Generalized conformal transformation and inflationary attractors

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We investigate the inflationary attractors in models of inflation inspired from the general conformal transformation of general scalar-tensor theories to the Einstein frame. The coefficient of the conformal transformation in our study depends on both the scalar field and its kinetic term. Therefore, the relevant scalar-tensor theories display the subset of the class I of the degenerate higher order scalar-tensor theories in which both the scalar field and its kinetic term can nonminimally couple to gravity. We find that if the conformal coefficient Ω takes a multiplicative form such that $\Omega \equiv w(\phi)W(X)$, where X is the kinetic term of the field ϕ, the theoretical predictions of the proposed model can have a usual universal attractor independent of any functions of $W(X)$. For the case where Ω takes an additive form, such that $\Omega \equiv w(\phi) + k(\phi)\Xi(X)$, we find that there are new ξ attractors in addition to the universal ones. We analyze the inflationary observables of these models and compare them to the latest constraints from the Planck Collaboration. We find that the observable quantities associated to these new ξ attractors do not satisfy the constraints from Planck data at a strong coupling limit.

I. INTRODUCTION

The mechanism of cosmic inflation is a conceivable framework when one wants to describe the Universe at very early times. It can nicely address a number of issues of standard big bang cosmology. More concretely, it paves the way for the treatment of primordial fluctuations resulting in the large scale structures and the anisotropy in the temperature of the cosmic microwave background (CMB) observed today. In the simplest version of the models, we require the presence of a scalar degree of freedom (inflaton), either as a fundamental scalar field, e.g., a Higgs field [1–6], or a composite field [7–12] (or even incorporated into gravity itself), in general, as an effective scalar degree of freedom. More recently, a broad class of inflationary models, dubbed cosmological attractors [13–16], has attracted a lot of attention. Cosmological attractor scenarios for the inflationary models have been developed in the past few years [17–21].

Interestingly, the cosmological α attractors constitute most of the existing inflationary models with plateaulike potentials. These include the Starobinsky model and some generalized versions of the Higgs inflation. Regarding the α attractors, the flatness of the inflaton potential is achieved

and protected by the existence of a pole in the kinetic term of the scalar field. Moreover, at large-field values, any nonsingular inflaton potential acquires a universal plateau-like form when performing the (conformal) transformation. Regarding the hyperbolic geometry and the flatness of the Kahler potential in the supergravity context, the universal behaviors of these theories make very similar cosmological predictions preserving good agreement with the current observational data [22]. This class of models has certain universal predictions for the important cosmological observables, i.e., scalar spectral index (n_s) and tensor-to-scalar ratio (r). It has been shown that the nonminimal coupling between inflaton and gravity in the strong coupling limit can lead to an attractor, in which the observational quantities are the same as the universal α attractors [14,15]. The general consideration for the relations between the inflationary attractor due to the non-minimal coupling, namely ξ attractors, and the α attractors is presented in [17].

In the present work, we extend analysis in the existing literature by considering the cases where the nonminimal coupling is also in the form of nonminimal kinetic coupling such that the term $k(\phi)f(X)R$ appears in the action. Here, $k(\phi)$ is an arbitrary function of the inflaton ϕ, $f(X)$ is an arbitrary function of X, $X = -\partial_\mu\phi\partial^\mu\phi/2$ is the kinetic term of the inflaton field, and R is the Ricci scalar. In general, such nonminimal coupling arises by applying the general

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conformal transformation, in which the conformal coefficient depends on both the scalar field and its kinetic term, to the Einstein-Hilbert action. In Sec. II, we construct an action in the Einstein frame that is conformally equivalent to scalar-tensor theories with a general nonminimal coupling using the general conformal transformation. We investigate inflationary attractors in the presence of the general nonminimal coupling using the action in the Einstein frame based on the assumption that observable quantities are frame invariant. The two-case scenarios are considered. In Sec. III, we concentrate on the multiplicative form of the generalized conformal factor, i.e., $\Omega = \Omega(X, \phi) = w(\phi)W(X)$. We show whether the attractors found in the literature can exist in our models and then review some essential ideas of the inflationary attractors as well as calculations of cosmological observables, i.e., n_s and r considering both hyperbolic tangent potential and exponential potential. In Sec. IV, we choose the additive form of the generalized conformal factor, i.e., $\Omega = \Omega(X, \phi) = w(\phi) + k(\phi)\Xi(X)$, which in some situations can be viewed as a generalization of the multiplicative form models. We compute the cosmological observables for the hyperbolic tangent potential and consider theoretical predictions in the weak and strong coupling limits, which are equivalent to large and small α limits in our setup, respectively. In Sec. IV C, we compare the obtained results of the cosmological observables with recent Planck 2015 data. Finally, we present our conclusion in the last section.

II. GENERAL CONFORMAL TRANSFORMATION AND ACTION IN THE EINSTEIN FRAME

Let us first consider a general conformal transformation in which the relation between a new metric, $\tilde{g}_{\mu\nu}$, and the old one, $g_{\mu\nu}$, takes the form,

$$\tilde{g}_{\mu\nu} = \Omega(X, \phi)g_{\mu\nu}. \tag{1}$$

According to this transformation, the determinant between the two metrices yields

$$\mathcal{J}_g \equiv \frac{\sqrt{-\tilde{g}}}{\sqrt{-g}} = \Omega^2, \tag{2}$$

and a relation between kinetic terms in different frames is

$$\tilde{X} \equiv -\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{X}{\Omega}. \tag{3}$$

Applying the transformation in Eq. (1) to the Einstein-Hilbert action,

$$S_E = \int d^4x\sqrt{-\tilde{g}}\frac{1}{2}\tilde{R}, \tag{4}$$

we get [23]

$$S_J = \int d^4x\sqrt{-g}\left[\frac{1}{2}\Omega R + \frac{3}{4}\frac{\Omega_\phi^2}{\Omega}\partial_\alpha\phi\partial^\alpha\phi \right. $$
$$\left. + \frac{3}{2\Omega}\Omega_\phi\Omega_X\partial_\alpha\phi\partial^\alpha X + \frac{3}{4}\frac{\Omega_X^2}{\Omega}\partial_\alpha X\partial^\alpha X\right], \tag{5}$$

where subscripts ϕ and X denote derivative with respect to ϕ and X, respectively. Here, we set the reduced Planck mass $M_P = (8\pi G)^{-1/2} = 1$. We now add the kinetic term $-\sqrt{-\tilde{g}}h(\phi, X)\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi/2$ to the Einstein-frame action in Eq. (4). Under the transformation given in Eq. (1), this kinetic term gives $-\sqrt{-g}\Omega h g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi/2$ in the Jordan-frame action. Let us define the kinetic term of scalar field in the Jordan Frame as $-\sqrt{-g}G(\phi, X)g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi/2$. Hence, we have

$$-G(\phi, X) = \frac{3}{2}\frac{\Omega_\phi^2}{\Omega} - \Omega h(\phi, X), \tag{6}$$

and therefore,

$$h = \frac{G + 3\Omega_\phi^2/(2\Omega)}{\Omega}. \tag{7}$$

Based on the above analysis, we conclude that under the transformation given in Eq. (1), the action in the Einstein frame,

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2}\tilde{R} - \frac{G\Omega + 3\Omega_\phi^2/2}{2\Omega^2}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right], \tag{8}$$

becomes

$$S_J = \int d^4x\sqrt{-g}\left[\frac{1}{2}\Omega R - \frac{1}{2}G(\phi, X)\partial_\mu\phi\partial^\mu\phi \right. $$
$$\left. + \frac{3}{2\Omega}\Omega_\phi\Omega_X\partial_\alpha\phi\partial^\alpha X + \frac{3}{4}\frac{\Omega_X^2}{\Omega}\partial_\alpha X\partial^\alpha X\right]. \tag{9}$$

The potential term for the scalar field in the Einstein frame can be obtained by adding the term $-\Omega^2 V_E(\phi)$ in the Jordan-frame action. Thus, under the general conformal transformation, the action in the Jordan frame,

$$S_J = \int d^4x\sqrt{-g}\left[\frac{1}{2}\Omega R + G(\phi, X)X - \Omega^2 V_E(\phi) \right. $$
$$\left. + \frac{3}{2\Omega}\Omega_\phi\Omega_X\partial_\alpha\phi\partial^\alpha X + \frac{3}{4}\frac{\Omega_X^2}{\Omega}\partial_\alpha X\partial^\alpha X\right], \tag{10}$$

is equivalent to the Einstein-frame action,

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2}\tilde{R} + \frac{G\Omega + 3\Omega_\phi^2/2}{\Omega^2}\tilde{X} - V_E(\phi)\right]. \tag{11}$$

We note that the coefficients $G(\phi, X)$ and $\Omega(\phi, X)$ in the above Einstein-frame action depend on the kinetic terms X in general. We will consider in the subsequent sections the cases where the X-dependent terms in the Einstein-frame action can cancel each other or can be transformed to $\tilde{X}$.

The combination of the second and third terms in the action (10) is the Lagrangian of K inflation, which can be defined as $L_2 \equiv GX - \Omega^2 V_E(\phi)$. Using the definition $3\Omega_\phi \Omega_X/(2\Omega) \equiv F(\phi, X) + F_X(\phi, X)X$, the third term in the action can be integrated by parts yielding the cubic Galileon term $-F\Box\phi$. The fourth term in the action is a subset of the degenerate higher order scalar-tensor theories (DHOST), so that it does not lead to Ostrogradski instability [24–29]. Due to the existence of this term, the theory described by the action (10) belongs to the class I of the DHOST theory in which the Laplacian instabilities emerging from the negative sound speed of the cosmological perturbations disappear [29]. Moreover, this theory satisfies the conditions for which the propagation speed of gravitational waves equals to speed of light [29,30].

In principle, physical quantities predicted from the inflationary model described by action (10) are the same as those obtained from the action in Eq. (11). However, to explicitly verify this statement, the predictions such as spectral indices and the tensor-to-scalar ratio of the perturbation amplitudes from the DHOST theories have to be studied; this we leave for a future investigation. Although the theoretical predictions are expected to be frame invariant, the comparisons between predictions from inflationary models and results from observational data require a relation between the predicted quantities and the number of e-foldings of inflation, which is frame dependent [31,32]. To investigate how the number of e-foldings depends on the frame, we suppose that the background metric is the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric given by

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \qquad (12)$$

where a is the cosmic scale factor and δ_{ij} is the Kronecker delta. Therefore, Eq. (1) yields

$$\tilde{a} = \sqrt{\Omega(X, \phi)}\, a, \qquad (13)$$

where $\tilde{a}$ and a are the cosmic scale factors in the Einstein and Jordan frames, respectively. Hence, the relation between the number of e-foldings for different frames is given by

$$\tilde{N} = N + \frac{1}{2}\ln\left(\frac{\Omega_{\text{end}}}{\Omega_N}\right), \qquad (14)$$

where the subscript $_{\text{end}}$ denotes the evaluation at the end of inflation, while the subscript $_N$ represents the quantities evaluated at the horizon crossing at the e-folding N of the

observed CMB modes. Since the function Ω describes nonminimal coupling in the Jordan frame, it can be generally written in the form,

$$\Omega(X, \phi) = 1 + \xi F(\phi, X), \qquad (15)$$

where ξ is the nonminimal coupling constant. Inserting Eq. (15) into Eq. (14), we get

$$\tilde{N} = N + \frac{1}{2}\ln\left(\frac{1 + \xi F_{\text{end}}}{1 + \xi F_N}\right)$$

$$= N + \begin{cases} \frac{1}{2}\xi(F_{\text{end}} - F_N) & \text{for } \xi \ll 1 \\ \frac{1}{2}\ln\left(\frac{F_{\text{end}}}{F_N}\right) & \text{for } \xi \gg 1 \end{cases}. \qquad (16)$$

It clearly follows from the above equation that if ϕ is supposed to slowly evolve during inflation, we can make an approximation $\tilde{N} \simeq N$, when ξ is sufficiently small. In the case of a large ξ limit, an approximation $\tilde{N} \simeq N$ is valid when the function F changes slowly during inflation. To estimate how much the function F changes during inflation, we compute evolution equations for the background Universe. Starting from the action given in Eq. (10), we insert the metric in Eq. (12) into the action and vary the action with respect to components of the metric. Here, varying the action with respect to a, we obtain

$$0 = 2\dot{H}\Omega + 3H^2\Omega + \ddot{\phi}\Omega_\phi + \dot{\phi}^2\Omega_X + \ddot{\phi}\,\dot{\phi}\,\Omega_X$$

$$+ 2\dot{\phi}H(\Omega_\phi + \ddot{\phi}\Omega_X) - \Omega^2 V_E$$

$$+ GX + 2\Omega_{\phi\phi}X + 4\ddot{\phi}\Omega_{\phi X}X - \frac{3}{\Omega}\Omega_\phi\Omega_X X\ddot{\phi}$$

$$- \frac{3}{2\Omega}\Omega_X^2 X\ddot{\phi}^2 + 2\ddot{\phi}^2\Omega_{XX}X, \qquad (17)$$

where $H \equiv \dot{a}/a$ is the Huble parameter and a dot denotes derivative with respect to time t. To vary the action with respect to the (00) component of the metric, we introduce an auxiliary function $\eta(t)$ in which we find the replacement of $-dt^2$ in Eq. (12) with $-\eta^2(t)dt^2$. Vary the action with respect to η, setting $\eta = 1$ in the obtained result, and then eliminating $\ddot{a}$ from the resulting evolution equation by Eq. (17), we get

$$0 = -2\Omega^4 V_E + 2\Omega^3\Omega_X V_E X$$

$$- 3\Omega_X(-4\Omega_\phi^2 + 2\ddot{\phi}\Omega_\phi\Omega_X + \ddot{\phi}^2\Omega_X^2)X^2 - 2\Omega^2 X(G + 2G_X X)$$

$$+ 6H^2\Omega^2(\Omega - \Omega_X X) + 6\dot{\phi}H\Omega(\Omega_\phi + \ddot{\phi}\Omega_X)(\Omega - \Omega_X X)$$

$$+ 3\Omega X(2\ddot{\phi}\Omega_\phi\Omega_X + \ddot{\phi}^2\Omega_X^2 - 4\Omega_\phi\Omega_{\phi X}X + 2G\Omega_X X). \qquad (18)$$

Combining Eq. (17) with Eq. (18), we can write the expression for $-\dot{H}/H^2$, which is the slow-roll parameter in terms of dimensionless parameters as

$$-\frac{\dot{H}}{H^2} = x_6(2x_4x_6 - 3x_2^2x_6 - x_2(1 - x_6 - x_7))$$
$$-\frac{1}{2}(x_1 - x_5 - x_1x_6 + 6x_1x_2x_6 - 4x_3x_6)$$
$$-\frac{1}{1-x_2}\left[\frac{3}{2}x_1(x_1x_2 - x_3) - (1 - 2x_2)x_{g0} - x_{g0}x_{g2}\right],$$
(19)

where

$$x_1 \equiv \frac{\Omega_\phi \dot{\phi}}{H\Omega}, \qquad x_2 \equiv \frac{\Omega_X X}{\Omega}, \qquad x_3 \equiv \frac{\Omega_{\phi X}\dot{\phi}X}{H\Omega},$$

$$x_4 \equiv \frac{\Omega_{XX}X^2}{\Omega}, \qquad x_5 \equiv \frac{\Omega_{\phi\phi}\dot{\phi}^2}{H^2\Omega}, \qquad x_6 \equiv \frac{\ddot{\phi}}{\dot{\phi}H},$$

$$x_7 \equiv \frac{\dddot{\phi}}{\ddot{\phi}H}, \qquad x_{g0} \equiv \frac{GX}{H^2\Omega}, \qquad x_{g2} \equiv \frac{G_X X^2}{H^2\Omega}.$$
(20)

At the leading order, Eq. (19) gives

$$-\frac{\dot{H}}{H^2} \simeq -x_2x_6 - \frac{1}{2}x_1 + \frac{1}{2}x_5 + \frac{1 - 2x_2}{1 - x_2}x_{g0}$$

$$= -\frac{\dot{\Omega}}{2H\Omega} + \left(\frac{\Omega_{\phi\phi}}{\Omega} + \frac{1 - 2x_2}{1 - x_2}\frac{G}{\Omega}\right)\frac{X}{H^2}. \quad (21)$$

The above equation suggests that $-\dot{\Omega}/(2H\Omega)$ as well as the remaining term on the right-hand side of (21) should be in the same order as $-\dot{H}/H^2$, i.e., $\dot{\Omega}/(2\Omega) \lesssim \dot{H}/H$. Hence, for a large ξ limit, we have $\dot{F}/(2F) \lesssim \dot{H}/H$, implying $F \sim H^{2s}$, where $s \lesssim 1$. Inserting this result into Eq. (16), we get

$$\tilde{N} \simeq N + s\ln\left(\frac{H_{\text{end}}}{H_N}\right). \quad (22)$$

From the PLANCK results [33], we have $H_N \lesssim 2.7 \times 10^{-5}M_p$. Suppose that the Hubble parameter during inflation is almost constant. Hence, we can approximately ignore the second term on the rhs of Eq. (22), and therefore, we have $\tilde{N} \simeq N$. Hence, the predicted quantities in terms of number of *e*-foldings from the action in Eqs. (10) and (11) are approximately the same in the strong ($\xi \gg 1$) and weak ($\xi \ll 1$) limits.

In the following consideration, we will investigate the attractor of the theoretical predictions from the inflationary model described by the action (11). Based on the discussion in the preceding paragraph, the inflationary attractors in the models described by the action (11) should imply the same attractors appearing in the subclass of DHOST theories described by the action (10) in the strong and weak coupling limits. These attractors are consequences of a general nonminimal coupling associated with a general conformal transformation, which are the main interests of

this work. Actually, nonminimal coupling can also be associated with another type of frame transformation called disformal transformation [23]. Some of subclasses of DHOST theories can be transformed to the Einstein frame using the disformal transformation. The kinetic terms of the scalar field in the resulting action in the Einstein frame should also take a noncanonical form. Hence, in this section, we consider the general conformal transformation between the Jordan and Einstein frames to ensure that the Einstein action used in our calculation represents the effects of nonminimal coupling associated with the general conformal transformation.

III. MULTIPLICATIVE FORM

We first consider the case where Ω has an multiplicative form, such that

$$\Omega(\phi, X) = w(\phi)W(X). \quad (23)$$

To make our consideration independent of the form of $W(X)$, we set $G(\phi, X) = g(\phi)W(X)$ and then Eq. (11) becomes

$$S_E = \int d^4x \sqrt{-\tilde{g}}\left[\frac{1}{2}\tilde{R} - \frac{gw + 3w_\phi^2/2}{2w^2}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V_E(\phi)\right]. \quad (24)$$

For suitable choices of field redefinition, inflationary models described by the above action should have a usual inflationary attractor such as those found in the literature. In terms of the canonical normalized field ψ, the above action takes the form,

$$S_E = \int d^4x \sqrt{-\tilde{g}}\left[\frac{1}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi - V(\psi)\right], \quad (25)$$

where

$$d\psi^2 = \left(\frac{gw + 3w_\phi^2/2}{w^2}\right)d\phi^2. \quad (26)$$

Since the action (24) is similar to the action in the Einstein frame for scalar-tensor theories with a nonminimal coupling term $w(\phi)$, we set $w(\phi) \equiv 1 + \xi f(\phi)$ with a dimensionless coupling constant ξ and an arbitrary function $f(\phi)$. To obtain the exact relation between ψ and $w(\phi)$, the relation between $w(\phi)$ and the kinetic coupling $g(\phi)$ is supposed to satisfy the following condition [17]:

$$g(\phi) = \frac{1}{4\xi}\left(\frac{w_\phi^2}{w}\right), \quad (27)$$

then Eq. (26) gives $\psi = \sqrt{3\alpha/2}\ln w(\phi)$. This yields

$$w(\phi) = \exp(\sqrt{2/3\alpha}\psi), \qquad (28)$$

where $\alpha = 1 + (6\xi)^{-1}$. Based on the above exact relation between ψ and $w(\phi)$, the action (25) will be independent from $w(\phi)$ if $V_E(\phi)$ is a function of $w(\phi)$. The slow roll parameters, ϵ, η, and the number of e-foldings, N, have the same forms as the standard slow-roll paradigm, and they read

$$\epsilon = \frac{1}{2}\left(\frac{1}{V}\frac{dV}{d\psi}\right)^2, \quad \eta = \left(\frac{1}{V}\frac{d^2V}{d\psi^2}\right), \quad N = \int_{\psi_{\text{end}}}^{\psi_N} d\psi \frac{V}{dV/d\psi},$$
$$(29)$$

where ψ_{end} is the value of ψ at the end of inflation, and ψ_N is the value of ψ at given N.

We can test our predictions with the experimental results by using the relative strength of the tensor perturbation, i.e., the tensor-to-scalar ratio r and the spectral index of curvature perturbation n_s. In terms of the slow-roll parameters, these observables are written as

$$r = 16\epsilon_N, \qquad n_s = 1 - 6\epsilon_N + 2\eta_N. \qquad (30)$$

Regarding the relation in Eq. (28), we consider

$$V_E(\phi) = V_0\left[\frac{w(\phi) - 1}{w(\phi) + 1}\right]^n, \qquad (31)$$

which leads to

$$V(\psi) = V_0\tanh^n\left(\frac{\psi}{\sqrt{6\alpha}}\right), \qquad (32)$$

which is a well-known attractor potential, and explicitly note that Ref. [34] gives r and n_s shown below. Since the potential takes the form of hyperbolic tangent, this class of models is called the T model [34,35]. Having used the effective potential in Eq. (32), the observable quantities given in Eq. (30) can be written in terms of N as [34]

$$r = \frac{12n\alpha}{nN^2 + \mathcal{G}(\alpha)N + 3n\alpha/4}, \qquad (33)$$

$$n_s = \frac{n(4(-2 + N)N - 3\alpha) + 4(-1 + N)\sqrt{3\alpha(n^2 + 3\alpha)}}{4nN^2 + 3n\alpha + 4N\sqrt{3\alpha(n^2 + 3\alpha)}}$$
$$(34)$$

$$= \frac{1 - \frac{2}{N} - \frac{3\alpha}{4N^2} + \frac{1}{2N}\left(1 - \frac{1}{N}\right)\mathcal{G}(\alpha)}{1 + \frac{1}{2N}\mathcal{G}(\alpha) + \frac{3\alpha}{4N^2}}, \qquad (35)$$

where $\mathcal{G}(\alpha) = \sqrt{3\alpha}\sqrt{(3\alpha + n^2)}$. To the lowest order in the slow-roll approximation, the inflationary predictions in terms of the number of e-foldings in the Einstein frame parameters for this model read

$$n_s = 1 - \frac{2n + 4}{4N + n}, \qquad r = \frac{16n}{4N + n} \quad \text{for } \alpha \gg 1 \ \& \ \alpha \gg n,$$
$$(36)$$

$$n_s = 1 - \frac{2}{N}, \qquad r = \frac{12\alpha}{N^2} \quad \text{for } \alpha \ll 1. \qquad (37)$$

The above expressions for n_s and r in the large and small α limits are computed by treating α as a free parameter which controls the slope of $V(\psi)$. From the definition of α in terms of the coupling constant ξ, we take $\alpha \to \infty$ in the weak coupling $\xi \ll 1$ limit and $\alpha \to 1$ in the strong $\xi \gg 1$ limit. We will see in the numerical investigation displaying in Fig. (1) that in the strong coupling limit ($\alpha = 1$), the observable quantities converge to the universal attractor regime in Eq. (37) [17,34,35]. This regime corresponds to the part of the $n_s - r$ plane favored by the Planck data [36]. For the small coupling limit, the predictions converge to Eq. (36) if n is replaced by $2n$. Moreover, regarding the relation in Eq. (28), the potential of the field ψ takes the exponential form, namely the E model [34,35], if we set $V_E(\phi) = V_0[1 - w^{-1}(\phi)]^n$. This form of $V_E(\phi)$ yields

$$V(\psi) = V_0\left[1 - \exp(-\sqrt{2/3\alpha}\psi)\right]^n. \qquad (38)$$

For this form of the potential, it is difficult to write the time-varying parts of the inflationary predictions r and n_s solely in terms of the number of e-folding as in Eqs. (33) and (35). Hence, we consider the inflationary predictions for this case in the large and small α limits. In the large α limit, the above potential coincides with the simplest chaotic inflation model with a ψ^n potential. In the limit $\alpha \gg 1$, i.e., $\sqrt{2/3\alpha} \ll 1$, we have

$$V(\psi) = V_0\left[1 - \exp\left(-\sqrt{2/3\alpha}\psi\right)\right]^n$$
$$= V_0\left[1 - \exp\left(-\sqrt{2/3\alpha}\psi\right)\right]^n$$
$$\simeq \frac{2}{3\alpha}V_0\psi^n \equiv \tilde{V}_0\psi^n. \qquad (39)$$

For this potential, the slow-roll parameters take the form,

$$\epsilon = \frac{n^2}{2\psi^2}, \qquad \eta = \frac{n(n - 1)}{\psi^2}. \qquad (40)$$

Slow-roll inflation terminates when $\epsilon = 1$, so the field value at the end of inflation reads

$$\epsilon = \frac{n^2}{2\psi^2} = 1 \to \psi_{\text{end}} = \frac{n}{\sqrt{2}}. \qquad (41)$$

The number of e-foldings for the change of the field ψ from ψ_N to ψ_{end} is given by

$$N + N_e = \frac{\psi_N^2}{2n} \quad \text{with} \quad N_e = \frac{\psi_{\text{end}}^2}{2n}. \tag{42}$$

Therefore, in terms of N, the values of n_s and r for the large α limit are given by

$$n_s = 1 - \frac{2n + 4}{4N + n}, \qquad r = \frac{16n}{4N + n}, \quad \text{for } \alpha \gg 1. \tag{43}$$

However, in the small α limit, i.e., $\alpha \ll 1$, the potential in Eq. (38) becomes

$$V(\psi) \simeq V_0 \left[1 - n \exp\left(-\sqrt{2/3\alpha} \psi \right) \right]. \tag{44}$$

For this potential, the slow-roll parameters are

$$\epsilon \simeq \frac{n^2}{3\alpha (e^{\sqrt{\frac{2}{3}}\sqrt{\frac{1}{\alpha}}\psi})^2}, \qquad \eta = -\frac{2n e^{-\sqrt{\frac{2}{3}}\sqrt{\frac{1}{\alpha}}\psi}}{3\alpha}. \tag{45}$$

Slow-roll inflation terminates when $\epsilon = 1$, so the field value at the end of inflation reads

$$\epsilon(\psi_{\text{end}}) = 1 = \frac{n^2 e^{-2\sqrt{\frac{2}{3}}\sqrt{\frac{1}{\alpha}}\psi}}{3\alpha} \rightarrow \psi_{\text{end}} = \sqrt{\frac{3\alpha}{8}} \ln\left(\frac{n^2}{3\alpha} \right). \tag{46}$$

The number of e-foldings for the change of the field ψ from ψ_N to ψ_{end} is given by

$$N = \int_{\psi_{\text{end}}}^{\psi_N} d\psi \frac{V}{dV/d\psi} \simeq \frac{3\alpha e^{\sqrt{\frac{2}{3}}\sqrt{\frac{1}{\alpha}}\psi_N}}{2n} - N_e \quad \text{with} \quad N_e = \frac{\sqrt{3\alpha}}{2}. \tag{47}$$

Therefore, in terms of N, the values of n_s and r for the small α limit are given by [34,35]

$$n_s = 1 - \frac{2}{N}, \qquad r = \frac{12\alpha}{N^2} \quad \text{for } \alpha \ll 1. \tag{48}$$

It follows from Eqs. (43) and (48) that when α is sufficiently large or small, the predictions for the E model also converge to the attractor given in Eq. (36) or the universal attractor given in Eq. (37), respectively. Both the T model and E model have the same α attractors because the potentials for the T model and E model have the same asymptotic behavior when $\alpha \ll 1$ and $\alpha \gg 1$. We conclude that the α attractors can be achieved from our multiplicative form models, where the conformal factor can be separated into two parts as in Eq. (23) and $G = g(\phi)W(X)$. Moreover, the attractors do not depend on the function $W(X)$ in this case. Notice that in this section we just showed that the general scalar-tensor theories we considered are equivalent to Einstein gravity with a canonical scalar. Therefore, it is clearly possible to choose a potential of any form, including previously studied attractors [13–18]. We note that the

results present in this section are a slight generalization from Ref. [16].

IV. ADDITIVE FORM

Let us now consider the case where Ω has an additive form, i.e.,

$$\Omega = w(\phi) + k(\phi)\Xi(X), \tag{49}$$

where $k(\phi)$ and $\Xi(X)$ are dimensionless. For this case, Eq. (11) becomes

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2}\tilde{R} + \left(\frac{G}{\Omega} + \frac{3}{2} \left(\frac{w_\phi + k_\phi\Xi}{w + k\Xi} \right)^2 \right) \tilde{X} - V_E(\phi) \right]. \tag{50}$$

This action can be reduced to Eq. (24) if $k(\phi) = k_1 w(\phi)$ and $G = g(\phi)(1 + k_1\Xi)$, where k_1 is a constant. Hence, the above action is a possible generalization of the action in Eq. (24). When Ω is separated as in Eq. (49), $w(\phi)$ will represent nonminimal coupling and $k(\phi)\Xi(X)$ will represent the nonmimimal kinetic coupling between ϕ and gravity. In an analogy to the consideration in Sec. III, we set $w(\phi) \equiv 1 + \xi f(\phi)$ and $k(\phi) \equiv \xi_k f_k(\phi)$, where ξ and ξ_k are dimensionless constants while $f(\phi)$ and $f_k(\phi)$ are arbitrary functions. In the weak nonminimal kinetic coupling limit, i.e., $|k(\phi)\Xi(X)| \ll |w(\phi)|$, the kinetic terms of ϕ in the action (50) becomes

$$\left(\frac{G}{\Omega} + \frac{3}{2} \left(\frac{w_\phi + k_\phi\Xi}{w + k\Xi} \right)^2 \right) \tilde{X} = \left(\frac{G}{w} + \frac{3}{2} \left(\frac{w_\phi}{w} \right)^2 \right) \tilde{X}. \tag{51}$$

The above kinetic term is similar to that in Eq. (24). Hence, when the nonminimal kinetic coupling is weak, the usual attractor discussed in the previous section can exist. In the limit where the nonminimal kinetic coupling is strong but the nonminimal coupling is weak, i.e., $|k(\phi)\Xi(X)| \gg |w(\phi)|$ and $w(\phi) \simeq 1$, the kinetic terms of ϕ in the action (50) becomes

$$\left(\frac{G}{\Omega} + \frac{3}{2} \left(\frac{w_\phi + k_\phi\Xi}{w + k\Xi} \right)^2 \right) \tilde{X} = \left(\frac{G}{k\Xi} + \frac{3}{2} \left(\frac{k_\phi}{k} \right)^2 \right) \tilde{X}. \tag{52}$$

Thus, the usual α attractor can exist if $G \equiv g(\phi)\Xi(X)$ and

$$g(\phi) = \frac{1}{\xi_k} \left(\frac{k_\phi^2}{k} \right). \tag{53}$$

In general, when both the nonminimal and nonminimal kinetic couplings are not weak, the action in Eq. (50) depends on the kinetic term X in the Jordan frame. The kinetic term X can be eliminated from this action using Eq. (3) to convert X to $\tilde{X}$ as

$$\tilde{X} = \frac{X}{w(\phi) + k(\phi)\Xi(X)},$$

$$\Rightarrow \quad k(\phi)\Xi(X)\tilde{X} + w(\phi)\tilde{X} - X = 0. \qquad (54)$$

For the simplest case where $\Xi(X) \equiv X/\Lambda$ and Λ is constant with dimension of mass[4], the above equation yields

$$X = \frac{\tilde{X}w}{1 - k\tilde{X}/\Lambda}. \qquad (55)$$

Therefore,

$$\Omega = \frac{w}{1 - k\tilde{X}/\Lambda}. \qquad (56)$$

Inserting Eqs. (55) and (56) into Eq. (50), we get

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{2}\tilde{R} + \left[\frac{G}{w}\left(1 - k\frac{\tilde{X}}{\Lambda}\right)\right.\right.$$
$$\left.\left. + \frac{3}{2w^2}\left(w_\phi + \frac{1}{\Lambda}(k_\phi w - w_\phi k)\tilde{X}\right)^2\right]\tilde{X} - V_E(\phi)\right\}. \qquad (57)$$

In principle, the function G can be chosen such that the Lagrangian in the above action is a linear function of $\tilde{X}$. Consequently, we will obtain exactly the same inflationary attractor as discussed in the previous section. For such choices of G, the term $1 + kX/\Lambda$ will appear in the denominator of G in the Jordan frame, and therefore, the Lagrangian of scalar field L_2 does not take a usual form for k inflation. In the following discussion, we will see that if G is a polynomial function of X, the action can contain a nonlinear $\tilde{X}$ term, and consequently, the inflationary predictions have different attractors compared with Eqs. (36) and (37).

To perform further analysis, it is necessary to specify forms of $w(\phi)$, $k(\phi)$, and $G(\phi, X)$. For simplicity, one may write these functions in concrete forms or keep one of them generic and then write the other two functions in terms of it. Here, we consider the second possibility by writing $G(\phi, X) = g(\phi)\gamma(\phi, X)$, where

$$\gamma(\phi, X) \equiv f_0 + f_1(\phi)\frac{X}{\Lambda} + f_2(\phi)\frac{X^2}{\Lambda^2}\cdots, \qquad (58)$$

where all coefficients $f_0, f_1, f_2, \ldots$ are dimensionless and f_0 is constant. Similarly to Eq. (27), $g(\phi)$ is written in terms of $w(\phi)$ as

$$g(\phi) = \alpha^2\frac{w_\phi^2}{w}, \qquad (59)$$

where $\alpha \equiv 1/(2\sqrt{\xi})$. For the case where the nonminimal kinetic coupling disappears, the action in Eq. (57) will not depend on the form of $w(\phi)$ if we can write the action in terms of a new field variable ψ similar to that in Eq. (28). When both the nonminimal and nonminimal kinetic couplings appear in the action, it is also possible to write the action in Eq. (57) in the form independent of the form of $w(\phi)$ by choosing a suitable relation between $k(\phi)$ and $w(\phi)$. Let us define

$$k(\phi) \equiv \kappa\alpha^2\left(\frac{w_\phi}{w}\right)^2, \qquad (60)$$

where κ is constant, so that the action (57) can be written as

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{2}\tilde{R} + \left[\gamma(\phi, X)\left(1 - \kappa\frac{X_\psi}{\Lambda}\right)\right.\right.$$
$$\left.\left. + \frac{3}{2\alpha^2}\left(1 - 3\kappa\frac{X_\psi}{\Lambda} + 2\kappa\frac{w_{\phi\phi}w}{w_\phi^2}\frac{X_\psi}{\Lambda}\right)^2\right]X_\psi - V(\psi)\right\}, \qquad (61)$$

where $X_\psi \equiv -\tilde{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi/2$ and ψ is defined via

$$w(\phi) = \exp\left(\frac{\psi}{\alpha}\right). \qquad (62)$$

It can be seen that the action in Eq. (61) still depends on the form of $w(\phi)$ unless $w_{\phi\phi}w/w_\phi^2$ is constant. The constancy of the ratio $w_{\phi\phi}w/w_\phi^2$ is possible for various forms of $w(\phi)$, for example, $w \sim e^{\xi\phi}$, $w \sim \cosh(\xi\phi)$, etc., and also $w = (1 + \xi\phi^p)$ with a large coupling constant ξ. Moreover, this ratio is expected to be nearly constant for an arbitrary form of $w(\phi)$ when ϕ slowly varies with time. Hence, it is reasonable to suppose that the ratio $w_{\phi\phi}w/w_\phi^2$ is constant and can be quantified by

$$\frac{w_{\phi\phi}w}{w_\phi^2} = \lambda, \qquad (63)$$

where λ is a constant parameter. Inserting the above relation into Eq. (61), we get

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{2}\tilde{R} + \left[\gamma(\phi, X)\left(1 - \kappa\frac{X_\psi}{\Lambda}\right)\right.\right.$$
$$\left.\left. + \frac{3}{2\alpha^2}\left(1 - 3\kappa\frac{X_\psi}{\Lambda} + 2\kappa\lambda\frac{X_\psi}{\Lambda}\right)^2\right]X_\psi - V(\psi)\right\}. \qquad (64)$$

Firstly, we consider the case where $\gamma(\phi, X)$ is constant, but not equal to $-3/(2\alpha^2)$. For this case, the slow-roll evolution of ψ during inflation suggests that the $\tilde{X}^2$ term and $\tilde{X}^3$ term in the action (64) can be neglected. Consequently, the theoretical predictions from the inflationary model described by the action (64) obeys the attractor in Eqs. (36) and (37) under a suitable redefinition of the parameter α.

For the case of $\gamma(\phi, X) = -3/(2\alpha^2)$, the linear X_ψ term in the action (64) disappears and then under the slow roll approximation, the kinetic term of ψ is proportional to X_ψ^2. Hence, the action becomes

$$S_E \simeq \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2}\tilde{R} + \frac{X_\psi^2}{\Lambda_2} - V(\psi) \right\}, \qquad (65)$$

where

$$\Lambda_2 \equiv \left[\frac{3\kappa}{2\alpha^2}(4\lambda - 5) \right]^{-1} \Lambda. \qquad (66)$$

The observable quantities for this case will be discussed in the subsequent studies. Another interesting form of $\gamma(X)$ is the form where $\gamma(X)$ is a linear function of X as

$$\gamma(\phi, X) = \frac{3}{2\alpha^2} \left[(5 - 4\lambda)\frac{X}{\Lambda}\frac{k(\phi)}{w(\phi)} - 1 \right]. \qquad (67)$$

The above equation can be written in terms of X_ψ as

$$\gamma = \frac{3}{2\alpha^2} \left[(5 - 4\lambda)\kappa \frac{X_\psi}{\Lambda - \kappa X_\psi} - 1 \right]. \qquad (68)$$

Inserting this relation into Eq. (64), we get

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2}\tilde{R} + \frac{X_\psi^3}{\Lambda_3^2} - V(\psi) \right\}, \qquad (69)$$

where

$$\Lambda_3 \equiv \left[\frac{3\kappa^2}{2\alpha^2}(2\lambda - 3)^2 \right]^{-1} \Lambda. \qquad (70)$$

The observable predictions from the inflationary models described by the actions in Eqs. (65) and (69) have different attractors from those given in Eqs. (36) and (37). We will study this attractor in the following considerations. In general, it is also possible to set $\gamma(\phi, X) \propto X^m$, where $m \geq 2$. Nevertheless, it leads to the term proportional to X_ϕ^{m+2}, which is negligible in the slow roll limit. To compute the observable quantities, we use the slow-roll approximation in which the evolution equations derived from the actions (65) and (69) can be written as

$$H^2 \simeq \frac{1}{3}V(\psi), \quad \text{and} \quad (\psi')^{2q-1} = -A_q \Lambda_q^{q-1} \frac{1}{V^q}\frac{dV}{d\psi}, \qquad (71)$$

where H is a Hubble parameter, a prime denotes the derivative with respect to $\ln a$, a is a cosmic scale factor, $A_q \equiv 6^{q-1}/q$, and $q = 2, 3$ for X_ϕ^2 and X_ϕ^3 models, respectively. Since the form of the equation of motion for the scalar field ψ is different from that for the usual canonical normalized field, we have to compute the slow

roll parameter ϵ and number of e-foldings N from their definitions,

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \text{and} \quad N \equiv \int H dt. \qquad (72)$$

Using Eq. (71), the relations in Eq. (72) can be written as

$$\epsilon = \frac{A_q^{1/(2q-1)}}{2} \Lambda_q^{(q-1)/(2q-1)} \frac{1}{V^{(3q-1)/(2q-1)}} \left(\frac{dV}{d\psi} \right)^{2q/(2q-1)}, \qquad (73)$$

and

$$N = A_q^{1/(1-2q)} \Lambda_q^{(1-q)/(2q-1)} \int_{\psi_{\text{end}}}^{\psi_N} d\psi \frac{V^{q/(2q-1)}}{(dV/d\psi)^{1/(2q-1)}}. \qquad (74)$$

For the inflaton with noncanonical kinetic terms, the spectral index and the tensor-to-scalar ratio are given by [37,38]

$$n_s = 1 - 2\epsilon - \psi' \frac{d}{d\psi}(\ln \epsilon) - \psi' \frac{d}{d\psi}(\ln c_s^2), \qquad (75)$$

$$r = 16c_s\epsilon, \qquad (76)$$

where $c_s \equiv \sqrt{(\partial P/\partial X_\psi)/(\partial \rho/\partial X_\psi)} = 1/\sqrt{2q-1}$ is the propagation speed of the scalar perturbations, $P = X_\psi^q/\Lambda^{q-1} - V(\psi)$ is the pressure of ψ, and $\rho \equiv (2q-1)X_\psi^q/\Lambda^{q-1} + V(\psi)$ is the energy density of ψ.

A. Hyperbolic tangent potential (T model)

To obtain the potential for the T model, we set $V_E(\phi) = V_0\Lambda_q[(w-1)/(w+1)]^n$ to obtain

$$V(\psi) = V_0\Lambda_q \left[\tanh\left(\frac{\psi}{2\alpha} \right) \right]^n, \qquad (77)$$

where V_0 is a dimensionless constant which is supposed to be order of unity. In the following analytical analysis, we will restrict ourselves to the case $n = 2$ in which the analytical expressions for n_s and r in terms of the number of e-foldings can be straightforwardly obtained. This restriction will be relaxed when a numerical analysis is performed in Sec. IV B. Substituting this potential into Eqs. (73) and (74) and then setting $n = 2$, we get

$$\epsilon = \frac{1}{2}\left(\frac{A_q}{V_0^{(q-1)}\alpha^{2q}} \right)^{1/(2q-1)}$$
$$\times \left[\sinh^2\left(\frac{\psi}{2\alpha} \right) \cosh^{2/(2q-1)}\left(\frac{\psi}{2\alpha} \right) \right]^{-1}, \qquad (78)$$

$$N = (2q-1)\left[\frac{V_0^{q-1}\alpha^{2q}}{A_q}\cosh^2\left(\frac{\psi}{2\alpha} \right) \right]^{1/(2q-1)} \Big|_{\psi_{\text{end}}}^{\psi_N}. \qquad (79)$$

In this case, the values of the field ψ at the end of inflation cannot be computed analytically from the relation $\epsilon = 1$. Hence, we define

$$N_e \equiv (2q - 1)\left[\frac{V_0^{q-1}\alpha^{2q}}{A_q}\cosh^2\left(\frac{\psi_{\text{end}}}{2\alpha}\right)\right]^{1/(2q-1)}, \quad (80)$$

so that Eq. (79) can be written as

$$N + N_e = A\cosh^{2/(2q-1)}\left(\frac{\psi_N}{2\alpha}\right), \quad (81)$$

where

$$A \equiv (2q - 1)\left[\frac{V_0^{q-1}\alpha^{2q}}{A_q}\right]^{1/(2q-1)}. \quad (82)$$

Equation (78) can be written in terms of the number of e-foldings using Eq. (81) as

$$\epsilon_N = \frac{(2q-1)A^{(2q-1)}}{2(N+N_e)((N+N_e)^{2q-1} - A^{2q-1})}. \quad (83)$$

Using Eqs. (78), (81), (75) and (76) can be written as

$$n_s = 1 - \frac{2q}{N+N_e} - \frac{(4q-2)A^{(2q-1)}}{(N+N_e)[(N+N_e)^{2q-1} - A^{2q-1}]}, \quad (84)$$

$$r = 16c_s\frac{(2q-1)A^{(2q-1)}}{2(N+N_e)[(N+N_e)^{2q-1} - A^{2q-1}]}. \quad (85)$$

For $\alpha \gg 1$ or equivalently, in the weak coupling limit $\xi \ll 1$, the condition $\epsilon = 1$ at the end of inflation yields $\psi_{\text{end}} \simeq \alpha\sqrt{(4q-2)/A}$, and therefore,

$$N_e \simeq \left(1 + \frac{1}{2A}\right)A \simeq A + \frac{1}{2}. \quad (86)$$

Substituting the above relation into Eqs. (85) and (84), we get

$$n_s \simeq 1 - \frac{8q-4}{(4q-2)N + (2q-1)},$$

$$r \simeq 16\frac{\sqrt{2q-1}}{(4q-2)N + (2q-1)}. \quad (87)$$

Interestingly, the theoretical predictions in this limit do not depend on κ which controls the relative strength between the nonminimal and nonminimal kinetic couplings. In contrast, for $\alpha \ll 1$ or equivalently, the strong coupling $\xi \gg 1$ limit, $\epsilon = 1$ gives $e^{2q\psi_{\text{end}}/((2q-1)\alpha)} \simeq 2^{4q/(2q-1)}(2q-1)/(2A)$, so that

$$N_e \simeq \left(\frac{5}{2}\right)^{1/2q}A^{(2q-1)/2q} \propto \alpha, \quad (88)$$

and consequently,

$$n_s \simeq 1 - \frac{2q}{N} + \mathcal{O}\left(\frac{\alpha}{N^2}\right),$$

$$r \simeq \frac{8}{3}(2q-1)^{2q-1/2}\frac{V_0^{(q-1)}\alpha^{2q}}{N^{2q}} + \mathcal{O}\left(\frac{\alpha^{2q+1}}{N^{2q+1}}\right). \quad (89)$$

Again, the inflationary predictions do not depend on κ. Note that r is independent of κ because the coefficient of the potential defined in Eq. (77) is in the form of $V_0\Lambda_q$ with a constant V_0. Instead of setting V_0 to be constant, if we set $\tilde{V}_0 \equiv V_0\Lambda_q$ to be a constant independent of κ and α, the expression for r will depend on κ when V_0 is replaced by $\tilde{V}_0$.

It follows from Eqs. (87) and (89) that at the $\alpha \gg 1$ and $\alpha \ll 1$ limits, the expressions for observable quantities, n_s and r, converge to the forms that are similar to Eqs. (36) and (37) up to some constant factors. The existence of these convergences does not depend on the form of $w(\phi)$ but of course depends on the relation among $w(\phi)$, $k(\phi)$, and $G(\phi, X)$. Moreover, Eqs. (87) and (89) are computed in the large and small α limits, in which various potentials take similar forms, especially the potentials for the T model and E model described in the previous sections. Hence, the convergence of the observable quantities to Eqs. (87) or (89) at a asymptotic value of α can imply the inflationary attractor.

In more general cases, where n is not restricted to be two, the number of e-foldings will depend on hypergeometric functions so that it is not possible to write ϵ in terms of the number of e-foldings. In this situation, it is difficult to write analytic expressions for n_s and r in terms of the number of e-foldings.

B. Theoretical predictions for large and small α limits

As mentioned previously, the potentials of the T model and E model have the same asymptotic behavior in the large and small α limits. Since the inflationary attractors are characterized by these asymptotic behaviors, we investigate in this section inflationary predictions for the models described by Eq. (71) in the large and small α limits instead of repeating the calculations in the previous section for the E model.

In the limit $\alpha \gg 1$, the potential in Eq. (77) becomes

$$V(\psi) \simeq V_0\Lambda_q\left(\frac{\psi}{2\alpha}\right)^n. \quad (90)$$

Replacing the potential in Eq. (77) by this approximated potential, it can be shown that ϵ and N are given by

$$\epsilon = \frac{1}{2}\left(\frac{A_q n^{2q} \psi^{n-2q-nq}}{V_0^{(q-1)}(2\alpha)^{n-nq}}\right)^{1/(2q-1)}, \qquad (91)$$

$$N = \frac{2q-1}{2q+nq-n}\left[\frac{V_0^{q-1}(2\alpha)^{n-nq}}{nA_q\psi_N^{n-2q-nq}}\right]^{1/(2q-1)} - \frac{n}{2}\frac{2q-1}{2q+nq-n}. \qquad (92)$$

Combining the above two equations, we can write ϵ in terms of the number of e-foldings as

$$\epsilon_N = \frac{n}{2}\frac{2q-1}{2q+nq-n}\left[N + \frac{n}{2}\frac{2q-1}{2q+nq-n}\right]^{-1}. \qquad (93)$$

Inserting the above results into Eqs. (75) and (76), we get

$$\begin{aligned} n_s &= 1 - 2\frac{2q+3nq-2n}{2(2q+nq-n)N + n(2q-1)}, \\ r &= \frac{16}{\sqrt{2q-1}}\frac{2qn-n}{2(2q+nq-n)N + n(2q-1)}. \end{aligned} \qquad (94)$$

In the limit $\alpha \ll 1$, the potential in Eq. (77) becomes

$$V(\psi) \simeq V_0\Lambda_q(1 - 2ne^{-\psi/\alpha}), \qquad (95)$$

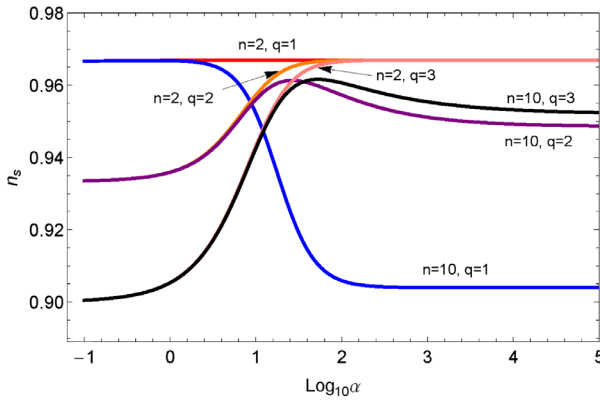and therefore, we have

$$\epsilon = \frac{1}{2}B\left[\frac{e^{-2q\psi/\alpha}}{(1-2ne^{-\psi/\alpha})^{3q-1}}\right]^{1/(2q-1)} \simeq \frac{1}{2}Be^{-2q\psi/(2q-1)\alpha}, \qquad (96)$$

$$\begin{aligned} N &= \frac{2n}{B}\left((2q-1) - \frac{nq}{1-q}e^{-\psi/\alpha}\right)e^{\psi/(2q-1)\alpha}\bigg|_{\psi_{\text{end}}}^{\psi_N} \\ &\simeq \frac{2n}{B}(2q-1)e^{\psi_N/(2q-1)\alpha} - (2q-1)\alpha\left(\frac{V_0^{q-1}}{2A_q}\right)^{1/2q}, \end{aligned} \qquad (97)$$

where

$$B \equiv \left[\frac{A_q(2n)^{2q}}{V_0^{q-1}\alpha^{2q}}\right]^{1/(2q-1)}. \qquad (98)$$

In terms of the number of e-foldings, ϵ can be written as

$$\epsilon_N \simeq \frac{1}{2}(2q-1)^{2q}\alpha^{2q}\frac{V_0^{q-1}}{A_q}\left[N + (2q-1)\alpha\left(\frac{V_0^{q-1}}{2A_q}\right)^{1/2q}\right]^{-2q}. \qquad (99)$$

Hence, for this case, Eqs. (75) and (76) yield

$$n_s \simeq 1 - \frac{2q}{N}, \qquad r \simeq (2q-1)^{2q}\frac{8}{\sqrt{2q-1}}\frac{V_0^{q-1}}{A_q}\frac{\alpha^{2q}}{N^{2q}}. \qquad (100)$$

Equations (94) and (100) are the generalization of the attractor in Eqs. (87) and (89). These equations will become the attractors in Eqs. (36) and (37) if $q=1$ for a suitable redefinition of α. We will see in the numerical investigation that due to the factor $2q$ in the expression for n_s in Eq. (100), the value of n_s in the small α limit is less than the observational bound and values from universal attractor at large number of e-foldings, e.g., at $N = 60$. This puts a tight constraint on the inflationary attractor in the strong coupling $\xi \gg 1$ limit for the models where $q > 1$.

In order to compute the observable quantities from the inflationary models whose dynamics are governed by evolution equations in Eq. (71) and potential is given by Eq. (77), we integrate Eq. (71) and compute the observable quantities numerically for various values of q, n, and α.

In Fig. 1, we plot the predictions of the model in the $n_s - \log(\alpha)$ and $r - \log(\alpha)$ plane for various values of the parameters n and q. From Fig. 1, we discover that our results for $\alpha < O(10)$ with any values of n and q show an attractor behavior, but with only $q = 1$ display an universal attractor given in Eq. (37). From our definition of $\alpha = 1/(2\sqrt{\xi})$, we see that the attractor can be achieved when $\xi > O(10^{-3})$ which is in agreement with Ref. [14]. In
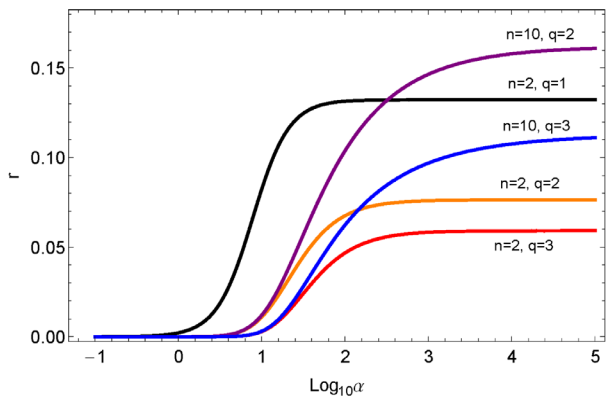

FIG. 1. The plots show how n_s and r evolve with the changing of α. In all plots, n_s and r are evaluated at $N = 60$.
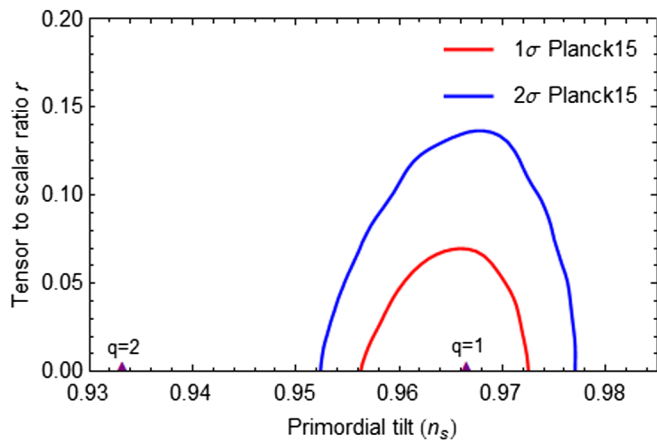
FIG. 2. In the case of small values of α, we compare the theoretical predictions in the $(n_s - r)$ plane for small α with the Planck 2015 results for TT, TE, EE, +lowP, and assuming ΛCDM + r [22].

addition, from Fig. 1, in the case of a large α the attractor can be achieved when $\xi < O(10^{-4})$.

C. Contact with recent Planck data

In this section, we compare our results in Eqs. (94) and (100) with Planck 2015 data. Note that the potentials of the

T model and E model have the same asymptotic behavior in the large and small α limits. In the small α limit, we compared our results with the Planck 2015 measurement by placing the predictions in the $(n_s - r)$ plane with different values of q while keeping $N = 60$, illustrated in Fig. 2. We notice that with $q = 1$ our results lie within 2σ C.L. of Planck 2015 contours. However, when $q > 1$, the results are far outside 2σ C.L. of Planck 2015 contours. In addition, from the right panel of Fig. 1, for various values of n and q at a strong coupling limit, our model provides $r < 0.064$ in precise agreement with the improved value recently reported in [33].

However, in the large α limits, with $n = 2$, our results lie within 1σ C.L. of Planck 2015 contours for $q = 1$ and 2, while within 2σ C.L. of Planck 2015 contours for $q = 3$, illustrated in the upper-left panel of Fig. 3. Moreover, our results lie far outside 2σ C.L. of Planck 2015 when $q = 1$, $n = 4$, but lie within 1σ C.L. of Planck 2015 when $q = 3$ with $n = 4$, displayed in the upper-right panel of Fig. 3.

In the lower-panel of Fig. 3 with $n = 5$, we observe that when $q = 2$ the results lie outside 1σ C.L. of Planck 2015 contours, while with $q = 3$, our results lie inside 1σ C.L. of Planck 2015. In addition, we can deduce that when $q > 3$, in this case the results lie well within 2σ C.L. of Planck 2015 contours. Interestingly, we conclude that the greater
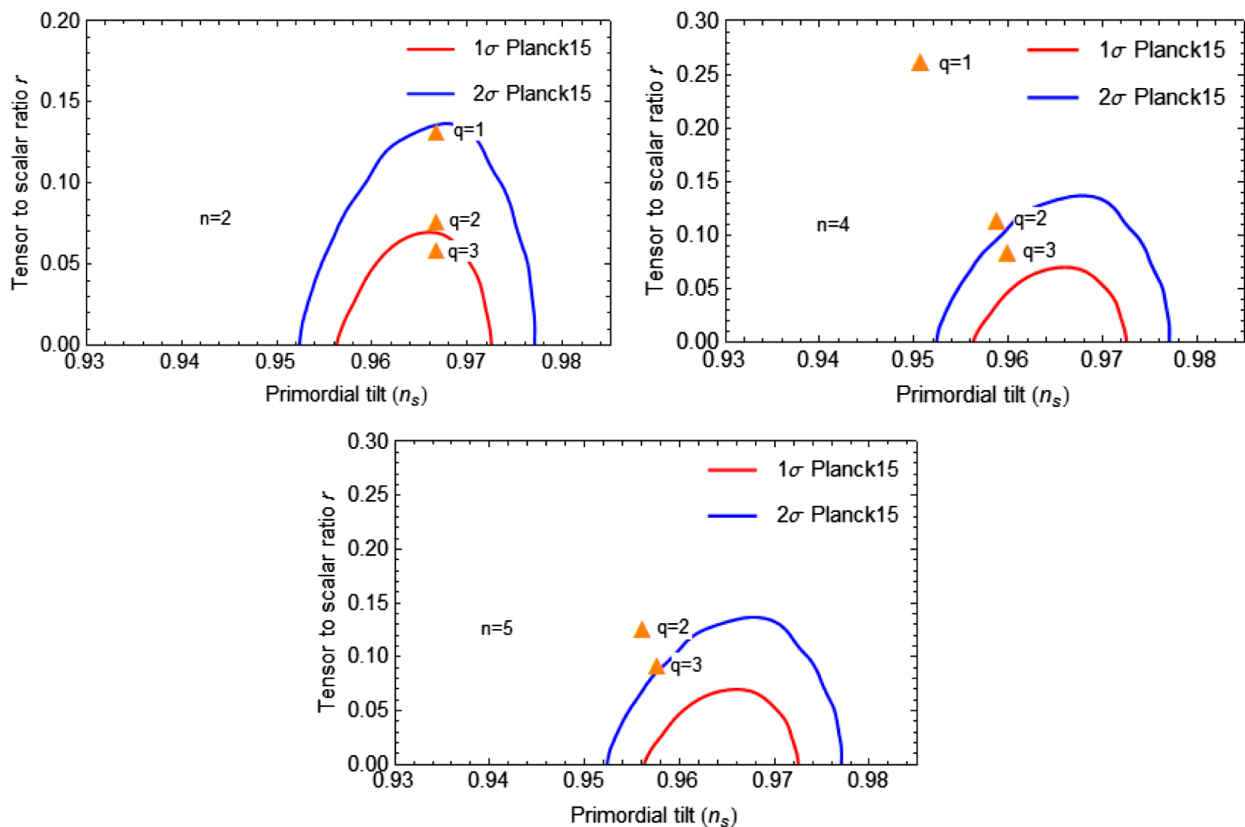

FIG. 3. In the case of large values of α, we compare the theoretical predictions in the $(n_s - r)$ plane for large α with Planck015 results for TT, TE, EE, +lowP, and assuming ΛCDM + r [22].

the value that q takes, the better the results that lie well within 2σ C.L. of Planck 2015 contours for any n.

V. CONCLUSION

Among viable inflationary models, the α attractors, in light of the presently existing CMB data, have received particular attention. In the present work, we investigated the inflationary attractors in models of inflation inspired by the general conformal transformation of general scalar-tensor theories to the Einstein frame. Since the coefficient of the conformal transformation in our study depends on both the scalar field and its kinetic term, the nonminimal coupling in the presence of both the field and its kinetic term can appear in the action in the Jordan frame. This action presents a subset of the class I of the DHOST theories, and therefore, the theories associated to this action are free from the Ostrogradski instability.

In our analysis, we concentrated on the inflationary models in the Einstein frame. Nevertheless, according to a brief consideration in Sec. II, the number of e-foldings in the Einstein frame is approximately equal to that in the Jordan frame in the strong $\xi \gg 1$ and weak $\xi \ll 1$ coupling limits. Hence, the observational quantities in terms of the number of e-foldings are approximately frame invariant, and consequently, the inflationary attractors in the Einstein frame should imply the existence of the same attractors in the Jordan frame in the strong and weak coupling limits.

We considered the two-case scenarios. We first concentrated on the multiplicative form of the generalized conformal factor, i.e., $\Omega = \Omega(X, \phi) = w(\phi)W(X)$. The action in the Einstein frame does not depend on $W(X)$ if the coefficient $G(\phi, X)$ of the kinetic term of ϕ in the Jordan frame takes the form $G(\phi, X) = g(\phi)W(X)$. Based on this setting, we have proposed the models which are designed specifically to be the T-model and E-model actions. The main finding from the multiplicative form model is that the usual α attractors can be achieved from models constructed by the generalized conformal transformation. From our definition of α in terms of the coupling constant ξ, we have $\alpha \to \infty$ in the weak coupling limit and $\alpha \to 1$ in the strong coupling limit. In the strong coupling limit, the predictions converge to the universal attractor regime in Eq. (37) [17,34,35], which corresponds to the part of the $n_s - r$ plane favored by the Planck data [36]. For the small coupling limit, the predictions converge to Eq. (36) if n is replaced by $2n$.

In addition, we have chosen the additive form of the generalized conformal factor, i.e., $\Omega = \Omega(X, \phi) = w(\phi) + k(\phi)X$. We also compute the cosmological observables for the T-model potential. We have found that in our choice of the relation among the functions of the coefficients, the inflationary predictions do not depend on both $w(\phi)$ and the relative strength between the nonminimal

kinetic and usual nonminimal couplings. However, in some choices of the relation among the functions of the coefficients, the kinetic term of the redefined field that governs the dynamics of inflation takes a nonlinear form, e.g., X_ψ^2 and X_ψ^3. In these situations, the inflationary predictions converge to new attractors given by Eqs. (94) and (100) in the weak and strong coupling limits, respectively. For the additive form of the conformal factor, the parameter α is defined such that the weak and strong coupling limits are equivalent to the large and small α, respectively. From our numerical calculation, we discovered that the attractor can be achieved for the strong coupling limit and the weak one when $\xi > O(10^{-3})$ and $\xi < O(10^{-4})$, respectively.

We confronted the obtained results of the cosmological observables with the recent Planck 2015 data. More concretely, in the small α limit, we compared our results given in Eq. (100) with the Planck 2015 measurement by placing the predictions in the $(n_s - r)$ plane with different values of q while keeping $N = 60$, as illustrated in Fig. 2. We notice that with $q = 1$ our results lie within 1σ C.L. of Planck 2015 contours. However, when $q > 1$, the results are not satisfied with the observational bound of the Planck 2015 contours. However, in the large α limits given in Eq. (94), with $n = 2$, our results lie within 1σ C.L. of Planck 2015 contours for $q = 1$ and 2, while within 2σ C.L. of Planck 2015 contours for $q = 3$, illustrated in the upper-left panel of Fig. 3. Moreover, our results lie far outside 2σ C.L. of Planck 2015 contours when $q = 1$, $n = 4$, but lie within 1σ C.L. of Planck 2015 contours when $q = 3$ with $n = 4$, as displayed in the upper-right panel of Fig. 3. Notice that the greater values that q takes, the better results lie well within 2σ C.L. of Planck 2015 contours for $n = 4$. In the lower-panel of Fig. 3 with $n = 5$, we observe that when $q = 2$, the results lie outside 1σ C.L. of Planck 2015 contours, while $q = 3$ our results lie inside 1σ C.L. of Planck 2015 contours. In addition, we can deduce that when $q > 3$, in this case, the results lie well within 2σ C.L. of Planck 2015 contours.

Notice that we started in Sec. II by considering a generic form of coefficients for conformal transformation and restricted our subsequent discussions focusing on two-case scenarios taking the multiplicative and additive forms of the conformal coefficient for simplicity. More precisely, the multiplicative form model is chosen in such a way that the standard α attractors can be recovered in the models where the conformal coefficient depends on the kinetic term of scalar field. Moreover, the new inflationary attractors can be achieved by choosing the additive form model which can be viewed as the extension of the multiplicative form model. However, the additive form of the conformal coefficient is restricted such that the exact relation between the kinetic terms of the scalar field in the Jordan and Einstein frames can be obtained. In the simplest case, this relation is presented in Eq. (55). Hence, there should be inflationary attractors other than those present in this work

if this restriction is relaxed. We will leave this interesting topic for our future investigation.

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