PHYSICAL REVIEW D **88,** 124006 (2013)

Generalized quasidilaton theory

Antonio De Felice*

ThEP's CRL, NEP, The Institute for Fundamental Study, Naresuan University, Phitsanulok 65000, Thailand and Thailand Center of Excellence in Physics, Ministry of Education, Bangkok 10400, Thailand

A. Emir Gümrükçüoğlu† and Shinji Mukohyama‡

Kavli Institute for the Physics and Mathematics of the Universe (WPI), Todai Institutes for Advanced Study, University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8583, Japan

(Received 23 September 2013; published 2 December 2013)

Recently the first example of a unitary theory of Lorentz-invariant massive gravity allowing for stable self-accelerating de Sitter solutions was found, extending the quasidilaton theory. In this paper we further generalize this new action for the quasidilaton field by introducing general Lagrangian terms which are consistent with the quasidilaton symmetry while leading to second order equations of motion. We find that the structure of the theory, compared to the simplest stable example, does not change on introducing these new terms.

DOI: 10.1103/PhysRevD.88.124006 PACS numbers: 04.50.Kd, 03.50.−z, 14.70.Kv

I. INTRODUCTION

The search for a consistent gravitational action which would lead to a massive graviton has been pushed forward recently in several directions. Recently, a nonlinear completion of the Fierz-Pauli model [1], which is free of the Boulware-Deser (BD) ghost [2], was introduced by de Rham-Gabadadze-Tolley (dRGT) [3,4] and revitalized the research on this topic. However, shortly thereafter, the homogeneous and isotropic solutions of this theory were found to suffer from ghost instabilities. Specifically, the self-accelerating branch solutions [5] suffer from a nonlinear ghost [6], while the remaining branch solutions have a linear ghost instability [7] of the type found in [8].

There have been several attempts to improve the stability of cosmological solutions. One possibility consists of introducing inhomogeneous and/or anisotropic background configurations either of the physical metric or of the Stückelberg fields [9–18]. In this approach, the deformations may stay in a hidden sector, giving a standard Friedmann-Lemaître-Robertson-Walker (FLRW) form to the physical metric, or through the recovery of the FLRW universe only in the observable patch. Many such deformed backgrounds still have stability issues [13,14,19,20], although in the anisotropic Friedmann solution of Ref. [17], pertubations may be stable [18].

A second approach consists of introducing new degrees of freedom to the dRGT action, in addition to the already existing gravitational ones.[1] One possibility is the BD ghost-free bimetric theory, where a second dynamical metric is introduced and the interaction between the two metrics is tuned such that the BD ghost is removed by construction [22–24]. The properties of the cosmology has been studied in [25–29]. As in the dRGT theory, on self-accelerating FLRW solutions three degrees of freedom have vanishing quadratic kinetic terms and thus render those cosmological solutions unstable at the nonlinear level. On the other hand, in the so-called normal branch without self-acceleration, all degrees of freedom are dynamical. Reference [28] found a gradient instability in this branch of solutions, although there may be cases where stable cosmologies are possible [29,30].

Another example of this approach is to introduce a single scalar field, interacting with the graviton mass term. For instance, the parameters of the dRGT theory can be promoted to vary with a dynamical scalar field [10,31]. The freedom in how these functions vary may allow for different types of cosmologies [32–34], and the instabilities of the usual massive gravity can potentially be avoided [35]. However, the stability condition forbids self-accelerating de Sitter solutions in this class of theories. This is because, whenever the extra scalar field stops rolling, the system reduces back to the original dRGT theory and thus suffers from the above-mentioned instabilities.

Recently the first example of a unitary theory of Lorentz-invariant massive gravity with stable self-accelerating de Sitter solutions was presented in [36]. The theory is an extension of the quasidilaton theory originally introduced in [37]: in addition to the pure gravitational degrees of freedom, the Lagrangian is endowed with an extra scalar field, which has a nontrivial coupling with the massive graviton, and is supposed to cure some of the unexpected pathological behavior of the original dRGT theory on homogeneous and isotropic manifolds. The action of the

*antoniod@nu.ac.th
†emir.gumrukcuoglu@ipmu.jp
‡shinji.mukohyama@ipmu.jp

[1] We note that all the extensions discussed here reduce to Fierz-Pauli theory [1] in the linear level. For alternative theories which are not connected to the Fierz-Pauli Lagrangian, see [21].

system is invariant under the so-called quasidilaton global symmetry. However, it was shown in [35,38] that self-accelerating de Sitter solutions in the original quasidilaton theory are always plagued with a ghost degree of freedom. The theory was thus extended in [36] by allowing for a new coupling (consistent with the quasidilaton symmetry) between the quasidilaton scalar field and the Stückelberg fields. It is this coupling that prevents ill-defined behavior from happening and renders the self-accelerating solutions stable. Moreover, the inclusion of the new coupling does not spoil the existence of the primary constraint which removes the BD degree [39].

The goal of this paper is to further generalize this new action for the quasidilaton field by introducing general Lagrangian terms which are consistent with the quasidilaton symmetry while leading to second order equations of motion. Some of these general terms are known in the literature as a subset of the general Horndeski action [40–42]. We find that the structure of the theory, compared to the simplest stable example provided in [36], does not change on introducing these new Horndeski terms. In particular, the no-ghost condition for the degree which is cured by the inclusion of the new coupling essentially keeps its original form upon introduction of the Horndeski terms. The main modification appears in the expressions for the speed of propagation and the no-ghost conditions of the remaining degrees, which can all be satisfied within a non-null set of parameter space.

The paper is organized as follows. In Sec. II, we present the model we consider and in Sec. III we summarize the evolution equations of the background. In Sec. IV, we introduce perturbations to the metric and quasidilaton field and study their stability, along with several examples. We conclude with Sec. V, where we summarize our results.

II. THE MODEL

Let us consider the quasidilaton action, which can be written as follows [36,37]:

$$S = \int d^4x\left[\sqrt{-g}\mathcal{L} + M_{\rm Pl}^2 m_g^2 \xi e^{4\sigma/M_{\rm Pl}}\sqrt{-\det\tilde{f}}\right], \quad (1)$$

where we have introduced the following expression:

$$\mathcal{L} = \mathcal{L}_{\rm G} + \mathcal{L}_{\rm H}. \quad (2)$$

Here, $\mathcal{L}_G$ represents the quasidilaton dRGT Lagrangian, that is

$$\mathcal{L}_{\rm G} = \frac{M_{\rm Pl}^2}{2}[R - 2\Lambda + 2m_g^2(\mathcal{L}_2 + \alpha_3\mathcal{L}_3 + \alpha_4\mathcal{L}_4)], \quad (3)$$

where

$$\mathcal{L}_2 \equiv \frac{1}{2}([\mathcal{K}]^2 - [\mathcal{K}^2]), \quad (4)$$

$$\mathcal{L}_3 \equiv \frac{1}{6}([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]), \quad (5)$$

$$\mathcal{L}_4 \equiv \frac{1}{24}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]). \quad (6)$$

Here, by square brackets, we indicate a trace operation, whereas $\mathcal{K}$ is the following tensor

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - e^{\sigma/M_{\rm Pl}}\left(\sqrt{g^{-1}\tilde{f}}\right)^\mu{}_\nu, \quad (7)$$

and

$$\tilde{f}_{\mu\nu} \equiv \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b - \frac{\alpha_\sigma}{M_{\rm Pl}^2 m_g^2}e^{-2\sigma/M_{\rm Pl}}\partial_\mu\sigma\partial_\nu\sigma. \quad (8)$$

This form of the quasidilaton dRGT Lagrangian is consistent with the *quasidilaton symmetry*, which is defined as follows [37]:

$$\sigma \to \sigma + \sigma_0, \qquad \phi^a \to e^{-\sigma_0/M_{\rm Pl}}\phi^a. \quad (9)$$

Furthermore the Lagrangian is also invariant under a Poincaré transformation in the space of the Stückelberg fields, as follows:

$$\phi^a \to \phi^a + c^a, \qquad \phi^a \to \Lambda^a{}_b\phi^b. \quad (10)$$

The second term in Eq. (8) gives a nontrivial interaction term between the quasidilaton field, and the metric tensor, which is capable, as we will see later on, to make the scalar perturbation sector stable. In fact, we will show that the term proportional to α_σ is of crucial importance for making the quasidilaton action free of ghosts.

Finally, we consider a general shift-symmetric Horndeski Lagrangian, $\mathcal{L}_{\rm H}$, in the form

$$\mathcal{L}_{\rm H} = P(\mathcal{X}) - G_3(\mathcal{X})\Box\sigma + \mathfrak{L}_4 + \mathfrak{L}_5, \quad (11)$$

where

$$\mathcal{X} \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma, \quad (12)$$

$$\mathfrak{L}_4 \equiv G_{4,\mathcal{X}}(\Box\sigma)^2 - G_{4,\mathcal{X}}(\nabla_\mu\nabla_\nu\sigma)(\nabla^\mu\nabla^\nu\sigma) + G_4(\mathcal{X})R, \quad (13)$$

$$\mathfrak{L}_5 \equiv -\frac{G_{5,\mathcal{X}}}{6}(\Box\sigma)^3 + \frac{G_{5,\mathcal{X}}}{2}(\nabla_\mu\nabla_\nu\sigma)(\nabla^\mu\nabla^\nu\sigma)\Box\sigma - \frac{G_{5,\mathcal{X}}}{3}(\nabla^\mu\nabla_\nu\sigma)(\nabla^\nu\nabla_\alpha\sigma)(\nabla^\alpha\nabla_\mu\sigma) + G_5(\mathcal{X})G_{\mu\nu}(\nabla^\mu\nabla^\nu\sigma), \quad (14)$$

where the free functions P, G_3, G_4, and G_5 are functions of $\mathcal{X}$ only, and the subscript ", $\mathcal{X}$" denotes differentiation with respect to $\mathcal{X}$. This Lagrangian, which is invariant under a quasidilaton symmetry transformation, has been constructed in order to lead to, at most, second order equations of motion.

III. THE BACKGROUND

On the background—where $\sigma = \bar{\sigma}(t)$ and $\phi^a = x^a$—the extended fiducial metric reduces to

$$\tilde{f}_{00} = -(\dot{\phi}^0)^2 - \frac{\alpha_\sigma}{M_{\rm Pl}^2 m_g^2} e^{-2\bar{\sigma}/M_{\rm Pl}} \dot{\bar{\sigma}}^2, \qquad \tilde{f}_{ij} = \delta_{ij}. \quad (15)$$

Then we can define the positive background variable n, such that

$$(\dot{\phi}^0)^2 \equiv n(t)^2 - \frac{\alpha_\sigma}{M_{\rm Pl}^2 m_g^2} e^{-2\bar{\sigma}/M_{\rm Pl}} \dot{\bar{\sigma}}^2. \quad (16)$$

In other words, we have that, on the background

$$\eta_{ab}\partial_\mu \phi^a \partial_\nu \phi^b = \mathrm{diag}\left(-n^2 + \frac{\alpha_\sigma}{M_{\rm Pl}^2 m_g^2} e^{-2\bar{\sigma}/M_{\rm Pl}} \dot{\bar{\sigma}}^2, 1, 1, 1\right). \quad (17)$$

Having introduced the variable n, the background for the fiducial metric $\tilde{f}_{\mu\nu}$ is expressed in the following form:

$$\tilde{f}_{\mu\nu} = \mathrm{diag}(-n(t)^2, 1, 1, 1). \quad (18)$$

For the background physical metric, we adopt the flat FLRW ansatz

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \delta_{ij} dx^i dx^j. \quad (19)$$

We find it convenient to define two background variables X and r as follows:

$$\bar{\sigma} = M_{\rm Pl} \ln(aX), \qquad r \equiv \frac{n}{N} a. \quad (20)$$

We consider here a to be dimensionless, so as α_σ, n, N, X, ω, and r. Also, $[\phi^0] = M^{-1}$, $[H] = M$, and $[\sigma] = M$. In this case, for convenience, we can replace the background variables $(\bar{\sigma}, n)$ by means of (X, r).

In the following we will consider self-accelerating de Sitter solutions for this model. It is then possible to search for solutions which admit the following quantities to be constants:

$$H \equiv \frac{\dot{a}}{Na}, \quad (21)$$

$$X \equiv \frac{e^{\sigma/M_{\rm Pl}}}{a}, \quad (22)$$

$$r \equiv \frac{n}{N} a. \quad (23)$$

Then we have, on the flat FLRW background (19), that

$$\mathcal{X} = \frac{1}{2}\frac{\dot{\sigma}^2}{N^2} = \frac{1}{2} M_{\rm Pl}^2 H^2 = \text{constant}. \quad (24)$$

The Friedmann equation reads

$$\Lambda = 3H^2 + \frac{P}{M_{\rm Pl}^2} - H^2 P_{,\mathcal{X}} - 3G_{3,\mathcal{X}} H^4 M_{\rm Pl} + [3(X-1)(X-2) - (X-1)^2(X-4)\alpha_3 - (X-1)^3\alpha_4] m_g^2 - 12 G_{4,\mathcal{X}} H^4 - 6H^6 M^2 G_{4,\mathcal{X}\mathcal{X}} + 6G_4 \frac{H^2}{M_{\rm Pl}^2} - H^8 M^3 G_{5,\mathcal{X}\mathcal{X}} - 5G_{5,\mathcal{X}} H^6 M_{\rm Pl}. \quad (25)$$

Looking for the condition of a positive background effective gravitational constant we can imspose the relation

$$\frac{\partial \Lambda}{\partial (H^2)} > 0, \quad (26)$$

which leads to the condition

$$P_{,\mathcal{X}} < 6 - 6H^2(5G_{5,\mathcal{X}} H^2 + 2G_{3,\mathcal{X}}) M_{\rm Pl} + \frac{12 G_4}{M_{\rm Pl}^2} - H^2(48 H^2 G_{4,\mathcal{X}\mathcal{X}} + P_{,\mathcal{X}\mathcal{X}}) M_{\rm Pl}^2 - 42 H^2 G_{4,\mathcal{X}} - M_{\rm Pl}^5 H^8 G_{5,\mathcal{X}\mathcal{X}\mathcal{X}} - 6 M_{\rm Pl}^4 H^6 G_{4,\mathcal{X}\mathcal{X}\mathcal{X}} - H^4(13 H^2 G_{5,\mathcal{X}\mathcal{X}} + 3 G_{3,\mathcal{X}\mathcal{X}}) M_{\rm Pl}^3. \quad (27)$$

In the case $P_{,\mathcal{X}} = \omega$, and in the absence of the other Horndeski terms, one finds the condition $\omega < 6$. If, for example, we add a rather simple Horndeski term, namely, the cubic Galileon term $G_3 = -\tilde{g}_3 \mathcal{X}/(M_{\rm Pl} m_g^2)$, we find $\omega < 6 + 12\tilde{g}_3 H^2/m_g^2$.

Besides the Friedmann equation, there are two other independent equations. We can choose them, for example, to be the second Einstein equation and the equation of motion for the scalar field σ. The variation of the action with respect to the Stückelberg fields does not introduce new independent equations of motion. On solving the above-mentioned three independent equations of motion, we can constrain three parameters. One constraint is given by the Friedmann equation (25). Another one can be written through the Bianchi identities (or equivalently, through the equation of motion of the Stückelberg fields) as

$$3(X-1) - 3(X-1)^2\alpha_3 + (X-1)^3\alpha_4 + \xi X^3 = 0. \quad (28)$$

Notice that this is the term which multiplies the quantity α_σ in the equation of motion for the scalar field σ. Therefore α_σ never enters the equations of motion for the self accelerating backgrounds of this model. Finally, we can write the σ equation of motion which gives the last independent constraint as

$$(X-1)(r-1)X^2 m_g^2 \alpha_3 = 3H^4 M_{\rm Pl} G_{3,\mathcal{X}} + 2(r-1)X^2 m_g^2 + 6H^6 M_{\rm Pl}^2 G_{4,\mathcal{X}\mathcal{X}} + 3H^6 M_{\rm Pl} G_{5,\mathcal{X}} + H^8 M_{\rm Pl}^3 G_{5,\mathcal{X}\mathcal{X}} + 6H^4 G_{4,\mathcal{X}} + H^2 P_{,\mathcal{X}} + \frac{X^4(r-1)\xi m_g^2}{X-1}. \quad (29)$$

In the following, we shall use equations (25), (28), and (29) in order to replace the constants Λ, α_3, and α_4 in terms of the other constants/parameters of the model.

IV. PERTURBATIONS

In order to make sure that the de Sitter solutions are stable and do not lead to pathological degrees of freedom, we need to study the behavior of the perturbations' fields around such backgrounds.

A. Scalar perturbations

We work here in the unitary gauge, where the Stückelberg fields are not perturbed. This choice completely fixes the gauge for the scalar, vector, and tensor modes.

As for the scalar sector we introduce the metric in the form

$$\delta g_{00} = -2N^2\Phi, \qquad \delta g_{0i} = Na\partial_i B, \qquad \delta g_{ij} = a^2\left[2\delta_{ij}\Psi + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial_l\partial^l\right)E\right], \tag{30}$$

whereas the dilaton field is perturbed as

$$\sigma = \bar{\sigma} + M_{\rm Pl}\delta\sigma. \tag{31}$$

In order to simplify the analysis we introduce the following quantities, which are constant on the self-accelerating background:

$$G_3(\mathcal{X}) \equiv g_3 M_{\rm Pl}, \quad G_{3,\mathcal{X}} \equiv \frac{g_{3x}}{M_{\rm Pl}H^2}, \quad G_{3,\mathcal{X}\mathcal{X}} \equiv \frac{g_{3xx}}{M_{\rm Pl}^3H^4}, \quad G_4(\mathcal{X}) \equiv M_{\rm Pl}^2 g_4,$$
$$G_{4,\mathcal{X}} \equiv \frac{g_{4x}}{H^2}, \quad G_{4,\mathcal{X}\mathcal{X}} \equiv \frac{g_{4xx}}{M_{\rm Pl}^2H^4}, \quad G_{4,\mathcal{X}\mathcal{X}\mathcal{X}} \equiv \frac{g_{4xxx}}{M_{\rm Pl}^4H^6}, \quad G_5(\mathcal{X}) \equiv \frac{M_{\rm Pl}}{H^2}g_5, \tag{32}$$
$$G_{5,\mathcal{X}} \equiv \frac{g_{5x}}{M_{\rm Pl}H^4}, \quad G_{5,\mathcal{X}\mathcal{X}} \equiv \frac{g_{5xx}}{M_{\rm Pl}^3H^6}, \quad G_{5,\mathcal{X}\mathcal{X}\mathcal{X}} \equiv \frac{g_{5xxx}}{M_{\rm Pl}^5H^8}, \quad P(\mathcal{X}) \equiv pM_{\rm Pl}^2H^2,$$
$$P_{,\mathcal{X}} \equiv p_x, \quad P_{,\mathcal{X}\mathcal{X}} \equiv \frac{p_{xx}}{M_{\rm Pl}^2H^2}, \quad \alpha_\sigma \equiv \frac{m_g^2X^2}{H^2}\bar{\alpha}, \quad m_g^2\xi \equiv \bar{\xi}H^2.$$

On using these variables, together with Eq. (16), we find that, on the self-accelerating backgrounds,

$$\left(\frac{\dot{\phi}^0}{n}\right)^2 = 1 - \frac{\bar{\alpha}}{r^2} > 0, \tag{33}$$

which implies

$$\bar{\alpha} < r^2. \tag{34}$$

This condition defines a set of consistent background variables. Although this condition does not constrain any parameter space in the simplest case (as we will see later on), in general, it will restrict the allowed parameter space for more general models.

1. No-ghost conditions

On expanding the action at second order in the perturbation fields, we can integrate out the fields B and Φ, as usual. Furthermore, because of the structure of the gravitational Lagrangian, we find that, on introducing the field redefinition

$$\delta\sigma = \Psi + \bar{\delta\sigma}, \tag{35}$$

the field Ψ also becomes an auxiliary field. After integrating out the field Ψ (the would-be Boulware-Deser ghost), the theory only admits two propagating scalar fields. By studying the property of the kinetic matrix in the total Lagrangian $\mathcal{L} \ni K_{11}|\dot{\bar{\delta\sigma}}|^2 + K_{22}|\dot{E}|^2 + K_{12}(\dot{\bar{\delta\sigma}}^\dagger\dot{E} + \text{H.c.})$, we find that, in order to remove any ghost degree of freedom, for any k mode, we require the following two conditions to hold:

$$K_{22} = \frac{a^4\gamma_1 H k^4 M_{\rm Pl}^2}{36\dot{a}}\left[(\bar{\alpha}-1)\gamma_4\frac{k^2}{a^2H^2} + 3\gamma_2\gamma_3\right]\times\left[(\bar{\alpha}-1)\gamma_5^2\frac{k^2}{a^2H^2} + \gamma_2\gamma_3\right]^{-1} > 0,$$
$$\det(K_{IJ}) = \left[\frac{\bar{\alpha}\gamma_4(r-1)^2\frac{k^2}{a^2H^2} + 3\gamma_2\gamma_3(r^2-\bar{\alpha})}{(\bar{\alpha}-1)\gamma_5^2\frac{k^2}{a^2H^2} + \gamma_2\gamma_3}\right]\times\left[\frac{a^{10}\gamma_1\gamma_2H^4k^2M_{\rm Pl}^4}{8\dot{a}^2(r-1)^2r^2}\right] > 0, \tag{36}$$

where

$$\gamma_1 \equiv 1 - 2g_{4x} - g_{5x} + 2g_4,$$
$$\gamma_2 \equiv 3g_{3x} + 6g_{4x} + 3g_{5x} + 6g_{4xx} + g_{5xx} + p_x,$$
$$\gamma_{2x} \equiv 3g_{3xx} + 6g_{4xx} + 3g_{5xx} + 6g_{4xxx} + g_{5xxx} + p_{xx},$$
$$\gamma_3 \equiv 6 + 12g_4 - 9(2\gamma_1 + \gamma_5) - (\gamma_2 + \gamma_{2x}),$$
$$\gamma_4 \equiv 3\gamma_5^2 - 2\gamma_1\gamma_3,$$
$$\gamma_5 \equiv g_{3x} + 8g_{4x} + 5g_{5x} + 4g_{4xx} + g_{5xx} - 4g_4 - 2. \tag{37}$$

We notice that

$$\frac{\partial\Lambda}{\partial(H^2)} = \frac{\gamma_3}{2} > 0, \tag{38}$$

so that we need to impose the following conditions:

$$K_{22} > 0, \qquad \det(K_{IJ}) > 0, \qquad \gamma_3 > 0, \tag{39}$$

and, by requiring the result to be independent of the value of the wave vector k, we need to further impose for K_{22}

$$\frac{(\bar{\alpha}-1)\gamma_4}{\gamma_2} > 0, \qquad \frac{(\bar{\alpha}-1)}{\gamma_2} > 0, \qquad \gamma_1 > 0, \tag{40}$$

which together impose $\gamma_4 > 0$. Using (34), the positivity of the determinant yields

$$\frac{\bar{\alpha}\gamma_4}{\gamma_2} > 0. \tag{41}$$

Collecting all the conditions, the allowed parameter region is

$$\begin{aligned} &\gamma_1 > 0, \qquad \gamma_2 > 0, \qquad \gamma_3 > 0, \\ &\gamma_4 > 0, \qquad r > 1, \qquad 1 < \bar{\alpha} < r^2. \end{aligned} \tag{42}$$

Notice, though, that for the general model we need a positive (nonzero, in particular) value for $\bar{\alpha}$ (i.e. α_σ) in order not to have ghosts in the scalar sector. This no-ghost condition does not depend on any of the new Horndeski terms, so that this same condition applies, unchanged, to the simplest [36], as well as to the most complicated, theory of these models.

2. Speed of propagation

In order to find the speed of propagation for the scalar modes, we find it convenient to diagonalize the kinetic matrix K_{IJ} by defining the fields q_1 and q_2 as

$$\bar{\delta}_s \equiv k q_1, \qquad E \equiv \frac{q_2}{k^2} - \frac{K_{12}}{K_{22}} k q_1. \tag{43}$$

The k dependence in this field redefinition has been introduced so that, for the new kinetic matrix, the diagonal elements tend to finite (nonzero) values for large k's.

The new kinetic matrix $\mathcal{T}_{IJ}$ can be written, without approximations, as

$$\mathcal{L} \ni \mathcal{T}_{11}(t,k)|\dot{q}_1|^2 + \mathcal{T}_{22}(t,k)|\dot{q}_2|^2, \tag{44}$$

and, when the no-ghost conditions hold we consistently find

$$\mathcal{T}_{11} > 0 \quad \text{and} \quad \mathcal{T}_{22} > 0. \tag{45}$$

For large momenta (with respect to H and m_g), the structure of the equations of motion for the total Lagrangian can be approximated as

$$\mathcal{T}_{11}\ddot{q}_1 + k\mathcal{B}\dot{q}_2 = 0, \qquad \mathcal{T}_{22}\ddot{q}_2 - k\mathcal{B}\dot{q}_1 + k^2\mathcal{C}q_2 = 0, \tag{46}$$

with

$$\begin{aligned} \mathcal{T}_{11} &\approx \frac{9}{2}\frac{\gamma_2\bar{\alpha}H^3M_{\rm Pl}^2a^6}{r^2\dot{a}(\bar{\alpha}-1)}, \\ \mathcal{T}_{22} &\approx \frac{a^4\gamma_1\gamma_4HM_{\rm Pl}^2}{36\dot{a}\gamma_5^2}, \\ \mathcal{B} &\approx \frac{a^3HM_{\rm Pl}^2\bar{\alpha}\gamma_1\gamma_2}{2r(1-\bar{\alpha})\gamma_5}, \\ \mathcal{C} &\approx -\frac{\dot{a}M_{\rm Pl}^2}{36(\bar{\alpha}-1)\gamma_5^2H}\{2\gamma_1^2[(\bar{\alpha}-1)\gamma_5+\gamma_2] \\ &\quad + (\bar{\alpha}-1)\gamma_5^2\gamma_6\}, \end{aligned} \tag{47}$$

where now $\mathcal{T}_{11}$, $\mathcal{T}_{22}$, $\mathcal{B}$, and $\mathcal{C}$ are k independent, as only their leading order term in a large k expansion has been considered here. All other terms in the equations of motion are suppressed by inverse powers of $k/(aH)$ and/or $k/(am_g)$. We note that this approximation breaks down when $\bar{\alpha} - 1 = \mathcal{O}(aH/k)$. Otherwise, the large k expansion employed here is justified deep inside the horizon.

Then, the speed of propagation of one of the two scalar modes reduces to

$$c_s^2 = \frac{\mathcal{B}^2 + \mathcal{C}\mathcal{T}_{11}}{\mathcal{T}_{11}\mathcal{T}_{22}}\frac{a^2}{N^2} = \frac{2\gamma_1^2(\gamma_2-\gamma_5) - \gamma_5^2\gamma_6}{\gamma_1\gamma_4}, \tag{48}$$

where

$$\gamma_6 \equiv 2g_4 + 1. \tag{49}$$

The other scalar mode has vanishing sound speed, and this property is not affected by the introduction of the new Horndeski terms. Hence, the scalar sector does not have any instabilities whose time scales are parametrically shorter than the background cosmological time scale if the condition (42) and

$$c_s^2 > 0 \tag{50}$$

are satisfied.

B. Tensor perturbations

The action for the tensor perturbation modes reduces to

$$\mathcal{L}_{GW} = \frac{M_{\rm Pl}^2}{8}a^3N\gamma_1\left[\frac{|\dot{h}_{ij}|^2}{N^2} - \left(c_G^2\frac{k^2}{a^2} + M_{GW}^2\right)|h_{ij}|^2\right]. \tag{51}$$

The no-ghost condition for the tensor modes is then

$$\gamma_1 > 0. \tag{52}$$

The speed of propagation, for large k, for the tensor modes becomes

$$c_G^2 = \frac{\gamma_6}{\gamma_1}, \tag{53}$$

and its mass is

$$M_{GW}^2 = \frac{(r-1)X^3 m_g^2}{\gamma_1(X-1)} + \frac{(r-1)X^4 H^2 \bar{\xi}}{\gamma_1(X-1)^2} + \frac{(Xr+r-2)\gamma_2 H^2}{\gamma_1(r-1)(X-1)}. \quad (54)$$

Since M_{GW} is generically of order $|m_g| \simeq H$, the tensor sector does not have any instabilities whose time scales are parametrically shorter than the background cosmological time scale if

$$c_G^2 > 0. \quad (55)$$

C. Vector perturbations

The reduced action for the vector modes reads as

$$\mathcal{L}_V = \frac{M_{\rm Pl}^2}{16} a^3 N \gamma_1 \left[\frac{Q_V |\dot{E}_i|^2}{N^2} - k^2 M_{GW}^2 |E_i|^2\right]. \quad (56)$$

As for the vector modes, after imposing $\gamma_1 > 0$, we have the no-ghost condition

$$Q_V \equiv \frac{2\gamma_2 k^2}{\gamma_1 (r^2-1)\frac{k^2}{H^2 a^2} + 2\gamma_2} > 0, \quad (57)$$

which is always satisfied, for the no-ghost parameter space allowed by the scalar modes. The speed of propagation, for large k, is

$$c_V^2 = \frac{\gamma_1 M_{GW}^2 (r^2-1)}{2H^2 \gamma_2}. \quad (58)$$

This expression has an interesting consequence: since $r > 1$ and $\gamma_{1,2} > 0$, the absence of gradient instability in the vector sector fixes the squared-mass of the tensor modes to be positive, i.e.

$$M_{GW}^2 > 0. \quad (59)$$

D. Allowed parameter space

The above conditions all together give the following allowed parameter space:

$$\gamma_1 > 0 \wedge M_{GW}^2 > 0 \wedge 0 < \gamma_3 < 3\gamma_5^2/(2\gamma_1) \wedge 0 < \gamma_6 < 2\gamma_1^2(\gamma_2 - \gamma_5)/\gamma_5^2 \wedge 1 < \bar{\alpha} < r^2 \wedge r > 1 \wedge \gamma_2 > 0. \quad (60)$$

This is tantamount to saying that the set of parameter space for the models which have a well-behaved stable late time de Sitter solution, is not empty.

E. Examples

1. K-essencelike case

The first example consists of setting to zero any Horndeski term, as well as the nonderivative coupling ($\xi = 0$). In this case we find

$$\gamma_1 = 1, \qquad \gamma_2 = p_x, \qquad \gamma_3 = 6 - p_x - p_{xx}, \qquad \gamma_5 = -2, \qquad \gamma_6 = 1, \quad (61)$$

so that the allowed parameter space (60) becomes

$$p_x > 0 \wedge 0 < p_x + p_{xx} < 6 \wedge M_{GW}^2 > 0 \wedge r > 1 \wedge 1 < \bar{\alpha} < r^2, \quad (62)$$

or, explicitly

$$r > 1 \wedge 1 < \bar{\alpha} < r^2 p_x > 0 \wedge 0 < p_x + p_{xx} < 6 \wedge \left[\left(0 < X < 1 \wedge m_g^2 < -\frac{H^2 p_x (r + rX - 2)}{X^3 (r-1)^2}\right) \vee \left(X > 1 \wedge m_g^2 > -\frac{H^2 p_x (r + rX - 2)}{X^3 (r-1)^2}\right)\right]. \quad (63)$$

The speed of propagation of one scalar mode is

$$c_s^2 = \frac{p_x}{p_x + p_{xx}}, \quad (64)$$

whereas the other scalar mode has vanishing speed of propagation. The speed of propagation of the tensor modes is unity, whereas the one of the vector modes reduces to

$$c_V^2 = \frac{M_{GW}^2 (r^2-1)}{2H^2 p_x}, \quad (65)$$

where

$$M_{GW}^2 = \frac{(r-1)X^3 m_g^2}{X-1} + \frac{(Xr+r-2)p_x H^2}{(r-1)(X-1)}. \quad (66)$$

For the simplest case, $p_{xx} = 0$, and $p_x = \omega$, we confirm the results as given in [36].

2. Horndeski case

Let us consider one of the easiest generalizations from the Horndeski Lagrangian. Setting g_4, g_5, their derivatives and ξ to zero, let us assume $p_x = \omega$, $p_{xx} = 0$, and $g_{3xx} = 0$. Then, we have

$$\gamma_1 = 1, \qquad \gamma_2 = \omega + 3g_{3x}, \qquad \gamma_3 = 6 - \omega - 12g_{3x}, \qquad \gamma_5 = g_{3x} - 2, \qquad \gamma_6 = 1, \quad (67)$$

so that we find the following set of allowed parameter space:

$$r>1\wedge 12g_{3x}+\omega<6\wedge 1<\bar{\alpha}<r^2\wedge\Bigg[\bigg(0<X<1\wedge m_g^2<-\frac{H^2(\omega+3g_{3x})(r+rX-2)}{X^3(r-1)^2}\bigg)\vee\bigg(X>1\wedge m_g^2>-\frac{H^2(\omega+3g_{3x})(r+rX-2)}{X^3(r-1)^2}\bigg)\Bigg]\wedge[(\omega+3g_{3x}>0\wedge 0<g_{3x}<2/3)\vee(-8-2\sqrt{19}<g_{3x}\leq -1\wedge g_{3x}(g_{3x}-8)<2\omega)\vee(-1<g_{3x}\leq 0\wedge 2\omega+3g_{3x}(4+g_{3x})>0)], \quad (68)$$

which implies $-2(4+\sqrt{19})<g_{3x}<2/3$.

In this case, the speed of the propagating scalar mode is

$$c_s^2=1-\frac{4g_{3x}(1+g_{3x})}{2\omega+3g_{3x}(4+g_{3x})}, \quad (69)$$

which is superluminal if $-1<g_{3x}<0$. The vector modes propagate with speed

$$c_V^2=\frac{M_{GW}^2(r^2-1)}{2H^2(\omega+3g_{3x})}, \quad (70)$$

where

$$M_{GW}^2=\frac{(r-1)X^3m_g^2}{X-1}+\frac{(Xr+r-2)(\omega+3g_{3x})H^2}{(r-1)(X-1)}. \quad (71)$$

The tensor modes, on the other hand, propagate with unity speed of propagation.

3. Vanishing bare cosmological constant

Let us consider the case of a vanishing bare cosmological constant. In this case, if a de Sitter solution exists, the system will be self-accelerating. The condition $\Lambda=0$, sets a constraint between m_g^2 and the other variables, as in

$$\frac{m_g^2}{H^2}=\frac{\frac{\gamma_2[X(rX-2)+1]}{(r-1)X^2}-(2\gamma_1+\gamma_6+p+\bar{\xi}X^2-2g_{4x})}{(X-1)^2}. \quad (72)$$

On inserting such relation in the expression for m_g^2 into the expression of M_{GW}^2, we find

$$M_{GW}^2=\frac{H^2}{\gamma_1(r-1)(X-1)^3}[\gamma_2(r^2X^3-3rX^2+r+3X-2)-(2\gamma_1+\gamma_6+\gamma_7)(r-1)^2X^3], \quad (73)$$

where

$$\gamma_7=p+\bar{\xi}X-2g_{4x}. \quad (74)$$

Then, on defining

$$\Gamma\equiv(2\gamma_1+\gamma_6+\gamma_7)(r-1)^2X^3-\gamma_2\{r[X^2(rX-3)+1]+3X-2\}, \quad (75)$$

we find the following allowed parameter space:

$$0<\gamma_3<\frac{3\gamma_5^2}{2\gamma_1}\wedge 0<\gamma_6<\frac{2\gamma_1^2(\gamma_2-\gamma_5)}{\gamma_5^2}\wedge r>1\wedge 1<\bar{\alpha}<r^2\wedge\{\gamma_1>0\wedge\gamma_2>0\wedge[(\Gamma<0\wedge X>1)\vee(\Gamma>0\wedge 0<X<1)]\}. \quad (76)$$

V. DISCUSSION AND CONCLUSIONS

We have studied a form for the quasidilaton action, which generalizes the recent ghost-free quasidilaton action introduced in [36]. We have found that all these model possess, in general, self-accelerating solutions. The background dynamics of the de Sitter solutions does *not* depend on α_σ. Therefore the background evolution is exactly the same as the original quasidilaton case already introduced in [37]. Nonetheless, this same parameter heavily affects the stability of the perturbation fields.

We have found that the background condition $(\dot{\phi}^0/n)^2>0$ implies

$$\frac{\alpha_\sigma H^2}{m_g^2}<r^2X^2. \quad (77)$$

This condition is restrictive enough to make the general model have the same structure of the simplest allowed case introduced in [36]. Furthermore, the no-ghost conditions for the scalar sector impose, in general,

$$r>1,\qquad X^2<\frac{\alpha_\sigma H^2}{m_g^2}<r^2X^2, \quad (78)$$

so that the parameter α_σ/m_g^2 needs to be positive (different from zero, in particular).

A coupling similar to α_σ exists also in the Dirac-Born-Infeld (DBI) Galileon coupled to massive gravity (DBI massive gravity) [43]. Among various Lagrangian terms of the generalized quasidilaton theory considered in the present paper, some are allowed in the DBI massive gravity as well but some are forbidden. Specifically, the parameter space (60) does not include a region corresponding to the DBI massive gravity.

In the allowed parameter space defined in (60), all the expected perturbation modes are well behaved: they possess positive kinetic energy and non-negative squared speed of propagation. One scalar mode has always zero speed of propagation. To give a positive (nonzero, in particular) mass to the graviton, makes the vector modes propagate (i.e. $c_V^2 > 0$). This behavior is quite different from GR, and, as such, it may lead to some interesting constraints/phenomenology.

ACKNOWLEDGMENTS

We thank Kurt Hinterbichler, Chunshan Lin, and Mark Trodden for useful comments. The work of A. E. G. and S. M. was supported by WPI Initiative, MEXT, Japan. S. M. also acknowledges the support by Grant-in-Aid for Scientific Research 24540256 and 21111006.

[1] M. Fierz and W. Pauli, Proc. R. Soc. A **173**, 211 (1939).
[2] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368 (1972).
[3] C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010).
[4] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011).
[5] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, J. Cosmol. Astropart. Phys. 11 (2011) 030.
[6] A. De Felice, A. E. Gumrukcuoglu, and S. Mukohyama, Phys. Rev. Lett. **109**, 171101 (2012).
[7] M. Fasiello and A. J. Tolley, J. Cosmol. Astropart. Phys. 11 (2012) 035.
[8] A. Higuchi, Nucl. Phys. **B282**, 397 (1987).
[9] A. H. Chamseddine and M. S. Volkov, Phys. Lett. B **704**, 652 (2011).
[10] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, and A. J. Tolley, Phys. Rev. D **84**, 124046 (2011).
[11] K. Koyama, G. Niz, and G. Tasinato, Phys. Rev. Lett. **107**, 131101 (2011).
[12] K. Koyama, G. Niz, and G. Tasinato, Phys. Rev. D **84**, 064033 (2011).
[13] P. Gratia, W. Hu, and M. Wyman, Phys. Rev. D **86**, 061504 (2012).
[14] M. Wyman, W. Hu, and P. Gratia, Phys. Rev. D **87**, 084046 (2013).
[15] T. Kobayashi, M. Siino, M. Yamaguchi, and D. Yoshida, Phys. Rev. D **86**, 061505 (2012).
[16] M. S. Volkov, Phys. Rev. D **86**, 061502 (2012).
[17] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, Phys. Lett. B **717**, 295 (2012).
[18] A. De Felice, A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, J. Cosmol. Astropart. Phys. 05 (2013) 035.
[19] N. Khosravi, G. Niz, K. Koyama, and G. Tasinato, J. Cosmol. Astropart. Phys. 08 (2013) 044.
[20] G. D'Amico, Phys. Rev. D **86**, 124019 (2012).
[21] V. A. Rubakov, arXiv:hep-th/0407104; S. L. Dubovsky, J. High Energy Phys. 10 (2004) 076; V. A. Rubakov and P. G. Tinyakov, Phys. Usp. **51**, 759 (2008); D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo, Phys. Rev. D **86**, 101502 (2012); D. Comelli, F. Nesti, and L. Pilo, Phys. Rev. D **87**, 124021 (2013); arXiv:1305.0236; C. Lin, arXiv:1305.2069; arXiv:1307.2574; D. Comelli, F. Nesti, and L. Pilo, arXiv:1307.8329.
[22] S. F. Hassan and R. A. Rosen, J. High Energy Phys. 02 (2012) 126.
[23] S. F. Hassan and R. A. Rosen, J. High Energy Phys. 04 (2012) 123.
[24] K. Hinterbichler and R. A. Rosen, J. High Energy Phys. 07 (2012) 047.
[25] M. von Strauss, A. Schmidt-May, J. Enander, E. Mortsell, and S. F. Hassan, J. Cosmol. Astropart. Phys. 03 (2012) 042.
[26] D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo, J. High Energy Phys. 03 (2012) 067; 06 (2012) 020(E).
[27] Y. Akrami, T. S. Koivisto, and M. Sandstad, J. High Energy Phys. 03 (2013) 099.
[28] D. Comelli, M. Crisostomi, and L. Pilo, J. High Energy Phys. 06 (2012) 085.
[29] A. De Felice, T. Nakamura, and T. Tanaka, arXiv:1304.3920.
[30] A. De Felice, A. E. Gumrukcuoglu, S. Mukohyama, N. Tanahashi, and T. Tanaka (work in progress).
[31] Q.-G. Huang, Y.-S. Piao, and S.-Y. Zhou, Phys. Rev. D **86**, 124014 (2012).
[32] D.-J. Wu, Y.-S. Piao, and Y.-F. Cai, Phys. Lett. B **721**, 7 (2013).
[33] K. Hinterbichler, J. Stokes, and M. Trodden, Phys. Lett. B **725**, 1 (2013).
[34] G. Leon, J. Saavedra, and E. N. Saridakis, Classical Quantum Gravity **30**, 135001 (2013).
[35] A. E. Gumrukcuoglu, K. Hinterbichler, C. Lin, S. Mukohyama, and M. Trodden, Phys. Rev. D **88**, 024023 (2013).
[36] A. De Felice and S. Mukohyama, arXiv:1306.5502.
[37] G. D'Amico, G. Gabadadze, L. Hui, and D. Pirtskhalava, Phys. Rev. D **87**, 064037 (2013).
[38] G. D'Amico, G. Gabadadze, L. Hui, and D. Pirtskhalava, Classical Quantum Gravity **30**, 184005 (2013).
[39] S. Mukohyama, arXiv:1309.2146.
[40] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974).
[41] C. Deffayet, S. Deser, and G. Esposito-Farese, Phys. Rev. D **80**, 064015 (2009).
[42] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, Phys. Rev. Lett. **108**, 051101 (2012).
[43] G. Gabadadze, K. Hinterbichler, J. Khoury, D. Pirtskhalava, and M. Trodden, Phys. Rev. D **86**, 124004 (2012).