EBI) gravity in metric formalism by replacing the electromagnetic tensor field strength with the Ricci tensor, $R_{\mu\nu}(g)$ and adding an arbitrary tensor field $X_{\mu\nu}$ to eradicate the appearing of ghost terms in this theory [43]. Their action is written as

$$S_{\rm DG} = \int d^4x \sqrt{|\, g_{\mu\nu} + bR_{\mu\nu}(g)\, |}. \tag{4.3}$$

In 2004, D.N. Vollick applied for the first time the Palatini variational approach to the EBI gravity [44, 45]. This preliminary attempt aimed to eradicate ghost terms appearing in metric formulation. There are two versions of Vollick's EBI action

$$S_{\rm V1} = \frac{1}{\kappa^2 b} \int d^4x \left[\sqrt{|g_{\mu\nu} + bR_{\mu\nu}(\Gamma)|} - \sqrt{|\, g_{\mu\nu}|} \right] \tag{4.4}$$

and

$$S_{\rm V2} = \frac{1}{\kappa^2 b} \int d^4x \left[\sqrt{|g_{\mu\nu} + bR_{\mu\nu}(\Gamma) + \kappa^2\, b \nabla_\mu \phi \nabla_\nu \phi + \kappa^2\, b\, \xi\, g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi|} - \sqrt{|\, g_{\mu\nu}|} \right], \tag{4.5}$$

where ξ is a constant.

In 2007, Máximo Bañados [17] worked on his formulation of the EBI action:

$$S_{\rm EBI} = \frac{1}{16\pi G_{\rm N}} \int d^4x \left[\sqrt{|g_{\mu\nu}|}\, R(g) + \frac{2}{\alpha l^2} \sqrt{|\, g_{\mu\nu} - l^2 K_{(\mu\nu)}(\Gamma)|} \right] + S_m(g_{\mu\nu}, \Psi), \tag{4.6}$$

where α is dimensionless constant, l is the dimension of length, $K_{\mu\nu}(\Gamma)$ is the auxiliary Ricci tensor, and $S_m(g_{\mu\nu}, \Psi)$ is the matter field action.

In 2009, Máximo Bañados [46] introduced the new form of the EBI action that contains the cosmological constant in λ term[1] . This is called the Eddington-inspired-Born-Infeld gravity or EiBI gravity for short

$$S_{\rm EiBI} = \frac{2}{b\kappa^2}\int d^4x\left[\sqrt{|g_{\mu\nu}+bR_{(\mu\nu)}(\Gamma)|}-\lambda\sqrt{|g_{\mu\nu}|}\right]+S_m(g_{\mu\nu},\Psi), \qquad (4.7)$$

where $\kappa^2 = 8\pi G_{\rm N}$ and the dimensionless constant $\lambda = 1 + b\Lambda$. This implies that the dimension of $[\Lambda] = [M_{\rm Pl}^2]$. Here $R_{\mu\nu}(\Gamma) = R_{(\mu\nu)}(\Gamma)$ is the symmetric part of the Ricci tensor built form the connection in the standard formulation of EiBI (see the footnote of [47]). For pre-metric case , we can show that Eq.(4.7) reduces to the Eddington purely affine theory that is expressed in Eq.(4.1). Some of cosmological implications of EiBI gravity are shown as follow.

At very early universe, the the critical density $\rho_{\rm B}$ appears automatically from the modified Friedmann equation without the contribution of the inflaton field in the de Sitter phase of expansion [49]. It is found that the Big Rip singularity is unavoidable in the EiBI phantom model with providing the greater cosmic time comparing to the standard GR [50]. The low energy version of EiBI gravity can be expressed by assuming that $b \ll 1$ [44, 48],

$$\sqrt{|q_{\mu\nu}|} = \sqrt{|g_{\mu\nu}+bR_{\mu\nu}(\Gamma)|} \simeq \sqrt{|g_{\mu\nu}|}\left[1+\frac{1}{2}bR+\frac{1}{8}b^2R^2-\frac{1}{4}b^2R_{\alpha\beta}R^{\alpha\beta}+\mathcal{O}(b^3)\right]. \qquad (4.8)$$

The Lagrange density becomes

$$\mathcal{L}_{\rm EiBI} = \frac{2}{\kappa^2 b}\sqrt{|g_{\mu\nu}|}\left[1+\frac{1}{2}bR+\frac{1}{8}b^2R^2-\frac{1}{4}b^2(R_{\alpha\beta}R^{\alpha\beta})+\mathcal{O}(b^3)-1-b\Lambda\right], \qquad (4.9)$$

---

[1]It should be noted that this theory includes the cosmological constant existed long before the evidence of the accelerated expansion of the present universe. Non-linear coupling between the vacuum energy and matter field is shown in the Hubble parameter of EiBI gravity (see section 5.4).

Hence, the EiBI's Lagrangian density for low energy regime is

$$
\begin{aligned}
\mathcal{L}_{\text{EiBI}} &= \frac{2}{\kappa^2 b}\sqrt{|g_{\mu\nu}|}\left[1+\frac{1}{2}bR+\frac{1}{8}b^2R^2-\frac{1}{4}b^2(R_{\alpha\beta}R^{\alpha\beta})+\mathcal{O}(b^3)-1-b\Lambda\right], \\
\mathcal{L}_{\text{EiBI}} &= \frac{1}{\kappa^2}\sqrt{|g_{\mu\nu}|}\left[R+\frac{1}{4}bR^2-\frac{1}{2}b(R_{\alpha\beta}R^{\alpha\beta})-\Lambda+\mathcal{O}(b^2)\right], \qquad (4.10)
\end{aligned}
$$

At galactic scale, EiBI free parameter (b) plays an important role to explain dark matter density profile and the existent of dark matter by rotation curve [51]. Non-linearly coupled between gravity and matter in the EiBI gravity is expected to play an important role in the high density regions inside the compact relativistic stars [52]. The pressureless stars is composed of non-interacting particles responsible for the self-gravitating dark matter [51]. It is also found that the positive EiBI parameter, i.e. $b > 0$, shows a finite constant pressure region of compact stars whereas the negative of EiBI free parameter,i.e. $b < 0$, leads to prohibit an equilibrium of stellar structures [53]. EiBI gravity affects the physics of oscillating stars by showing that neutron star oscillate with lower frequency than GR for $b > 0$. In contrast, relativistic stars oscillate with the high frequency than GR for $b < 0$. The appearing of curvature singularity on the surface of the polytropic stars is one of concerned problems for EiBI gravity [54].

4.2 EiBI Palatini action and equation of motions

In 2009, Máximo Bañados[46] introduced EiBI action on a Palatini formalism. The important of this formulation is to propose the metric $g_{\mu\nu}$ and the connection $\Gamma^{\lambda}_{\mu\nu}$ to be independent objects. The action of this approach is

$$
S_{\text{EiBI}}(g,\Gamma) = \frac{2}{b\kappa^2}\int d^4x\left[\sqrt{|g_{\mu\nu}+bR_{(\mu\nu)}(\Gamma)|}-\lambda\sqrt{|g_{\mu\nu}|}\right]+S_m(g_{\mu\nu},\Psi), \qquad (4.11)
$$

where Ψ is the collecting of matter fields. To get the equation of motions in Palatini formalism, we will vary the EiBI action with respect to $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ separately.

Before taking variation of Eq.(4.11) with respect to the metric $g_{\mu\nu}$, it is convenient but not mandatory to define term in square root to be $q_{\mu\nu} \equiv g_{\mu\nu} +$

$bR_{\mu\nu}(\Gamma)$. Hence, we can write

$$\delta_{\mathrm{g}} S_{\mathrm{EiBI}}(g,\Gamma) = \frac{2}{b\kappa^2}\int d^4x \left[\delta_g\sqrt{|q_{\mu\nu}|} - \lambda\delta_g\sqrt{|g_{\mu\nu}|}\right] + \int d^4x\,\delta\left[\sqrt{|g_{\mu\nu}|}\mathcal{L}(g_{\mu\nu},\Psi)\right] = 0\,, \tag{4.12}$$

where the subscript g denotes the variation with the metric $g_{\mu\nu}$. Substituting the relation

$$\delta_g\sqrt{|q_{\mu\nu}|} = \frac{1}{2}\frac{|q_{\mu\nu}|}{\sqrt{|q_{\mu\nu}|}}q^{\mu\nu}\delta_g q_{\mu\nu} = \frac{1}{2}\sqrt{|q_{\mu\nu}|}q^{\mu\nu}\delta_g q_{\mu\nu}\,. \tag{4.13}$$

into Eq.(4.12), we then get

$$\delta_g S_{\mathrm{EiBI}}(g,\Gamma) = \frac{2}{b\kappa^2}\int d^4x \left[\frac{1}{2}\sqrt{|q_{\mu\nu}|}\,q^{\mu\nu} - \frac{\lambda}{2}\sqrt{|g_{\mu\nu}|}\,g^{\mu\nu} + \frac{b\kappa^2}{2}\sqrt{|g_{\mu\nu}|}\,T^{\mu\nu}\right]\delta g_{\mu\nu}\,, \tag{4.14}$$

where $\delta_g q_{\mu\nu} = \delta_g g_{\mu\nu} + \cancel{b\delta_g R_{\mu\nu}(\Gamma)} = \delta_g g_{\mu\nu}$ is validation in Palatini formulation. Multiplying both sides of Eq.(4.14) by $b\kappa^2/\sqrt{|g_{\mu\nu}|}$, the first field equation of EiBI action can be expressed as

$$\frac{\sqrt{|q_{\mu\nu}|}}{\sqrt{|g_{\mu\nu}|}}q^{\mu\nu} - \lambda g^{\mu\nu} = -b\kappa^2 T^{\mu\nu}_{(\mathrm{m})}\,, \tag{4.15}$$

Restoring back the definition of $q_{\mu\nu}$, the first field equation of EiBI gravity is written in full form as

$$\frac{\sqrt{|g_{\mu\nu} + bR_{\mu\nu}|}}{\sqrt{|g_{\mu\nu}|}}[(g+bR)^{-1}]^{\mu\nu} - \lambda g^{\mu\nu} = -b\kappa^2 T^{\mu\nu}_{(\mathrm{m})}, \tag{4.16}$$

where the inverse metric found in literatures can be represented in several forms, i.e.

$$q^{\mu\nu} = [q^{-1}]^{\mu\nu} = [(g+bR)^{-1}]^{\mu\nu} = [\frac{1}{(g+bR)}]^{\mu\nu}. \tag{4.17}$$

We can interpret the left - hand side of the EiBI field equation as the modified Einstein tensor

$$\tilde{G}^{\mu\nu} \equiv -\frac{\sqrt{|g_{\mu\nu} + bR_{\mu\nu}|}}{b\sqrt{|g_{\mu\nu}|}}[(g+bR)^{-1}]^{\mu\nu} + \frac{\lambda g^{\mu\nu}}{b} = 8\pi G_{\mathrm{N}} T^{\mu\nu}_{(m)}\,. \tag{4.18}$$

To get the second field equations, we take variation the EiBI action with respect to $\Gamma^{\lambda}_{\mu\nu}$,

$$\delta_\Gamma S_{\rm EiBI}(g,\Gamma) = \frac{2}{b\kappa^2}\int d^4x \left[\delta_\Gamma(\sqrt{|q_{\mu\nu}|}) - \lambda\delta_\Gamma(\sqrt{|g_{\mu\nu}|})\right] + \delta_\Gamma S_m(g_{\mu\nu},\psi)\,. \quad (4.19)$$

Substituting

$$\begin{aligned}
\delta_\Gamma|q_{\mu\nu}| &= |q_{\mu\nu}|q^{\mu\nu}\delta_\Gamma q_{\mu\nu} \;=\; |q_{\mu\nu}|\,q^{\mu\nu}\delta_\Gamma\Big[\,g_{\mu\nu}+bR_{\mu\nu}(\Gamma)\Big], \\
&= |q_{\mu\nu}|\,q^{\mu\nu}\Big[\,\cancel{\delta_\Gamma g_{\mu\nu}} + b\,\delta_\Gamma R_{\mu\nu}(\Gamma)\Big] \\
&= |q_{\mu\nu}|q^{\mu\nu}\delta_\Gamma b R_{\mu\nu}(\Gamma), \qquad (4.20)\\
\delta_\Gamma(\sqrt{|g_{\mu\nu}|} &= 0\,, \qquad (4.21)\\
\delta_\Gamma S_m(g_{\mu\nu},\Psi) &= 0 \qquad (4.22)
\end{aligned}$$

into Eq.(4.19). To show the full derivation explicitly, the first term in a square bracket of Eq.(4.19) becomes

$$\begin{aligned}
\delta_\Gamma\sqrt{|q_{\mu\nu}|} = \frac{\delta\sqrt{|q_{\mu\nu}|}}{\delta|q_{\mu\nu}|}\delta_\Gamma|q_{\mu\nu}| &= \frac{1}{2}\frac{\delta|q_{\mu\nu}|}{\sqrt{|q_{\mu\nu}|}} = \frac{1}{2}\frac{|q_{\mu\nu}|}{}\sqrt{|q_{\mu\nu}|}\,q^{\mu\nu}\delta_\Gamma\, q_{\mu\nu}\,, \\
&= \frac{b}{2}\sqrt{|q_{\mu\nu}|}\,q^{\mu\nu}\delta_\Gamma R_{\mu\nu}(\Gamma)\,, \qquad (4.23)
\end{aligned}$$

where we use the relation

$$\begin{aligned}
\delta|q_{\mu\nu}| &= |q_{\mu\nu}|q^{\mu\nu}\delta q_{\mu\nu}, \qquad (4.24)\\
\delta_\Gamma q_{\mu\nu} &= \delta_\Gamma[g_{\mu\nu}+b\,R_{\mu\nu}(\Gamma)] = b\,\delta_\Gamma R_{\mu\nu}(\Gamma). \qquad (4.25)
\end{aligned}$$

to derive Eq.(4.23). Substituting Eq.(4.23) into Eq.(4.19), this gives

$$\delta_\Gamma S_{\rm EiBI}(g,\Gamma) = \frac{2}{b\kappa^2}\int d^4x\Big[\frac{b}{2}\sqrt{|q_{\mu\nu}|}\,q^{\mu\nu}\;\delta_\Gamma R_{\mu\nu}(\Gamma)\Big] = 0. \qquad (4.26)$$

For brevity, we define $\sqrt{|q_{\mu\nu}|}\,q^{\mu\nu} = \tilde{q}^{\mu\nu}$. Therefore Eq.(4.26) becomes

$$\begin{aligned}
\delta_\Gamma S_{\rm EiBI} &= \frac{2}{b\kappa^2}\int d^4x\,\frac{b}{2}\,\Big[\tilde{q}^{\mu\nu}\delta_\Gamma R_{\mu\nu}(\Gamma)\Big]\,, \\
&= \frac{1}{\kappa^2}\int d^4x\,\Big[\tilde{q}^{\mu\nu}\,\delta_\Gamma R_{\mu\nu}(\Gamma)\Big]. \qquad (4.27)
\end{aligned}$$

The variation of the symmetric Palatini Ricci tensor can be expressed as (see appendix A: for derivation)

$$\begin{aligned}\delta_\Gamma R_{\mu\nu}(\Gamma) &= \delta_\Gamma R_{(\mu\nu)}(\Gamma)\,,\\ &= \nabla^\Gamma_\lambda \delta\Gamma^\lambda_{\ \mu\nu} - \nabla^\Gamma_\nu \delta\Gamma^\lambda_{\ \mu\lambda}. \end{aligned}\tag{4.28}$$

It can be shown that

$$\begin{aligned}\delta_\Gamma S_{\rm EiBI}(g,\Gamma) &= \frac{1}{\kappa^2}\int d^4x\,[\tilde{q}^{\mu\nu}\nabla^\Gamma_\lambda\delta\Gamma^\lambda_{\ \mu\nu} - \tilde{q}^{\mu\nu}\nabla^\Gamma_\nu\delta\Gamma^\lambda_{\ \mu\lambda}]\,,\\ &= \frac{1}{\kappa^2}\int d^4x\,\Big[\cancel{\nabla^\Gamma_\lambda(\tilde{q}^{\mu\nu}\delta\Gamma^\lambda_{\ \mu\nu})} - (\nabla^\Gamma_\lambda\tilde{q}^{\mu\nu})\delta\Gamma^\lambda_{\ \mu\nu}\cancel{-\nabla^\Gamma_\nu(\tilde{q}^{\mu\nu}\delta\Gamma^\lambda_{\ \mu\lambda})}\\ &\quad +(\nabla^\Gamma_\nu\tilde{q}^{\mu\nu})\delta\Gamma^\lambda_{\ \mu\lambda}\Big],\\ &= \frac{1}{\kappa^2}\int d^4x\,\Big[-(\nabla^\Gamma_\lambda\tilde{q}^{\mu\nu})\delta\Gamma^\lambda_{\ \mu\nu} + (\nabla^\Gamma_\nu\tilde{q}^{\mu\nu})\delta\Gamma^\lambda_{\ \mu\lambda}\Big]\,,\\ &= \frac{1}{\kappa^2}\int d^4x\,\left[-(\nabla^\Gamma_\lambda\tilde{q}^{\mu\nu}) + (\nabla^\Gamma_\alpha\tilde{q}^{\mu\alpha})\delta^\nu_\lambda\right]\delta\Gamma^\lambda_{\ \mu\nu}\,, \end{aligned}\tag{4.29}$$

where the surface terms can be neglected. Using the fact that the variation $\delta\Gamma^\lambda_{\ \mu\nu}$ is arbitrary , one may therefore write

$$-(\nabla^\Gamma_\lambda\tilde{q}^{\mu\nu}) + (\nabla^\Gamma_\alpha\tilde{q}^{\mu\alpha})\delta^\nu_\lambda = 0. \tag{4.30}$$

Setting $\nu = \lambda$, Eq.(4.30) becomes

$$-(\nabla^\Gamma_\lambda\tilde{q}^{\mu\lambda}) + 4(\nabla^\Gamma_\lambda\tilde{q}^{\mu\lambda}) = 3(\nabla^\Gamma_\lambda\tilde{q}^{\mu\lambda}) = 0 \tag{4.31}$$

and then we put back to Eq.(4.30). Hence we get

$$-\nabla^\Gamma_\lambda\tilde{q}^{\mu\nu} = -\nabla^\Gamma_\lambda\left[\sqrt{|q_{\mu\nu}|}q^{\mu\nu}\right] = 0. \tag{4.32}$$

The above relation is noting but the metric compatibility (Levi-Civita connection) for a new metric $q_{\mu\nu}$. We further proof that

$$\begin{aligned}\nabla^\Gamma_\lambda\left[\sqrt{|q_{\mu\nu}|}q^{\mu\nu}\right] &\equiv (\nabla^\Gamma_\lambda\sqrt{|q_{\mu\nu}|})q^{\mu\nu} + \nabla^\Gamma_\lambda q^{\mu\nu}\sqrt{|q_{\mu\nu}|}\,,\\ &= \left[\partial_\lambda\sqrt{|q_{\mu\nu}|} - \Gamma^\sigma_{\lambda\sigma}\sqrt{|q_{\mu\nu}|}\right]q^{\mu\nu} + (\nabla^\Gamma_\lambda q^{\mu\nu})\sqrt{|q_{\mu\nu}|}\,,\\ &= \left[\partial_\lambda\sqrt{|q_{\mu\nu}|} - \frac{1}{\sqrt{|q_{\mu\nu}|}}\left(\partial_\lambda\sqrt{|q_{\mu\nu}|}\right)\sqrt{|q_{\mu\nu}|}\right]q^{\mu\nu} + (\nabla^\Gamma_\lambda q^{\mu\nu})\sqrt{|q_{\mu\nu}|}\,,\\ &= 0 + (\nabla^\Gamma_\lambda q^{\mu\nu})\sqrt{|q_{\mu\nu}|} = 0. \end{aligned}\tag{4.33}$$

Resulting from the non-vanishing of $\sqrt{|q_{\mu\nu}|}$, the condition that $(\nabla^{\Gamma}_{\rho} q^{\mu\nu}) = 0$ must be obeyed. Hence, the connection can be constructed from the auxiliary metric $q_{\mu\nu}$ with the metric compatible relation (see also appendix C),

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} q^{\lambda\tau} \left[\partial_{\mu} q_{\nu\tau} + \partial_{\nu} q_{\mu\tau} - \partial_{\tau} q_{\mu\nu}\right]. \tag{4.34}$$

The auxiliary metric can be written from the two independent fields with EiBI free parameter,

$$q_{\mu\nu} = q_{(\mu\nu)} = g_{\mu\nu} + bR_{(\mu\nu)}(\Gamma)\,. \tag{4.35}$$

Care must be taken to avoid circular logic of the relation $q_{\mu\nu} = g_{\mu\nu} + bR_{\mu\nu}(\Gamma)$(see figure 5.2).

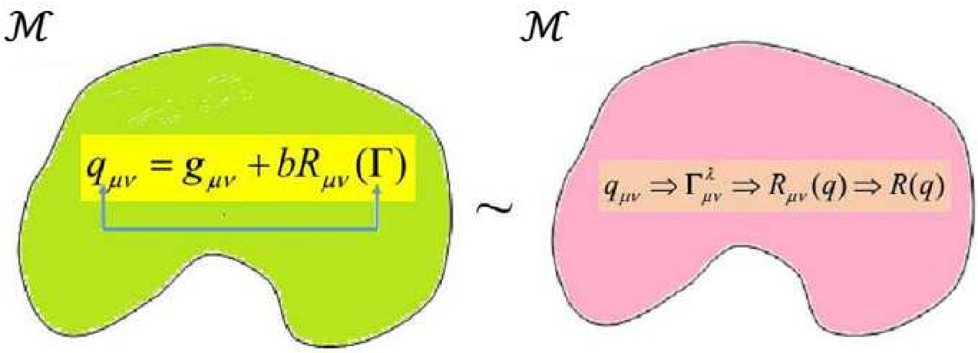

Figure 4.1: Two independent objects on the EiBI manifold($\mathcal{M}$)

It is important to note that when $T^{\mu\nu}_{(\mathrm{m})} = 0$ for vacuum solution , the first field equations of EiBI gravity become

$$\frac{\sqrt{|q_{\mu\nu}|}}{\sqrt{|g_{\mu\nu}|}} q^{\mu\nu} - \lambda g^{\mu\nu} = 0. \tag{4.36}$$

We find that

$$\sqrt{|q_{\mu\nu}|} q^{\mu\nu} = \lambda \sqrt{|g_{\mu\nu}|} g^{\mu\nu}, \tag{4.37}$$

It can be shown that the EiBI can be reduced to the Einstein field equations for vacuum case. In this case we set $T^{\mu\nu}_{(\mathrm{m})} = 0$ and assume $b \ll 1$, Eq.(4.36) becomes

$$\sqrt{|g_{\mu\nu}|}\Big[1 + \frac{1}{2}bR + \frac{1}{8}b^2R^2 - \frac{1}{4}b^2R_{\alpha\beta}R^{\alpha\beta} + \mathcal{O}(b^3)\Big]\Big[g^{\mu\nu} - bR^{\mu\nu} + b^2R^{\mu}{}_{\alpha}R^{\alpha\nu} + \mathcal{O}(b^3)\Big]$$
$$-\lambda\sqrt{|g_{\mu\nu}|}g^{\mu\nu} = 0 \quad (4.38)$$

Cancelling $\sqrt{|g_{\mu\nu}|}$ both side of Eq(4.38) and neglecting term $\mathcal{O}(b^3)$, we obtain

$$g^{\mu\nu} - bR^{\mu\nu} + b^2R^{\mu}{}_{\alpha}R^{\alpha\nu} + \frac{b}{2}Rg^{\mu\nu} - \frac{b^2}{2}RR^{\mu\nu} + \frac{1}{8}b^2R^2g^{\mu\nu} - \frac{1}{4}b^2R_{\alpha\beta}R^{\alpha\beta}g^{\mu\nu} - \lambda g^{\mu\nu} = 0 \quad (4.39)$$

With using the definition of the Einstein tensor, Eq.(4.39) can be written more elegantly as

$$-b(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}) + (g^{\mu\nu} + b^2R^{\mu}{}_{\alpha}R^{\alpha\nu} - \frac{b^2}{2}RR^{\mu\nu} + \frac{1}{8}b^2R^2g^{\mu\nu} - \frac{1}{4}b^2R_{\alpha\beta}R^{\alpha\beta}g^{\mu\nu}) - \lambda g^{\mu\nu} = 0 \quad (4.40)$$

Taking the EiBI parameter out of the parentheses and returning $\lambda = 1 + b\Lambda$ into Eq.(4.40), this yields

$$-b + b^2(R^{\mu}{}_{\alpha}R^{\alpha\nu} - \frac{R}{2}R^{\mu\nu} + \frac{1}{8}R^2g^{\mu\nu} - \frac{1}{4}R_{\alpha\beta}R^{\alpha\beta}g^{\mu\nu}) + \cancel{g^{\mu\nu}} - \cancel{g^{\mu\nu}} - b\Lambda g^{\mu\nu} = 0. \quad (4.41)$$

Cancelling the EiBI free parameter "b" in each term of Eq.(4.41), this gives

$$G^{\mu\nu} - b(R^{\mu}{}_{\alpha}R^{\alpha\nu} - \frac{R}{2}R^{\mu\nu} + \frac{1}{8}R^2g^{\mu\nu} - \frac{1}{4}R_{\alpha\beta}R^{\alpha\beta}g^{\mu\nu}) - \Lambda g^{\mu\nu} = 0. \quad (4.42)$$

With the condition that $b \ll 1$, this also reproduce the EiBI field equations for vacuum case,

$$G^{\mu\nu}(\Gamma) = \Lambda g^{\mu\nu}. \quad (4.43)$$

We would like to end this section with some opinions about the conformation form of the new metric $q_{\mu\nu}$. Up to my knowledge we cannot simply perform neither the conformal transformation as we do with $f(R)$ gravity and scalar tensor theory[56] nor the disformal transformation[57] of the new metric $q_{\mu\nu} = g_{\mu\nu} + bR_{\mu\nu}(\Gamma)$. The new metric depends on the causality metric $g_{\mu\nu}$ and the independent connection $\Gamma^{\lambda}_{\mu\nu}$ which both objects are on the EiBI manifold. It seems that nowhere in EiBI's literature discussed about this point.

4.3 EiBI cosmology

As derived from the previous section, the EiBI field equations may shed some light on on the quantum effect near quantum gravity regime. Usually, most modified gravity theories keep the matter-gravity coupling to be linear, i.e. $G_{\mu\nu} \sim T_{\mu\nu}$.

Even though both the energy-momentum tensor and the Einstein tensor are divergenceless quantity, there have no reasons why the matter-gravity coupling should be limited only linear form [58]. The EiBI gravity is shown to have the non-linear form of the matter-gravity coupling. Let us recall the first field equation of EiBI gravity here again

$$\frac{\sqrt{|g_{\mu\nu} + bR_{\mu\nu}(\Gamma)|}}{\sqrt{|g_{\mu\nu}|}} \left[(g + bR(\Gamma))^{-1}\right]^{\mu\nu} - \lambda g^{\mu\nu} = -b\kappa^2 T^{\mu\nu}, \tag{4.44}$$

where $\sqrt{|g_{\mu\nu}|} = a^3$ from $g_{\mu\nu} = \mathrm{diag}\,(-1, a^2, a^2, a^2)$.

The covariant form for the second field equations

$$\nabla_\lambda(\sqrt{|q_{\mu\nu}|} q^{\mu\nu}) = 0. \tag{4.45}$$

The constraint equation implies the auxiliary metric tensor [56],

$$q_{\mu\nu} = g_{\mu\nu} + bR_{\mu\nu}(\Gamma). \tag{4.46}$$

In Palatini formalism, the auxiliary metric derived from the constraint equation in Eq.(4.45) affects the background spacetime geometry. The modified form of FLRW metric can be expressed as follows

$$ds_{\mathrm{q}}^2 = -U^2(t)dt^2 + \frac{a^2V^2(t)}{1-kr^2}dr^2 + a^2V^2(t)r^2\left[\sin^2\theta d\phi^2 + (d\phi)^2\right], \tag{4.47}$$

where the subscript q denotes for the auxiliary metric $q_{\mu\nu}$. The Levi-Civita connection or the Christoffel symbols can be constructed directly from the new metric $q_{\mu\nu}$,

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} q^{\rho\sigma}(\partial_\mu q_{\sigma\nu} + \partial_\nu q_{\mu\sigma} - \partial_\sigma q_{\mu\nu}). \tag{4.48}$$

Using Eq.(4.48), the non-vanishing Christoffel symbols can be listed below

$$\begin{aligned}
\Gamma^t{}_{tt} &= \frac{\dot{U}}{U} && (4.49)\\
\Gamma^t{}_{rr} &= \frac{a^2V^2}{U^2}\frac{1}{(1-kr^2)}[H+\frac{\dot{V}}{V}] && (4.50)\\
\Gamma^r{}_{rr} &= \frac{kr}{(1-kr^2)} && (4.51)\\
\Gamma^r{}_{rt} &= \Gamma^\theta{}_{t\theta} = \Gamma^\phi_{t\phi} = \Big[H+\frac{\dot{V}}{V}\Big] && (4.52)\\
\Gamma^\theta{}_{r\theta} &= \Gamma^\phi{}_{r\phi} = \frac{1}{r} && (4.53)\\
\Gamma^t_{\phi\phi} &= \frac{a^2V^2r^2}{U^2}\Big[H+\frac{\dot{V}}{V}\Big] = \frac{\tilde{\Upsilon}^2}{U^2}\Big[H+\frac{\dot{V}}{V}\Big] && (4.54)\\
\Gamma^t_{\theta\theta} &= \frac{a^2V^2r^2\sin^2\theta}{U^2}\Big[H+\frac{\dot{V}}{V}\Big] = \frac{\tilde{\Upsilon}^2\sin^2\theta}{U^2}\Big[H+\frac{\dot{V}}{V}\Big] && (4.55)\\
\Gamma^r_{\theta\theta} &= -r(1-kr^2)\sin^2\theta && (4.56)\\
\Gamma^r_{\theta\theta} &= -r(1-kr^2) && (4.57)\\
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta && (4.58)\\
\Gamma^\phi_{\theta\phi} &= \cot\theta\,. && (4.59)
\end{aligned}$$

By setting k=0 for spatially flat universe , the modified FLRW metric Eq.(4.47) can be written as

$$ds^2_{\mathrm{q}} = -U^2(t)dt^2 + a(t)^2V^2(t)\delta_{ij}dx^idx^j. \tag{4.60}$$

For simplicity to compare with the notions used in paper of Cho and Kim[59] and derive of the component of the Ricci tensor, we redefine $X^2 \equiv U^2$ and $Y^2 \equiv a^2V^2$. We rewrite Eq.(4.82),

$$ds^2_q = -X^2(t)dt^2 + Y^2\delta_{ij}dx^idx^j. \tag{4.61}$$

$$\begin{aligned}
q_{00} &= -X^2 = -U^2\,, && (4.62)\\
q_{ij} &= Y^2\delta_{ij} = a^2V^2\delta_{ij}\,. && (4.63)
\end{aligned}$$

The square root of the determinant of the metric $q_{\mu\nu}$ is written as

$$\sqrt{|q_{\mu\nu}|} = \sqrt{a^6U^2V^6} = a^3UV^3. \tag{4.64}$$

The non-vanishing Christoffel symbols for spatially flat EiBI universe can be expressed as

$$\Gamma^0{}_{00} = \frac{\dot{U}}{U} = \frac{\dot{X}}{X}, \tag{4.65}$$

$$\Gamma^0{}_{ij} = \frac{a^2V^2}{U^2}\left[H + \frac{\dot{V}}{V}\right]\delta_{ij} = \frac{Y\dot{Y}}{X^2}\delta_{ij}, \tag{4.66}$$

$$\Gamma^i{}_{j0} = \left[H + \frac{\dot{V}}{V}\right]\delta^i_j = \frac{\dot{Y}}{Y}\delta^i_j, \tag{4.67}$$

where we use

$$\dot{X} = \dot{U} \quad \text{and} \quad \frac{\dot{\mathrm{Y}}}{\mathrm{Y}} = \mathrm{H} + \frac{\dot{\mathrm{V}}}{\mathrm{V}}. \tag{4.68}$$

to interchange between the notions of (U,V) and (X,Y).

The non-vanishing components of Ricci tensor are

$$R_{00}(\Gamma) = 3\Big[-\frac{d}{dt}(\frac{\dot{Y}}{Y}) - (\frac{\dot{Y}}{Y})^2 + \frac{\dot{X}}{X}\frac{\dot{Y}}{Y}\Big], \tag{4.69}$$

$$R_{11}(\Gamma) = R_{22}(\Gamma) = R_{33}(\Gamma) = \frac{Y^2}{X^2}\Big[\frac{d}{dt}(\frac{\dot{Y}}{Y}) + \frac{\dot{Y}}{Y}(\frac{3\dot{Y}}{Y} - \frac{\dot{X}}{X})\Big], \tag{4.70}$$

Three forms of energy momentum tensor for perfect fluid can be expressed as follow:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\, g^{\mu\nu}, \tag{4.71}$$

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\, g_{\mu\nu}, \tag{4.72}$$

$$T^\mu_\nu = (\rho + p)u^\mu u_\nu + p\, g^\mu_\nu, \tag{4.73}$$

where the expression for the four-velocity are $u^\mu = (1, 0, 0, 0)$ and $u_\mu = (-1, 0, 0, 0)$.

The energy momentum shows in metric forms as follows:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{p}{a^2} & 0 & 0 \\ 0 & 0 & \frac{p}{a^2} & 0 \\ 0 & 0 & 0 & \frac{p}{a^2} \end{pmatrix}, \qquad T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & a^2 p & 0 & 0 \\ 0 & 0 & a^2 p & 0 \\ 0 & 0 & 0 & a^2 p \end{pmatrix},$$

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{4.74}$$

With the energy-momentum tensor expressed above and the first field equations of the EiBI gravity,

$$\frac{\sqrt{|q_{\mu\nu}|}}{\sqrt{|g_{\mu\nu}|}} q^{\mu\nu} - \lambda g^{\mu\nu} = -b\kappa^2 T^{\mu\nu}. \tag{4.75}$$

The time-time component of the EiBI field equations is

$$\frac{V^3}{U} - \lambda = b\kappa^2 \rho\,. \tag{4.76}$$

The space-space component of the EiBI field equations is

$$\lambda - UV = b\kappa^2 p\,. \tag{4.77}$$

Two reciprocal forms of Eq.(4.76) and Eq.(4.77) are

$$X^2 \equiv U^2(\rho, p) = \frac{(\lambda - b\kappa^2 p)^{3/2}}{(\lambda + b\kappa^2 \rho)^{1/2}} \tag{4.78}$$

$$\frac{Y^2}{a^2} \equiv V^2(\rho, p) = \sqrt{(\lambda + b\kappa^2 \rho)(\lambda - b\kappa^2 p)} \tag{4.79}$$

respectively, so we can write

$$q_{00} = -X^2 = -U^2 = -(\lambda - \kappa^2 b p)^{3/2} (\lambda + \kappa^2 b \rho)^{-1/2}\,, \tag{4.80}$$

$$q_{ij} = Y^2\delta_{ij} = a^2V^2\delta_{ij} = a^2\Big[(\lambda - \kappa^2 bp)(\lambda + \kappa^2 b\rho)\Big]^{1/2}\delta_{ij}\,. \tag{4.81}$$

The modified FLRW metric can be rewritten as

$$ds_{\rm q}^2 = -\frac{(\lambda - b\kappa^2 p)^{3/2}}{(\lambda + b\kappa^2\rho)^{1/2}}\,dt^2 + a(t)^2\sqrt{(\lambda + b\kappa^2\rho)(\lambda - b\kappa^2 p)}\,\delta_{ij}dx^i dx^j. \tag{4.82}$$

From the second field equations

$$q_{\mu\nu} = g_{\mu\nu} + bR_{\mu\nu}(\Gamma)\,. \tag{4.83}$$

The time - time component of the second field equation can be easy to derived with parameters X and Y which is expressed in this form

$$\begin{aligned} q_{00} - g_{00} &= bR_{00}(\Gamma)\,, \\ -X^2 + 1 &= 3b\left[-\frac{d}{dt}(\frac{\dot{Y}}{Y}) - (\frac{\dot{Y}}{Y})^2 + \frac{\dot{X}\dot{Y}}{XY}\right]. \end{aligned} \tag{4.84}$$

The space - space components of the second field equation are

$$\begin{aligned} q_{ij} - g_{ij} &= bR_{ij}(\Gamma)\,, \\ Y^2 - a^2 &= b\frac{Y^2}{X^2}\left[\frac{d}{dt}(\frac{\dot{Y}}{Y}) + (\frac{\dot{Y}}{Y})(\frac{3\dot{Y}}{Y} - \frac{\dot{X}}{X})\right]. \end{aligned} \tag{4.85}$$

From Eq.(4.78) and Eq.(4.79), it can be shown that

$$a^2U^2V^2 \equiv X^2Y^2 = \frac{a^2(\lambda - b\kappa^2 p)^{3/2}}{\cancel{(\lambda + b\kappa^2\rho)^{1/2}}}\Big[(\lambda - b\kappa^2 p)\cancel{(\lambda + b\kappa^2\rho)}\Big]^{1/2} = a^2(\lambda - b\kappa^2 p)^2\,. \tag{4.86}$$

Next, our aim is to get rid of term $\frac{d}{dt}(\frac{\dot{Y}}{Y})$ by multiplying Eq.(4.84) with $Y^3/3X^2$ and adding it to Eq.(4.85) with helping of Eq.(4.86). we find that

$$(\frac{\dot{Y}}{Y})^2 = \frac{1}{6b}\left(1 + 2X^2 - \frac{3X^4}{(\lambda - b\kappa^2 p)^2}\right). \tag{4.87}$$

Using Eq.(4.84) and Eq.(4.85) where Eq.(4.86) has also been used, this shows

$$\frac{d}{dt}(\frac{\dot{Y}}{Y}) = \frac{\dot{X}\dot{Y}}{XY} - \frac{1}{2b}\left[1 - \frac{X^4}{(\lambda - b\kappa^2 p)^2}\right]. \tag{4.88}$$

The definition of an effective EoS parameter is

$$w_{\rm eff} = \frac{\Sigma_i {\rm p_i}}{\Sigma_i\rho_i}\,. \tag{4.89}$$

It is for typographical convenience to write $w = w_{\rm eff}$, and we will clarify this setting when necessary.

The time derivative of Y is directly derived from Eq.(4.81)

$$\dot{Y} = \dot{a}\Big[(\lambda - bw\kappa^2\rho)(\lambda + b\kappa^2\rho)\Big]^{1/4} + \frac{a}{4}\frac{\Big[(-3b(\lambda - bw\kappa^2\rho)H\rho(1+w)) - b(\lambda + b\kappa^2\rho)(\dot{w}\rho + \dot{\rho}w)\Big]}{\Big[(\lambda - bw\kappa^2\rho)(\lambda + b\kappa^2\rho)\Big]^{3/4}} . \tag{4.90}$$

Replacing $\dot{\rho} = -3H\rho(1+w)$ to Eq.(4.90) with assuming[1] that $\dot{w} = 0$, we then get

$$\frac{\dot{Y}}{Y} = H\left[1 - \frac{3}{4}\frac{b\kappa^2(\rho+p)}{(\lambda + b\kappa^2\rho)} + \frac{3}{4}\frac{wb\kappa^2(\rho+p)}{(\lambda - b\kappa^2 p)}\right], \tag{4.91}$$

$$(\frac{\dot{Y}}{Y})^2 = H^2\left[1 - \frac{3}{4}\frac{b\kappa^2\rho(1+w)(\lambda - w\lambda - b\kappa^2\rho)}{(\lambda + b\kappa^2\rho)(\lambda - bw\kappa^2\rho)}\right]^2 = H^2F^2(\rho, w), \tag{4.92}$$

where we define

$$F(\rho, w, b) \equiv \left[1 - \frac{3b\kappa^2\rho(1+w)(1 - w - b\kappa^2\rho)}{4(1 + b\kappa^2\rho)(1 - bw\kappa^2\rho)}\right] . \tag{4.93}$$

Another way to write $(\dot{Y}/Y)^2$ comes from Eq.(4.87). That is

$$(\frac{\dot{Y}}{Y})^2 = \frac{1}{6b}\left[1 + 2\frac{(\lambda - bw\kappa^2\rho)^{3/2}}{(\lambda + b\kappa^2\rho)^{1/2}} - 3\frac{(\lambda - bw\kappa^2\rho)}{(\lambda + b\kappa^2\rho)}\right], \tag{4.94}$$

$$= \frac{1}{6b}\left[1 + 2U^2 - 3\frac{U^2}{V^2}\right] = \frac{G(\rho, w)}{6}, \tag{4.95}$$

where we define

$$G(\rho, w, b) \equiv \frac{1}{b}\left[1 + 2U^2 - 3\frac{U^2}{V^2}\right] . \tag{4.96}$$

The Hubble parameter of EiBI gravity is [46]

$$H^2 = \frac{G}{6F^2} . \tag{4.97}$$

[1]The effective equation of state parameter is assumed to be not changed abruptly during the evolution of the universe. Therefore it is reasonable to set $\dot{w}_{\rm eff} = 0$ at the stage of derivation.

After some manipulations, the Hubble parameter for flat universe represents in nonlinearly coupled of matter fields,

$$H^2_{\rm EiBI} = \frac{(\lambda - bw\kappa^2\rho)^2}{6b}\frac{\left[(\lambda + b\kappa^2\rho)^2 + 2(\lambda - bw\kappa^2\rho)^2(\lambda + b\kappa^2\rho)^{3/2} - 3(\lambda - bw\kappa^2\rho)(\lambda + b\kappa^2\rho)\right]}{\left[(\lambda - bw\kappa^2\rho)(\lambda + b\kappa^2\rho) - \frac{3}{4}b\kappa^2\rho(1+w)(\lambda - b\kappa^2\rho - w\lambda - bw\kappa^2\rho)\right]^2}. \tag{4.98}$$

For simplicity we can set $\lambda = 1 + b\Lambda \simeq 1$, the EiBI's Hubble parameter is according to

$$H^2_{\rm EiBI} = \frac{(1 - bw\kappa^2\rho)^2}{6b}\frac{\left[(1 + b\kappa^2\rho)^2 + 2(1 - bw\kappa^2\rho)^2(1 + b\kappa^2\rho)^{3/2} - 3(1 - bw\kappa^2\rho)(1 + b\kappa^2\rho)\right]}{\left[(1 - bw\kappa^2\rho)(1 + b\kappa^2\rho) - \frac{3}{4}b\kappa^2\rho(1+w)(1 - b\kappa^2\rho - w - bw\kappa^2\rho)\right]^2}. \tag{4.99}$$

It should be noted that the total energy density

$$\rho = \sum \rho_i = \rho_{\rm r} + \rho_{\rm b} + \rho_{\rm dm} + \rho_\Lambda \tag{4.100}$$

and the total pressure

$$p = \sum p_i = p_{\rm r} + p_{\rm b} + p_{\rm dm} + p_\Lambda. \tag{4.101}$$

The subscripts $\rm r, b$, dm, and Λ denote for radiation, baryonic matter, dark matter and cosmological constant respectively.
The EiBI's Hubble parameter for the radiation dominated epoch can be obtained by setting $w = \frac{1}{3}$ and $\rho = \rho_{\rm r}$. This is

$$H^2_{\rm rad} = \frac{1}{3b}\frac{(1 + b\kappa^2\rho_{\rm r})(3 - b\kappa^2\rho_{\rm r})^2}{(3 + b\kappa^2\rho_r)}\left[b\kappa^2\rho_{\rm r} - 1 + \frac{1}{3\sqrt{3}}\sqrt{(1 + b\kappa^2\rho_{\rm r})(3 - b\kappa^2\rho_{\rm r})^3}\right]. \tag{4.102}$$

Using the condition at bouncing point $H = 0$ and assuming that $w = \frac{1}{3}$, the critical density automatically appears as

$$\rho_{\rm B} = \frac{3}{\kappa^2 b} \quad \text{for b} > 0 \tag{4.103}$$

and

$$\rho_{\rm B} = \frac{1}{\kappa^2|b|} \quad \text{for b} < 0\,. \tag{4.104}$$

This form of the Hubble parameter at radiation dominated shown in Eq.(4.102) prohibits us to write the effective energy density at this energy level. Avilino[60], nevertheless, derived the EiBI's Hubble parameter at low energy level with the condition that $b\kappa^2\rho_m \ll 1$,

$$H^2_{\rm Low} = \frac{\kappa^2\rho_{\rm m}}{3} + \frac{\Lambda}{3} + \frac{b}{8}\Big[(\kappa^2\rho_{\rm m} + \Lambda)^2(w+1)(1-3w)\Big] + \mathcal{O}(b^2, b\Lambda, b\Lambda^2). \quad (4.105)$$

We thus write the low energy version of the effective density energy,

$$\rho_{\rm eff} = \rho_{\rm m} + \frac{\Lambda}{\kappa^2} + \rho_{\rm EiBI}\,, \quad (4.106)$$

The energy density of EiBI fluid is mandatory to define as

$$\rho_{\rm EiBI} = \frac{3b}{8\kappa^2}\Big[(\kappa^2\rho_m + \Lambda)^2(w+1)(1-3w)\Big]. \quad (4.107)$$

To be not confused, we would like to stress here that $\rho_{\rm m} = \rho_{\rm baryon} + \rho_{\rm dm}$ where the subscripts "baryon" and "dm" denote for ordinary matters and dark matter respectively. It can be noted that the EiBI free parameter b stimulates the gravitational interaction between $\rho_{\rm m}$ and ρ_Λ. The energy density of EiBI fluid tends to be zero at radiation dominated epoch and the present universe [1] as well where the effective EoS parameters $w = \frac{1}{3}$ (no effect at late time universe) and $w \simeq -1$(this affects late time universe) respectively.

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[1]For ΛCDM model, the effective EoS parameter

$$w_{\rm eff} \equiv \frac{p_{\rm tot}}{\rho_{\rm tot}} = \frac{p_{\rm m} + p_\Lambda}{\rho_{\rm m} + \rho_\Lambda} = \frac{-\frac{\Lambda}{\kappa^2}}{\rho_{\rm m} + \frac{\Lambda}{\kappa^2}} = \frac{-\Lambda}{\kappa^2\rho_{\rm m} + \Lambda}\,. \quad (4.108)$$

CHAPTER V

NMDC-PALATINI GRAVITY AND COSMOLOGY

5.1 Introduction to Non - Minimal Derivative Coupling theory

The non-minimal coupling between derivative of scalar field and the geometrical objects in gravitational action can act as a source of acceleration of the universe. The derivative coupling represented by the coupling function $f(\phi, \phi_{,\mu}, \phi_{,\mu\nu}, \ldots)$ is not the new story in physics but it is found in QED theory which the derivative coupling term between the vector potential A_μ and the scalar field requires the U(1) gauge invariance. It is worth to note that coupling terms like $T^\phi R$ and $T^\phi_{\mu\nu} R^{\mu\nu}$ in some modified gravity models are in fact the non-minimally derivative coupling terms [61]. Non-minimal derivative coupling to Ricci scalar also appears in low energy versions of higher dimensional theories and in Weyl anomaly of $N = 4$ conformal supergravity [62, 63]. Other forms of coupling terms apart from $R\phi_{,\mu}\phi^{;\mu}$ and $R^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$ are shown to be unimportant [64]. The gravitational action which have only MNDC terms and a free canonical kinetic term is found to posses de Sitter phase of expansion [65]. Furthermore, NMDC models with two different coupling constants κ_1 and κ_2 show their interesting physical interpretation of Higgs, quadratic potentials, inflation driven, and dark energy [66, 67, 68]. It is not surprised that in special case by adding the NMDC terms, i.e. $\kappa_1 R\phi_{,\mu}\phi^{;\mu}$ and $\kappa_2 R^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$, and redefining the NMDC coupling constant to be single value, i.e.$\kappa = -2\kappa_1 = \kappa_2$, it can lead to the expression of the Einstein tensor which is kinetically coupled to field derivative as $\kappa G_{\mu\nu}\phi^{;\mu}\phi^{;\nu}$. This term indeed appears in subclass of Hondeski action[69], i.e. $\mathcal{L}_5 = G_5(\phi, X)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi$ where $X \equiv g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. Suskov showed that the divergent free object like the Einstein tensor kinetically coupled to scalar field gives a good dynamical theory by contribution in equation of motions of second-order derivative in $g_{\mu\nu}$ and ϕ. In spatially flat FLRW universe, choosing the positive NMDC coupling constant ($\kappa > 0$) supports the existence of quasi-de Sitter phase appeared at very early stage, whereas the negative NMDC coupling constant ($\kappa < 0$) advocates the appearing of initial singularity at very early stage. The expansion phase at very late time universe always occurs whether or not the NMDC coupling parameter is negative or positive signs with the scale factor $a \propto t^{1/3}$[70]. Inflationary phase is always possible with $\kappa > 0$ and $V =$ const. Besides, other types of expansions can generate by including $V =$ const and allowing for phantom phase by change sign of the free kinetic term [71]. The less steep potential than quadratic is required to generate inflationary phase [73]. By proposing the matter term and a constant potential, there has the transferable phase to change from inflation to matter domination phase without reheating and this establishes a direct connection between inflationary phase and soft-inflation at late time universe [74]. The heavy particles creation rate is actually found to decrease with increasing values of the coupling strength to inflaton field [75]. Without a free kinetic term in NMDC model, the existence of superluminal sound speed is undeniable [72]. This model does not give phantom crossing phase by setting $V(\phi) = 0$ and keeping $\kappa G_{\mu\nu}\phi^{;\mu}\phi^{;\nu}$ and free kinetic term in the model. By including positive potential and $\kappa > 0$ with confined the Hubble parameter, there is no limit of $\dot{\phi}$ [73]. The full form of NMDC action in metric formalism with adding potential term in the gravitational action is expressed as

$$S(g) = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R(g)}{8\pi G_{\mathrm{N}}} - \left(\varepsilon g_{\mu\nu} + \kappa G_{\mu\nu}(g)\right)\phi^{;\mu}\phi^{;\nu} - 2V(\phi)\right] + S_{\mathrm{m}}\,. \qquad (5.1)$$

Inflation and perturbation analysis of the model with constant potential are investigated with observational data [76]. Some attempts try to work out for resemble forms of NMDC gravity are presented, see such as [77, 78, 79, 80, 85, 86, 87, 88, 89, 90, 91, 92] and for recent review we refer to [93]. It is interesting to introduce the new form of NMDC action in metric formalism. Inspired by one coupling parameter of Granda's model [83], the new non-minimal derivative coupling to Ricci scalar gravity is proposed by transforming all the field value to its logarithm,

$\phi' \to \phi' = \mu \ln \phi$. The field logarithm form action is[94]

$$S(g) \;=\; \int \mathrm{d}^4x\sqrt{-g}\Big[\frac{1}{16\pi G_{\mathrm{N}}}R(g) - \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}\xi R(g)\partial_\mu\phi'\partial^\mu\phi' - V(\phi')\Big] + S_{\mathrm{m}}. \tag{5.2}$$

By restoring the transformation back, $\phi' = \mu \ln \phi$, we then get

$$S(g) \;=\; \int \mathrm{d}^4x\sqrt{-g}\Big[\frac{1}{16\pi G_{\mathrm{N}}}R(g) - \frac{1}{2}\left(\frac{\mu^2}{\phi^2}\right)\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\xi R(g)\left(\frac{\mu^2}{\phi^2}\right)\partial_\mu\phi\partial^\mu\phi - V(\phi)\Big] + S_{\mathrm{m}}. \tag{5.3}$$

The important point of this action is the contribution of term like $\frac{\mu^2}{\phi^2}$. This term is found to increase with decreasing values of ϕ. This re-scaling field also affects the dimensional setting of scalar field and the coupling strength of the action shown in Eq.(5.2). The Friedmann equation in this case is

$$H^2 \simeq \frac{8\pi G_{\mathrm{N}}}{3}\rho_{\mathrm{m}} + \frac{8\pi G_{\mathrm{N}}}{6}\Big[(1 + 12\xi\dot{H} - 18\xi H^2)\frac{\mu^2}{\phi^2}\dot{\phi}^2 + 2V(\phi') - 12\xi H\dot{\phi}\frac{dV\phi}{d\phi}\Big]. \tag{5.4}$$

All possible setting of parameters and potential forms in NMDC action Eq.(5.1) above are under metric formalism. We know that only Einstein-Hilbert action is shown an equivalence between metric and Palatini formalism. Gravity models beyond Einstein general relativity which have coupling terms between scalar field or its derivative to basic variables for gravity under Palatini formulation, of course, give different field equations compared to work in metric formulation. The metric tensor and the connection field are suggested to be independent dynamical objects under Palatini formalism. Hence the Palatini connection is not constructed from the metric $g_{\mu\nu}$ [24, 96, 97, 98]. Recently the NMDC-Palatini version was explored by suggesting two different coupling parameters; the first one is $\kappa_1 R(\Gamma)\phi_{,\mu}\phi^{,\mu}$ and the second one is $\kappa_2 R^{\mu\nu}(\Gamma)\phi_{,\mu}\phi_{,\nu}$. Phantom crossing with oscillating equation of state parameter is allowed in that Palatini model [99]. In this work, we derived the field equations for NMDC gravity in Palatini formalism by redefining two NMDC coupling constant κ_1 and κ_2 to be a single coupling constant κ in the same form of

Eq.(5.1). Capozziello suggested that the constraint equations derived from Palatini formalism relates to the conformal transformation[56] . More specifically, the generalised conformal transformations, which is first introduced by J. D. Bekenstein [57], consist of the conformal factor and the disformal factor. To make it clear meaning, this conformal transformation is a change of local units of length between two points which leaves global shapes invariant with shrink or stretch all spacetime direction equally, whereas the disformal transformation changes the units of length along the direction of gradient of scalar field for this reason the global shapes are distorted [57]. It is speculative in the next section that this work[100] is indeed the conformal transformation.

5.2 NMDC-Palatini action and field equations

The Einstein frame expression of Suskov's NMDC action in the metric formalism is [1]

$$S(g) = \int \mathrm{d}^4x\sqrt{-g}\left\{R(g) - \left[\varepsilon g_{\mu\nu} + \kappa\left(R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g)\right)\right]\phi^{,\mu}\phi^{,\nu} - 2V(\phi)\right\} + S_{\mathrm{m}}\left[g_{\mu\nu},\Psi\right], \tag{5.5}$$

where Ψ denotes the collecting of matter fields. Throughout this work we set $c = 1$ and $8\pi G_{\mathrm{N}} = 1$. In Palatini formalism, the NMDC action is expressed in the form [99]

$$S(g,\Gamma) = \int \mathrm{d}^4x\sqrt{-g}\left\{\tilde{R}(\Gamma) - \left[\varepsilon g_{\mu\nu} + \kappa_1 g_{\mu\nu}\tilde{R}(\Gamma) + \kappa_2\tilde{R}_{\mu\nu}(\Gamma)\right]\phi^{,\mu}\phi^{,\nu} - 2V(\phi)\right\} + S_{\mathrm{m}}\left[g_{\mu\nu},\Psi\right], \tag{5.6}$$

where tilde symbol signifies variables in Palatini formalism. We can set $\kappa = \kappa_2 = -2\kappa_1$ in the same spirit with Suskov and we define the Einstein tensor in Palatini

[1] The reason why we call the action in Eq.(5.1) the Einstein frame action is because the Ricci scalar $R(\Gamma)$ in the Einstein-Hilbert action does not coupling to any forms of scalar field.

formalism as

$$\tilde{G}_{\mu\nu}(\Gamma) = \tilde{R}_{\mu\nu}(\Gamma) - \frac{1}{2}g_{\mu\nu}\tilde{R}(\Gamma). \tag{5.7}$$

Hence the NMDC-Palatini action becomes

$$S(g,\Gamma) \;=\; \int \mathrm{d}^4x\sqrt{-g}\left\{\tilde{R}(\Gamma) - \left[\varepsilon g_{\mu\nu} + \kappa G_{\mu\nu}(\Gamma)\right]\phi^{,\mu}\phi^{,\nu} - 2V(\phi)\right\} + S_{\mathrm{m}}[g_{\mu\nu},\Psi]\,. \tag{5.8}$$

The Palatini Ricci tensor can be constructed from the independent connection by the following relation

$$\tilde{R}_{\mu\nu}(\Gamma) \;=\; \tilde{R}^{\lambda}_{\mu\lambda\nu}(\Gamma) \;=\; \partial_\lambda\Gamma^{\lambda}{}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\lambda}. \tag{5.9}$$

Taking trace of the the Palatini Ricci tensor, we get the Palatini Ricci scalar

$$\tilde{R}(\Gamma) = g^{\mu\nu}\tilde{R}_{\mu\nu}(\Gamma). \tag{5.10}$$

Varying the Palatini NMDC action in Eq. (5.8) with respect to the metric,

$$\begin{aligned}
\delta_{\mathrm{g}}S(g,\Gamma) \;=&\; \int d^4x\delta_{\mathrm{g}}\Big\{\sqrt{-g}\tilde{R}(\Gamma) - \sqrt{-g}\left[\varepsilon g_{\mu\nu} - \frac{\kappa}{2}g_{\mu\nu}\tilde{R}(\Gamma) + \kappa\tilde{R}_{\mu\nu}(\Gamma)\right]\phi^{,\mu}\phi^{,\nu} - 2\sqrt{-g}V(\phi)\Big\} \\
&+\delta_{\mathrm{g}}S_m \\
=&\; \int d^4x\Bigg\{\delta_{\mathrm{g}}(\sqrt{-g}\tilde{R}(\Gamma)) - \epsilon\delta_{\mathrm{g}}\left(g_{\mu\nu}\sqrt{-g}\phi^{,\mu}\phi^{,\nu}\right) + \frac{\kappa}{2}\Big[\delta_{\mathrm{g}}\left(\sqrt{-g}g_{\mu\nu}g^{\alpha\beta}\tilde{R}_{\alpha\beta}(\Gamma)\phi^{,\mu}\phi^{,\nu}\right)\Big] \\
&-\delta_{\mathrm{g}}\Big(\sqrt{-g}\kappa R_{\mu\nu}(\Gamma)g^{\mu\alpha}\phi_{,\alpha}\phi^{,\nu}\Big) - 2\left(\delta_{\mathrm{g}}\sqrt{-g}V(\phi)\right)\Bigg\} + \delta_{\mathrm{g}}S_m = 0.
\end{aligned} \tag{5.11}$$

It is important to note that the relation

$$\delta_g R_{\mu\nu}(\Gamma) = \frac{\delta R_{\mu\nu}(\Gamma)}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = 0 \tag{5.12}$$

is applied only in Palatini-formalism since the Ricci tensor $R_{\mu\nu}(\Gamma)$ is independent of the metric tensor $g_{\mu\nu}$. For conciseness, we will drop a subscript g which represents the variation with respect to the metric $g_{\mu\nu}$. Our aim is to show explicitly how to get the first field equation, so let us start from the first term in the braces on the right hand side of Eq.(5.11),

$$\delta(\sqrt{-g}\tilde{R}(\Gamma)) = (\delta\sqrt{-g})\tilde{R}(\Gamma) + \sqrt{-g}(\delta\tilde{R}(\Gamma)) = \sqrt{-g}\left[-\frac{1}{2}g_{\alpha\beta}\tilde{R}(\Gamma) + \tilde{R}_{\alpha\beta}(\Gamma)\right]\delta g^{\alpha\beta}. \tag{5.13}$$

The second term in the braces on the right hand side of Eq.(5.11) is

$$
\begin{aligned}
-\epsilon\delta(g_{\mu\nu}\sqrt{-g}\phi^{,\mu}\phi^{,\nu}) &= -\epsilon\Big[(\delta g_{\mu\nu})\sqrt{-g}\phi^{,\mu}\phi^{,\nu} + (\delta\sqrt{-g})g_{\mu\nu}\phi^{,\mu}\phi^{,\nu} + g_{\mu\nu}\sqrt{-g}\delta(\phi^{,\mu}\phi^{,\nu})\Big], \\
&= -\epsilon\Big[-\sqrt{-g}(\delta g^{\alpha\beta})g_{\alpha\mu}\,g_{\nu\beta}\phi^{,\mu}\phi^{,\nu} - \frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}g_{\mu\nu}\phi^{,\mu}\phi^{,\nu} \\
&\quad +\sqrt{-g}(2\phi_{,\alpha}\phi_{,\beta})\delta g^{\alpha\beta}\Big],
\end{aligned} \tag{5.14}
$$

where we use $\delta g_{\mu\nu} = -g_{\mu\lambda}g_{\nu\tau}\delta g^{\lambda\tau}$ and

$$
\begin{aligned}
g_{\mu\nu}\sqrt{-g}\delta(\phi^{,\mu}\phi^{,\nu}) &= g_{\mu\nu}\sqrt{-g}\left[\phi_{,\sigma}\phi^{,\nu}\delta g^{\mu\sigma} + \phi_{,\lambda}\phi^{,\mu}\delta g^{\nu\lambda}\right], \\
&= \sqrt{-g}\left[g_{\mu\nu}\phi_{,\sigma}\phi^{,\nu}\delta g^{\mu\sigma} + g_{\mu\nu}\phi_{,\lambda}\phi^{,\mu}\delta g^{\nu\lambda}\right], \\
&= \sqrt{-g}\left[g_{\mu\lambda}\phi_{,\nu}\phi^{,\lambda} + g_{\rho\nu}\phi_{,\mu}\phi^{,\rho}\right]\delta g^{\mu\nu}, \\
&= \sqrt{-g}\left[\phi_{,\alpha}\phi_{\beta} + \phi_{,\alpha}\phi_{\beta}\right]\delta g^{\alpha\beta}, \\
&= \sqrt{-g}\left[2\phi_{,\alpha}\phi_{\beta}\right]\delta g^{\alpha\beta}.
\end{aligned} \tag{5.15}
$$

between the steps of calculation. The first term becomes

$$
\begin{aligned}
-\epsilon\delta(g_{\mu\nu}\sqrt{-g}\phi^{,\mu}\phi^{,\nu}) &= -\epsilon\Big[-\delta g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\sqrt{-g} - \frac{1}{2}\sqrt{-g}g_{\alpha\beta}\phi_{,\nu}\phi^{,\nu}\delta g^{\alpha\beta} \\
&\quad +\sqrt{-g}\,[2\phi_{,\alpha}\phi_{\beta}]\delta g^{\alpha\beta}\Big], \\
&= \sqrt{-g}\Big[\epsilon\phi_{,\alpha}\phi_{,\beta} + \frac{\epsilon}{2}g_{\alpha\beta}\phi_{,\nu}\phi^{\nu} - 2\epsilon\phi_{,\alpha}\phi_{\beta}\Big]\delta g^{\alpha\beta}.
\end{aligned} \tag{5.16}
$$

We aim to write time - time component of the first term here, i.e.

$$
\epsilon\phi_{,\alpha}\phi_{,\beta} + \frac{\epsilon}{2}g_{\alpha\beta}\phi_{,\nu}\phi^{\nu} - 2\epsilon\phi_{,\alpha}\phi_{\beta}. \tag{5.17}
$$

We hence get

$$
\Big[\epsilon\dot{\phi}^2 + \frac{\epsilon}{2}\dot{\phi}^2 - 2\epsilon\dot{\phi}^2\Big] = \Big[-\frac{\epsilon}{2}\dot{\phi}^2\Big]. \tag{5.18}
$$

The third term in the braces on the right hand side of Eq.(5.11) is

$$\begin{aligned}\frac{\kappa}{2}\Big[\delta(\sqrt{-g}g_{\mu\nu}g^{\alpha\beta}\tilde{R}_{\alpha\beta}(\Gamma)\phi^{,\mu}\phi^{,\nu})\Big] &= \frac{\kappa}{2}\Big[(\delta\sqrt{-g})g_{\mu\nu}\tilde{R}(\Gamma)\phi^{,\mu}\phi^{,\nu} + \sqrt{-g}(\delta g_{\mu\nu})\tilde{R}(\Gamma)\phi^{,\mu}\phi^{,\nu} \\ &+ \sqrt{-g}g_{\mu\nu}(\delta g^{\alpha\beta})\tilde{R}_{\alpha\beta}(\Gamma)\phi^{,\mu}\phi^{,\nu} \\ &+ \sqrt{-g}\tilde{R}(\Gamma)\delta(g^{\mu\alpha}g_{\mu\nu})\phi_{,\alpha}\phi^{,\nu}\Big] \\ &+ \frac{\kappa}{2}\Big[\sqrt{-g}g_{\mu\nu}\tilde{R}(\Gamma)2\delta(\phi^{,\mu}\phi^{,\nu})\Big].\end{aligned} \tag{5.19}$$

Using relations $\delta g_{\mu\nu} \equiv -g_{\alpha\mu}g_{\beta\nu}\delta g^{\alpha\beta}$ and

$$\phi^{,\mu}\phi^{,\nu}\delta g_{\mu\nu} = -g_{\alpha\mu}g_{\beta\nu}\phi^{,\mu}\phi^{\nu}\delta g^{\alpha\beta} = -\phi_{,\alpha}\phi_{,\beta}\delta g^{\alpha\beta} \tag{5.20}$$

, the third term becomes

$$\begin{aligned}\frac{\kappa}{2}\left[\delta(\sqrt{-g}g_{\mu\nu}g^{\alpha\beta}\tilde{R}_{\alpha\beta}(\Gamma)\phi^{,\mu}\phi^{,\nu})\right] &= \frac{\kappa}{2}\left[-\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}g_{\mu\nu}\phi^{,\mu}\phi^{,\nu}\tilde{R}(\Gamma) - \sqrt{-g}\tilde{R}(\Gamma)\phi_{,\alpha}\phi_{,\beta}\delta g^{\alpha\beta}\right. \\ &\left. +\sqrt{-g}g_{\mu\nu}\tilde{R}_{\alpha\beta}(\Gamma)\phi^{,\mu}\phi^{\nu}\delta g^{\alpha\beta} + \sqrt{-g}\,\tilde{R}(\Gamma)(2\phi_{,\alpha}\,\phi_{,\beta})\,\delta g^{\alpha\beta}\right], \\ &= \sqrt{-g}\Big[-\frac{\kappa}{4}g_{\alpha\beta}\phi_{,\nu}\phi^{,\nu}\tilde{R}(\Gamma) - \frac{\kappa}{2}\tilde{R}(\Gamma)\phi_{,\alpha}\phi_{,\beta} \\ &+\frac{\kappa}{2}\tilde{R}_{\alpha\beta}(\Gamma)\phi_{,\lambda}\phi^{,\lambda} + \frac{\kappa}{2}\tilde{R}(\Gamma)(2\phi_{,\alpha}\,\phi_{,\beta})\Big]\delta g^{\alpha\beta}\,, \\ &= \sqrt{-g}\Big[-\frac{\kappa}{4}g_{\alpha\beta}\tilde{R}(\Gamma)\phi_{,\lambda}\phi^{,\lambda} - \frac{\kappa}{2}\tilde{R}(\Gamma)\phi_{,\alpha}\phi_{,\beta} \\ &+ \frac{\kappa}{2}\tilde{R}_{\alpha\beta}(\Gamma)\phi_{,\lambda}\phi^{,\lambda} + \kappa\tilde{R}(\Gamma)\phi_{,\alpha}\phi_{,\beta}\Big]\delta g^{\alpha\beta}\,, \\ &= \sqrt{-g}\left[\frac{\kappa}{2}\Big(\tilde{R}_{\alpha\beta}(\Gamma) - \frac{1}{2}g_{\alpha\beta}\tilde{R}(\Gamma)\Big)\phi_{,\lambda}\phi^{,\lambda}\right. \\ &\left. -\frac{\kappa}{2}\tilde{R}(\Gamma)\phi_{,\alpha}\phi_{,\beta} + \kappa\tilde{R}(\Gamma)\phi_{,\alpha}\phi_{,\beta}\right]\delta g^{\alpha\beta}\,, \\ &= \sqrt{-g}\left[\frac{\kappa}{2}\tilde{G}_{\alpha\beta}(\Gamma)\phi^{,\lambda}\phi_{,\lambda} + \frac{\kappa}{2}\tilde{R}(\Gamma)\phi_{,\alpha}\phi_{,\beta}\right]\delta g^{\alpha\beta}.\end{aligned} \tag{5.21}$$

The fourth term in the braces on the right-hand side of Eq.(5.11) is

$$
\begin{aligned}
-\delta\Big[\sqrt{-g}\kappa R_{\mu\nu}(\Gamma)g^{\mu\alpha}\phi_{,\alpha}\phi^{,\nu}\Big] &= -\Big[(\delta\sqrt{-g})\kappa\tilde{R}_{\mu\nu}(\Gamma)g^{\mu\alpha}\phi_{,\alpha}\phi^{,\nu} + \kappa\sqrt{-g}\tilde{R}_{\mu\nu}(\Gamma)(\delta g^{\mu\alpha})\phi_{,\alpha}\phi^{,\nu} \\
&\quad + \sqrt{-g}\,\kappa\tilde{R}_{\mu\nu}(\Gamma)\,\delta(\phi^{,\mu}\phi^{,\nu})\Big], \\
&= -\Big[-\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}\kappa\tilde{R}_{\mu\nu}(\Gamma)g^{\mu\alpha}\phi_{,\alpha}\phi^{,\nu} \\
&\quad + \sqrt{-g}\,\kappa R_{\mu\nu}(\Gamma)\phi_{,\alpha}\phi^{,\nu}\delta g^{\mu\alpha} + \sqrt{-g}\,\kappa\tilde{R}_{\mu\nu}(\Gamma)(\phi_{,\sigma}\phi^{,\nu}\delta g^{\mu\sigma} \\
&\quad + \phi_{,\lambda}\phi^{,\mu}\delta g^{\nu\lambda})\Big] \\
&= \sqrt{-g}\,\Big[\frac{1}{2}g_{\alpha\beta}\,\kappa\tilde{R}_{\mu\nu}(\Gamma)\phi^{,\mu}\phi^{,\nu} - \kappa\tilde{R}_{\beta\nu}(\Gamma)\phi_{,\alpha}\phi^{,\nu} \\
&\quad - \kappa\tilde{R}_{\alpha\nu}(\Gamma)\phi_{,\beta}\phi^{,\nu} - \kappa\tilde{R}_{\nu\alpha}(\Gamma)\phi_{,\beta}\phi^{,\nu}\Big]\delta g^{\alpha\beta}, \\
&= \sqrt{-g}\Big[\frac{\kappa}{2}g_{\alpha\beta}\tilde{R}_{\mu\nu}(\Gamma)\phi^{,\mu}\phi^{,\nu} - \kappa\tilde{R}_{\beta\nu}(\Gamma)\phi_{,\alpha}\phi^{,\nu} \\
&\quad - 2\kappa R_{\alpha\nu}(\Gamma)\phi_{,\beta}\phi^{,\nu}\Big]\delta g^{\alpha\beta}.
\end{aligned} \tag{5.22}
$$

The fifth term for arbitrary potentials is

$$
\begin{aligned}
-2\Big[\delta\sqrt{-g}V(\phi)\Big] &= 2V(\phi)\frac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\,\delta g^{\alpha\beta}, \\
&= \sqrt{-g}\,\Big[g_{\alpha\beta}V(\phi)\Big]\,\delta g^{\alpha\beta}.
\end{aligned} \tag{5.23}
$$

Putting Eq.(5.13) - Eq.(5.23) together , the variation of NMDC-Palatini action with respect to the metric $g_{\mu\nu}$ can be written as

$$
\begin{aligned}
\delta_{\rm g}S(g,\Gamma) = 0 &= \int d^4x\sqrt{-g}\Bigg[\bigg(\tilde{R}_{\alpha\beta}(\Gamma) - \frac{1}{2}g_{\alpha\beta}\tilde{R}(\Gamma)\bigg) + \frac{\kappa}{2}\tilde{R}_{\mu\nu}(\Gamma)g_{\alpha\beta}\phi^{,\mu}\phi^{,\nu} \\
&\quad - \kappa\tilde{R}_{\beta\nu}(\Gamma)\phi_{,\alpha}\phi^{,\nu} - 2\kappa R_{\alpha\nu}(\Gamma)\phi_{,\beta}\phi^{,\nu} - \frac{\kappa}{4}g_{\alpha\beta}\tilde{R}(\Gamma)\phi_{,\lambda}\phi^{,\lambda} - \frac{\kappa}{2}\tilde{R}(\Gamma)\phi_{,\alpha}\phi_{,\beta} \\
&\quad + \frac{\kappa}{2}\tilde{R}_{\alpha\beta}(\Gamma)\phi_{,\lambda}\phi^{,\lambda} + \kappa\tilde{R}(\Gamma)\phi_{,\alpha}\phi_{,\beta} + \frac{\epsilon}{2}g_{\alpha\beta}\phi_{,\nu}\phi^{,\nu} + \epsilon\phi_{,\alpha}\phi_{,\beta} - 2\epsilon g_{\alpha\nu}\phi_{,\beta}\phi^{,\nu} \\
&\quad + g_{\alpha\beta}V(\phi) - T_{\alpha\beta}\Bigg]\delta g^{\alpha\beta}.
\end{aligned} \tag{5.24}
$$

Hence, the first field equation can be written as

$$
\begin{aligned}
T_{\mu\nu} &= \tilde{G}_{\mu\nu}(\Gamma) + \Bigg[\frac{\kappa}{2}\tilde{G}_{\mu\nu}(\Gamma)\phi_{,\lambda}\phi^{,\lambda} + \frac{\kappa}{2}\tilde{R}_{\alpha\beta}(\Gamma)g_{\mu\nu}\phi^{,\alpha}\phi^{,\beta} - \kappa\tilde{R}_{\nu\lambda}(\Gamma)\phi_{,\mu}\phi^{,\lambda} \qquad (5.25) \\
&\quad + \frac{\kappa}{2}\tilde{R}(\Gamma)\phi_{,\mu}\phi_{,\nu} - 2\kappa\tilde{R}_{\mu\lambda}(\Gamma)\phi_{,\nu}\phi^{,\lambda} + \frac{\varepsilon}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha} - \varepsilon\phi_{,\mu}\phi_{,\nu} + g_{\mu\nu}V(\phi)\Bigg],
\end{aligned}
$$

where the definition of the energy-momentum tensor for matter field is

$$T^{(\mathrm{m})}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m[g_{\kappa\lambda},\Psi]}{\delta g^{\mu\nu}}. \tag{5.26}$$

Next, we will show explicitly how to take variation of the NMDC - Palatini action with respect to the independent connection $\Gamma^{\lambda}{}_{\mu\nu}$, and redefine the NMDC coupling constants κ_1 and κ_2 to be a single coupling constant (κ) as shown in Eq.(5.8). Taking variation the the NMDC - Palatini action with respect to the independent connection $\Gamma^{\lambda}{}_{\mu\nu}$, this shows that

$$\begin{aligned}\delta_\Gamma S(g,\Gamma) \;=\; & \int d^4x\Big[\delta_\Gamma(\sqrt{-g}\,R(\Gamma)) - \cancel{\sqrt{-g}\,\epsilon\, g_{\mu\nu}\,\phi^{,\mu}\phi^{,\nu}\,\delta_\Gamma(1)} - \kappa_1\, g_{\mu\nu}\,\phi^{,\mu}\phi^{,\nu}\sqrt{-g}\,\delta_\Gamma R(\Gamma) \\ & -\kappa_2\phi^{,\mu}\phi^{,\nu}\,(\delta_\Gamma R_{\mu\nu}(\Gamma))\sqrt{-g} - \cancel{2V(\phi)\sqrt{-g}\,\delta_\Gamma(1)}\Big], \end{aligned} \tag{5.27}$$

where a subscript Γ denotes performing variation the gravitational action with respect to the connection field. It is convenient to be calculated separately for each term by defining terms as follows:

Term A is

$$\begin{aligned} A \;\equiv\; & \int d^4x\sqrt{-g}(\delta_\Gamma R(\Gamma))\,, \\ =\; & \int d^4x\sqrt{-g}\Big[\cancel{(\delta_\Gamma g^{\mu\nu})R_{\mu\nu}(\Gamma)} + g^{\mu\nu}(\nabla^\Gamma_\lambda\delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu\delta\Gamma^\lambda_{\mu\lambda})\Big]\,, \\ =\; & \int d^4x\Big[\sqrt{-g}g^{\mu\nu}(\nabla^\Gamma_\lambda\delta\Gamma^\lambda_{\mu\nu}) - \sqrt{-g}g^{\mu\nu}(\nabla^\Gamma_\nu\delta\Gamma^\lambda_{\mu\lambda})\Big]\,, \\ =\; & \int d^4x\Big[\cancel{\nabla^\Gamma_\lambda(\tilde g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu})} - \cancel{\nabla^\Gamma_\nu(\tilde g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda})} - (\nabla^\Gamma_\lambda\tilde g^{\mu\nu})\delta\Gamma^\lambda_{\mu\nu} + (\nabla^\Gamma_\nu\tilde g^{\mu\nu})\delta\Gamma^\lambda_{\mu\lambda}\Big]\,, \\ =\; & \int d^4x\Big[-\nabla^\Gamma_\lambda\tilde g^{\mu\nu} + \delta^\nu_\lambda\nabla^\Gamma_\nu\tilde g^{\mu\nu}\Big]\delta\Gamma^\lambda_{\mu\nu}\,, \end{aligned} \tag{5.28}$$

where we define $\tilde g^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$. Term B is

$$B \equiv -\int d^4x\sqrt{-g}\epsilon g_{\mu\nu}\phi^{,\mu}\phi^{,\nu}\cancel{(\delta_\Gamma[1])} = 0. \tag{5.29}$$

Term C is

$$
\begin{aligned}
C &\equiv -\int d^4x\sqrt{-g}\kappa_1 g_{\mu\nu}\phi^{,\mu}\phi^{,\nu}\delta_\Gamma R(\Gamma),\\
&= -\int d^4x\kappa_1\phi^{,\mu}\phi^{,\nu}\sqrt{-g}\,g_{\mu\nu}\delta_\Gamma R(\Gamma),\\
&= -\int d^4x\kappa_1\phi^{,\mu}\phi^{,\nu}\sqrt{-g}\,g_{\mu\nu}\Big[g^{\alpha\beta}\nabla^\Gamma_\lambda\delta\Gamma^\lambda_{\alpha\beta} - g^{\alpha\beta}\nabla^\Gamma_\beta\delta\Gamma^\lambda_{\alpha\lambda}\Big],\\
&= \int d^4x\Big[\nabla^\Gamma_\lambda\left(\sqrt{-g}\,g_{\mu\nu}\kappa_1\phi^{,\mu}\phi^{,\nu}g^{\alpha\beta}\right)\delta\Gamma^\lambda_{\alpha\beta} - \nabla^\Gamma_\beta\left(\sqrt{-g}\,g_{\mu\nu}\kappa_1\phi^{,\mu}\phi^{,\nu}g^{\alpha\beta}\right)\delta\Gamma^\lambda_{\alpha\lambda}\Big],\\
&= \int d^4x\Big[\nabla^\Gamma_\lambda\left(\sqrt{-g}\,g_{\mu\nu}\kappa_1\phi^{,\mu}\phi^{,\nu}g^{\alpha\beta}\right) - \delta^\beta_\lambda\nabla^\Gamma_\sigma\left(\sqrt{-g}\,g_{\mu\nu}\kappa_1\phi^{,\mu}\phi^{,\nu}g^{\alpha\sigma}\right)\Big]\delta\Gamma^\lambda_{\alpha\beta}.
\end{aligned}
\tag{5.30}
$$

It is convenient to redefine Eq.(5.30) to

$$
\int d^4x\Big[\nabla^\Gamma_\lambda\left(\sqrt{-g}\,g_{\alpha\beta}\kappa_1\phi^{,\alpha}\phi^{,\beta}g^{\mu\nu}\right) - \delta^\nu_\lambda\nabla^\Gamma_\sigma\left(\sqrt{-g}\,g_{\alpha\beta}\kappa_1\phi^{,\alpha}\phi^{,\beta}g^{\mu\sigma}\right)\Big]\delta\Gamma^\lambda_{\mu\nu} \tag{5.31}
$$

Term D is

$$
\begin{aligned}
D &\equiv -\int d^4x\sqrt{-g}\,(\delta_\Gamma R_{\mu\nu}(\Gamma))\kappa_2\phi^{,\mu}\phi^{,\nu}\\
&= -\int d^4x\,\sqrt{-g}\,g^{\mu\sigma}\,(\delta_\Gamma\tilde{R}_{\mu\nu}(\Gamma))\kappa_2\phi_{,\sigma}\phi^{,\nu}\\
&= -\int d^4x\,\sqrt{-g}\,g^{\mu\sigma}(\nabla^\Gamma_\lambda\delta\Gamma^\lambda_{\mu\nu} - \nabla^\Gamma_\nu\delta\Gamma^\lambda_{\mu\lambda})\kappa_2\phi_{,\sigma}\phi^{,\nu}\\
&= \int d^4x\,\Big[\nabla^\Gamma_\lambda(\sqrt{-g}\,g^{\mu\sigma}\kappa_2\phi_{,\sigma}\phi^{,\nu}) - \delta^\nu_\lambda\nabla^\Gamma_\tau(\sqrt{-g}\,g^{\mu\sigma}\kappa_2\phi_{,\sigma}\phi^{,\tau})\Big]\delta\Gamma^\lambda_{\mu\nu}
\end{aligned}
\tag{5.32}
$$

Term E is

$$
\int d^4x\,(-2V(\phi)\,\sqrt{-g}\,\delta_\Gamma(1)) = 0 \tag{5.33}
$$

The independent constraint equations from term A,C, and D add up together to give rise the simplest constraint equations for the gravity model

$$
\begin{aligned}
&-\nabla^\Gamma_\lambda\tilde{g}^{\mu\nu} + \delta^\nu_\lambda\nabla^\Gamma_\nu\tilde{g}^{\mu\nu} + \nabla^\Gamma_\lambda\left(\sqrt{-g}\,g_{\alpha\beta}\kappa_1\phi^{,\alpha}\phi^{,\beta}g^{\mu\nu}\right) - \delta^\nu_\lambda\nabla^\Gamma_\sigma\left(\sqrt{-g}\,g_{\alpha\beta}\kappa_1\phi^{,\alpha}\phi^{,\beta}g^{\mu\sigma}\right)\\
&+\nabla^\Gamma_\lambda(\sqrt{-g}\,g^{\mu\sigma}\kappa_2\phi_{,\sigma}\phi^{,\nu}) - \delta^\nu_\lambda\nabla^\Gamma_\tau(\sqrt{-g}\,g^{\mu\sigma}\kappa_2\phi_{,\sigma}\phi^{,\tau}) = 0.
\end{aligned}
\tag{5.34}
$$

By setting $\lambda = \nu$, the above constraint equations becomes

$$
3\nabla^\Gamma_\sigma\Big[\sqrt{-g}g^{\mu\sigma} - \sqrt{-g}g_{\alpha\beta}\kappa_1\phi^{,\alpha}\phi^{,\beta}g^{\mu\sigma} - \sqrt{-g}\,g^{\mu\tau}\kappa_2\phi_{,\tau}\phi^{,\sigma}\Big] = 0, \tag{5.35}
$$

Substituting Eq.(5.35) to Eq.(5.34), we then get

$$\nabla_\lambda^\Gamma\Big[\sqrt{-g}g^{\mu\nu} - \sqrt{-g}g_{\alpha\beta}\kappa_1\phi^{,\alpha}\phi^{,\beta}g^{\mu\nu} - \sqrt{-g}\,g^{\mu\sigma}\kappa_2\phi_{,\sigma}\phi^{,\nu}\Big] = 0, \tag{5.36}$$

It can further simplifies to

$$\nabla_\lambda^\Gamma\Big[\sqrt{-g}\left(g^{\mu\nu} - g_{\alpha\beta}\kappa_1\phi^{,\alpha}\phi^{,\beta}g^{\mu\nu} - \, g^{\mu\sigma}\kappa_2\phi_{,\sigma}\phi^{,\nu}\right)\Big] = 0, \tag{5.37}$$

$$\nabla_\lambda^\Gamma\Big[\sqrt{-g}\,(g^{\mu\nu} - \kappa_1\phi_{,\alpha}\phi^{,\alpha}g^{\mu\nu} - \kappa_2\,g^{\mu\sigma}\phi_{,\sigma}\phi^{,\nu})\Big] = 0, \tag{5.38}$$

$$\nabla_\lambda^\Gamma\Big[\sqrt{-g}g^{\mu\nu}\,(1 - \kappa_1\phi_{,\alpha}\phi^{,\alpha} - \kappa_2\delta_\nu^\sigma\phi_{,\sigma}\phi^{,\nu})\Big] = 0, \tag{5.39}$$

$$\nabla_\lambda^\Gamma\Big[\sqrt{-g}g^{\mu\nu}\,(1 - \kappa_1\phi_{,\alpha}\phi^{,\alpha} - \kappa_2\phi_{,\alpha}\phi^{,\alpha})\Big] = 0. \tag{5.40}$$

How to derive Eq.(5.36)- Eq.(5.40) looks similar to results shown in ref [99]. Next, we interest to reduce two NMDC coupling constants to a single constant by setting $\kappa_1 = -\frac{\kappa}{2}$ and $\kappa_2 = \kappa$. We then get

$$\nabla_\lambda^\Gamma\left\{\sqrt{-g}\left[g^{\mu\nu}\left(1 + \frac{\kappa}{2}\phi^{,\alpha}\phi_{,\alpha} - \kappa\phi^{,\alpha}\phi_{,\alpha}\right)\right]\right\} = 0\,, \tag{5.41}$$

$$\nabla_\lambda^\Gamma\left\{\sqrt{-g}\left[g^{\mu\nu}\left(1 - \frac{1}{2}\kappa\phi^{,\alpha}\phi_{,\alpha}\right)\right]\right\} = 0, \tag{5.42}$$

By defining

$$f \;=\; 1 - \frac{1}{2}\,\kappa\phi^{,\alpha}\phi_{,\alpha}\,, \tag{5.43}$$

Eq.(5.42) can be written as

$$\nabla_\lambda^\Gamma\left(\sqrt{-g}\,g^{\mu\nu}f\right) = 0. \tag{5.44}$$

The conformal metric $h_{\mu\nu}$ is related to the metric $g_{\mu\nu}$ by the transformation factor f

$$h_{\mu\nu} = fg_{\mu\nu} = (1 - \frac{1}{2}\kappa\phi^{,\alpha}\phi_{,\alpha})g_{\mu\nu}\,, \tag{5.45}$$

and its inverse

$$h^{\mu\nu} = f^{-1}\,g^{\mu\nu}. \tag{5.46}$$

Therefore, we get $\sqrt{-g} = \sqrt{-h}\, f^{-2}$. Substituting two expression for $g^{\mu\nu}$ and $\sqrt{-g}$ into Eq.(5.44), we obtain

$$\begin{aligned}
\nabla^{\Gamma}_{\lambda}\left(\sqrt{-g}g^{\mu\nu}f\right) &= \nabla^{\Gamma}_{\lambda}\left(f^{-2}\sqrt{-h}\, g^{\mu\nu}f\right) , \\
&= \nabla^{\Gamma}_{\lambda}\left(f^{-1}\sqrt{-h}\, g^{\mu\nu}\right), \\
&= \nabla^{\Gamma}_{\lambda}\left(\sqrt{-h}f^{-1}\, g^{\mu\nu}\right), \\
&= \nabla^{\Gamma}_{\lambda}\left(\sqrt{-h}\, h^{\mu\nu}\right) = 0.
\end{aligned} \tag{5.47}$$

By solving Eq.(5.47), we get

$$\begin{aligned}
\nabla^{\Gamma}_{\lambda}\left(\sqrt{-h}h^{\mu\nu}\right) &= (\nabla^{\Gamma}_{\lambda}\sqrt{-h})h^{\mu\nu} + (\nabla^{\Gamma}_{\lambda}h^{\mu\nu})\sqrt{-h} \\
&= (\partial_{\lambda}\sqrt{-h} - \Gamma^{\rho}{}_{\rho\lambda}\sqrt{-h})h^{\mu\nu} + (\nabla^{\Gamma}_{\lambda}h^{\mu\nu})\sqrt{-h} = 0.
\end{aligned} \tag{5.48}$$

Since $h^{\mu\nu}$ and $\sqrt{-h}$ do not vanish, we have to set $(\partial_{\lambda}\sqrt{-h} - \Gamma^{\rho}{}_{\lambda\rho}\sqrt{-h}) = 0$ and $(\nabla^{\Gamma}_{\lambda}h^{\mu\nu}) = 0$. In fact two solutions above represent the metric compatibility relation of $h_{\mu\nu}$, so we can write

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}h^{\lambda\sigma}(\partial_{\mu}h_{\sigma\nu} + \partial_{\nu}h_{\sigma\mu} - \partial_{\sigma}h_{\mu\nu}). \tag{5.49}$$

We perform conformal transformation in order to connect the standard metric tensor and the conformal metric in modified gravity [32, 57, 101, 102, 103].

$$h_{\mu\nu} \equiv \alpha(\phi, X)g_{\mu\nu} + \beta(\phi, X)\phi_{,\mu}\phi_{,\nu}\,, \tag{5.50}$$

where $\alpha(\phi, X)$ and $\beta(\phi, X)$ represent conformal and disformal factors respectively. In general, the conformal and disformal factors depend on the field kinetic term, $X = g^{\mu\nu}\nabla_{\mu}\phi\nabla_{\nu}\phi$. Eq.(5.50) shows conformal transformation from metric $g_{\mu\nu}$ to the new metric $h_{\mu\nu}$. It does not perform disformal transformation. The conformal part of Eq. (5.50) can be written as

$$\begin{aligned}
h_{\mu\nu} = \alpha(\phi, X)g_{\mu\nu} &= \alpha\left(\phi, g^{\sigma\lambda}\nabla_{\sigma}\phi\nabla_{\lambda}\phi\right) g_{\mu\nu} \\
&= \left[\alpha_1(\phi) + \alpha_2(\phi)g^{\sigma\lambda}\nabla_{\sigma}\phi\nabla_{\lambda}\phi\right] g_{\mu\nu} \\
&= \left[\alpha_1(\phi) + \alpha_2(\phi)\phi^{,\sigma}\phi_{,\sigma}\right] g_{\mu\nu}
\end{aligned} \tag{5.51}$$

Comparing the result above with Eq.(5.45), we find that $\alpha_1(\phi) = 1$ and $\alpha_2(\phi) = -\frac{\kappa}{2}$ then it is suffice to say for now that Palatini NMDC gravity obeys the conformal transformation. Using these relations above, the action Eq. (5.8) is therefore expressed in conformal frame as

$$S(h) = \int \mathrm{d}^4x\sqrt{-h}\left\{\frac{\tilde{R}(h)}{f^2} - \left[\frac{\varepsilon h_{\mu\nu}}{f^3} + \kappa\frac{\tilde{G}_{\mu\nu}(h)}{f^2}\right]\phi^{,\mu}\phi^{,\nu} - \frac{2V(\phi)}{f^2}\right\} + S_{\mathrm{m}}\left(\frac{h_{\mu\nu}}{f}, \Psi\right). \tag{5.52}$$

It is worth to noted that the kinetic form of scalar field enters to matter action to couple with ordinary matter.

The relation between the energy momentum tensor in conformal frame $\tilde{T}_{\mu\nu}$ and in ordinary frame $T_{\mu\nu}$ can be derived as follows[23](p.185)

$$\begin{aligned}\tilde{T}^{(\mathrm{m})}_{\mu\nu} &= -\frac{2}{\sqrt{-h}}\frac{\delta\mathcal{L}_{\mathrm{m}}(g_{\kappa\lambda},\Psi)}{\delta h^{\mu\nu}} = -\frac{2}{f^2\sqrt{-g}}\frac{\delta\mathcal{L}_{\mathrm{m}}(g_{\kappa\lambda},\Psi)}{\delta(f^{-1}g^{\mu\nu})} \\ &= -\frac{2}{f\sqrt{-g}}\frac{\delta\mathcal{L}_{\mathrm{m}}(g_{\kappa\lambda},\Psi)}{\delta g^{\mu\nu}} = f^{-1}T^{(\mathrm{m})}_{\mu\nu}\,. \end{aligned} \tag{5.53}$$

From Eq.(5.46) and Eq.(5.53), it is easy to proof that

$$\tilde{T}^{\mu\nu}_{(\mathrm{m})} = f^{-3}T^{\mu\nu}_{(\mathrm{m})}\,. \tag{5.54}$$

The trace of the conformal energy-momentum tensor is

$$\tilde{T}^{(\mathrm{m})} = f^{-2}T^{(\mathrm{m})} = f^{-2}(-\rho_{\mathrm{m}} + 3p_{\mathrm{m}}). \tag{5.55}$$

5.3 NMDC-Palatini cosmology

The NMDC-Palatini field equations derived in previous section can be applied to FLRW metric with assumption that scalar field depends on time only, i.e. $\phi = \phi(t)$. This indicates that[1]

$$f(\dot{\phi}) = 1 - \frac{\kappa}{2}g^{00}\frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}t} = 1 + \frac{\kappa}{2}\dot{\phi}^2\,. \tag{5.56}$$

[1]Even though $-\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)$ is the Lorentz -invariant quantity, the kinetic energy $\frac{1}{2}\dot{\phi}^2$ and the gradient energy $\frac{1}{2}(\nabla\phi)^2$ do not keep Lorentz invariant anymore.

In Eq.(5.51), the new metric $h_{\mu\nu}$ is globally preserved with the Lorentz signature (-,+,+,+) in order to reduce to a Minkowski metric. This shows the values of conformal factor $\alpha_2(\phi) = -\kappa/2$ and the NMDC coupling strength $-2/\dot{\phi}^2 < \kappa \leq \infty$ in order to keep the values of the conformal factor lie in a range $0 < f(\dot{\phi}) \leq \infty$. For fast-rolling field, the coupling strength must be small but for slowly-rolling field, the coupling strength is allowed to be large in the model. It is interesting to note that gravitons in conformal frame travel slower than photon because of the effect of the conformal factor [57]. The conformal metric can be expressed in metric form as

$$h_{\mu\nu} = \begin{pmatrix} -1-\frac{\kappa}{2}\dot{\phi}^2 & 0 & 0 & 0 \\ 0 & a^2(1+\frac{\kappa}{2}\dot{\phi}^2) & 0 & 0 \\ 0 & 0 & a^2(1+\frac{\kappa}{2}\dot{\phi}^2) & 0 \\ 0 & 0 & 0 & a^2(1+\frac{\kappa}{2}\dot{\phi}^2) \end{pmatrix}. \tag{5.57}$$

of which there is a relation $\sqrt{-h} = \sqrt{-g}f^2$. Hence $\nabla^{\Gamma}_{\lambda}(\sqrt{-h}h^{\mu\nu}) = \nabla^{\Gamma}_{\lambda}(\sqrt{-g}g^{\mu\nu}f)$. The Levi-Civita connection $\Gamma^{\lambda}_{\mu\nu}(h)$ is constructed from the conformal metric $h_{\mu\nu}$ as

$$\Gamma^{\lambda}_{\mu\nu}(h) = \frac{1}{2}h^{\lambda\sigma}\left(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right). \tag{5.58}$$

Following e.g. [104, 105, 106], effective gravitational coupling of Palatini NMDC gravity can be expressed as

$$G_{\text{eff}} = \frac{f^2}{8\pi} = \frac{1}{8\pi}\Big(1+\frac{\kappa}{2}\dot{\phi}^2\Big)^2. \tag{5.59}$$

This leads to modification of the entropy of black hole's apparent horizon for this theory[1] as $S_{\text{AH}} = A/\left[4(1+\frac{\kappa}{2}\dot{\phi}^2)^2/8\pi\right]$. Additionally, the effective gravitational

[1]For the sake of clarity, the gravitational coupling strength of NMDC-Palatini in Jordan frame is

$$G_{\text{eff}} = G_{\text{N}}(1+\frac{\kappa}{2}\dot{\phi}^2)^2. \tag{5.60}$$

coupling strength and its time derivative can be used to test the correctness of this model by limiting the range of observation by the relation

$$\frac{\dot{G}_{\rm eff}}{G_{\rm eff}} = \frac{2\kappa\dot{\phi}\ddot{\phi}}{(1+\frac{\kappa}{2}\dot{\phi}^2)}. \tag{5.61}$$

It can be divided into two considered cases . The first one is the kinetic form of scalar field plays a dominance role with the condition $\frac{\kappa\dot{\phi}^2}{2} \gg 1$.

$$\frac{\dot{G}_{\rm eff}}{G_{\rm eff}} \simeq \frac{4\ddot{\phi}}{\dot{\phi}} \tag{5.62}$$

and the second case shows an insignificant of the kinetic form of scalar field by the condition that $\frac{\kappa\dot{\phi}^2}{2} \ll 1$

$$\frac{\dot{G}_{\rm eff}}{G_{\rm eff}} \simeq 2\kappa\dot{\phi}\ddot{\phi}\ . \tag{5.63}$$

So far we have argued that $\dot{\phi}\ddot{\phi} \neq 0$ even in the late-time evolution of the universe. For $\Gamma = \Gamma(h, \partial h)$ therefore the field equation (5.25) expressed as function of the new metric, for instance, $\tilde{R}_{\mu\nu}(\Gamma)$ is hence $\tilde{R}_{\mu\nu}(h)$ but for brevity we express them as $\tilde{R}_{\mu\nu}(h)$. Other terms follow similar argument. The energy-momentum tensor obeys the relation $\tilde{T}_{\mu\nu} = f^{-1}T_{\mu\nu}$ as shown in Eq.(5.53). It can be shown that $T^{\mu\nu}_{(\rm m)}$ is conserved covariant, i.e. $\nabla_\mu T^{\mu\nu}_{(\rm m)} = 0$,whereas $\tilde{T}^{\mu\nu}_{(\rm m)}$ does not. This is

$$\begin{aligned}
\nabla^{\Gamma}_{\mu}\tilde{T}^{\mu\nu}_{(\rm m)} &= -\tilde{T}^{(\rm m)}h^{\mu\nu}\nabla^{\Gamma}_{\nu}(\ln\sqrt{f}) = -\tilde{T}^{(\rm m)}h^{00}\frac{\partial\ln\sqrt{f}}{\partial t} \\
&= -\frac{T^{(\rm m)}}{2f^4}\dot{f} = -\frac{(-\rho_{\rm m}+3p_{\rm m})\kappa\dot{\phi}\ddot{\phi}}{(1+\frac{\kappa}{2}\dot{\phi}^2)^4} \\
&\simeq -(-\rho_{\rm m}+3p_{\rm m})(\kappa\dot{\phi}\ddot{\phi} - \cancel{2\kappa^2\dot{\phi}^3\ddot{\phi}}\ ) \ \text{ where } \frac{\kappa\dot{\phi}^2}{2} \ll 1 \\
&\simeq (\rho_{\rm m}-3p_{\rm m})\kappa\dot{\phi}\ddot{\phi},
\end{aligned} \tag{5.64}$$

where $2\kappa^2\dot{\phi}^3\ddot{\phi}$ is negligibly small relative to $\kappa\dot{\phi}\ddot{\phi}$. Recall that ρ_m and p_m signify for the total energy density and the total pressure respectively. Having set the zero of $T^{(\rm m)} = 0$ during radiation dominated epoch to Eq.(5.64), the conserved covariant of $\tilde{T}^{\mu\nu}_{(\rm m)}$ vanishes away. It is because the field velocity and the field acceleration temporarily ceased to couple to the trace of matter fields at that time.

Considering time - time component of the field equation,

$$T_{00} = \tilde{G}_{00}(h) - \frac{\kappa}{2}\tilde{G}_{00}(h)\dot{\phi}^2 + \frac{5\kappa}{2}\tilde{R}_{00}(h)\dot{\phi}^2 + \frac{\kappa}{2}\tilde{R}(h)\dot{\phi}^2 - \Big(\frac{\varepsilon}{2}\dot{\phi}^2 + V(\phi)\Big). \quad (5.65)$$

The Ricci tensor for the new metric $h_{\mu\nu}$ in n dimensions is related to the usual Ricci tensor by the following formula [23]

$$\begin{aligned} \tilde{R}_{\sigma\nu}(h) &= R_{\sigma\nu}(g) - \Big[(n-2)\delta^{\alpha}_{\sigma}\delta^{\beta}_{\nu} + g_{\sigma\nu}g^{\alpha\beta}\Big]\frac{1}{\sqrt{f}}(\nabla^g_\alpha\nabla^g_\beta\sqrt{f}) \\ &+ \Big[2(n-2)\delta^{\alpha}_{\sigma}\delta^{\beta}_{\nu} - (n-3)g_{\sigma\nu}g^{\alpha\beta}\Big]\frac{1}{f}(\nabla^g_\alpha\sqrt{f})(\nabla^g_\beta\sqrt{f}), \quad (5.66) \end{aligned}$$

where ∇^g_λ is the usual covariant derivative constructed from $g_{\mu\nu}$. First and second-order time derivatives of $\sqrt{f}$ are

$$\begin{aligned} \nabla^g_0\sqrt{f} &= \frac{1}{2}\frac{\dot{f}}{\sqrt{f}}, \quad (5.67) \\ \nabla^g_0\nabla^g_0\sqrt{f} &= \frac{\partial}{\partial t}\Big(\frac{\partial\sqrt{f}}{\partial t}\Big) - \Gamma^0_{00}\frac{\partial\sqrt{f}}{\partial t} \\ &= \frac{1}{2}\left(\frac{\ddot{f}}{f^{1/2}} - \frac{\dot{f}^2}{2f^{3/2}}\right), \quad (5.68) \end{aligned}$$

respectively. The Ricci tensor and the Ricci scalar for flat FLRW universe are

$$\begin{aligned} R_{00}(g) &= -3(\dot{H} + H^2), \qquad R_{11}(g) = R_{22}(g) = R_{33}(g) = a^2(\dot{H} + 3H^2), \\ R(g) &= 6(\dot{H} + 2H^2). \quad (5.69) \end{aligned}$$

Substituting these results into Eq. (5.66), one finds

$$\begin{aligned} \tilde{R}_{00}(h) &= R_{00}(g) - \frac{3}{2}\Big(\frac{\ddot{f}}{f} - \frac{2\dot{f}^2}{f^2}\Big) \quad (5.70) \\ &= -3(\dot{H} + H^2) - \frac{3}{2}\Big(\frac{\ddot{f}}{f} - \frac{2\dot{f}^2}{f^2}\Big), \quad (5.71) \\ \tilde{R}_{ij}(h) &= R_{ij}(g) + \frac{a^2\ddot{f}}{2f}. \quad (5.72) \end{aligned}$$

The space-space components of the Ricci tensor for FLRW metric are

$$\tilde{R}_{11}(g) = \tilde{R}_{22}(h) = \tilde{R}_{33}(h) = a^2(\dot{H} + 3H^2) + \frac{a^2\ddot{f}}{2f}. \quad (5.73)$$

As we shall see, the first and second-order time derivative of the conformal factor can be expressed in term of $\dot{\phi}, \ddot{\phi}$ and $\dddot{\phi}$, i.e.

$$\dot{f} = \kappa\dot{\phi}\,\ddot{\phi}\,, \qquad \ddot{f} = \kappa\left(\ddot{\phi}^2 + \dot{\phi}\,\dddot{\phi}\right). \tag{5.74}$$

Replacing $\kappa\dot{\phi}^2 = 2(f-1)$, therefore

$$\begin{aligned}
\dot{f} &= 2(f-1)\frac{\ddot{\phi}}{\dot{\phi}}\,, &(5.75)\\
\ddot{f} &= 2\dot{f}\frac{\ddot{\phi}}{\dot{\phi}} + 2(f-1)\frac{\dddot{\phi}}{\dot{\phi}} - 2(f-1)(\frac{\ddot{\phi}}{\dot{\phi}})^2\\
&= 4(f-1)(\frac{\ddot{\phi}}{\dot{\phi}})^2 + 2(f-1)\frac{\dddot{\phi}}{\dot{\phi}} - 2(f-1)(\frac{\ddot{\phi}}{\dot{\phi}})^2\\
&= 2(f-1)(\frac{\ddot{\phi}}{\dot{\phi}})^2 + 2(f-1)\frac{\dddot{\phi}}{\dot{\phi}}\\
&= 2(f-1)\Big[(\frac{\ddot{\phi}}{\dot{\phi}})^2 + \frac{\dddot{\phi}}{\dot{\phi}}\Big] &(5.76)
\end{aligned}$$

The Ricci scalar under the conformal transformation is [23]

$$\tilde{R}(h) = f^{-1}R(g) - 2(n-1)g^{\alpha\beta}f^{-3/2}\left(\nabla_\alpha\nabla_\beta\sqrt{f}\right) - (n-1)(n-4)g^{\alpha\beta}f^{-2}\left(\nabla_\alpha\sqrt{f}\right)\left(\nabla_\beta\sqrt{f}\right). \tag{5.77}$$

Performing calculation in four dimensions and neglecting the last term on the right hand-side of Eq. (5.77), we obtain

$$\tilde{R}(h) = \frac{1}{f}R(g) + 3\left(\frac{\ddot{f}}{f^2} - \frac{\dot{f}^2}{2f^3}\right) = \frac{6}{f}\left(\dot{H} + 2H^2\right) + 3\left(\frac{\ddot{f}}{f^2} - \frac{\dot{f}^2}{2f^3}\right). \tag{5.78}$$

Using $\tilde{T}_{\mu\nu} = f^{-1}T_{\mu\nu}, \tilde{\rho}_{\rm m} = f^{-2}\rho_{\rm m}$, $\tilde{p}_{\rm m} = f^{-2}p_{\rm m}$, Eqs. (5.71), (5.72) and (5.78) in Eqs. (5.7) and (5.25), the time-time component of NMDC field equation becomes

$$\begin{aligned}
T_{00} = \rho_{\rm m} &= \dot{H}\Big[12f + \frac{6}{f} - 18\Big] + H^2\Big[12f + \frac{12}{f} - 21\Big]\\
&\quad -\frac{3}{2}(1-f)\Big(\frac{4\ddot{f}}{f} - \frac{8\dot{f}^2}{f^2}\Big) - \frac{3\ddot{f}}{2f} + \frac{3\ddot{f}}{f^2} + \frac{3\dot{f}^2}{f^2} - \frac{3\dot{f}^2}{2f^3}\\
&\quad -\Big(\frac{\epsilon}{2}\dot{\phi}^2 + V(\phi)\Big)\,, &(5.79)\\
\rho_{\rm tot} &= \dot{H}\Big[12f + \frac{6}{f} - 18\Big] + H^2\Big[12f + \frac{12}{f} - 21\Big]\\
&\quad -\frac{3}{2}(1-f)\Big(\frac{4\ddot{f}}{f} - \frac{8\dot{f}^2}{f^2}\Big) - \frac{3\ddot{f}}{2f} + \frac{3\ddot{f}}{f^2} + \frac{3\dot{f}^2}{f^2} - \frac{3\dot{f}^2}{2f^3}
\end{aligned} \tag{5.80}$$

where $\rho_{\rm tot} \equiv \rho_{\rm m} + \rho_\phi$ and $\rho_\phi = \varepsilon(\dot\phi^2/2) + V(\phi)$.

The space-space components of NMDC gravity can be written as

$$T_{11} = a^2 p_{\rm m} \;=\; \tilde R_{11}(h) - \frac{1}{2}a^2 \tilde R(h) + \Bigg[(f-1)\Big(\tilde R_{11}(h) - \frac{1}{2}a^2\tilde R(h) + a^2 \tilde R_{00}(h)\Big)\Bigg] - a^2\left(\frac{\varepsilon\dot\phi^2}{2} - V(\phi)\right) \tag{5.81}$$

Multiplying both sides of Eq.(5.81) with $\frac{1}{a^2}$ and using Eq.(5.71), Eq.(5.72) and Eq.(5.78), one finds

$$p_{\rm tot} = (p_{\rm m} + p_\phi) \;=\; \dot H\,(4f-6) + H^2\,(6f-9) - \frac{3}{2}(1-f)\left(\frac{\ddot f}{f} - \frac{2\dot f^2}{f^2}\right) + \frac{\ddot f}{f} - \frac{3\ddot f}{2f} + \frac{3\dot f^2}{4f^2}\,, \tag{5.82}$$

where the pressure of scalar field is $p_\phi = \frac{\varepsilon\dot\phi^2}{2} - V(\phi)$.

For brevity, one can introduce the following variables

$$A \equiv 4f-6, \tag{5.83}$$

$$B \equiv 6f-9, \tag{5.84}$$

$$C \equiv -\frac{3}{2}(1-f)(\frac{\ddot f}{f} - \frac{2\dot f^2}{f^2}) + \frac{\ddot f}{f} - \frac{3\ddot f}{2f} + \frac{3\dot f^2}{4f^2}, \tag{5.85}$$

$$D \equiv 12f + \frac{6}{f} - 18, \tag{5.86}$$

$$E \equiv 12f + \frac{12}{f} - 21, \tag{5.87}$$

$$F \equiv -\frac{3}{2}(1-f)(\frac{4\ddot f}{f} - \frac{8\dot f^2}{f^2}) - \frac{3\ddot f}{2f} + \frac{3\ddot f}{f^2} + \frac{3\dot f^2}{f^2} - \frac{3\dot f^2}{2f^3}\,. \tag{5.88}$$

The effective equation of state parameter is therefore

$$w_{\rm eff} \equiv \frac{p_{\rm tot}}{\rho_{\rm tot}} = \frac{A\dot H + BH^2 + C}{D\dot H + EH^2 + F} \tag{5.89}$$

If we set $f = 1$ for GR limit thus the total pressure , $p_{\rm tot} = p_{\rm m} + p_\phi$, Eq.(5.82) reduces to $p_{\rm tot} = -2\dot H - 3H^2$.

Furthermore, the effective equation of state parameter can be reduced to

$$w_{\rm eff} \equiv \frac{p_{\rm tot}}{\rho_{\rm tot}} = -1 - \frac{2\dot H}{3H^2}. \tag{5.90}$$

For the total energy density can be written explicitly in Eq.(5.80) or $\rho_{\rm tot}(\dot{H}, H, f, \dot{f}, \ddot{f})$ and the total pressure in Eq.(5.82)or $p_{\rm tot}(\dot{H}, H, f, \dot{f}, \ddot{f})$, it is worth to calculate the Hubble parameter in terms of these variables, i.e. $H^2(\rho_{\rm tot}, w_{\rm eff}, A, B, .., F)$. From Eq.(5.89), we obtain

$$\dot{H} = \frac{[(B - Ew_{\rm eff})H^2 - Fw_{\rm eff} + C]}{Dw_{\rm eff} - A} \tag{5.91}$$

Substituting Eq.(5.91) into Eq.(5.80), the Hubble parameter can be written as the following form:

$$H^2(\phi, \dot{\phi}, \ddot{\phi}) = \frac{\rho_{\rm tot}}{3} \frac{\left[1 - \frac{(C - Fw_{\rm eff})D}{(Dw_{\rm eff} - A)\rho_{\rm tot}} - \frac{F}{\rho_{\rm tot}}\right]}{\left[\frac{(B - Ew_{\rm eff})D}{3(Dw_{\rm eff} - A)} + \frac{E}{3}\right]} . \tag{5.92}$$

In GR limit, f=1 then $A = -2$, $B = -3$, $E = 3$ and $C = D = F = 0$, Eq.(5.92) can be reduced to

$$H^2 = \frac{\rho_{\rm tot}}{3}. \tag{5.93}$$

The accelerated equation is derived from

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = \rho_{\rm tot} \frac{\left[1 + \frac{(Dw_{\rm eff} - A)}{(B - Ew_{\rm eff})}\right] \left[1 - \frac{(C - Fw_{\rm eff})D}{(Dw_{\rm eff} - A)\rho_{\rm tot} - \frac{F}{\rho_{\rm tot}}}\right]}{\left[D + E\frac{(Dw_{\rm eff} - A)}{(B - Ew_{\rm eff})}\right]} - \frac{(Fw_{\rm eff} - C)}{(Dw_{\rm eff} - A)} . \tag{5.94}$$

In the GR limit, the expression recovers the usual acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho_{\rm tot} + 3p_{\rm tot}\right) . \tag{5.95}$$

By using the Euler- Lagrange equation for the scalar field

$$\frac{\delta \mathcal{L}}{\delta \phi^a} = \frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}\right] = 0, \tag{5.96}$$

the modified Klien - Gordon equation of NMDC theory

$$\ddot{\phi}\left[-\varepsilon + \frac{\kappa}{2}\left(\tilde{R}(h) - \tilde{R}_{00}(h)\right)\right] - \kappa \dot{\phi} \nabla_0^h \tilde{R}_{00}(h) + \frac{\kappa}{2} \dot{\phi} \nabla_0^h \tilde{R}(h) - 3\varepsilon H \dot{\phi} - V_{,\phi} = 0, \tag{5.97}$$

where the covariant derivative of a scalar filed is

$$\nabla_\mu^h \phi = \nabla_\mu^g \phi = \partial_\mu \phi \tag{5.98}$$

and

$$\nabla^h_\mu \nabla^h_\nu \phi = \nabla^g_\mu \nabla^g_\nu \phi - (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu - g_{\mu\nu} g^{\alpha\beta}) \frac{1}{\sqrt{f}} (\nabla^g_\alpha \sqrt{f})(\nabla^g_\beta \sqrt{f}). \tag{5.99}$$

One finds that

$$\ddot{\tilde{\phi}} = \ddot{\phi} - \frac{\dot{f}^2}{4f^{3/2}}. \tag{5.100}$$

The explicit form of the modified Klein-Gordon equation of scalar field in NMDC gravity is

$$\left(\ddot{\phi} - \frac{\dot{f}^2}{4f^{3/2}}\right) \quad \left\{-\varepsilon + \frac{\kappa}{2}\left[\left(\frac{R(g)}{f} + 3(\frac{\ddot{f}}{f^2} - \frac{\dot{f}^2}{2f^3})\right) - \left(R_{00}(g) - \frac{3}{2}(\frac{\ddot{f}}{f} - \frac{2\dot{f}^2}{f^2})\right)\right]\right\}$$
$$+\frac{\kappa}{2}\dot{\phi}\nabla^g_0\left[\frac{R(g)}{f} + 3(\frac{\ddot{f}}{f^2} - \frac{\dot{f}^2}{2f^3})\right] - 3\varepsilon H\dot{\phi} - V_{,\phi} = 0. \tag{5.101}$$

Substituting relation in Eq.(5.69) into Eq.(5.101), we get

$$\left(\ddot{\phi} - \frac{\dot{f}^2}{4f^{3/2}}\right) \quad \left\{-\varepsilon + \frac{\kappa}{2}\left[\left(\frac{6\dot{H} + 12H^2}{f} + 3(\frac{\ddot{f}}{f^2} - \frac{\dot{f}^2}{2f^3})\right) - \left(-3(\dot{H} + H^2) - \frac{3}{2}(\frac{\ddot{f}}{f} - \frac{2\dot{f}^2}{f^2})\right)\right]\right\}$$
$$+\frac{\kappa}{2}\dot{\phi}\nabla^g_0\left[\frac{6\dot{H} + 12H^2}{f} + 3(\frac{\ddot{f}}{f^2} - \frac{\dot{f}^2}{2f^3})\right] - 3\varepsilon H\dot{\phi} - V_{,\phi} = 0. \tag{5.102}$$

By setting $\varepsilon = 1$ hence $f = 1$ in GR limit, we recover the usual Klien-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \tag{5.103}$$

In case of $0 < |\dot{\phi}| \ll 1$ and $|\dddot{\phi}| \ll |\ddot{\phi}| \ll |\dot{\phi}|$, i.e. $0 \sim |\ddot{f}| \ll |\dot{f}| \ll |f|$, neglecting $\dddot{\phi}$, $\dot{\phi}^4\ddot{\phi}^2$, $\dot{\phi}\,\dddot{\phi}$ terms and using binomial approximation, $(1 - \kappa\dot{\phi}^2/2)^{-1} \simeq 1 + \kappa\dot{\phi}^2/2$ therefore $1/f \simeq 2 - f$, one can approximate that

$$A \simeq -2 + 2\kappa\dot{\phi}^2, \tag{5.104}$$

$$B \simeq -3 + 3\kappa\dot{\phi}^2, \tag{5.105}$$

$$D \simeq 3\kappa\dot{\phi}^2, \tag{5.106}$$

$$E \simeq 3, \tag{5.107}$$

$$C \simeq F \simeq 0\,. \tag{5.108}$$

It can be noticed that $\frac{A}{B} = \frac{2}{3}$. Expressing in term of the field velocity and neglecting terms with higher power than $\dot{\phi}^2$,

$$\begin{aligned} \dot{H} + H^2 = \frac{\ddot{a}}{a} &\simeq \rho_{\rm tot} \frac{[B - A + w_{\rm eff}(D - E)]}{[BD - AE]}, \\ &\simeq -\frac{1}{6}\rho_{\rm tot}\left[\frac{1 + 3w_{\rm eff} - \kappa\dot{\phi}^2 - 3w_{\rm eff}\kappa\dot{\phi}^2}{1 - \frac{5}{2}\kappa\dot{\phi}^2}\right], \\ &\simeq -\frac{1}{6}\rho_{\rm tot}\left[1 + \frac{7}{2}\kappa\dot{\phi}^2 + 3w_{\rm eff}(1 + \frac{3}{2}\kappa\dot{\phi}^2)\right]. \end{aligned} \tag{5.109}$$

We can straightforwardly obtain the acceleration condition,

$$w_{\rm eff} < -\frac{1}{3}\left[\frac{1 + \frac{7}{2}\kappa\dot{\phi}^2}{1 + \frac{3}{2}\kappa\dot{\phi}^2}\right] \simeq -\frac{1}{3}\left(1 + 2\kappa\dot{\phi}^2\right). \tag{5.110}$$

In the slow-roll regime, the modified Friedmann equation (5.92) is approximated,

$$\begin{aligned} H^2 &\simeq \frac{\rho_{\rm tot}}{3}\left[\frac{3(Dw_{\rm eff} - A)}{BD - EA}\right], \\ &\simeq \frac{\rho_{\rm tot}}{3}\left[\frac{1 - \kappa\dot{\phi}^2 + w_{\rm eff}(\frac{3}{2}\kappa\dot{\phi}^2)}{1 - \frac{5}{2}\kappa\dot{\phi}^2}\right], \\ &\simeq \frac{\rho_{\rm tot}}{3}\left[1 + \frac{3}{2}\kappa\dot{\phi}^2(1 + w_{\rm eff})\right]. \end{aligned} \tag{5.111}$$

To be not confused, we would like to stress here again that $\rho_{\rm tot} = \rho_{\rm m} + \rho_\phi$. The Klein-Gordon equation (5.102) can be approximated in the slow-roll regime as

$$\begin{aligned} &\ddot{\phi}\Big\{ -\varepsilon + \frac{\kappa}{2}\left[\frac{1}{f}\left(6\dot{H} + 12H^2\right) + 3\left(\dot{H} + H^2\right)\right]\Big\} + \frac{\kappa}{2}\dot{\phi}\nabla_0^g\left[\frac{1}{f}\left(6\dot{H} + 12H^2\right)\right] \\ &-3\varepsilon H\dot{\phi} - V_{,\phi} = 0. \end{aligned} \tag{5.112}$$

This can approximated further to

$$\begin{aligned} &\ddot{\phi}\left\{\varepsilon - \frac{9\kappa}{2}\dot{H}\left(1 - \kappa\dot{\phi}^2\right) - \frac{3\kappa}{2}H^2\left(5 - 6\kappa\dot{\phi}^2\right)\right\} + 3H\dot{\phi}\Bigg[\varepsilon - \\ &\left(\frac{\ddot{H}}{H} + 4\dot{H}\right)\kappa\left(1 - \frac{\kappa\dot{\phi}^2}{2}\right)\Bigg] + V_{,\phi} = 0. \end{aligned} \tag{5.113}$$

5.4 Non-minimal derivative coupling - Palatini model and inflation

We start form the Friedmann equation of NMDC-Palatini gravity

$$H^2 \simeq \frac{\rho_{\rm tot}}{3}\left[1+\frac{3}{2}\kappa\dot{\phi}^2(1+w_{\rm eff})\right]. \tag{5.114}$$

We consider only scalar field dominated by setting $\rho_{\rm tot} = \rho_\phi$ and $w_{\rm eff} = w_\phi = p_\phi/\rho_\phi$. Hence Eq.(5.114) becomes

$$\begin{aligned}
H^2 &\simeq \frac{\rho_\phi}{3}\left[1+\frac{3}{2}\kappa\dot{\phi}^2(1+\frac{p_\phi}{\rho_\phi})\right], \\
&\simeq \frac{1}{3}\rho_\phi + \frac{\rho_\phi}{2}\kappa\dot{\phi}^2\Big(1+\frac{p_\phi}{\rho_\phi}\Big), \\
&\simeq \frac{1}{3}\Big(\frac{\varepsilon\dot{\phi}^2}{2}+V(\phi)\Big)+\frac{1}{2}\Big(\frac{\varepsilon\dot{\phi}^2}{2}+V(\phi)\Big)\kappa\dot{\phi}^2\left(1+\frac{\frac{\varepsilon\dot{\phi}^2}{2}-V(\phi)}{\frac{\varepsilon\dot{\phi}^2}{2}+V(\phi)}\right), \\
&\simeq \frac{1}{3}\Big(\frac{\varepsilon\dot{\phi}^2}{2}+V(\phi)\Big)+\frac{\kappa}{2}\left[\frac{\varepsilon^2\,\dot{\phi}^6}{2V(\phi)}+\varepsilon\dot{\phi}^4-\frac{\varepsilon^3\dot{\phi}^8}{4V^2(\phi)}-\frac{\varepsilon^2\dot{\phi}^6}{2V(\phi)}\right].
\end{aligned} \tag{5.115}$$

We set $\dot{\phi}^2 \ll V(\phi)$ at slow roll regime. Hence we get

$$H^2 \simeq \frac{1}{3}\frac{V(\phi)}{M_{\rm P}^2}. \tag{5.116}$$

For clarify, we restore back $8\pi G_{\rm N} = M_{\rm P}^{-2}$ in our calculation and it is useful to keep in mind that $\kappa = M^{-2} \leq M_{\rm P}^{-2}$. It is easy to get $\dot{H}$ from first time derivative of the Friedmann equation. This is

$$\begin{aligned}
2H\dot{H} &= \frac{V'(\phi)\dot{\phi}}{3M_{\rm P}^2} \\
\dot{H} &= \frac{V'(\phi)\dot{\phi}}{6HM_{\rm P}^2}
\end{aligned} \tag{5.117}$$

where $V'(\phi) = \frac{dV(\phi)}{d\phi}$. We get $\dot{\phi}$ from modified Klein-Gordon equation as shown in Eq.(5.113) at slow-roll regime by setting $\ddot{\phi} \simeq 0$ and using the slow-roll condition that $|\ddot{H}| \ll -H\dot{H} \ll H^3$. This gives

$$\dot{\phi} \simeq -\frac{V'(\phi)}{3H(\varepsilon - 4\kappa\dot{H})}. \tag{5.118}$$

Substituting Eq.(5.118) into Eq.(5.117), we have

$$\begin{aligned} \dot{H} &\simeq \frac{V'(\phi)}{6HM_{\rm P}^2}\left[\frac{-V'(\phi)}{3H(\varepsilon - 4\kappa\,\dot{H})}\right] \\ &\simeq -\frac{(V'(\phi))^2}{18H^2M_{\rm P}^2(\varepsilon - 4\kappa\,\dot{H})} \end{aligned} \tag{5.119}$$

During inflationary phase , this requires that $\dot{H} < 0$. This implies $(\varepsilon - 4\kappa\,\dot{H}) > 0$. The first slow-roll parameter is $\epsilon_{\rm v} = -\frac{\dot{H}}{H^2} \ll 1$. For NMDC-gravity model, we can write

$$\begin{aligned} \epsilon_{\rm v} \equiv -\frac{\dot{H}}{H^2} &\simeq \frac{(V'(\phi))^2}{18H^2M_{\rm P}^2(\varepsilon - 4\kappa\,\dot{H})}\frac{1}{H^2} \\ &\simeq \frac{(V'(\phi))^2}{6H^2V(\phi)(\varepsilon - 4\kappa\,\dot{H})}, \\ &\simeq \frac{M_{\rm P}^2}{2(\varepsilon - 4\kappa\,\dot{H})}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \end{aligned} \tag{5.120}$$

where $H^2 \simeq \frac{1}{3}\frac{V(\phi)}{M_{\rm P}^2}$ is used to derive last line of Eq.(5.120). It should be noticed that ϵ is positive by definition [107].

The second slow-roll parameter can be defined as

$$\delta \equiv \frac{\ddot{\phi}}{H\dot{\phi}} \ll 1 \tag{5.121}$$

To get $\ddot{\phi}$, we have to taking time derivative for $\dot{\phi}$. This is

$$\ddot{\phi} = -\frac{V''(\phi)\dot{\phi}}{3H(\varepsilon - 4\kappa\dot{H})} + \frac{V'(\phi)\dot{H}}{3H^2(\varepsilon - 4\kappa\dot{H})} - \frac{4\kappa\ddot{H}V'(\phi)}{3H(\varepsilon - 4\kappa\dot{H})^2}. \tag{5.122}$$

Hence, Eq.(5.123) can be expressed as

$$\delta = \frac{\ddot{\phi}}{H\dot{\phi}} \simeq -\frac{V''(\phi)\dot{\phi}}{3H^2(\varepsilon - 4\kappa\dot{H})\dot{\phi}} + \frac{V'(\phi)\dot{H}}{3H^2(\varepsilon - 4\kappa\dot{H})H\dot{\phi}} - \frac{4\kappa\ddot{H}V'(\phi)}{3H(\varepsilon - 4\kappa\dot{H})^2H\dot{\phi}}. \tag{5.123}$$

The first term on the right-hand side of Eq.(5.123) is

$$-\frac{V''(\phi)\dot{\phi}}{3H^2(\varepsilon - 4\kappa\dot{H})\dot{\phi}} = -\frac{V''(\phi)}{3H^2(\varepsilon - 4\kappa\dot{H})} \equiv -\eta_{\rm v}, \tag{5.124}$$

where we define

$$\eta_{\rm v} \equiv \frac{V''(\phi)}{3H^2(\varepsilon - 4\kappa\dot{H})} = \frac{M_{\rm P}^2}{(\varepsilon - 4\kappa\dot{H})}\frac{V''(\phi)}{V(\phi)}. \tag{5.125}$$

The second term on the right-hand side of Eq.(5.123) is

$$\begin{aligned}\frac{V'(\phi)\dot{H}}{3H^2(\varepsilon-4\kappa\dot{H})H\dot{\phi}} &= \frac{-V'(\phi)}{3H\dot{\phi}(\varepsilon-4\kappa\dot{H})}\left(-\frac{\dot{H}}{H^2}\right)\\ &= \frac{-V'(\phi)\,\epsilon_{\rm v}}{3H\dot{\phi}(\varepsilon-4\kappa\dot{H})}\\ &= \frac{-V'(\phi)\,\epsilon_{\rm v}}{3H(\varepsilon-4\kappa\dot{H})}\left(\frac{3H(\varepsilon-4\kappa\dot{H})}{-V'(\phi)}\right) = \epsilon_{\rm v}. \qquad (5.126)\end{aligned}$$

The third term on the right-hand side of Eq.(5.123) is

$$-\frac{4\kappa\ddot{H}V'(\phi)}{3H(\varepsilon-4\kappa\dot{H})^2H\dot{\phi}} = -\frac{4\kappa\ddot{H}V'(\phi)}{3H(\varepsilon-4\kappa\dot{H})^2H}\left(\frac{3H(\varepsilon-4\kappa\dot{H})}{-V'(\phi)}\right) = \frac{4\kappa\ddot{H}}{H(\varepsilon-4\kappa\dot{H})} \equiv \eta_{{\rm v},\kappa}\,. \qquad (5.127)$$

then

$$\eta_{{\rm v},\kappa} \simeq \frac{4\kappa}{(\varepsilon-4\kappa\dot{H})^3}\left[\frac{V''(V')^2}{18V} - \frac{V'^4}{36V^2}\right] \qquad (5.128)$$

Hence

$$\delta = -\eta_{\rm v} + \epsilon_{\rm v} + \eta_{{\rm v},\kappa} \qquad (5.129)$$

Consider spectral index $n_{\rm s} - 1 = -4\epsilon_{\rm v} - 2\delta$ [108]

$n_{\rm s} - 1 = -6\epsilon_{\rm v} + 2\eta_{\rm v} - 2\eta_{{\rm v},\kappa}$

$$n_{\rm s} - 1 = -\frac{3M_{\rm P}^2}{(\varepsilon-4\kappa\dot{H})}\left(\frac{V'}{V}\right)^2 + \frac{2M_{\rm P}^2}{(\varepsilon-4\kappa\dot{H})}\frac{V''}{V} - \frac{8\kappa}{(\varepsilon-4\kappa\dot{H})^3}\left[\frac{V''(V')^2}{18V} - \frac{(V')^4}{36V^2}\right] \qquad (5.130)$$

For power law potential, i.e. $V(\phi) = V_0\phi^n$, we can show that

$$\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \frac{n^2}{\phi^2} \qquad (5.131)$$

$$\frac{V''(\phi)}{V(\phi)} = \frac{n(n-1)}{\phi^2}. \qquad (5.132)$$

Hence,

$$n_{\rm s} - 1 \;=\; -\frac{M_{\rm P}^2\,[n(n+2)]}{(\varepsilon-4\kappa\dot{H})}\,\phi^{-2} \;-\; \frac{2\kappa V_0^2\,[n^3(n-2)]}{9(\varepsilon-4\kappa\dot{H})^3}\,\phi^{2n-4} \qquad (5.133)$$

The number of e-folds during inflation epoch is

$$N_{\rm I} = \ln(\frac{a_{\rm f}}{a_{\rm i}}) = \int_{t_{\rm i}}^{t_{\rm f}} H\,dt, \qquad (5.134)$$

where $a_{\rm i}$ and $a_{\rm f}$ denote for scale factor at starting inflationary phase and ending of inflationary phase respectively.

From Eq.(5.118)

$$\begin{aligned} \dot{\phi} &= \frac{-V'(\phi)}{3H(\varepsilon - 4\kappa\dot{H})}, \\ \frac{d\phi}{dt} &= \frac{-V'(\phi)H}{3H^2(\varepsilon - 4\kappa\dot{H})}, \\ &= \frac{-V'(\phi)HM_{\rm P}^2}{V(\phi)(\varepsilon - 4\kappa\dot{H})}. \end{aligned} \tag{5.135}$$

We can write

$$Hdt = \frac{-V(\phi)(\varepsilon - 4\kappa\dot{H})}{V'(\phi)M_{\rm P}^2}d\phi. \tag{5.136}$$

To simplify our calculation, we assume that $(\varepsilon - 4\kappa\dot{H})$ is almost constant during inflation era. Hence, the number of e-folds can be written as

$$\begin{aligned} N_{\rm I} \equiv \int_{t_{\rm i}}^{t_{\rm f}} H\,dt &\simeq \frac{(\varepsilon - 4\kappa\dot{H})}{M_{\rm P}^2}\int_{\phi_{\rm f}}^{\phi_{\rm i}} \frac{V(\phi)}{V'(\phi)}d\phi\,, \\ &\simeq \frac{(\varepsilon - 4\kappa\dot{H})}{M_{\rm P}^2}\int_{\phi_{\rm f}}^{\phi_{\rm i}} \frac{\phi}{n}d\phi\,, \\ &\simeq \frac{(\varepsilon - 4\kappa\dot{H})}{2nM_{\rm P}^2}\left(\phi_{\rm i}^2 - \phi_{\rm f}^2\right) \end{aligned} \tag{5.137}$$

Let $\tilde{\phi}_{\rm f}^2 = \frac{\phi_{\rm f}^2}{2nM_{\rm P}^2}(\varepsilon - 4\kappa\dot{H})$ where we have use Eq.(5.131) and $\phi_{\rm i} > \phi_{\rm f}$.

$$\phi^2 \equiv \phi_{\rm i}^2(n, N_{\rm I}) \simeq \frac{2nM_{\rm P}^2}{(\varepsilon - 4\kappa\dot{H})}\left(N_{\rm I} + \tilde{\phi}_{\rm f}^2\right) \tag{5.138}$$

For power law potentials with Eq.(5.120), we obtain

$$\epsilon_v = \frac{M_{\rm P}^2}{2(\varepsilon - 4\kappa\dot{H})}\left(\frac{n^2}{\phi^2}\right) \tag{5.139}$$

$$= \frac{n}{4\left[N_{\rm I} + \tilde{\phi}_{\rm f}^2\right]} \tag{5.140}$$

From Eq. (5.125)

$$\begin{aligned} \eta_v &= \frac{M_{\rm P}^2}{(\varepsilon - 4\kappa\dot{H})}\frac{n(n-1)}{\phi^2} \\ &= \frac{n-1}{2(N_{\rm I} + \tilde{\phi}_{\rm f}^2)} \end{aligned} \tag{5.141}$$

$$\eta_{v,\kappa} \;=\; \frac{\kappa V_0^2 \phi^{2n-4}}{9(\varepsilon - 4\kappa \dot{H})^3}\left[n^3(n-2)\right] \tag{5.142}$$

$$\eta_{v,\kappa} \;=\; \frac{\kappa V_0^2 2^{n-2} M_{\rm P}^{2n-4}}{9(\varepsilon - 4\kappa \dot{H})^{n+1}}(n-2)n^{n+1}\left(N_{\rm I} + \tilde{\phi}_{\rm f}^2\right)^{n-2} \tag{5.143}$$

For $n = 2$, $\eta_{V,\kappa} = 0$ and for $n = 4$,

$$\eta_{V,\kappa,n=4} \;=\; \frac{8192}{9}\frac{\kappa V_0^2 M_{\rm P}^4}{(\varepsilon - 4\kappa \dot{H})^5}\left(N_{\rm I} + \tilde{\phi}_{\rm f}^2\right)^2 \tag{5.144}$$

$$\eta_{V,\kappa,n=4} \;=\; \frac{128}{9}\frac{\kappa V_0^2}{(\varepsilon - 4\kappa \dot{H})^3}\left[\frac{64 N_{\rm I}^2 M_{\rm P}^4}{(\varepsilon - 4\kappa \dot{H})^2} + \frac{16 N_{\rm I} M_{\rm P}^2 \phi_{\rm f}^2}{(\varepsilon - 4\kappa \dot{H})} + \phi_{\rm f}^4\right] \tag{5.145}$$

Substituting Eq.(5.139) and Eq.(5.141) (5.142), the scalar spectral index is

$$n_{\rm s} \simeq 1 - \frac{3n}{2(N_{\rm I} + \tilde{\phi}_{\rm f}^2)} + \frac{n-1}{(N_{\rm I} + \tilde{\phi}_{\rm f}^2)} - 2^{n-1}\frac{\kappa V_0^2 M_{\rm P}^{2n-4}}{9(\varepsilon - 4\kappa \dot{H})^{n+1}}(n-2)n^{n+1}\left(N_{\rm I} + \tilde{\phi}_{\rm f}^2\right)^{n-2} \tag{5.146}$$

For $n = 2$

$$n_{\rm s} = 1 - \frac{2}{N_{\rm I} + \tilde{\phi}_{\rm f}^2} \tag{5.147}$$

For $n = 4$

$$n_{\rm s} = 1 - \frac{3}{(N_{\rm I} + \tilde{\phi}_{\rm f}^2)} - \frac{16384}{9}\frac{\kappa V_0^2 M_{\rm P}^4}{(\varepsilon - 4\kappa \dot{H})^5}\left(N_{\rm I} + \tilde{\phi}_{\rm f}^2\right)^2 \tag{5.148}$$

and the tensor to scalar ratio is [109]

$$r \simeq 16\epsilon_{\rm v} \tag{5.149}$$

CHAPTER VI

DYNAMICAL SYSTEM FOR EIBI GRAVITY

6.1 Introduction to Dynamical system and Linear stability theory

The dynamical system of any abstract system can be start from the dynamics of the simple pendulum system to the evolutionary of the entire Universe. Even though, it may seem nonsense that in some models of modified gravity only two or three simple ordinary differential equations(ODEs) can be used to describe the entire universe. With proposing the homogeneity and isotropy of universe[1], the complexity of non-linear equation of Einstein field equation reduces to ODEs which encapsule the evolution of any points of the system[111]. The important concept of dynamical system is composed of

1.State vectors or the phase space parameters. It can be a set of coordinate and momentum,

2.Mathematical rules describe the evolution of all point in phase space which is real phase space.

Let us denote the state vector $x = (x_1, x_2, ..., x_n) \in X,\ X \subseteq \mathbf{R}^n$. The evolution of the system in time is defined by a set of ordinary differential equations (ODEs).

$$\frac{dx}{dt} = \dot{x} = f(x), \tag{6.1}$$

where f(x) which is the tangent to the orbit through x can be interpreted as a vector field in real phase space $\mathbf{R}^n$[113]. The autonomous equations have a critical points or fixed points which satisfy

$$f(x_c) = 0. \tag{6.2}$$

[1]The homogeneity of the universe is more difficult to test than the isotropy of the universe [110].

In general, physical system in nature are described by non-linear autonomous system. For example, non-linear ODEs in two dimensions:

$$\dot{x} = a_1 x^2 + a_2 xy + a_3 \frac{x}{y}\,, \quad (6.3)$$

$$\dot{y} = a_4 xy^3 + a_5 y + a_6 x\,, \quad (6.4)$$

where $a_1, a_2, ..., a_6$ are constant.

The n-dimensional Jacobian matrix can be written as follows[112]:

$$J = \frac{\partial f_i(x)}{\partial x^j} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}. \quad (6.5)$$

It is quite easy to determine eigenvalues of 2×2 and 3×3 Jacobian matrix, but to compute eigenvalues for all possible critical points for Jacobian matrix which have dimensions greater than three is more difficult by its algebra. The information about the (in)stability of each fixed point is encapsuled in the eigenvalues of the Jacobian matrix J. The eigenvalues of Jacobian matrix for n-fixed points can be expressed by

$$\lambda_j = a_j + ib_j, \quad (6.6)$$

where $j = 1, 2, ..., n$. From Eq.(6.6), if $a_j \neq 0, x_c$ is called the hyperbolic fixed point, whereas $a_j = 0$ the eigenvalues can be reduced to $\lambda_j = ib_j$ then x_0 is called the non-hyperbolic fixed point.

However, for the sake of clarity, let us work out for two dimensional autonomous system given by

$$\dot{x} = f(x, y), \quad (6.7)$$

$$\dot{y} = g(x, y), \quad (6.8)$$

where both f and g are smooth functions of state vectors x and y.

We further assume that there has a hyperbolic critical point at (x_c, y_c) from two conditions

$$f(x_c, y_c) = 0, \tag{6.9}$$

$$g(x_c, y_c) = 0. \tag{6.10}$$

The Jacobian matrix of the two dimensions autonomous system is given by

$$J = \begin{pmatrix} f_{,x} & f_{,y} \\ g_{,x} & g_{,y} \end{pmatrix}_{|(x=x_c, y=y_c)} \tag{6.11}$$

,where comma denotes the partial derivative. Two eigenvalues of the Jacobian matrix obtained from $\det(J_{2\times2} - \lambda I_{2\times2}) = 0$ are expressed as follows:

$$\lambda_1 = \frac{1}{2}(f_{,x} + g_{,y}) + \frac{1}{2}\sqrt{(f_{,x} - g_{,y})^2 + 4f_{,y}g_{,x}} \quad , \tag{6.12}$$

$$\lambda_2 = \frac{1}{2}(f_{,x} + g_{,y}) - \frac{1}{2}\sqrt{(f_{,x} - g_{,y})^2 + 4f_{,y}g_{,x}} \; . \tag{6.13}$$

Taking linearized perturbation around x_c and y_c, these show

$$x = x_c + \delta x, \tag{6.14}$$

$$y = y_c + \delta y. \tag{6.15}$$

The evolution of dynamical system can be explained by

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = J\begin{pmatrix} \delta x \\ \delta y \end{pmatrix}, \tag{6.16}$$

where J is the Jacobi matrix defined in Eq.(6.11). The two eigenvalues λ_1 and λ_2 of the Jacobian matrix which are used to judge the (in)stability of the critical point (x_c, y_c) can be expressed as follows:

$$\lambda_1 = a_1 + ib_1, \tag{6.17}$$

$$\lambda_2 = a_2 + ib_2. \tag{6.18}$$

Hence, two solutions of Eq.(6.16) depending on both eigenvalues λ_1 and λ_2

$$\delta x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \tag{6.19}$$

and

$$\delta y = c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t}. \tag{6.20}$$

In principle, the stability analyse of autonomous system can be divided into 8 cases which we summarize in table 7.1

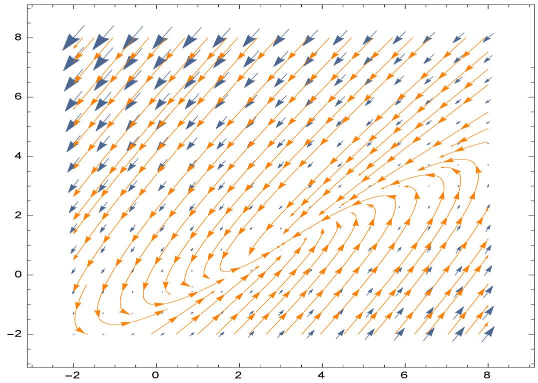

Figure 6.1: case 1 Asymtotic stable

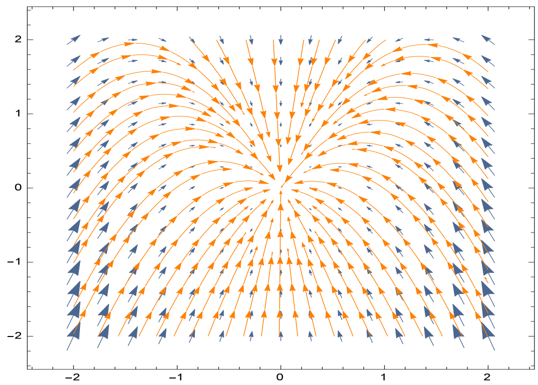

Figure 6.2: case 1 Stable fixed point

Table 6.1: Eight cases of linear stability analysis of autonomous system

Case	b	Eigenvalues	Description of the fixed point
1	b = 0	$a_1 < 0, a_2 < 0$	Asymptotically stable and $\lim_{t\to\ \infty}(x, y) = (x_0, y_0)$
2	b = 0	$a_1 > 0, a_2 > 0$	Unstable, Repelled from $\lim_{t\to\infty}(x, y) = (x_0, y_0)$
3	b = 0	$a_1 < 0, a_2 > 0$	Saddle point
4	b = 0	$a_1 = 0, a_2 > 0$	Fail of linear stability, non-hyperbolic
5	b = 0	$a_1 = 0, a_2 < 0$	Fail of linear stability, non-hyperbolic
6	$b\ \neq\ 0$	$\lambda_1 = a_1 + ib_1\ ; \lambda_2 = a_2 - ib_2$	With $a_j > 0$ and $b_j \neq 0$ where $j = 1, 2$. Spiral repellor
7	$b\ \neq\ 0$	$\lambda_1 = a_1 + ib_1\ ; \lambda_2 = a_2 - ib_2$	With $a_j < 0$ and $b_j \neq 0$ where $j = 1, 2$. Stable spiral
8	$b\ \neq\ 0$	$\lambda_1 = ib_1, \lambda_2 = -ib_2$	Solutions are oscillatory of $\sin(bt)$, $\cos(bt)$ and the point is called a centre

6.2 Linear stability for EiBI theory

From section 5.4, we bring up here again for the modified Friedmann equation at low energy regime of EiBI gravity [60, 114]

$$H^2 = \frac{\kappa^2\rho_{\rm m}}{3} + \frac{\Lambda}{3} + b\left[\frac{(\kappa^2\rho_{\rm m} + \Lambda)^2}{8}(1 + w_{\rm eff})(1 - 3w_{\rm eff})\right] + \mathcal{O}(b^2, b\Lambda, b\Lambda^2). \quad (6.21)$$

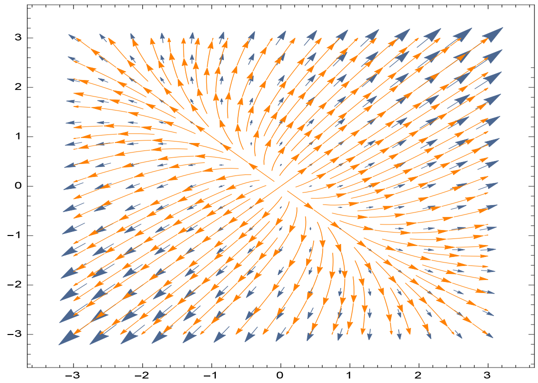

Figure 6.3: case 2 Unstable fixed point

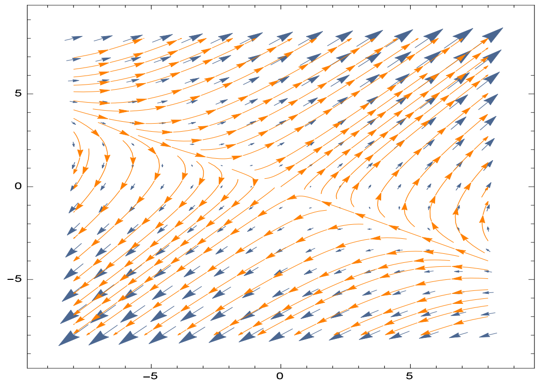

Figure 6.4: case 3 Saddle point

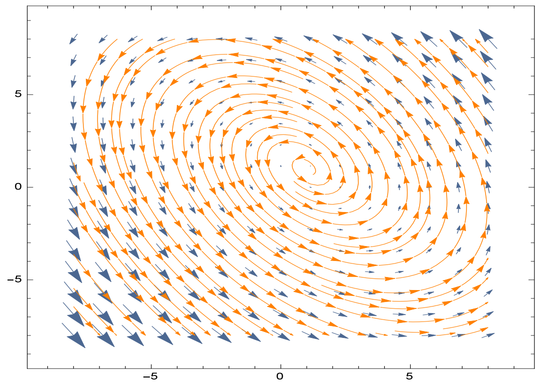

Figure 6.5: case 6 Unstable spiral

Dividing both sides by H^2, we get

$$1 = \frac{H^2}{H^2} = \frac{\kappa^2 \rho_{\rm m}}{3H^2} + \frac{\Lambda}{3H^2} + b\left[\frac{(\kappa^2\rho_{\rm m} + \Lambda)^2(1+w_{\rm eff})(1-3w_{\rm eff})}{8H^2}\right]. \tag{6.22}$$

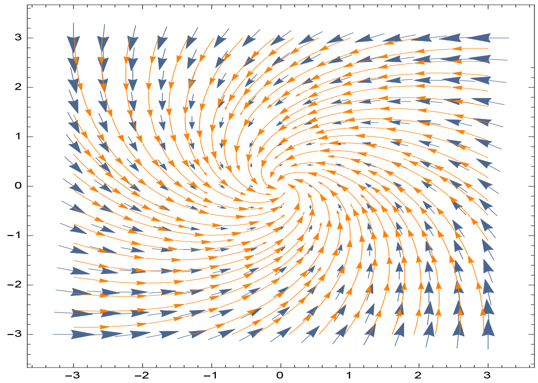

Figure 6.6: case 7 Spiral attractor

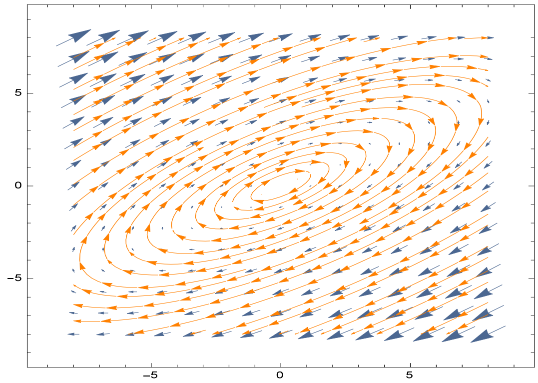

Figure 6.7: case 8 Center stable

The density parameter for (dark+ordinary) matter component is

$$\Omega_{\rm m} \equiv X = \frac{\kappa^2 \rho_{\rm m}}{3H^2} \,. \tag{6.23}$$

The density parameter for EiBI fluid can be defined as follows:

$$\begin{aligned} \Omega_{\rm EiBI} \equiv Y &= b\left[\frac{(\kappa^2\rho_{\rm m} + \Lambda)^2(1+w_{\rm eff})(1-3w_{\rm eff})}{8H^2}\right], \\ &= b\left[\frac{(\kappa^2\rho_{\rm m} + \Lambda)^2\gamma(4-3\gamma)}{8H^2}\right]. \end{aligned} \tag{6.24}$$

This term may shed some light on the interplay between $\rho_{\rm m}$ and Λ for the cosmological constant where b or EiBI parameter does the role here to be a coupling constant. It is worth to note that this term only appears whenever the effective EOS parameter is in the range of $-1 \leq w_{\rm eff} \leq \frac{1}{3}$. Otherwise, this gives the negative density parameter by its definition. The density parameter for the cosmological constant[1] can be defined as follow the reference[110].

$$\Omega_\Lambda \equiv Z \;=\; \frac{\Lambda}{3H^2} \;=\; \frac{\kappa^2\rho_\Lambda}{3H^2}\,. \tag{6.25}$$

where $\rho_\Lambda = \frac{\Lambda}{\kappa^2}$ is defined to be interchangeable between vacuum energy and cosmological constant point of view. The total density parameter can be defined as follow:

$$\Omega_{\rm tot} = 1 = \Omega_{\rm m} + \Omega_{\rm EiBI} + \Omega_\Lambda = X + Y + Z. \tag{6.26}$$

The effective EoS parameter of EiBI gravity at late time universe[2],

$$\begin{aligned} w_{\rm eff} &= \frac{p_{\rm m} + p_\Lambda}{\rho_{\rm m} + \rho_\Lambda} = \frac{-\Lambda}{\kappa^2\rho_{\rm m} + \Lambda}\,, \\ &= \frac{-Z}{X+Y} = \frac{X+Y-1}{1-Y}\,. \end{aligned} \tag{6.27}$$

For simplicity, we assume all fluids under EiBI gravity obey the continuity equation, i.e. $\dot{\rho}_{\rm i} = -3H\rho_{\rm i}(1+w_{\rm i})$. Hence there has no energy exchange between dark matter and vacuum energy. We also see that $\Omega_{\rm EiBI}$ can be written as

$$\Omega_{\rm EiBI} \;=\; \frac{9bH^2}{2}\frac{\left(1-\frac{3X^2}{4}\right)}{\left(1+9b\frac{H^2X}{2}\right)}\,. \tag{6.28}$$

From Eq.(6.21), the EiBI Hubble parameter at late time then becomes

$$\begin{aligned} H^2(t) \;=\; & H_0^2\Big[\Omega_{\rm m}(\frac{a}{a_0})^3 + \Omega_\Lambda + \frac{9bH_0^2}{8\kappa^2}(\Omega_{\rm m}^2(\frac{a}{a_0})^6 \\ & +2\Omega_{\rm m}\Omega_\Lambda(\frac{a}{a_0})^3 + \Omega_\Lambda^2)(1+w_{\rm eff})(1-3w_{\rm eff})\Big]\,. \end{aligned} \tag{6.29}$$

From observation point of view, we use the relationship between redshift and scale

---

[1]One of the most motivated essay about the cosmological constant is quoted from Eric V. Linder [115] " Space itself has the cosmological constant or the other name is the vacuum energy with negative pressure that could accelerate the expansion of the universe."

[2]We do not include the contribution form EiBI fluid ($\rho_{\rm EiBI}$ as shown in Eq.(4.107)) because the fluid is composed of dark matter and vacuum energy under controlling of EiBI parameter b.

factor $\frac{a}{a_0} = \frac{1}{1+z}$ to write

$$H^2(z) = H_0^2 \Bigg[\Omega_{\rm m}(z)(\frac{1}{1+z})^3 + \Omega_\Lambda(z) + \frac{9bH_0^2}{8\kappa^2} \Big[\Omega_{\rm m}^2(z)(\frac{1}{1+z})^6 + 2\Omega_{\rm m}(z)\Omega_\Lambda(z)(\frac{1}{1+z})^3 + \Omega_\Lambda^2(z) \Big] \left(1 + w_{\rm eff}(z)\right)\left(1 - 3w_{\rm eff}(z)\right) \Bigg] \quad (6.30)$$

so the observable function $E^2(z)$ for EiBI gravity can be expressed as

$$E^2(z) \equiv \frac{H^2(z)}{H_0^2} = \Omega_{\rm m0}(1+z)^3 + \Omega_{\Lambda 0} + \frac{9bH_0^2}{8\kappa^2} \Big[\Omega_{\rm m0}^2(1+z)^6 + 2\Omega_{\rm m0}\Omega_{\Lambda 0}(1+z)^3 + \Omega_{\Lambda 0}^2 \left(1 + w_{\rm eff}(z)\right)\left(1 - 3w_{\rm eff}(z)\right) \Big]. \quad (6.31)$$

Let us get back to setting autonomous system equations by taking time derivative of the EiBI parameter, this yields

$$2H\dot{H} = \frac{\kappa^2\dot{\rho}_{\rm m}}{3} + \frac{b}{8}\left[2(\kappa^2\rho_{\rm m} + \Lambda)\kappa^2\dot{\rho}_{\rm m}(1 + w_{\rm eff})(1 - 3w_{\rm eff})\right]. \quad (6.32)$$

Substituting the definition for X ,Y ,Z , and $\dot{\rho}_{\rm m} = -3H\rho_{\rm m}$ into Eq.(6.32), one finds

$$\begin{aligned} \frac{\dot{H}}{H} &= -\frac{3\Omega_m H}{2} - \frac{27b}{8}(\Omega_{\rm m}^2 + \Omega_{\rm m}\Omega_\Lambda)(1 + w_{\rm eff})(1 - 3w_{\rm eff})H^3, \\ &= -\frac{3XH}{2} - \frac{27b}{8}X(1 - Y)(1 + w_{\rm eff})(1 - 3w_{\rm eff})H^2. \end{aligned} \quad (6.33)$$

Having replaced $\frac{\ddot{a}}{a} = \dot{H} + H^2$ to Eq.(6.33), the accelerated equation can be performed as

$$\frac{\ddot{a}}{a} = (1 - \frac{3\Omega_{\rm m}}{2})H^2 - \frac{27bH^4}{8}(\Omega_{\rm m}^2 + \Omega_\Lambda\Omega_{\rm m})(1 + w_{\rm eff})(1 - 3w_{\rm eff}). \quad (6.34)$$

It can be noted that that last term of Eq.(6.34) which shows up due to the effect of EiBI gravity ,vanishes when $w_{\rm eff} = -1$ and $w_{\rm eff} = \frac{1}{3}$. This is why the effect of the EiBI term cannot play the important role today. For the present day universe where $w_{\rm eff} = -1$ and the condition for expansion phase is $\Omega_{\rm m} \le \frac{2}{3}$, the acceleration equation,

$$\frac{\ddot{a}}{a} = (1 - \frac{3\Omega_{\rm m}}{2})H^2. \quad (6.35)$$

For matter dominated universe where $w_{\rm eff} = 0$, this supports the decelerated phase of the universe. The acceleration equation in this case,

$$\frac{\ddot{a}}{a} = (1 - \frac{3\Omega_{\rm m}}{2})H^2 - \frac{27bH^4}{8}\big(\Omega_{\rm m}^2 + \Omega_{\rm m}\Omega_\Lambda\big). \tag{6.36}$$

The time derivative of the density parameter of matter[1],

$$\dot{\Omega}_{\rm m} \equiv \dot{X} = \frac{\kappa^2\dot{\rho}_{\rm m}}{3H^2} - \frac{2\kappa^2\rho_{\rm m}}{3H^2}\frac{\dot{H}}{H}. \tag{6.37}$$

Using the e-folding number $N = \ln a$ and $\dot{X} = HX' = H\frac{dX}{dN} = H\frac{dX}{d\ln a}$, we can show that

$$\Omega_{\rm m}' \equiv \frac{dX}{dN} \equiv X'(X,Y)\,, \tag{6.38}$$

$$X' = 3(1+w_{\rm eff})X^2 + \frac{12XY}{2-w_{\rm eff}} - 3(1+w_{\rm eff})X\,, \tag{6.39}$$

$$= \frac{-9X^2+12X^3-3X^4+12XY+9X^2Y-9X^3Y-24XY^2+12XY^3}{3-X-6Y+XY+3Y^2}. \tag{6.40}$$

Taking time derivative for density parameter of the EiBI fluid, this yields

$$\dot{\Omega}_{\rm EiBI} \equiv \dot{Y} \equiv \frac{2b}{8H^2}\Big[(\kappa^2\rho_{\rm m}+\Lambda)\kappa^2\dot{\rho}_{\rm m}(1+w_{\rm eff})(1-3w_{\rm eff})\Big] \tag{6.41}$$

$$-\frac{2b}{8H^2}(\frac{\dot{H}}{H})\Big[(\kappa^2\rho_{\rm m}+\Lambda)^2(1+w_{\rm eff})(1-3w_{\rm eff})\Big].$$

Using the relation $\dot{\rho}_{\rm m} = -3H\rho_{\rm m}$, Eq.(6.41) becomes

$$\dot{Y} = -\frac{27b(1+w_{\rm eff})(1-3w_{\rm eff})H^3}{4}\Big[(\Omega_{\rm m}^2 + \Omega_\Lambda\Omega_{\rm m})\Big] \tag{6.42}$$

$$-\frac{9bH^2(1+w_{\rm eff})(1-3w_{\rm eff})}{4}(\frac{\dot{H}}{H})\Big[\Omega_{\rm m}^2 + 2\Omega_{\rm m}\Omega_\Lambda + \Omega_\Lambda^2\Big],$$

Next, we use the expression for $\frac{\dot{H}}{H}$ in Eq.(6.33) and restore back the definition $X = \Omega_{\rm m},\ Y = \Omega_{\rm EiBI},$ and $Z = \Omega_\Lambda$. We obtain

$$\dot{Y} = -\frac{27b}{4}(1+w_{\rm eff})(1-3w_{\rm eff})H^3X(1-Y)$$

$$-\frac{9b}{4}(1+w_{\rm eff})(1-3w_{\rm eff})H^3\Big[-\frac{3X}{2} - \frac{27b}{8}X(1-Y)(1+w_{\rm eff})(1-3w_{\rm eff})H\Big],$$

$$\simeq \frac{27b}{4}(1+w_{\rm eff})(1-3w_{\rm eff})H^3X(Y-\frac{1}{2})\,, \tag{6.43}$$

[1] The total matter here comes form the existence of dark matter and ordinary matter owning to the fact that our knowledge that the matter sector does not vanishing at late time , i.e. $\rho_{\rm m} = \rho_{\rm dm} + \rho_{\rm b}$.

where the second line of Eq.(6.43) comes from the condition that $b^2 \ll 1$ at late time. We can show that

$$\begin{aligned}\dot{Y} = Y'H &\simeq \frac{27b}{4}(1+w_{\rm eff})(1-3w_{\rm eff})H^3X(Y-\frac{1}{2}) \\ Y' &\simeq \frac{27b}{4}(1+w_{\rm eff})(1-3w_{\rm eff})H^2X(Y-\frac{1}{2})\,. \end{aligned} \tag{6.44}$$

Substituting $H^2 = \frac{8Y}{9b(4-3X^2-4XY)}$ as showing in Eq.(6.28) , then we get

$$Y'(X,Y,w_{\rm eff}) \simeq \frac{6(1+w_{\rm eff})(1-3w_{\rm eff})XY(Y-\frac{1}{2})}{(4-3X^2-4XY)}\,. \tag{6.45}$$

To get rid of $w_{\rm eff}$, we have to substitute $w_{\rm eff} = \frac{X+Y-1}{1-Y}$ into Eq.(6.45). We therefore get

$$\begin{aligned}\Omega'_{\rm EiBI} = \frac{dY}{dN} = Y'(X,Y) &\simeq \frac{3X^2Y(2Y-1)(3X+4Y-4)}{(Y-1)^2(3X^2+4XY-4)}\,, \\ &\simeq \frac{12X^2Y-9X^3Y-36X^2Y^2+18X^3Y^2+24X^2Y^3}{3X^2+8Y+4XY-6X^2Y-4Y^2-8XY^2+3X^2Y^2+4XY^3-4}\,. \end{aligned} \tag{6.46}$$

Due to the fact that the variable Y in Eq.(6.46) cannot equal to unity, EiBI term has not been experienced dominated phase along the late time evolution of EiBI universe.

Solving $\Omega'_{\rm m} = X'(X,Y) = 0$ and $\Omega'_{\rm EiBI} = Y'(X,Y) = 0$ simultaneously, we get three meaningful fixed points as follows:

The first one is $(0,0,1)$ which is called cosmological constant dominated fixed point;

The second one is $(1,0,0)$ which is called dark matter dominated fixed point;

The third one is $(0,\frac{1}{2},\frac{1}{2})$ which is simply called the ΛEiBI fixed point.

The field plot of EiBI gravity can illustrate with the increase time direction of ODEs in Figure 7.8.

The Jacobian matrix for the dynamical system of the EiBI gravity is

$$J \equiv \begin{pmatrix} \frac{\partial X'(X,Y)}{\partial X} & \frac{\partial X'(X,Y)}{\partial Y} \\ \frac{\partial Y'(X,Y)}{\partial X} & \frac{\partial Y'(X,Y)}{\partial Y} \end{pmatrix}_{(X_c,Y_c)} \tag{6.47}$$

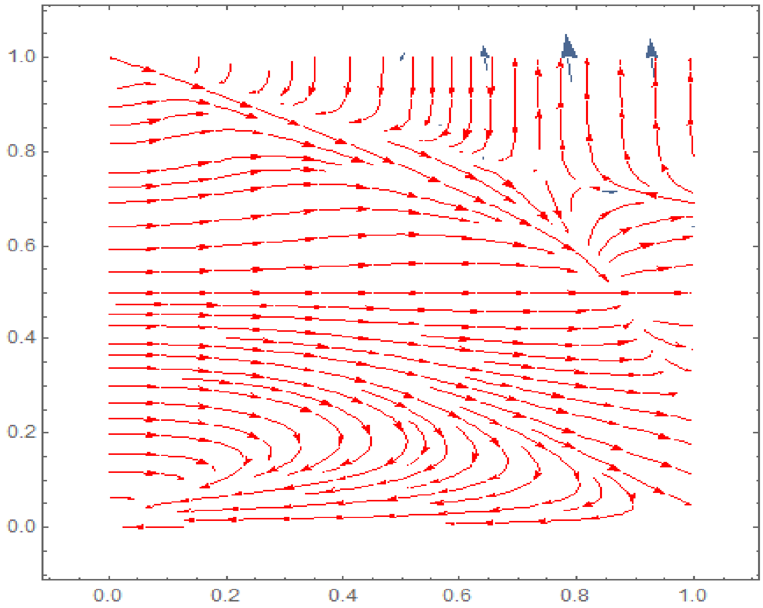

Figure 6.8: Field Plot for late time EiBI universe

,where (X_c, Y_c) is a critical point under consideration.

For EiBI gravity, the component of the Jacobian matrix can be listed as follows.

$$
\begin{aligned}
J_{11} &= \frac{\partial X'}{\partial X} = \frac{-3\left(3X^4 - 18(Y-1)^2 - 12(Y-1)^3Y + 2X^3(9Y-10) + 3X^2(9Y^2 - 22Y + 13)\right)}{(Y-1)(3Y+X-3)^2}\,, \\
J_{12} &= \frac{\partial X'}{\partial Y} = \frac{3X\left(X^4 + 12(Y-1)^4 + X^3(6Y-7) + X(Y-1)^2(8Y-13) + 3X^2(3Y^2 - 8Y + 5)\right)}{(Y-1)^2(3Y+X-3)^2}\,, \\
J_{21} &= \frac{\partial Y'}{\partial X} = \frac{3XY(2Y-1)\left(9X^3 - 32(Y-1) + 24X^2Y + 4X(4Y^2 - 4Y - 9)\right)}{(Y-1)^2(4XY + 3X^2 - 4)^2}\,, \\
J_{22} &= \frac{\partial Y'}{\partial Y} = \Big[-3X^2\Big(9X^3(3Y-1) + 12X^2(6Y^2 - 5Y + 1) + 16(2Y^3 - 6Y^2 + 5Y - 1) \\
&\quad +4X(4Y^3 - 4Y^2 - 9Y + 3)\Big)\Big]\Big/\Big[(Y-1)^3(4XY + 3X^2 - 4)^2\Big]\,.
\end{aligned}
$$

The eigenvalues of the Jacobian matrix are

$$
\lambda_1, \lambda_2 = \frac{1}{2}\left[J_{11} + J_{22} \pm \sqrt{[(J_{11} + J_{22})^2 - 4(J_{11}J_{22} - J_{12}J_{21})]}\right]. \tag{6.48}
$$

In general the solution around the fixed point can be expressed in the following form:

$$
\begin{aligned}
\delta X &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\,, \\
\delta Y &= c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t}\,.
\end{aligned} \tag{6.49}
$$

For the first fixed point (0,0,1), we find that the Jacobian matrix of this fixed point is

$$J_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.50}$$

which is the degenerate fixed point that using KCC method or Lyapunov function to identify the stability of this fixed point.

For the second fixed point $(1, 0, 0)$, the Jacobian matrix of this fixed point is

$$J_{(1,0)} = \begin{pmatrix} 3 & 6 \\ 0 & -3 \end{pmatrix}. \tag{6.51}$$

The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 3$, this is the saddle point therefore it is an unstable fixed point for dark matter dominated in EiBI universe.

For the third fixed point $(0, \frac{1}{2}, \frac{1}{2})$, the Jacobian matrix of this fixed point is

$$J_{(1,0)} \equiv \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.52}$$

Two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 0$. This is also a degenerate fixed point that using KCC method or Lyapunov function to identify the stability of this fixed point.

We summarize the stability analysis for three fixed points in table 6.2 below. Two

Table 6.2: Three critical points and their (in)stability of late time EiBI universe

Point	(X_c, Y_c)	Existence	Stability
1^{st}	(0 , 0, 1)	$\lambda_1 = 0$; $\lambda_2 = 0$	the degenerate fixed point,inconclusive
2^{nd}	(1, 0 , 0)	$\lambda_1 = -3$;$\lambda_2 = 3$	saddle point
3^{rd}	$(0, \frac{1}{2}, \frac{1}{2})$	$\lambda_1 = 2$; $\lambda_2 = 0$	the degenerate fixed point,inconclusive

degenerate fixed point found in this work will be evaluated their instability by Kosambi-Cartan-Chern theory and Lyapunov function in the coming section.

6.3 Kosambi-Cartan-Chern (KCC)theory for EiBI theory

The KCC theory and Lyapunov function (see section 7.3) provide us with important tools when confront with the appearing of degenerate fixed points which posses zero eigenvalue.

6.3.1 Finsler manifold

Let $\mathcal{M}$ be an smooth n-dimensional C^ω real analytic manifold, and $T_x\mathcal{M}$ denotes the tangent vector space (bundle) of $\mathcal{M}$ at $x \in \mathcal{M}$. Each element of $T\mathcal{M}$ has the form of function of $u = (x, y)$ in which $x \in \mathcal{M}$ and $y \in T_x\mathcal{M}$ be a point in $T\mathcal{M}$,where $x = (x^1, x^2, ..., x^n) \in \mathcal{M}$ be a local coordinate system on open subset $U \subset \mathcal{M}$, and $y = y^i \frac{\partial}{\partial x^i} = (y^1, y^2, ..., y^n) \in T_x\mathcal{M}$ where $\frac{\partial}{\partial x^i}$ refers to the induced coordinate bases vector for $T_x\mathcal{M}$. Indeed, the geometry of spacetime of two variable which x stands for position and y stands for velocity on Finslerian spacetime that served as a generalized geometric background which is extension form of Riemannian metric geometry [116],[117].

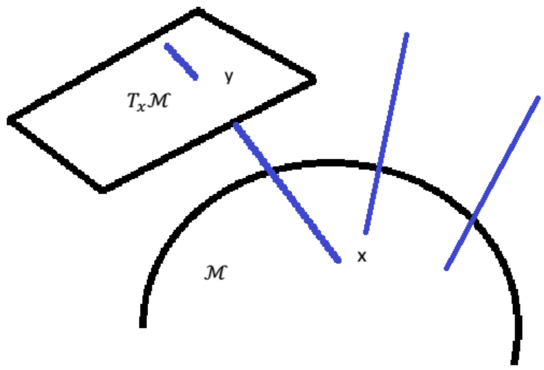

Figure 6.9: Finsler manifold

The Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = F_i, \tag{6.53}$$

where $y^i = \frac{dx^i}{dt}$, $i = 1, 2, ..., n$, $L = L(x, y)$ is the Lagrangian of M, and F_i are the external forces.

Theorem: Every set of dynamical systems can be expressed to a system of second-order differential equations define stability of a whole trajectory.

$$\frac{dx^i}{dt} = f(x^i) \qquad x^i = (x^1, x^2, ..., x^n). \tag{6.54}$$

The Euler-Lagrange equations Eq.(6.53) are equivalent to a system of second-order differential equations in the following form.

$$\frac{d^2x^i}{dt^2} + 2G^i(x, y) = 0, \qquad i = 1, 2, ..., n \ , \tag{6.55}$$

where $G^i(x, y)$ are smooth functions defined in a local system of coordinates on TM. $G^i(x, y)$ can be interpreted as the Newtonian force which includes friction forces.

Eq.(6.55) was discovered[1] by D.D. Kosambi [119] in 1933 and part of work of E. Cartan [120] in 1933 and revised to be an elegant form by S. Chern [121] in 1939. Substituting X', Y' from EiBI's autonomous system equations, i.e.

$$X'(X,Y) = \frac{-9X^2 + 12X^3 - 3X^4 + 12XY + 9X^2Y - 9X^3Y - 24XY^2 + 12XY^3}{3 - X - 6Y + XY + 3Y^2}, \tag{6.56}$$

$$Y'(X,Y) \simeq \frac{12X^2Y - 9X^3Y - 36X^2Y^2 + 18X^3Y^2 + 24X^2Y^3}{3X^2 + 8Y + 4XY - 6X^2Y - 4Y^2 - 8XY^2 + 3X^2Y^2 + 4XY^3 - 4}, \tag{6.57}$$

and taking the second derivative of X with respect to N, we have

$$\begin{aligned} X'' = \frac{dX'}{dN} &= \frac{\partial X'}{\partial X}\frac{dX}{dN} + \frac{\partial X'}{\partial Y}\frac{dY}{dN}, \\ &= \frac{\partial X'}{\partial X}X' + \frac{\partial X'}{\partial Y}Y', \end{aligned} \tag{6.58}$$

[1]See for more details about the historical background of Finsler spacetime in Ref[117].

where the expression for $\frac{\partial X'(X,Y)}{\partial X}$ and $\frac{\partial X'(X,Y)}{\partial Y}$ are

$$\frac{\partial X'}{\partial X} = -\frac{3\Big[3X^4 - 18X(Y-1)^2 - 12Y(Y-1)^3 + 2X^3(9Y-10) + 3X^2(9Y^2-22Y+13)\Big]}{3Y^2+XY-6Y-X+3},$$

$$\frac{\partial X'}{\partial Y} = \frac{3X(X^4 + 12(Y-1)^4 + X^3(6Y-7) + X(Y-1)^2(8Y-13) + 3X^2(3Y^2-8Y+5)}{(Y-1)^2(3Y+X-3)^2} \qquad (6.59)$$

respectively. Taking the second derivative of Y with respect to N, this shows

$$Y'' = \frac{dY'}{dN} = \frac{\partial Y'}{\partial X}\frac{dX}{dN} + \frac{\partial Y'}{\partial Y}\frac{dY}{dN}, = \frac{\partial Y'}{\partial X}X' + \frac{\partial Y'}{\partial Y}Y', \qquad (6.60)$$

where the expression for $\frac{\partial Y'(X,Y)}{\partial X}$ and $\frac{\partial Y'(X,Y)}{\partial Y}$ are

$$\frac{\partial Y'}{\partial X} = \frac{3XY(2Y-1)\Big[9X^3 - 32(Y-1) + 24X^2Y + 4X(4Y^2-4Y-9)\Big]}{(Y-1)^2(4XY+3X^2-4)^2}, \qquad (6.61)$$

$$\frac{\partial Y'}{\partial Y} = -3X^2\Big[9X^3(3Y-1) + 12X^2(6Y^2-5Y+1) + 16(2Y^3-6Y^2+5Y-1) + 4X(4Y^3-4Y^2-9Y+3)\Big] \Big/ (Y-1)^3(4XY+3X^2-4)^2.$$

From Eq.(6.55), we can express that $X'' + 2G^1(X,Y,X',Y') = 0$ and $Y'' + 2G^2(X,Y,X',Y') = 0$. $G^1(X,Y,X',Y')$ and $G^2(X,Y,X',Y')$ become

$$G^1(X,Y,X',Y') = -\frac{1}{2}\Big[\frac{\partial X'(X,Y)}{\partial X}\frac{dX}{dN} + \frac{\partial X'(X,Y)}{\partial Y}\frac{dY}{dN}\Big] = -\frac{1}{2}\Big[\frac{\partial X'(X,Y)}{\partial X}X' + \frac{\partial X'(X,Y)}{\partial Y}Y'\Big], \qquad (6.62)$$

$$G^2(X,Y,X',Y') = -\frac{1}{2}\Big[\frac{\partial Y'(X,Y)}{\partial X}\frac{dX}{dN} + \frac{\partial Y'(X,Y)}{\partial Y}\frac{dY}{dN}\Big] = -\frac{1}{2}\Big[\frac{\partial Y'(X,Y)}{\partial X}X' + \frac{\partial Y'(X,Y)}{\partial Y}Y'\Big] \qquad (6.63)$$

.

We can write G^1 and G^2 in term of variable X and Y only as

$$
\begin{aligned}
G^1(X,Y) \quad = \quad & -\frac{3}{2(Y-1)^2}\Bigg\{\Bigg[3X^3(1-2Y)^2Y^2(4Y+3X-4)\Big(9X^3-32(Y-1)+24X^2Y \\
& +4X(4Y^2-4Y-9)\Big)\Big/(Y-1)^2(4XY+3X^2-4)^3\Bigg] \qquad (6.64) \\
& +\Bigg[3X\Big(X^3-3X(Y-1)-4Y(Y-1)^2+X^2(3Y-4)\Big)\Big(3X^4-18X(Y-1)^2 \\
& -12Y(Y-1)^3+2X^3(9Y-10)+3X^2(9Y^2-22Y+13)\Big)\Bigg]\Big/(3Y+X-3)^3\Bigg\},
\end{aligned}
\tag{6.65}
$$

$$
\begin{aligned}
G^2(X,Y) \quad = \quad & \frac{9X^2Y(2Y-1)}{2(Y-1)^3(3X^2+4XY-4)^2}\Bigg\{\Big[\Big(X^3-3X(Y-1)-4(Y-1)^2Y \\
& +X^2(3Y-4)\Big)\Big(9X^3-32(Y-1)+24X^2Y+4X(4Y^2-4Y-9)\Big)\Big/(3Y+X-3)\Big] \\
& +\Big[X^2(4Y+3X-4)\Big(9X^3(3Y-1)+12X^2(6Y^2-5Y+1) \\
& +16(2Y^3-6Y^2+5Y-1)+4X(4Y^3-4Y^2-9Y+3)\Big) \\
& \Big/(Y-1)^2(3X^2+4XY-4)\Big]\Bigg\}.
\end{aligned}
\tag{6.66}
$$

A non-linear connection N^i_j on TM which plays the role of parallel transport in Finsler space can be defined as

$$
N^i_j \equiv \frac{\partial G^i(x^j,y^j,t)}{\partial y^j}. \tag{6.67}
$$

Here in Eq.(6.67), we set $x^j = \{X, Y\}$ and $y^j = \{X', Y'\}$.

$$N_1^1 = \frac{\partial G^1(X, Y, X', Y')}{\partial X'} = -\frac{1}{2}\frac{\partial X'(X, Y)}{\partial X}, \tag{6.68}$$

$$N_2^1 = \frac{\partial G^1(X, Y, X', Y')}{\partial Y'} = -\frac{1}{2}\frac{\partial X'(X, Y)}{\partial Y}, \tag{6.69}$$

$$N_1^2 = \frac{\partial G^2(X, Y, X', Y')}{\partial X'} = -\frac{1}{2}\frac{\partial Y'(X, Y)}{\partial X}, \tag{6.70}$$

$$N_2^2 = \frac{\partial G^2(X, Y, X', Y')}{\partial Y'} = -\frac{1}{2}\frac{\partial Y'(X, Y)}{\partial Y}. \tag{6.71}$$

The Berwald connection is

$$G_{jl}^i \equiv \frac{\partial N_j^i}{\partial y^l}, \tag{6.72}$$

where we define $y^l = \{X', Y'\}$.

Then the Berwald connection becomes

$$G_{11}^1 = \frac{\partial N_1^1}{\partial X'} = \frac{\partial N_1^1}{\partial X}\frac{\partial X}{\partial X'} + \frac{\partial N_1^1}{\partial Y}\frac{\partial Y}{\partial X'}, \tag{6.73}$$

$$G_{12}^1 = \frac{\partial N_1^1}{\partial Y'} = \frac{\partial N_1^1}{\partial X}\frac{\partial X}{\partial Y'} + \frac{\partial N_1^1}{\partial Y}\frac{\partial Y}{\partial Y'}, \tag{6.74}$$

$$G_{21}^1 = \frac{\partial N_2^1}{\partial X'} = \frac{\partial N_2^1}{\partial X}\frac{\partial X}{\partial X'} + \frac{\partial N_2^1}{\partial Y}\frac{\partial Y}{\partial X'}, \tag{6.75}$$

$$G_{22}^1 = \frac{\partial N_2^1}{\partial Y'} = \frac{\partial N_2^1}{\partial X}\frac{\partial X}{\partial Y'} + \frac{\partial N_2^1}{\partial Y}\frac{\partial Y}{\partial Y'}, \tag{6.76}$$

$$G_{11}^2 = \frac{\partial N_1^2}{\partial X'} = \frac{\partial N_1^2}{\partial X}\frac{\partial X}{\partial X'} + \frac{\partial N_1^2}{\partial Y}\frac{\partial Y}{\partial X'}, \tag{6.77}$$

$$G_{12}^2 = \frac{\partial N_1^2}{\partial Y'} = \frac{\partial N_1^2}{\partial X}\frac{\partial X}{\partial Y'} + \frac{\partial N_1^2}{\partial Y}\frac{\partial Y}{\partial Y'}, \tag{6.78}$$

where we use the reciprocal forms of $[\frac{\partial X}{\partial Y'}] = [\frac{\partial Y'}{\partial X}]^{-1}$, $[\frac{\partial Y}{\partial X'}] = [\frac{\partial X'}{\partial Y}]^{-1}$ and etc. The second KCC-invariant or the deviation curvature tensor can be defined as follows.

$$P^i_j \equiv -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l \frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j + \frac{\partial N^i_j}{\partial t}. \tag{6.79}$$

Theorem: The trajectories of Eq.(6.55) are Jacobi stable if and only if the real parts of the eigenvalues of the deviation tensor P^i_j are strictly negative everywhere, and Jacobi unstable, otherwise [112](p. 34-35).

For two dimensional case the curvature deviation tensor can be expressed in a matrix form as

$$P^i_j = \begin{pmatrix} P^1_1 & P^1_2 \\ P^2_1 & P^2_2 \end{pmatrix}, \tag{6.80}$$

The eigenvalues of this metric is given by

$$\lambda_\pm = \frac{1}{2}\Big[P^1_1 + P^2_2 \pm \sqrt{(P^1_1 - P^2_2)^2 + 4P^1_2 P^2_1} \Big] \tag{6.81}$$

A very powerful algebraic of the Routh-Hurwitz criteria [122] shows that the fixed point which have the negative real parts of eigenvalues of the deviation tensor P^i_j is the Jacobi stable fixed point.

$$P^1_1 + P^2_2 < 0, \text{ and } \mathrm{P}^1_1\mathrm{P}^2_2 - \mathrm{P}^1_2\mathrm{P}^2_1 > 0. \tag{6.82}$$

All possible components of the deviation curvature tensor are

$$\begin{aligned}
P^1_1(X,Y) &= -2\frac{\partial G^1}{\partial X} - 2G^1 G^1_{11} - 2G^2 G^1_{12} + X'\frac{\partial N^1_1}{\partial X} + Y'\frac{\partial N^1_1}{\partial Y} + N^1_1 N^1_1 + N^1_2 N^2_1 \,, \\
P^1_2(X,Y) &= -2\frac{\partial G^1}{\partial Y} - 2G^1 G^1_{21} - 2G^2 G^1_{22} + X'\frac{\partial N^1_2}{\partial X} + Y'\frac{\partial N^1_2}{\partial Y} + N^1_1 N^1_2 + N^1_2 N^2_2 \,, \\
P^2_1(X,Y) &= -2\frac{\partial G^2}{\partial X} - 2G^1 G^2_{11} - 2G^2 G^2_{12} + X'\frac{\partial N^2_1}{\partial X} + Y'\frac{\partial N^2_1}{\partial Y} + N^2_1 N^1_1 + N^2_2 N^2_1 \,, \\
P^2_2(X,Y) &= -2\frac{\partial G^2}{\partial Y} - 2G^1 G^2_{21} - 2G^2 G^2_{22} + X'\frac{\partial N^2_2}{\partial X} + Y'\frac{\partial N^2_2}{\partial Y} + N^2_1 N^1_2 + N^2_2 N^2_2 \,,
\end{aligned} \tag{6.83}$$

We found that the eigenvalues of the dark matter dominated fixed point from the the deviation curvature tensor,

$$P^i_{j\ (1,0)} = \begin{pmatrix} \frac{81}{4} & 18 \\ 0 & \frac{45}{4} \end{pmatrix}, \tag{6.84}$$

Table 6.3: List of Geometrical objects in KCC theory for EiBI gravity

Geomertical objects	Fixed point $(1,0)$	Fixed point $(0,\frac{1}{2})$	Fixed point $(0,0)$
$\frac{\partial G^1}{\partial X}$	-9	0	0
$\frac{\partial G^1}{\partial Y}$	-9	0	0
$\frac{\partial G^2}{\partial X}$	0	0	0
$\frac{\partial G^2}{\partial Y}$	$-\frac{9}{2}$	0	0
G^1	0	0	0
G^2	0	0	0
N_1^1	$-\frac{3}{2}$	-1	0
N_2^1	-3	0	0
N_1^2	0	0	0
N_2^2	$\frac{3}{2}$	0	0

are

$$\lambda_1 = \frac{81}{4} > 0, \tag{6.85}$$

$$\lambda_2 = \frac{45}{4} > 0. \tag{6.86}$$

While the eigenvalues of the ΛEiBI fixed point from the the deviation curvature tensor,

$$P^i_{j\ (0,\frac{1}{2})} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.87}$$

are

$$\lambda_1 = 1 > 0, \tag{6.88}$$

$$\lambda_2 = 0. \tag{6.89}$$

We also investigated the eigenvalues of the cosmological constant fixed point from the the deviation curvature tensor,

$$P^i_{j\ (0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.90}$$

are

$$\lambda_1 = 0, \tag{6.91}$$

$$\lambda_2 = 0. \tag{6.92}$$

It is worth to notice that for KCC theory can use to indicate that the second and third fixed point for dark matter dominated phase and ΛEiBI are unstable fixed point whereas we cannot evaluate the (in)stability of the first fixed point for cosmological constant dominated phase of the universe by the KCC theory. We

Table 6.4: KCC method for three fixed points of late time EiBI universe

Name of fixed point	Fixed point	Eigenvalues	Stability
Λ dominated	$(0, 0, 1)$	$\{0, 0\}$	KCC cannot identify
DM dominated	$(1, 0, 0)$	$\{\frac{81}{4}, \frac{45}{4}\}$	Jacobi unstable
ΛEiBI	$(0, \frac{1}{2}, \frac{1}{2})$	$\{1, 0\}$	Jacobi unstable

know that the cosmological constant or the vacuum dominated fixed point has to be stable phase at the asymptotic evolution of the universe. Nevertheless, we would like to confirm the stability of this fixed point by Lyapunov function in the next section.

6.4 Lyapunov functions

Even though linearization method is a great tools for indicating the stability of fixed points of dynamical systems. If a critical point is nonhyberbolic, the Lyapunov functions method may use to be an emergency back up to test the stability of these fixed point without solving ODEs [123, 124]. Typically, it is the most usual method be useable to define the global convergence in a dynamical system along the evolutionary dynamics and widely used to find the (in)stability of non-hyperbolc fixed points. This method is completely different to linear stability of fixed point in dynamical system. The main problem of this approach is no systematic way to get Lyapunov function. Even though, there are some methods like The Lotka-Voltera dynamic to obtain the general Lyapunov function for coupled system but the application is limited to some forms of dynamical system [125]. Additionally, the centre manifold method cannot apply to the analysis the fixed point which have zero eigenvalue because we unable to set the autonomous system equation of EiBI as the following form.

$$X' = AX + f(X, Y)\,, \tag{6.93}$$

$$Y' = BY + g(X, Y)\,, \tag{6.94}$$

where A is a $c \times c$ matrix which gives the eigenvalues to be zero real part and B is an $s \times s$ matrix which gives negative real part(see for more details in ref.[126]). In fact this method is inspired from the energy loss of dynamical system explaining a bizarre medium with highly nonlinear resistance in which an the end the system eventually halt its motion because the weird friction is draining of energy from the system[124].

Theorem: Let $\dot{x} = f(x)$ with $x \in X \subset R^n$ be a smooth autonomous system of equations with fixed point x_c. Let $V : R^n \mapsto R$ be a continuous function in a neighbourhood $\mathcal{U}$ of x_0, V is a Lyapunov function which behaves like the energy for the critical point x_c providing that

1. V is differentiable function in $\mathcal{U}$ with $V(x) > 0$ for all $x \neq x_c$, and $V(x_c = 0) = 0$.

2. $\dot{V} \leq 0$ at all state x. Additionally, any state $x \neq x_c$ where $\dot{V} = 0$, the system instantly moves to a state where $\dot{V} < 0$.

It can be noticed that the second requirement is the crucial one. This implies

$$\frac{d}{dt}V(x_1, x_2, ..., x_n) = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} ... + \frac{\partial V}{\partial x_n}\dot{x_n} = \frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 ... + \frac{\partial V}{\partial x_n}f_n, \quad (6.95)$$

where $x \in R^n$. Then

1. If $\dot{V}(x) \leq 0$ for all $x \in \mathcal{U}, x_c$ is stable.

2. If $\dot{V}(x) < 0$for all $x \in \mathcal{U}, x_c$ is asymptotically stable.

3. If $\dot{V}(x) > 0$ for all $x \in \mathcal{U}, x_c$ is unstable.

The crucial point to be summarized is that if there is $\dot{V} \leq 0$, then x_c is an asymptotically stable fixed point. Furthermore, if $\|x\| \to \infty$ and $V(x) \to \infty$ for all x, then x_c is said to be globally stable or globally asymptotically stable, respectively. If we be able to find a Lyapunov function that satisfy the critical point that Lyapunov stability theorem concerned, we could establish (asymptotic) stability without any reference to a solution of the ODEs.

Let us recall the ODE system of EiBI gravity,

$$\begin{aligned} X'(X,Y) &= \frac{-9X^2 + 12X^3 - 3X^4 + 12XY + 9X^2Y - 9X^3Y - 24XY^2 + 12XY^3}{3 - X - 6Y + XY + 3Y^2}, \\ Y'(X,Y) &\simeq \frac{12X^2Y - 9X^3Y - 36X^2Y^2 + 18X^3Y^2 + 24X^2Y^3}{3X^2 + 8Y + 4XY - 6X^2Y - 4Y^2 - 8XY^2 + 3X^2Y^2 + 4XY^3 - 4}. \end{aligned}$$

Our trial Lyapunov function for EiBI gravity is expressed in an ad hoc form as

$$V(X,Y) = aX^2 + cY^2 + fX^2Y^2 + h(X^2 + Y^2)X^2Y^2, \quad (6.96)$$

where $a, c,$ f, and h are positive constants. This function indicates that two condition for fixed point $(x_c, y_c) = (0,0)$,i.e. $V(x_c) = 0$ and $V(x) > 0$ are obeyed. The Lyapunov function of EiBI gravity plots in Figure 7.10 below. We have found that the generalized Lotka-Voltera method cannot construct Lyapunov function

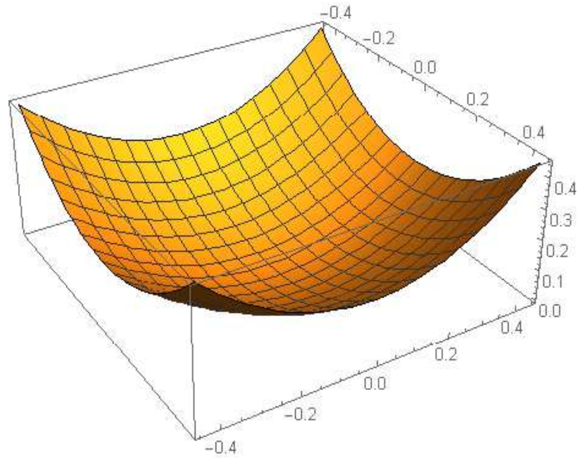

Figure 6.10: The Lyapunov function of EiBI gravity

for EiBI gravity V due to the appearing of complicated from numerator and denominator for the autonomous system.

From Eq.(6.96), we get

$$\frac{\partial V}{\partial X} = 2aX + 2fXY^2 + 2hX^3Y^2 + 2hXY^2(X^2+Y^2), \qquad (6.97)$$

$$\frac{\partial V}{\partial Y} = 2cY + 2fX^2Y + 2hX^2Y^3 + 2hX^2Y(X^2+Y^2). \qquad (6.98)$$

Defining the Lyapunov function

$$V' = \frac{dV}{dN} = \frac{\partial V}{\partial X}\frac{dX}{dN} + \frac{\partial V}{\partial Y}\frac{dY}{dN} = \frac{\partial V}{\partial X}X' + \frac{\partial V}{\partial Y}Y' \leq 0\,, \qquad (6.99)$$

we can use the fixed point of EiBI gravity at late time to see the energy loss of the system with our trial potential form of Lyapunov function,

$$V' = \frac{\partial V}{\partial X}X' + \frac{\partial V}{\partial Y}Y'. \qquad (6.100)$$

For simplicity, we set $a = c = f = h = 1$ and perturb around the fixed point $(0,0)$

by $(0+\epsilon, 0+\xi)$ where ϵ and ξ are a small positive value. This gives

$$
\begin{aligned}
V'(\epsilon,\xi) &= 6\epsilon^2\left[\frac{(\xi-1)(-\epsilon^3+\epsilon^2(4-3\xi)+3\epsilon(\xi-1)+4\xi(\xi-1)^2)(1+\xi^2(1+2\epsilon^2+\xi^2))}{(\epsilon+3\xi-3)(\xi-1)^2}\right. \\
&\quad \left.+\frac{\xi^2(2\xi-1)(4\xi+3\epsilon-4)(1+\epsilon^2(1+\epsilon^2+2\xi^2))}{(3\epsilon^2+4\epsilon\xi-4)(\xi-1)^2}\right], \\
&\simeq -6\epsilon^3+8\epsilon^2\xi > 0.
\end{aligned}
$$

This confirms that the first fixed point $(0,0)$ is an unstable fixed point.

CHAPTER VII

CONCLUSIONS, DISCUSSIONS, AND FUTURE PERSPECTIVES

This work ends with several final remarks, conclusions and future perspectives of both models of gravity, i.e. NMDC-Palatini and EiBI gravity respectively.

7.1 NMDC-Paltatini gravity and cosmology

7.1.1 Conclusions for NMDC-Palatini gravity

We have derived the field equations for NMDC gravity in Palatini formalism for the Einstein tensor non-minimally couples to the kinetic term of scalar field, i.e. $\kappa\tilde{G}_{\mu\nu}(\Gamma)\phi^{,\mu}\phi^{,\nu}$. The conformal metric automatically appears together with the conformal factor which depends on the time derivative of scalar field, i.e. $f(\dot{\phi}) = 1 + \frac{\kappa}{2}\dot{\phi}^2$ as the result of Palatini formulation. We nevertheless find it important to preserve the Lorentz signature of the conformal metric by limiting the values of NMDC coupling strength in the range of $-\frac{2}{\dot{\phi}^2} < \kappa \leq \infty$. It is found that gravitons travel slower than photon in the conformal frame. This shows that variation of graviton mass originates from field velocity. In conformal frame, the effective gravitational coupling is $G_{\rm eff} = G_{\rm N}(1 + \kappa\dot{\phi}^2/2)^2$ which leads to modification of the entropy of blackhole's apparent horizon to $S_{\rm AH} = A/[4G_{\rm N}(1 + \frac{\kappa}{2}\dot{\phi}^2)^2]$. It would however be interesting to estimate tiny values of $|\frac{\dot{G}_{\rm eff}}{G_{\rm eff}}| \simeq 2\kappa\dot{\phi}\ddot{\phi}$ for $\frac{\kappa}{2}\dot{\phi}^2 \ll 1$ from current observations. The modified Friedmann equations are found to be complicated with nonlinear interactions of matter fields, scalar field, and scalar field kinetic terms. Simplifying the field equations by considering the slow-roll regime, we see that the acceleration condition is modified to $w_{\rm eff} \simeq -(1/3)(1 + 2\kappa\dot{\phi}^2)$. It is shown that the quadratic power law potential fits very well with Inflationary stage of this theory.

7.1.2 Some discussions for NMDC-Palatini gravity

This is reasonable to construct the new metric from geometrical point of view by performing variation the action with respect to the independent connection and then we will inspect the form of new metric directly from the constraint equations. We still, therefore, do physics in a meaningful way.

The second time derivative of the conformal factor, i.e. $f(\dot{\phi}) = 1 + \frac{\kappa\dot{\phi}^2}{2}$, is undeniable to bring about the third time derivative of scalar field to equation of motions.

It is possible that the velocity and the acceleration of the homogenous scalar field is important in the universe than the scalar field itself. One point has to note here that the $\dot{H}$ is sourced by the kinetic energy density in General relativity theory [127](p.37),

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{\rm pl}^2}, \tag{7.1}$$

and then

$$\ddot{H} = -\frac{\dot{\phi}\ddot{\phi}}{M_{\rm pl}^2}. \tag{7.2}$$

The relations Eq.(7.1) and Eq.(7.2) may show the significance of the expression for $\dot{\phi}\ddot{\phi}$ in $G_{\rm eff}$ of NMDC-Palatini gravity. Quadratic potential eliminates an existence of the NMDC - term for inflation phase (see Eq.(5.146)and Eq.(5.149)). Nevertheless, it is good enough to calculate the spectral index ($n_{\rm s}$) and tensor to scalar ratio (r) within the acceptation values. On the contrary, the fourth power potential generates the large value for the third term of Eq.(5.148), so this form of potential does not suitable to explain inflationary epoch of NMDC-Palatini model.

7.1.3 Future perspectives for NMDC-Palatini gravity

The late time dynamical system will be investigated with some suitable potentials, e.g. the power law potential and a simple double well potential, i.e. $V(\phi) = \frac{1}{2}m^2\phi^2$ and $V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$ respectively.

7.2 EiBI gravity

We have derived EiBI field equations and have shown that the singularity avoidance is possible in this theory. The nonlinear coupling of matter field is shown in the modified Friedmann equation.

7.2.1 conclusions about late time evolution of EiBI gravity

We have analized the stability of late time evolution of EiBI cosmology. Three interested fixed points were calculated in this work by the notation $(\Omega_\Lambda, \Omega_{\rm m}, \Omega_{\rm EiBI})$. The first fixed point is $(0, 0, 1)$ stands for the vacuum dominated fixed point. The second fixed point is (1,0,0) represents for dark matter dominated fixed point. The third fixed point is $(0, 1/2, 1/2)$ denotes for the ΛEiBI fixed point.

The linear stability method evaluates the stability of dark matter dominated universe is an unstable phase. The KCC theory (or Jacobi stability) evaluates the stability of the ΛEiBI fixed point to an unstable point and also confirms that the dark matter dominated phase is Jacobi unstable. The Lyapunov function points out that the vacuum dominated universe is an unstable phase.

7.2.2 Some discussions about late time evolution of EiBI gravity

There was one serious drawback, however, from the definition of the density parameter for EiBI fluid $(\Omega_{\rm EiBI})$, the effective equation of state parameters in this fluid are allowed to lie only in the range $-1 \leq w_{\rm eff} \leq \frac{1}{3}$. Values of equation of state parameter beyond this range lead to the negative sign of the density parameter which is prohibit in flat universe. Since the upper bound of the effective equation of state parameter $w_{\rm rad} = \frac{1}{3}$ does not make sense at late time universe, so we have to ignore this value, it would be better to start the late time evolution phase from the dark matter dominated universe where $w_{\rm dm} = 0$.

The appearing of the unstable of vacuum dominated phase in the EiBI universe is concerned in our work. This may raise the question about our trial and error for the form of Lyapunov functions. Hence, it is possible to use other method, e.g. Center manifold theory , to specify the stability of the de Sitter fixed point.

7.2.3 Future perspectives of EiBI gravity

Event though the second version of Vollick's action is very impressive, it is not within the scope of this study and we will try to study the action in future. We also interest to work out with two modified forms of EiBI gravity as follows

$$L_1 = \sqrt{|g_{\mu\nu}(1 + R(\Gamma) + bR^2(\Gamma)) + bR(\Gamma)\nabla_\mu\phi\nabla_\nu\phi|}\,, \qquad (7.3)$$

$$L_2 = \sqrt{|g_{\mu\nu} + bR_{\mu\nu}(\Gamma) + bT^{(\phi)}_{\mu\nu}|}. \qquad (7.4)$$

The actions claimed above might furnish phenomenological values and let us deeply understand how do the Born-Infeld type of gravity works.

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APPENDIX

APPENDIX A VARIATIONAL APPROACH

We would like to proof the following identity that very important mathematical tools in modified gravity theories:

$$\begin{aligned}
\delta g_{\mu\nu} &= -g_{\mu\lambda}g_{\nu\tau}\delta g^{\tau\lambda} \\
\delta\sqrt{-g} &= +\frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \\
&= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \\
\delta R_{\mu\nu} &= \delta\Gamma^{\lambda}{}_{\mu\nu,\sigma} - \delta\Gamma^{\lambda}{}_{\mu\lambda,\nu} \\
\delta R &= [-R^{\mu\nu} + \nabla^{\mu}\nabla^{\nu} - g^{\mu\nu}\Box]\delta g_{\mu\nu} \\
&= [R_{\mu\nu} - \nabla_{\mu}\nabla_{\nu} + g_{\mu\nu}\Box]\delta g^{\mu\nu}\,.
\end{aligned} \tag{A.1}$$

Proof: The variation of the metric tensor

$$\delta g_{\mu\nu} = -g_{\mu\lambda}g_{\nu\tau}\delta g^{\tau\lambda}\,, \tag{A.2}$$

$$\delta g^{\mu\nu} = -g^{\mu\lambda}g^{\nu\tau}\delta g_{\tau\lambda}. \tag{A.3}$$

Let us start with the variation of the Kronecker delta

$$\begin{aligned}
\delta(g^{\mu\nu}g_{\mu\lambda}) &= \delta(\delta^{\nu}_{\lambda}) \;=\; 0\,, \\
g_{\nu\tau}g^{\mu\nu}\delta g_{\mu\lambda} + (\delta g^{\mu\nu})g_{\mu\lambda}g_{\nu\tau} &= 0\,.
\end{aligned} \tag{A.4}$$

Then we can write

$$\begin{aligned}
\delta^{\mu}_{\tau}\delta g_{\mu\lambda} &= -(\delta g^{\mu\nu})g_{\mu\lambda}g_{\nu\tau}\,, \\
\delta g_{\tau\lambda} &= -g_{\tau\nu}g_{\lambda\mu}(\delta g^{\mu\nu})\,.
\end{aligned} \tag{A.5}$$

Redefining indices, we get the variation of metric tensor

$$\delta g_{\mu\nu} = -g_{\mu\lambda}g_{\nu\tau}(\delta g^{\tau\lambda})\,. \tag{A.6}$$

Next, we aim to proof the variation of square root of absolute determinant of the metric tensor. This is

$$\delta\sqrt{-g} = +\frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}\,, \\ = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{A.7}$$

Let us start with the variation of determinant of any symmetric metric $A_{\mu\nu}$ represents by A

$$\delta A = AA^{\mu\nu}\delta A_{\mu\nu}\,. \tag{A.8}$$

The above formulae can be applied to the metric tensor $g_{\mu\nu}$ as follows

$$\delta g = \delta(\sqrt{-g}\sqrt{-g}) = 2\sqrt{-g}\delta\sqrt{-g} = gg^{\mu\nu}\delta g_{\mu\nu} = -gg_{\mu\nu}\delta g^{\mu\nu}\,. \tag{A.9}$$

Then it is easy to proof that

$$2\sqrt{-g}\,\delta\sqrt{-g} = g\,g^{\mu\nu}\delta g_{\mu\nu}$$
$$2\cancel{\sqrt{-g}}\,\delta\sqrt{-g} = \cancel{\sqrt{-g}}\sqrt{-g}g\,g^{\mu\nu}\delta g_{\mu\nu}$$
$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}$$
$$= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}\,.$$

The next target is the derivation of the variation of the connection $\Gamma^{\lambda}_{\mu\nu}$. Let us start from the definition of the Christoffel symbels or the Levi-Civita connections

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\tau}(g_{\mu\tau,\nu} + g_{\nu\tau,\mu} - g_{\mu\nu,\tau})\,. \tag{A.10}$$

Taking variation of the Christoffel symbels, this yields

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\delta g^{\lambda\tau}(g_{\mu\tau,\nu} + g_{\nu\tau,\mu} - g_{\mu\nu,\tau}) + \frac{1}{2}g^{\lambda\tau}(\delta g_{\mu\tau,\nu} + \delta g_{\nu\tau,\mu} - \delta g_{\mu\nu,\tau}). \tag{A.11}$$

Our task will be reduced if we work in local inertial frame (L.I.F) where $g_{\mu\nu;\nu} = g_{\mu\nu,\nu} = 0$ for the vanishing of the Christoffel symbols. The first term on the right-hand side of Eq.(A.11) can be neglected , one gets

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\tau}(\delta g_{\mu\tau,\nu} + \delta g_{\nu\tau,\mu} - \delta g_{\mu\nu,\tau})\,. \tag{A.12}$$

We always write

$$\delta\Gamma^{\lambda}_{\mu\nu} \;=\; \frac{1}{2}\eta^{\lambda\tau}(\delta g_{\mu\tau;\nu} + \delta g_{\nu\tau;\mu} - \delta g_{\mu\nu;\tau})\,. \tag{A.13}$$

The definition of the Riemann tensor

$$R^{\lambda}{}_{\mu\sigma\nu} \;=\; \Gamma^{\lambda}{}_{\mu\nu,\sigma} - \Gamma^{\lambda}{}_{\mu\sigma,\nu} + \Gamma^{\lambda}{}_{\rho\sigma}\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\rho\nu}\Gamma^{\rho}{}_{\mu\sigma}\,. \tag{A.14}$$

Working in L.I.F, we get the variation of the Riemann tensor,

$$\delta R^{\lambda}{}_{\mu\sigma\nu} \;=\; \delta\Gamma^{\lambda}{}_{\mu\nu,\sigma} - \delta\Gamma^{\lambda}{}_{\mu\sigma,\nu}\,. \tag{A.15}$$

By contracting the first and the third indices, we obtain the definition of variation of the Riemann tensor

$$\delta R^{\lambda}{}_{\mu\lambda\nu} \;=\; \delta R_{\mu\nu} \;=\; \delta\Gamma^{\lambda}{}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}{}_{\mu\lambda,\nu}. \tag{A.16}$$

Let us work out term by term by using the relation in Eq.(A.13). Then the first term and the second term on the right-hand side of Eq.(A.16) become

$$\delta\Gamma^{\lambda}_{\mu\nu,\lambda} \;=\; \frac{1}{2}\eta^{\lambda\tau}(\delta g_{\mu\tau,\nu\lambda} + \delta g_{\nu\tau,\mu\lambda} - \delta g_{\mu\nu,\tau\lambda}), \tag{A.17}$$

$$\delta\Gamma^{\lambda}_{\mu\lambda,\nu} \;=\; \frac{1}{2}\eta^{\nu\tau}(\delta g_{\mu\tau,\lambda\nu} + \delta g_{\lambda\tau,\mu\nu} - \delta g_{\mu\lambda,\tau\nu}), \tag{A.18}$$

respectively. The variation of the Ricci scalar can be derived from

$$\begin{aligned}\delta R \;&=\; \delta\left(g^{\mu\nu}R_{\mu\nu}\right),\\ &=\; \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\,.\end{aligned}$$

From the fact that $\delta R_{\mu\nu}(p) \;=\; \nabla_{\lambda}(\delta\Gamma^{\lambda}{}_{\mu\nu}(p)) - \nabla_{\nu}(\delta\Gamma^{\lambda}{}_{\mu\lambda}(p))$ point p in L.I.F, one gets

$$\delta R \;=\; \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\left[\nabla_{\lambda}(\delta\Gamma^{\lambda}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}_{\mu\lambda})\right]. \tag{A.19}$$

Eq. (A.19) can be written in more useful form as follows

$$\begin{aligned}\delta R \;&=\; \Big[-R^{\mu\nu} + \nabla^{\mu}\nabla^{\nu} - g^{\mu\nu}\nabla_{\nu}\nabla^{\nu}\Big]\delta g_{\mu\nu}\,,\\ &=\; \Big[R_{\mu\nu} - \nabla_{\mu}\nabla_{\nu} + g_{\mu\nu}\Box\Big]\delta g^{\mu\nu}, \end{aligned} \tag{A.20}$$

where $\Box = \nabla_\mu \nabla^\mu$. We use Eq.(A.1) to prove the above relation,

$$\begin{aligned}
\delta R &= \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\Big[\delta_\lambda(\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\lambda_{\mu\lambda})\Big], \\
&= -g^{\alpha\mu}g^{\beta\nu}\delta g_{\alpha\beta}R_{\mu\nu} + \Big[\nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda})\Big],
\end{aligned}$$

where $\nabla_\lambda g^{\mu\nu} = \nabla_\nu g^{\mu\nu} = 0$. Hence we have

$$\delta R = -R^{\mu\nu}\delta g_{\mu\nu} + \nabla_\lambda\Big[(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}) - (g^{\mu\lambda}\delta\Gamma^\alpha_{\mu\alpha})\Big]. \tag{A.21}$$

The tensor $g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}$ can be thought that the tensor rank $(1,0)$ or vector V^λ. Then it means that the covariant derivative can operate to vector V^λ as follows

$$\nabla_\lambda[g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\alpha_{\mu\alpha}] = \nabla_\lambda V^\lambda .$$

Substituting

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}[\delta g_{\mu\nu;\nu} + \delta g_{\nu\sigma;\mu} - \delta g_{\mu\nu;\sigma}] \tag{A.22}$$

into Eq.(A.22) This is equivalence of $\int_a^b dt\frac{df}{dt} = f(b) - f(a)$ Thus,

$$\begin{aligned}
\delta R &= -R^{\mu\nu}\delta g_{\mu\nu} + \nabla_\lambda\Bigg[g^{\mu\nu}(\frac{1}{2}g^{\lambda\sigma})\Big(\nabla_\nu\delta g_{\mu\sigma} + \nabla_\mu\delta g_{\nu\sigma} - \nabla_\sigma\delta g_{\mu\nu}\Big) \\
&\quad -g^{\mu\lambda}\frac{1}{2}g^{\alpha\gamma}\Big(\nabla_\alpha\delta g_{\mu\gamma} + \nabla_\mu\delta g_{\nu\alpha} - \nabla_\gamma\delta g_{\mu\alpha}\Big)\Bigg], \\
&= -R^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{2}\Bigg[g^{\mu\nu}g^{\lambda\sigma}\nabla_\lambda\nabla_\nu\delta g_{\mu\sigma} + g^{\mu\nu}g^{\lambda\sigma}\nabla_\lambda\nabla_\mu\delta g_{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma}\nabla_\lambda\nabla_\sigma\delta g_{\mu\nu} \\
&\quad -g^{\mu\lambda}g^{\alpha\gamma}\nabla_\lambda\nabla_\alpha\delta g_{\mu\gamma} - g^{\mu\lambda}g^{\alpha\gamma}\nabla_\lambda\nabla_\mu\delta g_{\alpha\gamma} + g^{\mu\lambda}g^{\alpha\gamma}\nabla_\lambda\nabla_\gamma\delta g_{\mu\alpha}\Bigg], \\
&= -R^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{2}\Bigg[\cancel{\nabla^\sigma\nabla^\mu\delta g_{\mu\sigma}} + \nabla^\sigma\nabla^\nu\delta g_{\nu\sigma} - g^{\mu\nu}\nabla_\lambda\nabla^\lambda\delta g_{\mu\nu} - \cancel{\nabla^\mu\nabla^\gamma\delta g_{\mu\gamma}} \\
&\quad -g^{\alpha\gamma}\nabla_\lambda\nabla^\lambda\delta_{\alpha\gamma} + \nabla^\mu\nabla^\alpha\delta g_{\mu\alpha}\Bigg], \\
&= -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\sigma\nabla^\mu\delta g_{\mu\sigma} - g^{\mu\nu}\nabla_\lambda\nabla^\lambda\delta g_{\mu\nu} , \\
&= -R^{\mu\nu}\delta g_{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - g^{\mu\nu}\Box\delta g_{\mu\nu} , \\
&= \Big[-R^{\mu\nu} + \nabla^\mu\nabla^\nu - g^{\mu\nu}\Box\Big]\delta g_{\mu\nu} .
\end{aligned} \tag{A.23}$$

By applying $\delta g_{\mu\nu} = -g_{\mu\lambda}g_{\nu\tau}\delta g^{\tau\lambda}$, Eq.(A.23) becomes

$$\delta R = \Big[R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\Big]\delta g^{\mu\nu}. \tag{A.24}$$

Therefore we have two choices of variation of Ricci scalar.

$$\delta R = \Big[-R^{\mu\nu} + \nabla^\mu\nabla^\nu - g^{\mu\nu}\Box\Big]\delta g_{\mu\nu} = \Big[R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\Big]\delta g^{\mu\nu} \tag{A.25}$$

Example: Variation of Einstein-Hilbert action in metric formalism

$$\begin{aligned}
\delta S_{\rm EH} &= \frac{1}{2\kappa^2}\int d^4x\,(\delta\mathcal{L}_{EH}),\\
&= \frac{1}{2\kappa^2}\int d^4x\Big[\sqrt{-g}\delta R + R\delta\sqrt{-g}\Big],\\
&= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[[-R^{\alpha\beta} + \nabla^{\alpha\beta} - g^{\alpha\beta}\Box]\delta g_{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}R\delta g_{\alpha\beta}\Big],\\
&= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big(-R^{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}R\Big)\delta g_{\alpha\beta} + \int d^4x\sqrt{-g}\Big[\cancel{\nabla^\alpha\nabla^\beta\delta g_{\alpha\beta}}\\
&\quad \cancel{-g^{\alpha\beta}\Box\delta g_{\alpha\beta}}\Big],\\
&= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\Big[-R^{\alpha\beta} + \frac{1}{2}g^{\alpha\beta}R\Big]\delta g_{\alpha\beta} = 0.
\end{aligned} \tag{A.26}$$

The variation $\delta g_{\alpha\beta}$ can be set arbitrarily, we thus recover Einstein's field equation in vacuum case:

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = 0. \tag{A.27}$$

We have proved the definition of energy momentum tensor as follows

$$S = \frac{1}{2\kappa^2}S_{\rm EH} + S_{\rm m} = \int d^4x\sqrt{-g}(\frac{1}{2\kappa^2}\mathcal{L}_{EH} + \mathcal{L}_M). \tag{A.28}$$

Taking the variation of the total action above, we get

$$\frac{1}{2\kappa^2}\frac{\delta(\sqrt{-g}\mathcal{L}_{EH})}{\delta g^{\mu\nu}} + \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = 0. \tag{A.29}$$

Rearranging Eq.(A.29) a little bit, we write

$$\frac{(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} = -2\kappa^2\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}. \tag{A.30}$$

Using the result from Eq.(A.26) above, one gets

$$\frac{\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}}{\delta g^{\mu\nu}} = -2\kappa^2\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}. \tag{A.31}$$

Using the definition of Einstein's field equation $G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R \;=\; \kappa^2 T^{\alpha\beta}$, we can write

$$\sqrt{-g}G_{\mu\nu} \equiv \sqrt{-g}\kappa^2 T_{\mu\nu} \;=\; -2\kappa^2 \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \,. \tag{A.32}$$

Then we obtain

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \;=\; -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \,. \tag{A.33}$$

The full form the definition of energy momentum tensor is

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}(g_{\alpha\beta}, \Psi)}{\delta g^{\mu\nu}} \,, \tag{A.34}$$

and again we use $\delta g_{\mu\nu} \;=\; -g_{\mu\lambda}g_{\nu\tau}\delta g^{\tau\lambda}$ to derive the definition of $T^{\mu\nu}$

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}(g_{\alpha\beta}, \Psi)}{\delta g_{\mu\nu}} \,. \tag{A.35}$$

Example2: We want to proof that $\nabla_t\nabla_t F \;=\; \ddot{F} - \Gamma^{\alpha}_{tt}\partial_\alpha F$ that operation can be found in $f(R)$ gravity.

$$\begin{aligned} \nabla_\mu\nabla_\nu F &= \nabla_\mu(\partial_\nu F) \;=\; \partial_\mu\partial_\nu F - \Gamma^{\alpha}_{\mu\nu}\partial_\alpha F \,. \\ \nabla_t\nabla_t F &= \nabla_t(\partial_t F) \;=\; \partial_t\partial_t F - \Gamma^{\alpha}_{tt}\partial_\alpha F \;=\; \ddot{F} - \Gamma^{\alpha}_{tt}\partial_\alpha F \,. \end{aligned}$$

APPENDIX B BORN-INFELD MATHEMATICAL OBJECTS

We want to proof for arbitrarily square matrix A

$$\delta(\det A) \;=\; (\det A)\mathrm{Tr}(\mathrm{A}^{-1}\delta\mathrm{A})\,. \tag{B.1}$$

Let us start from the relation between determinant and trace of the square matrix A

$$\det[e^{A}] \;=\; e^{\mathrm{Tr\,A}} \tag{B.2}$$

Taking natural logarithm both sides of Eq.(B.2), we get

$$\ln[\det e^{M}] \;=\; \mathrm{Tr\,A} \tag{B.3}$$

where the operation det and ln are commuted each other. Then we can write

$$\det\,(\ln e^{\mathrm{A}}) \;=\; \mathrm{Tr\,A} \tag{B.4}$$

Taking operation ln once again , this gives

$$\ln(\det\mathrm{A}) \;=\; \ln\,(\mathrm{Tr\,A}) \;=\; \mathrm{Tr}(\ln\ \mathrm{A}) \tag{B.5}$$

Taking the variation both sides of Eq.(B.5) and using the commutation between Tr and δ, this gives

$$\begin{aligned} \delta\ln\,(\det\mathrm{A}) &= \mathrm{Tr}\,(\delta\ln\mathrm{A}) \\ \frac{1}{\det\mathrm{A}}\,\delta(\det\mathrm{A}) &= \mathrm{Tr}\,(\mathrm{A}^{-1}\delta\mathrm{A}) \end{aligned} \tag{B.6}$$

We get a very useful relation

$$\delta(\det\mathrm{A}) \;=\; (\det\mathrm{A})\mathrm{Tr}(\mathrm{A}^{-1}\delta\mathrm{A}). \tag{B.7}$$

<u>Example 1</u>: By setting $\det\mathrm{A} = \mathrm{g} = |\mathrm{g}_{\mu\nu}|$ and using Eq.(B.7), we can write

$$\delta g \;=\; g g^{\mu\nu}\delta g_{\mu\nu}. \tag{B.8}$$

Example 2: By using Eq.(B.7) for EiBI gravity , we then show

$$\delta\Big|g_{\mu\nu}+bR_{\mu\nu}(\Gamma)\Big| \;=\; \Big|g_{\mu\nu}+bR_{\mu\nu}(\Gamma)\Big|\Big[g_{\mu\nu}+bR_{\mu\nu}(\Gamma)\Big]^{-1}\Big[\delta g_{\mu\nu}+b\delta R_{\mu\nu}(\Gamma)\Big], \quad \text{(B.9)}$$

where the inverse of metric $g_{\mu\nu}+bR_{\mu\nu}(\Gamma)$ is

$$\Big[g_{\mu\nu}+bR_{\mu\nu}(\Gamma)\Big]^{-1} \equiv \left[\frac{1}{g+bR}\right]^{\mu\nu} \equiv q^{\mu\nu}\,. \quad \text{(B.10)}$$

APPENDIX C THE CONNECTION FIELDS TRANSFERS THE METRIC FIELDS INTO ITSELF

Our aim here is to show that under metric compatibility $\nabla_c g_{ab}$ implies that [42](p.65-66).

$$\Gamma^d_{(ab)} = \frac{1}{2} g^{dc} (\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c}). \tag{C.1}$$

$$\nabla_c g_{ab} \equiv \frac{\partial g_{ab}}{\partial x^c} - g_{db}\Gamma^d_{ac} - g_{ad}\Gamma^d_{bc} = 0. \tag{C.2}$$

With a cyclic permutation of the subscripts abc, we can write

$$\begin{aligned} -\frac{1}{2}\left(\frac{\partial g_{ab}}{\partial x^c} - g_{db}\Gamma^d{}_{ac} - g_{ad}\Gamma^d{}_{bc}\right) &= 0 \\ +\frac{1}{2}\left(\frac{\partial g_{bc}}{\partial x^a} - g_{dc}\Gamma^d{}_{ba} - g_{bd}\Gamma^d{}_{ac}\right) &= 0 \\ +\frac{1}{2}\left(\frac{\partial g_{ca}}{\partial x^b} - g_{da}\Gamma^d{}_{cb} - g_{cd}\Gamma^d{}_{ab}\right) &= 0. \end{aligned} \tag{C.3}$$

By combining three terms of Eq.(C.3) together with the factor $\frac{1}{2}$ and $-\frac{1}{2}$ beyond the closed brackets, the most generalized for the non-symmetric affinities relation implying the symmetric of the metric tensor g_{ab} but not for Γ^d_{ac}. Thus we get

$$\frac{1}{2}(\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ca}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c}) - \frac{1}{2} g_{cd}(\Gamma^d_{ba} + \Gamma^d_{ab}) + \frac{1}{2} g_{ad}(\Gamma^d_{bc} - \Gamma^d_{cb}) + \frac{1}{2} g_{bd}(\Gamma^d_{ac} - \Gamma^d_{ca}) = 0. \tag{C.4}$$

Defining the relations

$$\begin{aligned} \Gamma^d_{(ab)} &= \frac{1}{2}(\Gamma^d_{ab} + \Gamma^d_{ba}) \\ \Gamma^d_{[ab]} &= \frac{1}{2}(\Gamma^d_{ab} - \Gamma^d_{ba}), \end{aligned} \tag{C.5}$$

Eq.(C.4) becomes

$$\frac{1}{2}(\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ca}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c}) - g_{cd}\Gamma^d_{(ab)} + g_{ad}\Gamma^d_{[bc]} + g_{bd}\Gamma^d_{[ac]} = 0. \tag{C.6}$$

Multiplying Eq.(C.4) by g^{ec} and using the relation $g^{ij}g_{ik} = \delta^j_k$, we therefore write

$$\frac{1}{2} g^{ec}(\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ca}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c}) - g^{ec} g_{cd}\Gamma^d_{(ab)} + g^{ec} g_{ad}\Gamma^d_{[bc]} + g^{ec} g_{bd}\Gamma^d_{[ac]} = 0. \tag{C.7}$$

By adding ${\Gamma^e}_{[ab]}$ both sides of Eq.(C.7), we get

$$\frac{1}{2}g^{ec}(\frac{\partial g_{bc}}{\partial x^a}+\frac{\partial g_{ca}}{\partial x^b}-\frac{\partial g_{ab}}{\partial x^c})-\delta^e_d\Gamma^d_{(ab)}+\Gamma^e_{[ab]}+g^{ec}g_{ad}\Gamma^d_{[bc]}+g^{ec}g_{bd}\Gamma^d_{[ac]} \;=\; {\Gamma^e}_{[ab]}. \quad \text{(C.8)}$$

We rewrite Eq.(C.8) as

$$\frac{1}{2}g^{ec}(\frac{\partial g_{bc}}{\partial x^a}+\frac{\partial g_{ca}}{\partial x^b}-\frac{\partial g_{ab}}{\partial x^c})+\Gamma^e_{[ab]}+g^{ec}g_{ad}\Gamma^d_{[bc]}+g^{ec}g_{bd}\Gamma^d_{[ac]} \;=\; {\Gamma^e}_{[ab]}+\Gamma^e_{(ab)}. \quad \text{(C.9)}$$

Defining ${\Gamma^e}_{[ab]}+\Gamma^e_{(ab)}={\Gamma^e}_{ab}$, we hence get

$$\frac{1}{2}g^{ec}(\frac{\partial g_{bc}}{\partial x^a}+\frac{\partial g_{ca}}{\partial x^b}-\frac{\partial g_{ab}}{\partial x^c})+\Gamma^e_{[ab]}+g^{ec}g_{ad}\Gamma^d_{[bc]}+g^{ec}g_{bd}\Gamma^d_{[ac]} \;={\Gamma^e}_{ab}. \quad \text{(C.10)}$$

For torsionless case, one can show that $\Gamma^e_{[ab]}=\Gamma^d_{[bc]}=\Gamma^d_{[ac]}=0$ and $\Gamma^e_{ab}=\Gamma^e_{(bc)}$. This shows that

$$\nabla_c g_{ab}=0 \quad \text{(C.11)}$$

is identical to

$$\Gamma^e_{(ab)}=\frac{1}{2}g^{ec}(\frac{\partial g_{ac}}{\partial x^b}+\frac{\partial g_{bc}}{\partial x^a}-\frac{\partial g_{ab}}{\partial x^c}). \quad \text{(C.12)}$$

This ends of proof.

BIOGRAPHY

BIOGRAPHY

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