# Perfect fluid in Lagrangian formulation due to generalized three-form field

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A Lagrangian formulation of perfect fluid due to a noncanonical three-form field is investigated. The thermodynamic quantities such as energy density, pressure and the four velocity are obtained and then analyzed by comparing with the k-essence scalar field. The nonrelativistic matter due to the generalized three-form field with the equation of state parameter being zero is realized while it might not be possible for the k-essence scalar field. We also found that nonadiabatic pressure perturbations can be possibly generated. The fluid dynamics of the perfect fluid due to the three-form field corresponds to the system in which the number of particles is not conserved. We argue that it is interesting to use this three-form field to represent the dark matter for the classical interaction theory between dark matter and dark energy.

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## I. INTRODUCTION

A theory of cosmological perturbations is one of the important issues in cosmology nowadays. It provides us a way to understand how astronomical structures at large scales are generated and evolve. Also, it can provide us the resulting signatures of the theoretical model to compare with observational data. The theory of cosmological perturbations for a perfect fluid have been developed and studied intensively at the level of the equations of motion, for example, the study of the perturbed Einstein field equations together with the equation of conservation of energy momentum tensor [1,2]. Besides the cosmological perturbations at the level of the equations of motion, a study of the cosmological perturbations at the Lagrangian level has been investigated. The advantage of the study at the Lagrangian level is that it is useful to find the perturbed dynamical field as well as derive closed evolution equations. This can be clearly seen by considering the cosmological perturbations in $f(R,G)$ gravity theories where there are two dynamical fields for scalar perturbations [3,4]. For the study in the Lagrangian approach, one can straightforwardly identify which fields are dynamical or auxiliary and then immediately obtain the closed evolution equations.

A Lagrangian formulation for a perfect fluid in general relativity has been constructed and developed for a long time [5–7]. The Lagrangian of the fluid is simply written as its pressure [6] or energy density [7]. The advantage of this formulation is that it naturally provides a consistent way to construct a covariant theory for dark energy and dark matter coupling. The study of dark energy and dark matter coupling has been widely investigated in order to describe a way out of the cosmic coincidence problem [8–12]. Moreover, the observation also provides a hint for the existence of the coupling [13]. However, in order to recover the standard thermodynamics equations, the Lagrangian must involve at least five independent functions. Even though this formulation can provide a consistent way for studying the perfect fluid in cosmology and is well known as a standard approach for the perfect fluid at the Lagrangian level, there might be disadvantage for this approach since the theory involves too many functions.

A simple Lagrangian approach for the perfect fluid has been investigated by using a noncanonical scalar field [14], namely the k-essence field [15–17]. It was found that the k-essence scalar field can provide a description of the perfect fluid with a constant equation of state parameter. Moreover, it was found that the cosmological perturbations of this kind of scalar field are equivalent to those in perfect fluid. However, it cannot be properly used to describe nonrelativistic matter with the equation of state parameter being zero since the Lagrangian is not finite. It was also found that the nonadiabatic pressure perturbations cannot be generated [18], nor can the vector mode of the perturbations be produced [19]. Besides the cosmological models due to the scalar field, a three-form field can be successfully used to describe both the inflationary models and dark energy models [20–30]. Even though there is a duality between the scalar field and three-form field [20], the cosmological models are significantly different in both background and perturbation levels. At the perturbation level, it is obvious to see that the three-form field can generate intrinsic vector perturbations while it is not possible for the scalar field. Therefore, it might be worthy to find an equivalence between the three-form field with a perfect fluid. In the present work, by mimicking the k-essence scalar field, we consider a generalized version of the three-form field and then find a possible Lagrangian form to describe the perfect fluid in the cosmological background. We found that a simple power-law of the canonical kinetic term can provide the constant equation of state parameter like in the case of the k-essence. The advantage of the three-form field compared with the scalar field is that it can provide a consistent description of the nonrelativistic matter field where its equation of state parameter satisfies $w = 0$. The stability issue is also investigated and found that the nonrelativistic matter field due to the three-form field is free-from ghost and Laplacian instabilities.

By using the equations of motion of the generalized three-form field, the thermodynamic quantities are identified and found that the perfect fluid due to the three-from field corresponds to fluid in which the number of particles is not conserved. By analyzing the speed of propagation of scalar perturbations and the adiabatic sound speed, we found that the nonadiabatic perturbations can possibly be generated. We argue that it is interesting to use this three-form field to represent the dark matter for the classical interaction theory between dark matter and dark energy.

This paper is organized as follows. In Sec. II, we propose a general form of the three-form field and then find the equation of motion as well as the energy momentum tensor. By working in the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, the energy density and the pressure as well as the equation of state parameter are found. Some specific forms of the Lagrangian satisfying the equations of motion are obtained and found that it can represent the nonrelativistic matter. We also investigate the stability issue by using the perturbed action at the second order in Sec. III. We found conditions to avoid ghost and Laplacian instabilities. In Sec. IV, we investigate the thermodynamic properties of the model. We begin this section with review of some important ideas of the Lagrangian formulation for the standard and k-essence scalar field and then find the thermodynamic properties due to the three-form fluid. Finally, the results are summarized and discussed in Sec. V.

## II. EQUATIONS OF MOTION AND ENERGY MOMENTUM TENSOR

Cosmological models due to a three-form field have been investigated not only in inflationary models but also dark energy models [20–30]. Moreover, at the end of the inflationary period, a viable model due to the three-form field for the reheating period have been investigated [31]. A consistent mechanism to generate large scale cosmological magnetic fields by using the three-form field has been studied [32]. Recently, a generalized inflationary model by considering two three-form fields was also investigated [29]. All investigations of cosmological models due to three-form are considered only in canonical form. Since the noncanonical form of scalar fields have been intensively investigated, it is interesting to investigate the cosmological model with a noncanonical form of the three-form field. In this section, we will consider a noncanonical form of the kinetic term of a three-form field, $A_{\alpha\beta\gamma}$, as follows

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + P(K, y)\right], \tag{1}$$

where the kinetic term and scalar quantity of the three-form field are expressed as

$$K = -\frac{1}{48}F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta}, \tag{2}$$

$$y = \frac{1}{12}A_{\alpha\beta\gamma}A^{\alpha\beta\gamma}, \tag{3}$$

$$F_{\mu\nu\rho\sigma} = \nabla_\mu A_{\nu\rho\sigma} - \nabla_\sigma A_{\mu\nu\rho} + \nabla_\rho A_{\sigma\mu\nu} - \nabla_\nu A_{\rho\sigma\mu}. \tag{4}$$

By varying the action with respect to the three-form field, the equations of motion of the three-form field can be written as

$$E_{\alpha\beta\gamma} = \nabla_\mu(P_{,K}F^\mu{}_{\alpha\beta\gamma}) + P_{,y}A_{\alpha\beta\gamma} = 0, \tag{5}$$

where the notation with subscript $P_{,x}$ denotes $P_{,x} = \partial_x P$. Due to the totally antisymmetric property of the tensor $F_{\mu\alpha\beta\gamma}$, one finds that there exists constraint equations as follows

$$\nabla_\mu(P_{,y}A^{\mu\alpha\beta}) = 0. \tag{6}$$

These equations suggest us that the conserved quantity is expressed in terms of three-form field. Note that for the k-essence scalar field, the conserved quantity is expressed in terms of one-form or vector quantity. We will discuss on this issue in detail in Sec. IV where we investigate the fluid dynamics. The energy momentum tensor can be obtained by varying the action of the three-form field with respect to the metric as

$$T_{\mu\nu} = \frac{1}{6}P_{,K}F_{\mu\rho\sigma\alpha}F_\nu{}^{\rho\sigma\alpha} - \frac{1}{2}P_{,y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} + Pg_{\mu\nu}. \tag{7}$$

For consistency of the derived equations, one can check that the conservation of the energy momentum tensor can be obtained up to the equation of motion as follows

$$\nabla_\mu T^\mu{}_\nu = \frac{1}{6}F_{\nu\alpha\beta\gamma}E^{\alpha\beta\gamma} = 0. \tag{8}$$

In order to capture the thermodynamics quantities such as the energy density and pressure due to the three-form field like the investigation in scalar field, let us consider a flat Friedmann-Lemaître-Robertson-Walker (FLRW) manifold whose metric element can be written as

$$ds^2 = -dt^2 + \gamma_{ij}dx^i dx^j = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j. \tag{9}$$

By using this form of the metric and the constraint equation in Eq. (6), the components of the three-form field, $A_{\alpha\beta\gamma}$, can be written as

$$A_{0ij} = 0, \qquad A_{ijk} = \epsilon_{ijk} X(t) = \sqrt{\gamma}\varepsilon_{ijk} X(t) = a^3 \varepsilon_{ijk} X(t), \tag{10}$$

where $\varepsilon_{ijk}$ is the three-dimensional Levi-Civita symbol with $\varepsilon_{123} = 1$. By using this form of the metric, the components of the energy momentum tensor can be expressed as

$$T^0_0 = P - 2KP_{,K}, \tag{11}$$

$$T^i_j = (P - 2KP_{,K} - 2yP_{,y})\delta^i_j. \tag{12}$$

By comparing these components of the energy momentum tensor of the three-form to one from the perfect fluid, the energy density and pressure of the three-form can be expressed as

$$\rho = 2KP_{,K} - P, \tag{13}$$

$$p = P - 2KP_{,K} - 2yP_{,y} = -\rho - 2yP_{,y}. \tag{14}$$

Note that we have used $y = X^2/2$ and $K = (\dot{X} + 3HX)^2/2$ where $H = \dot{a}/a$ is the Hubble parameter. From the energy density and the pressure above, the equation of state parameter of the three-form can be written as

$$w = \frac{p}{\rho} = -1 - \frac{2yP_{,y}}{\rho}. \tag{15}$$

The equation of motion of the three-form field in Eq. (5) can be written in the flat FLRW background as

$$(2KP_{,KK} + P_{,K})\dot{K} + 2KP_{,yK}\dot{y} - 2\sqrt{Ky}P_{,y} = 0. \tag{16}$$

From this point, one can check the validity of the derived equations by reducing the general form of the action to the canonical one as setting $P = K - V(y)$. As a result, we found that all equations can be reduced to the canonical one investigated in [20–30]. Substituting $\rho$ from Eq. (13) into Eq. (15), one obtains

$$2yP_{,y} + (1 + w)2KP_{,K} = (1 + w)P. \tag{17}$$

In order to find the form of $P$, one has to solve this equation. It is useful to solve this equation by considering a simple assumption such as taking the equation of state parameter to be a constant, $w = \text{const}$. By using the separation of variable method, the solution can be written as

$$P = P_0 K^\nu y^\mu, \tag{18}$$

where $P_0$ is an integration constant and $\mu$, $\nu$ are the exponent constants obeying the relation

$$\nu = \frac{1 + w - 2\mu}{2(1 + w)}, \quad \text{or} \quad w = -1 + \frac{2\mu}{1 - 2\nu}, \qquad \nu \neq \frac{1}{2}. \tag{19}$$

This form of the solution is very useful since one can interpret the three-form field as a nonrelativistic matter or dark matter by setting the equation of state parameter as $w = 0$ while it cannot be properly used for the k-essence scalar field case. We will show explicitly why we cannot properly use the k-essence scalar field for the nonrelativistic matter in Sec. IV. In order to study the covariant coupling form between dark matter and dark energy as suggested from the observation [13], one can use the three-form as the dark matter with the consistent covariant interaction forms. Moreover, it may be interpreted as dark radiation by setting $w = 1/3$. Note that, in the case of $\nu = 1/2$, it corresponds to the trivial solution since the energy density of the field vanishes. It is important to note that the late-time acceleration of the universe can also be achieved by setting $w = -1$. Even though this may not be distinguished to the cosmological constant at the background level, the cosmological perturbations due to this model of the three-form can be significantly deviated from the model of the cosmological constant.

Since the form of the Lagrangian $P$ is obtained by assuming a constant equation of state parameter, the dark energy model from this three-form field cannot be proposed to solve the coincidence problem. One may allow the equation of state to be varying in order to overcome this issue. One of interesting solutions is assuming that the equation of state parameter depends on the three-form field $w = w(y)$. In order to solve Eq. (17) to obtain a suitable form of $P$, one may choose the equation of state parameter such as $w = -1 + \lambda y$, where $\lambda$ is a constant. As a result, the solution can be written as

$$P = P_0 K^\nu e^{\frac{(1-2\nu)}{2}\lambda y}. \tag{20}$$

Naively, it is not difficult to obtain the dynamical dark energy due to the generalized three-form. One can set $\lambda$ be effectively small and find the condition to provide an evolution of $y$ such that it evolves from a large value to a small value. However, since it is not in the canonical form, the theory may suffer from instabilities. In this work, the stability issue will be investigated in the next section. The investigation of the dark energy model due to the generalized three-form is left in further work.

## III. STABILITY

In order to capture the stability conditions of the generalized three-form field, we may consider the perturbations of the field. Since the field minimally couples to the gravity, one has to take into account the metric perturbations. However, for simplicity but useful study, we will investigate the stabilities of the model only in a high-momentum limit. This will capture only some stability conditions. Nevertheless, this includes most of the necessary conditions as found in the canonical three-form field [27]. We leave the full investigation in further work where the cosmological perturbations are taken into account. For this purpose, the metric is held fixed as the Minkowski metric and the three-form field can be written as

$$A_{ijk} = \varepsilon_{ijk}(X(t) + \alpha(t,\vec{x})), \tag{21}$$

$$A_{0ij} = \varepsilon_{ijk}(\partial_k\beta(t,\vec{x}) + \beta_k(t,\vec{x})), \tag{22}$$

where $\alpha$ and $\beta$ are perturbed scalar fields and $\beta_k$ is a transverse vector obeying the relation $\partial_k\beta^k = 0$. This vector field will be responsible for the intrinsic vector perturbation of the three-form field. For the linear perturbations, the scalar and vector modes are decoupled and then they can be separately investigated. For the scalar modes, by expanding the action up to second order in the field, the second order action can be written as

$$S^{(2)} = \int d^4x\left(\frac{1}{2}\frac{\dot{Q}^2}{(P_{,K} + 2KP_{,KK})} - \frac{1}{2}P_{,y}(\partial\beta)^2 + \frac{1}{2}P_{,y}c_s^2\alpha^2\right), \tag{23}$$

$$\dot{Q} = (P_{,K} + 2KP_{,KK})\dot{\alpha} + 2\sqrt{Ky}P_{,K}P_{,y}\alpha - (P_{,K} + 2KP_{,KK})\partial^2\beta, \tag{24}$$

$$c_s^2 = 1 + \frac{2yP_{,yy}}{P_{,y}} - \frac{4KyP_{,Ky}^2}{P_{,y}(2KP_{,KK} + P_{,K})}. \tag{25}$$

One can see that the field $\beta$ is nondynamical so that one can eliminate it by using its equation of motion. By applying the Euler-Lagrange equation to the above action, the equation of motion for the field $\beta$ can be written as

$$(P_{,K} + 2KP_{,KK})\dot{\alpha} + 2\sqrt{Ky}P_{,Ky}\alpha - (P_{,K} + 2KP_{,KK})\partial^2\beta - P_{,y}\beta = 0. \tag{26}$$

From this equation of motion, we can replace the quantity $\dot{Q}$ as $\dot{Q} = P_{,y}\beta$. Note that this equation can be obtained by using the component $(0, i, j)$ of the covariant equation in Eq. (5). In order to find the solution for $\beta$, it is convenient to work in Fourier space so that the above equation can be algebraically solved. As a result, by substituting the solution of $\beta$ into the action in Eq. (23), the second order action for the scalar perturbations can be rewritten as

$$S^{(2)} = \int dtd^3k(F_1\dot{\alpha}^2 + F_2\dot{\alpha}\alpha + F_3\alpha^2), \tag{27}$$

where

$$F_1 = -\frac{P_{,y}(2KP_{,KK} + P_{,K})}{2(k^2(2KP_{,KK} + P_{,K}) - P_{,y})}, \tag{28}$$

$$F_2 = -\frac{2\sqrt{Ky}P_{,Ky}P_{,y}}{(k^2(2KP_{,KK} + P_{,K}) - P_{,y})}, \tag{29}$$

$$F_3 = \frac{(2yP_{,yy} + P_{,y})(2k^2KP_{,KK} + k^2P_{,K} - P_{,y}) - 4k^2KyP_{,Ky}^2}{2(k^2(2KP_{,KK} + P_{,K}) - P_{,y})}. \tag{30}$$

As we have discussed above, we will consider the stability conditions at high-momentum limit. Therefore, by taking the limit $k^2 \to \infty$, the second order action becomes

$$S^{(2)} = \int dtd^3kk^{-2}(-P_{,y})\left(\frac{1}{2}\dot{\alpha}^2 - \frac{1}{2}k^2c_s^2\alpha^2 - \frac{1}{2}m_A^2\alpha^2\right). \tag{31}$$

where

$$m_A^2 = \frac{d}{dt}\left(\frac{2\sqrt{Ky}P_{,Ky}}{(P_{,K} + 2KP_{,KK})}\right) - \frac{4KyP_{,Ky}^2}{(P_{,K} + 2KP_{,KK})^2}. \tag{32}$$

Therefore, the condition to avoid ghost instabilities can be written as

$$P_{,y} < 0. \tag{33}$$

This condition can be reduced to the canonical case by taking $P = K - V(y)$, which provides the result as $V_{,y} > 0$ consistently with the result in [27]. By finding the equation of motion of $\alpha$ from the action in Eq. (31), one finds that the equation is in the form of the massive wave equation of mass $M_A$ propagating with speed $c_s$ defined in Eq. (25). In order to avoid the Laplacian instability, one requires $c_s^2 \geq 0$ leading to the condition

$$1 + \frac{2yP_{,yy}}{P_{,y}} - \frac{4KyP_{,Ky}^2}{P_{,y}(2KP_{,KK} + P_{,K})} \geq 0. \tag{34}$$

To obtain a clear picture of this condition, one may specify the form of $P$. For the form with constant equation of state parameter, $P = P_0K^\nu y^\mu$, the sound speed square can be expressed as $c_s^2 = w$. Therefore, the three-form field can be interpreted as the nonrelativistic matter up to a perturbation level since $c_s^2 = 0$ and $w = 0$. Moreover, it is obvious that the nonrelativistic matter represented by the generalized three-form field is free from ghost and Laplacian instabilities. Note that the dark energy model with $w < -1/3$ for this form of the Lagrangian suffers from Laplacian instabilities since the sound speed square is negative.

For another simple form of the Lagrangian with $P = P_0 K^\nu e^{\frac{1-2\nu}{2}\lambda y}$, the sound speed square and the equation of state parameter read $c_s^2 = 1 + \lambda y$ and $w = -1 + \lambda y$. The no-ghost condition can be expressed as $P_0 \lambda (2\nu - 1) > 0$. At this point, it is possible to obtain a viable model of dark energy due to the generalized three-form field.

Now we will consider the vector mode of the perturbations by following the same steps as in the scalar one. As a result, the second order action for the vector perturbations can be written as

$$S^{(2)} = \int d^4x \left( -\frac{1}{2} P_{,y} \beta_i \beta^i \right). \tag{35}$$

From this action, one can see that the vector mode does not propagate at a linear level. One has to perform nonlinear perturbations in order to find the stability behavior of the perturbations. If there are propagating degrees of freedom, it implies that the perturbations are strongly coupled. If the vector modes still do not propagate at a nonlinear level, it may imply that the symmetry of the background metric does not allow the vector mode to propagate. We leave this investigation for further work. A condition to avoid the instabilities coincides with the condition obtained in the scalar mode.

In order to find the possibility of obtaining nonadiabatic perturbations due to the three-form field, one may find a difference between the speed of propagation of scalar perturbations, $c_s^2$, and the adiabatic sound speed, $c_a^2$. If these two kinds of sound speed are equal, there are no nonadiabatic perturbations while it provides the possibility to generate nonadiabatic perturbations if they are not equal [18]. The speed of propagation of scalar perturbations is found in Eq. (25). For the adiabatic sound speed, one can derive the equations as follows

$$c_a^2 \equiv \frac{\dot{p}}{\dot{\rho}} = 1 + 2 \frac{(P_{,y} + yP_{,yy})\dot{y} + P_{,Ky} y \dot{K}}{P_{,y}(\dot{y} - 2\sqrt{Ky})}, \tag{36}$$

$$= c_s^2 + \frac{4\sqrt{Ky}}{P_{,y}(\dot{y} - 2\sqrt{Ky})} \times \left( P_y + yP_{,yy} + \frac{yP_{,Ky}(P_{,y} - 2KP_{Ky})}{(P_{,K} + 2KP_{,KK})} \right). \tag{37}$$

Note that the second line of the above equation is obtained by using the equation of motion in Eq. (16). From this equation, one can see that the sound speed of scalar perturbations and the adiabatic sound speed are not generally equal. Therefore, it is possible to generate nonadiabatic perturbations from the generalized three-form field. This is one of the advantages of the generalized three-form field compare with the k-essence scalar field. It is of interest to find a condition for which $c_s^2$ and $c_a^2$ are the same. From Eq. (37), such a condition can be written as

$$P_{,y} + yP_{,yy} + \frac{yP_{,Ky}(P_{,y} - 2KP_{Ky})}{(P_{,K} + 2KP_{,KK})} = \partial_K \left( \frac{KP_{,K}}{P_{,y}} \right) = 0. \tag{38}$$

Note that the above equation is obtained by using the definition of the energy density and pressure expressed in Eq. (13) and Eq. (14) respectively. By following the calculation in [18], a generic Lagrangian for which $c_s^2$ and $c_a^2$ are the same can be written in the form as

$$P = f(Kg(y)), \tag{39}$$

where $f$ and $g$ are arbitrary functions. Surprisingly, this formula is exactly the same with the formula obtained in the scalar field case. Note that the Lagrangian forms considered in Eq. (18) and Eq. (20) belong to this form.

## IV. FLUID DYNAMICS DUE TO THREE-FROM FIELD

In order to compare the results with the standard description of the fluid dynamics for the perfect fluid, let us briefly review an important concept of the standard version for the fluid dynamics. Since the perfect fluid dynamics due to the noncanonical scalar filed or k-essence field have been intensively investigated and interpreted as a nonrelativistic matter field, for example, in the case of massive gravity theory [33,34], we will also review some important results of the k-essence scalar field before we further discuss the three-form field.

### A. Standard version and k-essence field

There are many approaches of the standard version for the perfect fluid Lagrangian. We will use the Brown formulation [7] since it is more useful and has been widely used for recent studies in dark energy and dark matter couplings [9–12]. The Lagrangian of the perfect fluid can be written in terms of the energy density with Lagrange multipliers as

$$S_m[g_{\mu\nu}, j^\mu, \varphi, s, \alpha_A, \beta_A] = \int d^4x (-\sqrt{-g}\rho + j^\mu(\varphi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha^A_{,\mu})), \tag{40}$$

where $\rho = \rho(n, s)$ is the energy density of the fluid, $n$ is a particle number density, $s$ is an entropy density per particle and $j^\mu$ are components of the particle number flux. The second term which is contracted with $j^\mu$ is the Lagrangian multiplier term with Lagrangian multiplier fields $\varphi$, $\theta$ and $\beta_A$ where $\alpha_A$ are the Lagrangian coordinates of the fluid with index $A$ running as 1,2,3. $j^\mu$ can be written in terms of the four velocity $u^\mu$ of the fluid as

$$j^\mu = \sqrt{-g} n u^\mu. \tag{41}$$

The four velocity satisfies the relation $u_\mu u^\mu = -1$ where $n = |j|/\sqrt{-g}$ and $|j| = \sqrt{-j^\mu g_{\mu\nu} j^\nu}$. The standard energy momentum tensor of the perfect fluid can be obtained by varying the action with respect to the metric $g_{\mu\nu}$ as

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{42}$$

where $p$ is the pressure of the fluid defined as

$$p \equiv n \frac{\partial \rho}{\partial n} - \rho. \tag{43}$$

By varying the action with respect to the Lagrangian multiplier fields $\theta$ and $\varphi$, the first law of thermodynamics and the conservation of the particle number can be obtained respectively [7] as

$$dp = n d\mu - T ds, \tag{44}$$

$$\partial_\nu j^\nu = 0. \tag{45}$$

where $T$ is a temperature and $\mu$ is a chemical potential defined as

$$\mu \equiv \frac{\rho + p}{n}. \tag{46}$$

From these equations of motion together with the conservation of the energy momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, all main thermodynamics equations can be obtained. For example, the conservation of the entropy density can be obtained by using a projection of the conservation equation of the energy momentum tensor along the fluid flow as follows

$$u_\nu \nabla_\mu T^{\mu\nu} = -\frac{\mu}{\sqrt{-g}} \partial_\nu j^\nu - u^\nu T \partial_\nu s = 0. \tag{47}$$

From these equations, in the viewpoint of field theory, all main thermodynamics equations can be obtained if one can identify the main thermodynamics quantities in terms of the field such as energy density, pressure, four velocity and chemical potential which give the form of energy momentum tensor as found in Eq. (42). We will show this procedure for instruction in the case of scalar field.

For the k-essence scalar field, we will follow [14] in which the action of the k-essence field can be written as

$$S_\phi = \int d^4x \sqrt{-g} P(K_\phi), \tag{48}$$

where $K_\phi = -\nabla_\mu \phi \nabla^\mu \phi / 2$ is the canonical kinetic term of the scalar field. The corresponding equations of motion of the scalar field can be expressed as

$$\nabla_\mu (P' \nabla^\mu \phi) = 0, \tag{49}$$

where prime denotes the derivative with respect to $K_\phi$. The energy momentum tensor of the scalar field can be written as

$$T_{\mu\nu} = P' \nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu} P. \tag{50}$$

By comparing this energy momentum tensor with that in the perfect fluid in Eq. (42), the energy density, pressure and the four velocity can be identified as

$$\rho_\phi = 2K_\phi P' - P, \tag{51}$$

$$p_\phi = P, \tag{52}$$

$$u^\mu = \frac{\nabla^\mu \phi}{\sqrt{2K_\phi}}. \tag{53}$$

Therefore, the particle number density can be obtained in order to satisfy the conservation of the particle flux as $n_\phi = \sqrt{2K_\phi} P'$ while the chemical potential reads $\mu_\phi = \sqrt{2K_\phi}$. Therefore, one can check that the equation of motion in Eq. (49) satisfies the equation of the conservation of the particle flux as follows

$$\begin{aligned} \sqrt{-g} \nabla_\mu (P' \nabla^\mu \phi) &= \partial_\mu (\sqrt{-g} P' \nabla^\mu \phi) \\ &= \partial_\mu (\sqrt{-g} n_\phi u^\mu) = \partial_\mu j^\mu_\phi = 0. \end{aligned} \tag{54}$$

As a result, all fluid dynamics equations can be derived by using the results in the standard version. Note that the first law of thermodynamics is adopted for the scalar field while in the case of the standard version, it is obtained from the equation of motion. It is important to note that the conservation of the particle flux does not hold if we generalize the Lagrangian of the scalar field as $P = P(K_\phi, \phi)$ since the equations of motion in Eq. (49) become $\nabla_\mu (P' \nabla^\mu \phi) = -\partial P / \partial \phi$. This is not so surprising since the simple scalar field, such as a quintessence field, is also equivalent to the system in which the particle flux is not conserved. This can be explicitly seen by taking $P = K_\phi - V(\phi)$. Note that a particular form the Lagrangian $P(K_\phi, \phi) = f(K_\phi g(\phi))$ still provides the conserved particle flux. This is due to a suitable field redefinition to provide the Lagrangian depending only on the kinetic term, $P = P(K_\phi)$ [18].

By taking the equation of state parameter to be constant, the form of the Lagrangian obeys a relation

$$P(1 + w_\phi) = 2 w_\phi K_\phi P'. \tag{55}$$

From this equation, one can find the exact form of the Lagrangian as

$$P = P_0 K_\phi^{\frac{1+w_\phi}{2w_\phi}}, \quad \text{where } w_\phi \neq 0. \tag{56}$$

It is obvious that one cannot properly use this form of the scalar field to describe the nonrelativistic matter since its equation of state parameter is zero, $w = 0$. This is one of drawbacks for the k-essence scalar field. As we have shown before, this does not happen in the case of the generalized three-form field.

### B. Generalized three-form field

As we have mentioned, one can find the equivalence between the energy momentum tensor of the three-form and the standard perfect fluid and then identify the fluid quantities such as $\rho$, $p$ and the four velocity $u^\mu$ in terms of the three-form field. By using these identifications, one can find the consequent thermodynamics equations of the three-form field as done in the scalar field case. The energy density and the pressure have been identified in Eq. (13) and Eq. (14) respectively. Now, we will identify the four velocity of the three-form field by comparing the energy momentum tensor of the perfect fluid in Eq. (42) and the energy momentum tensor of the three-form in Eq. (7). As a result, the relation of the four velocity and the three-form field can be written as

$$(\rho + p)u_\mu u_\nu = \frac{1}{6}P_{,K}F_{\mu\rho\sigma\alpha}F_\nu{}^{\rho\sigma\alpha} - \frac{1}{2}P_{,y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} + (2KP_{,K} + 2yP_{,y})g_{\mu\nu}. \tag{57}$$

Since $F_{\mu\nu\rho\sigma}$ is a totally symmetric rank-4 tensor in four-dimensional spacetime, it can be written in terms of a covariant tensor $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}$ where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol in four-dimensional spacetime. By using the components of the three-form field in Eq. (10), the field strength tensor can be written as

$$F_{\mu\nu\rho\sigma} = (\dot{X} + 3HX)\epsilon_{\mu\nu\rho\sigma} = \sqrt{2K}\epsilon_{\mu\nu\rho\sigma}. \tag{58}$$

By using this equation, the first term in the right-hand side of Eq. (57) can be rewritten as

$$\frac{1}{6}P_{,K}F_{\mu\rho\sigma\alpha}F_\nu{}^{\rho\sigma\alpha} = -2KP_{,K}g_{\mu\nu}. \tag{59}$$

Substituting this equation into Eq. (57), one obtains

$$(\rho + p)u_\mu u_\nu = -\frac{1}{2}P_{,y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} + 2yP_{,y}g_{\mu\nu},$$
$$u_\mu u_\nu = \frac{1}{4y}A_{\mu\rho\sigma}A_\nu{}^{\rho\sigma} - g_{\mu\nu}. \tag{60}$$

One can check that the relation $u_\mu u^\mu = -1$ valid from this relation. Since the tensor $u_\mu u_\nu$ is constructed from two three-form fields, it plays the role of symmetric rank-2 tensor $S_{\mu\nu}$ instead of outer product of two four velocity. Therefore, it is not trivial to find the form of the four velocity of the three-form field. However, one may expect that the four velocity may relate to the three-form field by the relation of the vector and the three-form in four dimensionality as $u^\mu \propto \epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}$. As a result, the four velocity of the fluid can be written in terms of the three-form field as

$$u^\mu = \frac{\epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}}{3!\sqrt{2y}}, \tag{61}$$

where the three-form field can be written in terms of the four velocity as

$$A^{\alpha\beta\gamma} = \sqrt{2y}\epsilon^{\mu\alpha\beta\gamma}u_\mu. \tag{62}$$

It is not trivial to find the conserved current density corresponding to the three-form field. Actually, there are no conserved quantities obtained from the invariance of the action under the shift of the field like the scalar field. However, one may find the conserved quantity from the constraint equation in Eq. (6) as follows

$$j^{\alpha\beta\gamma} = n^{\mu\alpha\beta\gamma}u_\mu = \sqrt{2y}P_{,y}\epsilon^{\mu\alpha\beta\gamma}u_\mu = P_{,y}A^{\alpha\beta\gamma}. \tag{63}$$

From this relation, the conserved quantity is now a three-form field instead of a vector field and the number density now is the four-form field instead of the scalar field. This equivalence comes from the Hodge duality in four-dimensional spacetime. One may obtain the effective particle number density as

$$n = \sqrt{\frac{n_{\mu\alpha\beta\gamma}n^{\mu\alpha\beta\gamma}}{4!}} = \sqrt{2y}P_{,y}. \tag{64}$$

Therefore, the usual particle flux for the three-form field can be written as

$$j^\mu = \sqrt{-g}nu^\mu = \sqrt{-g}P_{,y}\frac{\epsilon^{\mu\alpha\beta\gamma}A_{\alpha\beta\gamma}}{3!}. \tag{65}$$

This quantity does not trivially vanish due to the equation of motion in Eq. (16). Since $\partial_\mu j^\mu \neq 0$ together with Eq. (47), it is inferred that the entropy along the fluid flow is not conserved. The nonconservation of the particle flux for the three-form is due to the fact that the action is not invariant under shift of the field. In the scalar field case, the action is invariant under $\phi \to \phi + \xi$ where $\xi$ is a constant. For a general case of the scalar field with $P_\phi = P_\phi(K_\phi, \phi)$, this symmetry is also broken and then its dynamics will correspond to the nonconservation of the particle flux like in the three-form case. For the three-form, if we restrict our attention to the case where $P = P(K)$ which is invariant under shift of the field, the particle number density, $n \propto \rho + p \propto P_{,y}$, will always vanish. Also, the equation of state parameter is always equal to $-1$ which cannot be responsible for the nonrelativistic matter.

### C. Vector field duality

In order to complete our analysis, let us consider the thermodynamics interpretation in terms of the dual vector field. In four-dimensional spacetime, the three-form field is dual to the vector field via the Hodge duality, $A_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma\mu}V^\mu$. By using this duality the kinetic term $K$ and scalar function $y$ of the three-form field can be written in terms of the vector field, $V^\mu$, as

$$K = -\frac{1}{48}F^2 = \frac{1}{2}(\nabla_\mu V^\mu)^2, \tag{66}$$

$$y = \frac{1}{12}A_{\alpha\beta\gamma}A^{\alpha\beta\gamma} = -\frac{1}{2}V_\mu V^\mu. \tag{67}$$

Therefore, the action of the vector field is still in the same form as the one in Eq. (1) where $P = P(K, y)$. However, the dynamical fields are now the vector field and the metric. By varying the action with respect to the vector field, the equation of motion for the vector field can be written as

$$\nabla_\mu(P_{,K}\nabla_\rho V^\rho g^{\mu\nu}) + P_{,y}V^\nu = 0. \tag{68}$$

As we have done in the three-form case, the energy momentum tensor for the vector field can be obtained as

$$T_{\mu\nu} = -P_{,y}V_\mu V_\nu + (P - 2yP_{,y} - 2KP_{,K})g_{\mu\nu}. \tag{69}$$

Note that we have used the equation of motion for the vector in Eq. (68) to obtain this form of the energy momentum tensor. One can check that this energy momentum tensor is covariantly conserved up to the equation of motion as we expect. From this form of the energy momentum tensor, it is similar to one for the perfect fluid found in Eq. (42). By comparing $T_{\mu\nu}$ of the vector field in Eq. (69) to one of the perfect fluid in Eq. (42), we can identify the pressure of the vector field as follows

$$p = P - 2yP_{,y} - 2KP_{,K}. \tag{70}$$

This form of the pressure for the vector field coincides with one for the three-form field in Eq. (14). Now we have to identify the four velocity and the energy density of the vector field. Again, by comparing $T_{\mu\nu}$ of the vector field to one of the perfect fluid, we found that the four velocity, $u^\mu$ must be proportional to $V^\mu$. Therefore, one can write

$$u^\mu = \frac{V^\mu}{\sqrt{2y}}, \tag{71}$$

where the proportional function $\sqrt{2y}$ is obtained by using Eq. (67) and relation $u_\mu u^\mu = -1$. Note also that, by using Hodge duality, this four velocity is in the same form with one for the three-form case, Eq. (61). This suggests that the results obtained in terms of the three-form field in the previous section are trustable. The energy density can be obtained by evaluating $\rho = -T^0_0$. As a result, we have

$$\rho = 2KP_{,K} - P. \tag{72}$$

By using these thermodynamics quantities, one obtains the other quantities as done in the same manner in the previous section such as $n = -\sqrt{2y}P_{,y}$, $\mu = \sqrt{2y}$ and $j^\mu = \sqrt{-g}P_{,y}V^\mu$. Note that we do not need to consider the FLRW metric in order to find the thermodynamics quantities in the case of the vector field while we do in the case of the three-form field. One can see that all of the thermodynamics quantities obtained in terms of the vector field are the same as the results found in the three-form field case. This is due to the Hodge duality. As a result, this also implies that the particle number is not conserved as found in the three-form field case.

We observe that the condition of nonconservation of the entropy density along the fluid flow coincides with the condition of the generation of nonadiabatic perturbations even though these conditions come from different approaches. The conservation of the entropy density is derived from the background equation while the nonadiabatic perturbations are properties of the fluid at the perturbation level. This argument also holds in both the scalar field and the three-form field cases. Therefore, this may shed light on the interplay between the conserved quantities under the shift of the field and nonadiabatic perturbations. It is important to note that the conservation of the energy momentum tensor of the three-form is still valid, $\nabla_\mu T^\mu_\nu = 0$. The nonconservation quantities mentioned above are the thermodynamically effective quantities. As we have mentioned, the useful point of this three-form field is that it can represent the nonrelativistic matter field with $w = 0$. Therefore, one may interpret it as dark matter. This may be a useful approach for studies of dark energy and dark matter coupling since one can find the covariant interaction terms at the Lagrangian level and then the resulting closed evolution equations are obtained. This issue is of interest and we leave this detailed investigations for further work.

It is worthwhile to note that the generalized three-form field may be dual to the scalar field by introducing some nonminimal couplings to the gravity [20,23] or nontrivial terms into the Lagrangian. Here, we provide a simple example of the Lagrangian form in which the scalar duality is obtained,

$$\mathcal{L} = P(K, y) + \frac{1}{6}A_{\alpha\beta\gamma}\nabla_\mu F^{\mu\alpha\beta\gamma}. \tag{73}$$

The scalar duality may obtained by $F_{\mu\alpha\beta\gamma} = \phi\epsilon_{\mu\alpha\beta\gamma}$ and $A_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma\mu}V^\mu$. Therefore, the kinetic term of the three-form field is proportional to a function of the scalar field and then one obtains $P(K,y) = P(\phi, y)$. The additional term is proportional to $V^\mu \partial_\mu \phi$. Therefore, one can integrate out the field $V_\mu$ which turns out that $V_\mu \propto \partial_\mu \phi / P_{,y}$. This provides that $y$ is proportional to the kinetic term of the scalar field and then one obtains the scalar k-essence model as $P(K,y) = P(\phi,y) = P(\phi,X)$, where $X = -(\partial_\mu \phi)^2/2$.

## V. SUMMARY

A Lagrangian formulation of a perfect fluid is a powerful tool to study the dynamics of the universe, especially the interacting approach between dark energy and dark matter. A general description in this formulation invokes many functions and then it is not easy to handle. A k-essence scalar field can be used to describe the dynamics of the perfect fluid in cosmology. At the background level, even though the k-essence scalar field can be used to describe the perfect fluid with a constant equation of state parameter, it cannot be properly used for the nonrelativistic matter with $w_\phi = 0$. At the perturbation level, the k-essence scalar field cannot provide nonadiabatic perturbations as well as intrinsic vector perturbations.

In the present paper, we propose an alternative way to provide nonadiabatic perturbations and intrinsic vector perturbations by using a generalized three-form field. The investigation begun with proposing a general form of the action of the three-form field with a function depending on both the kinetic term and the field, $P = P(K,y)$, similar to the k-essence scalar field. Equations of motion and the energy momentum tensor of the three-form field in the covariant form have been calculated. By working in the FLRW background, the energy density and the pressure as well as the equation of state parameter are found. For the constant equation of the state parameter, an exact form of the Lagrangian reads $P = P_0 K^\nu y^\mu$ where $w = -1 + \frac{2\mu}{1-2\nu}$ and $\nu \neq 1/2$. Therefore, one can set $w = 0$ by choosing proper values of the parameters $\mu$ and $\nu$ and then use the generalized three-form field to represent the nonrelativistic matter. For a nonconstant equation of the state parameter, we also point out that it is possible to construct an alternative model of dark energy. The stability analysis of the model is also performed. We found the conditions to avoid ghost and Laplacian instabilities. For the fluid with $w = 0$, it is free from ghost and Laplacian instabilities. For some specific model of dark energy, we argue that, to avoid the superluminality, the equation of state parameter must be greater than $-1$. In other words, the viable model of dark energy from the generalized three-form field cannot provide the phantom phase of the universe. Note that the no-ghost condition we found in this paper can be trusted only in the high momentum limit. We leave the full investigation for further work where we investigate the cosmological perturbations and observational constraints. One of important problems found in scalar field quintessence is quantum mechanical consistency, by considering that the quantum fluctuation may alter the classical quintessence potential and then provide an instability of the model [35]. It is also of interest to study the quantum mechanical consistency for the three-form field model. We leave this investigation for further work.

Thermodynamics properties due to the generalized three-form field are also investigated. It is found that this model corresponds to a system with nonconservation of the particle flux. This leads to a nonconservation of the entropy density along the fluid flow. This is not so surprising since many models of dark energy, for example quintessence model, also correspond to the nonconservation of the particle flux. We also found some links between nonconservation of the entropy density along the fluid flow which is a thermodynamically effective quantity at the background level and the generation of nonadiabatic perturbations which is a property of the model at the perturbation level. This may shed light on the interplay between conserved quantities under shift of the field and nonadiabatic perturbations. We can argue that this is a useful approach for the study of dark energy and dark matter coupling classically since one can find the covariant interaction terms at the Lagrangian level and then the resulting closed evolution equations are obtained. This issue is of interest and we leave this detailed investigations for further work.

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