PHYSICAL REVIEW D **96**, 044044 (2017)

Pseudotopological quasilocal energy of torsion gravity

Sheng-Lan Ko,[1,2,*] Feng-Li Lin,[2,†] and Bo Ning[3,‡]

[1]*The Institute for Fundamental Study "The Tah Poe Academia Institute," Naresuan University, Phitsanulok 65000, Thailand*

[2]*Department of Physics, National Taiwan Normal University, Taipei 116, Taiwan*

[3]*Center for Theoretical Physics, College of Physical Science and Technology, Sichuan University, Chengdu 610064, People's Republic of China*

(Received 15 March 2017; published 30 August 2017)

Torsion gravity is a natural extension to Einstein gravity in the presence of fermion matter sources. In this paper we adopt Wald's covariant method of calculating the Noether charge to construct the quasilocal energy of the Einstein-Cartan-fermion system, and find that its explicit expression is formally independent of the coupling constant between the torsion and axial current. This seemingly topological nature is unexpected and is reminiscent of the quantum Hall effect and topological insulators. However, a coupling dependence does arise when evaluating it on shell, and thus the situation is pseudotopological. Based on the expression for the quasilocal energy, we evaluate it for a particular solution on the entanglement wedge and find agreement with the holographic relative entropy obtained before. This shows the equivalence of these two quantities in the Einstein-Cartan-fermion system. Moreover, the quasilocal energy in this case is not always positive definite, and thus it provides an example of a swampland in torsion gravity. Based on the covariant Noether charge, we also derive the nonzero fermion effect on the Komar angular momentum. The implications of our results for future tests of torsion gravity in gravitational-wave astronomy are also discussed.

DOI: 10.1103/PhysRevD.96.044044

I. INTRODUCTION

General covariance is the key ingredient in formulating the general theory of relativity, which can be thought of as an infinite-dimensional local gauge symmetry. However, due to such a huge gauge symmetry, it is then impossible to define any sensible local observable in the context of general relativity, such as the stress tensor associated with the gravitational field. Instead, some global or quasilocal quantities have been proposed, such as the total mass/energy of self-gravitating systems. The first quantity is the well-known Arnowitt-Deser-Misner (ADM) mass [1], which is useful in formulating black hole thermodynamics [2,3]. This inspired a search for the quasilocal energy [4] covering only a finite domain of the spacetime, which is closer to the classic notion of the local energy density. There are various ways of deriving the quasilocal energy; for a review, see Ref. [5]. Among them, the derivation based on Wald's formulation [6–10] has the advantage of obtaining a covariant quasilocal energy as a Noether charge associated with some time-like Killing vector.

Recently, the quasilocal energy in AdS space was proposed as being equivalent to the relative entropy of the dual conformal field theory (CFT) in the context of the Ryu-Takayanagi proposal for the holographic entanglement entropy [11,12]. In particular, the positive energy condition of the quasilocal energy was shown to be the same as the positivity of the relative entropy, which can then be used to constrain the swampland of the bulk gravity theory [13,14]. This implies a deep connection between the positive energy theorem [15–18] of gravity and the quantum information inequalities of CFTs. It was further shown that the holographic relative entropy for the usual Einstein gravity in anti–de Sitter (AdS) space can be constructed as Wald's quasilocal energy. We then expect that the proof in Ref. [19] for the positive energy condition of the quasilocal energy for flat space can be generalized to AdS space, and similarly the proof in Refs. [18,20] of a positive ADM mass in AdS space to the quasilocal energy.

On the other hand, the positive energy theorem for gravity theories other than Einstein gravity and its connection to the positivity of the relative entropy in dual CFTs has been less explored. In Ref. [21] we studied the holographic relative entropy of the deformed CFT in bulk Einstein-Cartan gravity, i.e., torsion gravity [22]. We first perturbatively solve the field equations of the Einstein-Cartan-fermion system up to the second order of Newton's constant. Based on this solution, we evaluate the variations of the modular Hamiltonian and entanglement entropy, and then use these to obtain the relative entropy of the dual CFT. Interestingly, we find that the resultant relative entropy is not always positive definite, which implies a swampland in the bulk torsion gravity possibly beyond the reach of the weak gravity conjecture [23].

*sheng-lank@nu.ac.th
†fengli.lin@gmail.com
‡ningbbo@gmail.com

In this paper we will explicitly construct the variation of the quasilocal energy for the Einstein-Cartan-fermion system, and demonstrate the equivalence between the holographic relative entropy and quasilocal energy in torsion gravity. To achieve this, we use Wald's formalism to derive the quasilocal energy not in its original formulation in terms of the metric, but rather in terms of the vielbein and spin connection. We then evaluate the derived quasilocal energy on the entanglement wedge for a particular perturbative solution and show the agreement with the holographic relative entropy obtained in Ref. [21]. Besides, our results also show that the quasilocal energy is not always positive (i.e., a swampland) even though the theory itself is consistent with the symmetry principle.

Moreover, the symplectic potential as well as the Noether charge are found to be *formally* independent of the torsion-fermion (axial current) coupling constant. Hence, there is no direct torsion contribution to the physical charges, such as the quasilocal energy or ADM quantities. The torsion infiltrates the value of physical charges only through the backreacted on-shell solution. This suggests that the physical charges constructed via Wald's formalism are "pseudotopological quantities," i.e., their values are somehow stable against the change of the torsion-fermion coupling. This is a reminiscent of the topological order in quantum Hall systems [24] or topological insulators [25], for which the physical quantities are insensitive to some coupling strength and there is a bulk-edge correspondence [26].

As a natural extension of Einstein gravity, torsion gravity calls for arenas to test its validity, and we think the results obtained and the techniques developed in this paper should be helpful for this purpose. For example, there have been more serious attempts to incorporate the torsion effect in cosmological models under the scrutiny of cosmic microwave background physics; see Ref. [27] for a review. Another arena is gravitational-wave astronomy, which is expected from future events similar to the recent LIGO discoveries of gravitational waves emitted from compact binaries [28–30]. Once there are enough events to reduce statistical uncertainties, one should be able to test the validity of Einstein gravity and some modified gravities such as torsion gravity. For example, an analysis of the constraints by the first two LIGO observations on physics beyond Einstein was already put forward in Ref. [31]. This calls for more precise theoretical templates of gravitational waves for compact binaries in order to fit the observed data. As most of the templates at this stage are done for Einstein gravity, there remains a lack of high-precision templates for the modified gravity theories. Our construction of the quasilocal energy, ADM mass, and angular momentum can be seen as a first step toward this challenging goal in torsion gravity. For example, one can use these quantities to construct an effective field theory for the coupling between torsion and spin for the post-Newtonian approximation, similar to that done for the coupling between spin and the spin connection in Einstein gravity [32–34]. Moreover, these conserved quantities can also serve as adiabatic invariants in the framework of the effective-one-body approach [35,36], which has been used to generate most of the waveform templates in Einstein gravity. One more challenging task is to generalize the Baumgarte-Shapiro-Shibata-Nakamura-Oohara-Kojima (BSSNOK) formulation of numerical gravity [37–39] to a first-order formulation in terms of vielbeins and spin connections, as the fermion couples to the spin connection but not to the metric. To this end, our extension of Wald's formalism to use local tetrads will be helpful.

The paper is organized as follows. In the next section, we will briefly review Wald's formalism and torsion gravity. Section III contains our main results. We first generalize Wald's formalism to the case with fermion matter and torsion coupling, and then compute the quasilocal energy of the entanglement wedge to compare with the relative entropy. We also discuss the effects of torsion and fermions on some ADM quantities, in particular the extension of the Komar angular momentum. In Sec. IV, we discuss the implications for gravitational-wave physics. We then conclude our paper in Sec. V. In the Appendix we give details about solving the deformed Killing vector field used to evaluate the quasilocal energy.

II. WALD FORMALISM AND TORSION GRAVITY

Before applying the Wald formalism to the Einstein-Cartan-fermion system to obtain our main result in Sec. III, we provide concise reviews of each separately.

A. Wald formalism for quasilocal energy

In this section, we briefly review the quasilocal energy defined via the covariant phase space formalism put forward by Wald and his collaborators [7,9]. The basic idea is to construct a covariant Noether current and charge associated with a time-like vector field inside a space-like subregion, and then relate this Noether charge to the quasilocal energy defined for this subregion.

Let us denote the subregion of the Cauchy surface by Σ and denote all the dynamical fields (including the metric[1] and matter fields) collectively as ϕ. In the following, a boldface letter denotes a differential form in the spacetime; for example, the Lagrangian is written as a 4-form $\mathbf{L}$. Generically, the variation of a covariant Lagrangian is written as

$$\delta\mathbf{L} = \mathbf{E}\delta\phi + d\mathbf{\Theta}(\phi, \delta\phi), \tag{1}$$

[1]The original formulation in the literature was developed for metric gravity; however, we will see in Sec. III A that the Wald formalism can be generalized to theories formulated with a vielbein.

where $\mathbf{E} = 0$ are the field equations and the surface term $\mathbf{\Theta}(\phi, \delta\phi)$, called the symplectic potential 3-form, is constructed covariantly and locally in terms of ϕ and $\delta\phi$. The following antisymmetrized variations of Θ give rise to the symplectic current 3-form:

$$\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\Theta(\phi, \delta_2\phi) - \delta_2\Theta(\phi, \delta_1\phi). \quad (2)$$

Note that the symplectic current is a bilinear functional of $\delta\phi_1$ and $\delta\phi_2$, and its volume integral is simply the symplectic form.

Given an arbitrary vector field ξ, we can formally associate with it a Hamiltonian H_ξ, with its variation satisfying

$$\delta H_\xi = \int_\Sigma \boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi\phi), \quad (3)$$

where $\mathcal{L}_\xi\phi$ is the Lie derivative of ϕ along the vector field ξ. If ξ is a time-like vector field, it is natural to interpret δH_ξ as the perturbation of the quasilocal energy contained in the subregion Σ [14].[2] Moreover, the existence of the full Hamiltonian H_ξ requires the following integrability condition:

$$0 = (\delta_1\delta_2 - \delta_2\delta_1)H_\xi = -\int_{\partial\Sigma} \xi \cdot \boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi). \quad (4)$$

On the other hand, we can associate with ξ a Noether current 3-form defined by

$$\mathbf{J}_\xi = \mathbf{\Theta}(\phi, \mathcal{L}_\xi\phi) - \xi \cdot \mathbf{L}. \quad (5)$$

It is straightforward to show that Noether current 3-form is closed on shell, i.e., $d\mathbf{J}_\xi = -\mathbf{E}\mathcal{L}_\xi\phi$, so that it can be written as [8]

$$\mathbf{J}_\xi = d\mathbf{Q}_\xi + \xi^\mu \mathbf{C}_\mu, \quad (6)$$

where $\mathbf{C}_\mu$ vanishes on shell. The spacetime 2-form $\mathbf{Q}_\xi$ is the Noether charge. A useful identity relates the symplectic current to the variation of the Noether current:

$$\boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi\phi) = \delta\mathbf{J}_\xi - d(\xi \cdot \mathbf{\Theta}(\phi, \delta\phi)), \quad (7)$$

with ϕ assumed to be on shell. Using this, the symplectic current can be written as

$$\boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi\phi) = d(\delta\mathbf{Q}_\xi - \xi \cdot \mathbf{\Theta}) + \xi^\mu \delta\mathbf{C}_\mu. \quad (8)$$

Hence, if we restrict to variations $\delta\phi$ that satisfy the field equations (so that $\delta\mathbf{C}_\mu = 0$), we obtain the expression for the variation of the quasilocal energy,

$$\delta H_\xi = \int_{\partial\Sigma} (\delta\mathbf{Q}_\xi - \xi \cdot \mathbf{\Theta}). \quad (9)$$

Notice that this is in the form of a surface integral. In general, $\partial\Sigma$ contains two parts: one at asymptotic infinity denoted by B, and the inner boundary into the bulk denoted by $\tilde{B}$. Moreover, the form of Eq. (9) suggests that the integrability condition (4) should be equivalent to the existence of some $\mathbf{K}$ such that

$$\delta(\xi \cdot \mathbf{K}) = \xi \cdot \mathbf{\Theta} \quad \text{on } \partial\Sigma. \quad (10)$$

If so, we then have the full quasilocal energy

$$H_\xi = \int_{\partial\Sigma} (\mathbf{Q}_\xi - \xi \cdot \mathbf{K}), \quad (11)$$

and the difference between the quasilocal energies of two geometries is given by

$$\Delta H_\xi = \Delta \int_{\partial\Sigma} (\mathbf{Q}_\xi - \xi \cdot \mathbf{K}). \quad (12)$$

In the case that ξ is a Killing vector field (i.e., $\mathcal{L}_\xi\phi = 0$), $\boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi\phi = 0) = 0$ and hence $\delta H_\xi = 0$. Then, Eq. (9) can be written in the form of the first law,

$$\int_B (\delta\mathbf{Q}_\xi - \xi \cdot \mathbf{\Theta}) = \int_{\tilde{B}} (\delta\mathbf{Q}_\xi - \xi \cdot \mathbf{\Theta}) \Leftrightarrow \delta\mathcal{E} = \frac{\kappa_s}{8\pi G_N} \delta\mathcal{A}. \quad (13)$$

The lhs is related to the variation of the canonical energy $\delta\mathcal{E}$ (such as ADM mass or the modular energy), and the rhs is related to the variation of the area $\delta\mathcal{A}$ of the inner boundary $\tilde{B}$. To make the first law manifest we should impose the following boundary conditions on ξ:

$$\xi|_B = \zeta, \quad (14)$$

$$\tilde{\nabla}^{[\mu}\xi^{\nu]}|_{\tilde{B}} = \kappa_s n^{\mu\nu}, \quad (15)$$

$$\xi|_{\tilde{B}} = 0, \quad (16)$$

where $\tilde{\nabla}^\nu$ is the Riemannian covariant derivative, ζ is the asymptotic time-like Killing vector field, κ_s is the surface gravity for the inner boundary $\tilde{B}$, and $n^{\mu\nu} := n^\mu_{(1)} n^\nu_{(2)} - n^\mu_{(2)} n^\nu_{(1)}$ is the unit binormal vector. For the black hole geometry, B is the full asymptotic boundary and $\tilde{B}$ is

[2] As commented in Ref. [14], this is a natural generalization of the Hamiltonian for the particle Lagrangian. In Refs. [10,13], δH_ξ was called the canonical energy for the second-order perturbation; however, this could be confused with the linearized ADM mass for which it was also called the canonical energy in Refs. [6,7]. Thus, we will simply call H_ξ the quasilocal energy to avoid confusion.

the black hole horizon. Thus, Eq. (13) will yield the first law of black hole thermodynamics. On the other hand, if we consider the asymptotic AdS space, then Σ is the entanglement wedge bounded by the asymptotic boundary disk B and the Ryu-Takayanagi minimal surface $\tilde{B}$ whose surface gravity κ_s is set to 2π. Thus, Eq. (13) yields the first law of entanglement thermodynamics for the dual CFT.

In this paper we will consider the case that ξ is not the Killing vector field for the background solution (i.e., $\mathcal{L}_\xi\phi \neq 0$), so that δH_ξ can be treated as the quasilocal energy for ϕ. Despite this, to preserve the asymptotic Killing symmetry we will impose the additional boundary condition

$$\mathcal{L}_\xi\phi|_B = 0. \tag{17}$$

If ξ is a time-like vector field, this requires ϕ to be asymptotically stationary. A specific example that we consider below is the holographic relative entropy, which is dual to the quasilocal energy (9) for the entanglement wedge in the AdS space for torsion gravity. This quasilocal energy is for the second-order stationary solution so that the background metric is the metric up to the first order, and hence $\mathcal{L}_\xi\phi = 0$ does not hold [though Eq. (17) still holds] to yield nonzero δH_ξ.

B. Torsion gravity

In this subsection we briefly review torsion gravity. We start with the Lagrangian for the Einstein-Cartan-fermion system,

$$\mathbf{L} = \frac{1}{2\kappa^2}\mathbf{L}_R + \mathbf{L}_M, \tag{18}$$

where $\kappa^2 := 8\pi G_N$ and the Einstein-Cartan part $\mathbf{L}_R$ and the fermion part $\mathbf{L}_M$ of the Lagrangian are, respectively, given by

$$\mathbf{L}_R = -e^\mu_a e^\nu_b R_{\mu\nu}{}^{ab}\boldsymbol{\epsilon} - 2\Lambda\boldsymbol{\epsilon} = (R - 2\Lambda)\boldsymbol{\epsilon}, \tag{19}$$

$$\mathbf{L}_M = -\frac{1}{2}[\bar{\psi}\gamma^\mu\nabla_\mu\psi - (\nabla_\mu\bar{\psi})\gamma^\mu\psi + 2m\bar{\psi}\psi]\boldsymbol{\epsilon}, \tag{20}$$

with the vielbein e^μ_a and the volume element $\boldsymbol{\epsilon} := \sqrt{-g}d^4x$.

The covariant derivative for the Dirac fermion field ψ and the curvature tensor used in Eqs. (19) and (20) are formally defined as usual in terms of the spin connection $\omega_\mu{}^a{}_b$, e.g., the Riemann tensor

$$R_{\mu\nu ab} = -\partial_\mu\omega_{\nu ab} + \partial_\nu\omega_{\mu ab} - \omega_{\mu ac}\omega_\nu{}^c{}_b + \omega_{\nu ac}\omega_\mu{}^c{}_b. \tag{21}$$

However, the spin connection now contains the torsion part. Explicitly, it can be divided into the following:

$$\omega_{\mu\nu\rho} := \omega_{\mu ab}e^a_\nu e^b_\rho = \tilde{\omega}_{\mu\nu\rho}(e) + K_{\mu\nu\rho}, \tag{22}$$

where the Riemannian part of the spin connection is given by

$$\tilde{\omega}_\mu{}^{ab}(e) = 2e^{\nu[a}\partial_{[\mu}e^{b]}_{\nu]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e^c_\sigma, \tag{23}$$

and the remaining part is the contorsion tensor $K_{\mu\nu\rho}$ which is related to the torsion tensor $S_{\mu\nu}{}^\rho := \frac{1}{2}(\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu})$ (where $\Gamma^\rho_{\mu\nu}$ is the affine connection) as

$$K_{\mu\nu\rho} = -(S_{\mu\nu\rho} - S_{\nu\rho\mu} + S_{\rho\mu\nu}). \tag{24}$$

In the following, we will work on the formalism developed in Ref. [40], in which the vielbein e^a_μ and the torsion tensor $S_{\mu\nu}{}^\rho$ were considered as independent fields.

Moreover, we can introduce the nonminimal coupling between the fermion and torsion in the following way:

$$\nabla_\mu\psi \to \overset{*}{\nabla}_\mu\psi := \partial_\mu\psi + \frac{1}{4}\tilde{\omega}_\mu{}^{ab}\gamma_{ab}\psi + \frac{\eta_t}{4}K_{\mu\nu\rho}\gamma^{\nu\rho}\psi, \tag{25}$$

and similarly for $\overset{*}{\nabla}_\mu\bar{\psi} := \nabla_\mu\bar{\psi} - \frac{\eta_t - 1}{4}K_{\mu\nu\rho}\bar{\psi}\gamma^{\nu\rho}$. It is called minimal coupling when $\eta_t = 1$. This amounts to adding the following interaction term:

$$-\frac{1}{4}(\eta_t - 1)\sqrt{-g}\bar{\psi}\gamma^{[\mu}\gamma^\nu\gamma^{\lambda]}\psi K_{\mu\nu\lambda}. \tag{26}$$

In Ref. [21], it was shown that there is a nontrivial constraint on η_t from the positivity of the holographic relative entropy. In this paper, we will show that the same constraint arises from the positivity of the quasilocal energy over the entanglement wedge.

By the variational principle we obtain the field equations for the action $\mathbf{L}$:

$$S^{\mu\nu\rho} = \eta_t\frac{\kappa^2}{4}\bar{\psi}\gamma^{\mu\nu\rho}\psi, \tag{27}$$

$$\overset{*}{\nabla}_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = 0, \qquad \gamma^\mu\overset{*}{\nabla}_\mu\psi + m\psi = 0, \tag{28}$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2(\bar{\Sigma}_{(\mu\nu)} + \eta_t\bar{\Sigma}_{[\mu\nu]}), \tag{29}$$

where $\bar{\Sigma}_{(\mu\nu)}$ and $\bar{\Sigma}_{[\mu\nu]}$ are the symmetric and antisymmetric parts of $\bar{\Sigma}_{(\mu\nu)}$, which is defined by

$$\bar{\Sigma}_{\mu\nu} := \frac{1}{2}[\bar{\psi}\gamma_\nu\overset{*}{\nabla}_\mu\psi - (\overset{*}{\nabla}_\mu\bar{\psi})\gamma_\nu\psi]. \tag{30}$$

To solve the field equations, one can split the Einstein tensor into the Riemannian and non-Riemannian parts. In Ref. [21], we did this to obtain the second-order perturbative solution in asymptotically AdS space to evaluate the holographic relative entropy. The solution is summarized as follows.

To set up the notation, the AdS metric rewritten using the Poincaré coordinate,

$$ds^2 = \frac{\ell^2}{z^2}(-dt^2 + dx^2 + dy^2 + dz^2). \tag{31}$$

We will expand the solution in terms of the dimensionless Newton constant

$$\mathrm{K} := \frac{\kappa^2 z_L^2}{\ell^4}, \tag{32}$$

where z_L is an IR cutoff; however, all os the physical quantities such as the quasilocal energy are independent of it.

Then, the fermion solution up to first order in K is

$$\psi = \begin{pmatrix} (\frac{z}{\ell^2})^{3/2+m\ell} a_+ \\ (\frac{z}{\ell^2})^{3/2-m\ell} a_- \end{pmatrix} + \frac{\mathrm{K}}{3}\begin{pmatrix} \Delta_+ (\frac{z}{\ell^2})^{9/2+m\ell} a_+ \\ -\Delta_- (\frac{z}{\ell^2})^{9/2-m\ell} a_- \end{pmatrix}, \tag{33}$$

where

$$\Delta_\pm = \frac{1}{4}(3\eta_t^2 + 2\mu_0 m^2\ell^2 \pm (3\mu_0 - 2)m\ell)\ell^5 \alpha\beta z_L^{-2}, \tag{34}$$

where μ_0 is the integration constant while solving the first-order metric. Besides, the $a_\pm$ are integration constant 2-spinors, and without loss of generality we will choose $a_\pm$ to be

$$a_+ = \{0, \alpha\}^{\mathrm{T}}, \qquad a_- = \{i\beta, 0\}^{\mathrm{T}}. \tag{35}$$

Due to the presence of the fermion solution (33), the AdS metric (31) is then backreacted into

$$ds^2 = \frac{\ell^2}{z^2}(-F(z)dt^2 + H(z)dtdx + dx^2 + dy^2 + G(z)dz^2), \tag{36}$$

with

$$F(z) = 1 + \mathrm{K}\left(\frac{2}{3} - \mu_0\right) m\alpha\beta \frac{z^3}{z_L^2} + \mathrm{K}^2 \frac{(2-3\mu_0)m^2\alpha^2\beta^2}{18}\frac{z^6}{z_L^4},$$
$$G(z) = 1 + \mathrm{K}\mu_0 m\alpha\beta \frac{z^3}{z_L^2} + \mathrm{K}^2 \frac{(\eta_t^2\ell^{-2} + 4\mu_0^2 m^2)\alpha^2\beta^2}{4}\frac{z^6}{z_L^4},$$
$$H(z) = \mathrm{K}^2 \frac{(2-3\mu_0)m\ell^{-1}\alpha^2\beta^2}{18}\frac{z^6}{z_L^4}. \tag{37}$$

III. TOPOLOGICAL QUASILOCAL ENERGY OF TORSION GRAVITY

In this section, we present our results starting from the derivation of the quasilocal energy for Einstein-Cartan-fermion system in Sec. III A. Then, the explicit computation of it on the entanglement wedge for the solution (36) and the comparison with the relative entropy are presented in Sec. III B. Finally, the torsion and fermion effects in the ADM mass and angular momentum are studied in Sec. III C.

Wald's covariant phase space formulation (reviewed in Sec. II A) is very powerful and has been widely applied to black hole thermodynamics and the study of holographic entanglement entropy. However, the discussions in the literature have been restricted to metric gravity. In Sec. III A, we show how the formalism can be extended to include fermions with torsion coupling. Readers who just want to skip ahead to the physical implications of the result may skip this technical subsection, although these techniques could be useful in the context of gravitational-wave physics of torsion gravity. Subsequently, we show that the quasilocal energy of torsion gravity is not always positive definite, but instead leads to a bound constraining the physical parameters of the theory. Remarkably, this means an innocent-looking theory which passed all the symmetry constraints might actually be pathological. The same bound was obtained in Ref. [21] in a different context by the holographic computation of the relative entropy. Our results provide a nontrivial example of a swampland beyond the reach of the grand symmetry principle and possibly the weak gravity conjecture [23].

In the end, we discuss the fermion and torsion effects on the ADM mass and angular momentum. We find that the angular momentum is extended by the axial current in the asymptotically flat space. The physical gauge-invariant quantities such as global charges are crucial in many aspects of gravity. For example, we expect that this extension term will play an important role in the canonical analysis of torsion gravity and hence deform the post-Newtonian expansion of the gravitational waveform as well as the adiabatic invariants in the framework of the effective-one-body approach [35]. This will be further explored in future works.

A. Derivation of quasilocal energy for torsion gravity

To proceed with Wald's formalism, we need to vary the action carefully while retaining all of the surface terms arising from the integration by parts. In this subsection, we first present the essential steps of extracting the symplectic potential by varying the action of the Einstein-Cartan-fermion system with respect to the independent fields e^a_μ, $S_{\mu\nu}{}^a$ and $\bar{\psi}$, ψ. To the best of our knowledge, this is the first discussion on the quasilocal energy of fermionic fields coupled to torsion in the literature, and thus we describe the details which could be useful for other explorations. Then, based on the result for the symplectic potential we derive the associated Noether charge and the variation of the quasilocal energy.

1. Summary of the results

Although the procedure seems straightforward, it is in fact quite tedious. Before sketching the detailed derivation, we first write down the result: the symplectic potential defined in Eq. (1) for the Einstein-Cartan-fermion system turns out to be

$$\Theta(\phi,\delta\phi) = \frac{1}{3!}\frac{1}{2\kappa^2}\epsilon_{\mu\rho_1\rho_2\rho_3}\bigg(g^{\mu\alpha}g^{\beta\gamma}(\tilde{\nabla}_\beta\delta g_{\alpha\gamma} - \tilde{\nabla}_\alpha\delta g_{\beta\gamma}) - \frac{\kappa^2}{2}\bar{\psi}\gamma^{\alpha\gamma\mu}\psi\delta e^a_\alpha e_{\gamma a} + \kappa^2(\delta\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta\psi)\bigg)dx^{\rho_1\rho_2\rho_3}, \tag{38}$$

where $dx^{\rho_1\rho_2\rho_3}$ is a shorthand for the wedge product $dx^{\rho_1}\wedge dx^{\rho_2}\wedge dx^{\rho_3}$. Recall that $\tilde{\nabla}^\nu$ is the Riemannian covariant derivative so that Eq. (38) is reduced to the symplectic potential for pure Einstein gravity once the fermion field is put to zero. Moreover, the last two terms are of subleading order in κ^2 compared to the first term.

Then, we use this symplectic potential to obtain the Noether charge associated with some vector field ξ. The result is

$$\mathbf{Q} = \frac{1}{2!}\frac{-1}{2\kappa^2}\epsilon_{\alpha\beta\rho_1\rho_2}\bigg(\tilde{\nabla}^\alpha\xi^\beta + \kappa^2\frac{1}{4}\bar{\psi}\gamma^{\alpha\beta\gamma}\psi\xi_\gamma\bigg)dx^{\rho_1\rho_2}. \tag{39}$$

Using the above, we can further obtain the quasilocal energy or its variation from Eqs. (11) and (9). Note that we have used the on-shell relation (27) in arriving at Eqs. (38) and (39) by replacing the torsion with the fermion bilinear and at the same time canceling the η_t dependence.

Before starting the derivation, we remark that the symplectic potential (38) and the Noether charge (39) are both formally independent of the torsion coupling η_t, as is the quasilocal energy. This is intriguing because it implies that the quasilocal energy—which is a physical quantity—is *formally* independent of the torsion coupling. The only way that torsion takes effect is through the backreacted geometry. This feature is analogous to the topological order observed in quantum Hall systems or topological insulators. This analogy is just formal, as the topological order is known to be due to the nontrivial patterns of many-body entanglement. On the other hand, the quasilocal energy is a classical quantity of gravity theory.

Also notice that η_t cannot be absorbed by field redefinitions. This is actually expected because the physical quantities—such as the relative entropy that was calculated in Ref. [21] and the quasilocal energy that will be computed later—depend explicitly on η_t. However, the appearance of the η_t dependence in the value of the quasilocal energy comes from the η_t dependence of the backreacted on-shell solution, though the formal expression for the quasilocal energy is independent of η_t.

2. Variation of the Einstein-Cartan action

We first consider the variation of the Einstein-Cartan action. The Ricci curvature can be expressed in terms of the covariant derivative of the spin connection. We can vary the Ricci curvature with respect to the vielbein and spin connection, and then relate the variation of the spin connection to those of the vielbein and torsion as follows:

$$e_{a\nu}e_{b\rho}\delta\omega_\mu{}^{ab} = \triangle^{\tau\sigma\lambda}_{\rho\mu\nu}\nabla_{[\tau}\delta e^a_{\sigma]}e_{a\lambda} + \triangle^{\tau\sigma\lambda}_{\rho\mu\nu}S_{\tau\sigma}{}^\eta\delta e^a_\eta e_{a\lambda} - \triangle^{\tau\sigma\lambda}_{\nu\rho\mu}e_{a\tau}\delta S_{\sigma\lambda}{}^a, \tag{40}$$

where

$$\triangle^{\tau\sigma\lambda}_{\rho\mu\nu} \coloneqq \delta^\tau_\rho\delta^\sigma_\mu\delta^\lambda_\nu - \delta^\tau_\nu\delta^\sigma_\rho\delta^\lambda_\mu + \delta^\tau_\mu\delta^\sigma_\nu\delta^\lambda_\rho. \tag{41}$$

Using the above relations and the fact that $\delta e^\mu_a R_\mu{}^a = -R_a{}^\mu\delta e^a_\mu$, we can arrange the variation of $\mathbf{L}_R$ as follows:

$$\begin{aligned}\delta\frac{1}{2\kappa^2}\mathbf{L}_R = &-\frac{1}{\kappa^2}(G_a{}^\mu + \Lambda e^\mu_a)\epsilon\delta e^a_\mu + d\bigg(\frac{1}{3!}\frac{1}{\kappa^2}\epsilon_{\mu\beta\gamma\delta}e^{a\mu}e^{b\nu}\delta\omega_{\nu ab}dx^{\beta\gamma\delta}\bigg) - \frac{2}{\kappa^2}\epsilon S^\mu{}_\rho{}^\rho g^{\eta\nu}(\triangle^{\tau\sigma\lambda}_{\eta\nu\mu}\nabla_{[\tau}\delta e^a_{\sigma]}e_{a\lambda} + \triangle^{\tau\sigma\lambda}_{\eta\nu\mu}S_{\tau\sigma}{}^\theta\delta e^a_\theta e_{a\lambda})\\ &+\frac{2}{\kappa^2}\epsilon(S^\nu{}_\sigma{}^\sigma g^{\rho\mu} - S^\rho{}_\sigma{}^\sigma g^{\mu\nu} + S^\mu{}_\sigma{}^\sigma g^{\nu\rho})e_{a\nu}\delta S_{\rho\mu}{}^a + \frac{1}{\kappa^2}S^{\mu\nu\rho}(\triangle^{\tau\sigma\lambda}_{\nu\rho\mu}\nabla_{[\tau}\delta e^a_{\sigma]}e_{a\lambda} + \triangle^{\tau\sigma\lambda}_{\nu\rho\mu}S_{\tau\sigma}{}^\eta\delta e^a_\eta e_{a\lambda})\epsilon\\ &-\frac{2}{\kappa^2}\epsilon S^{(\nu|\rho|\mu)}e_{a\nu}\delta S_{\rho\mu}{}^a - \frac{1}{\kappa^2}\epsilon S^{\mu\nu\rho}e_{a\nu}\delta S_{\rho\mu}{}^a.\end{aligned} \tag{42}$$

In the process, one should keep track of all surface terms arising from integration by parts. To this end, it is useful to use the modified divergence operator

$$\hat{\nabla}_\mu \coloneqq \nabla_\mu + 2S_{\mu\nu}{}^\nu, \tag{43}$$

which satisfies

$$\hat{\nabla}_\mu v^\mu = \partial_\mu v^\mu \tag{44}$$

but not the Leibniz rule. The v^μ in the above formula is a vector density. Therefore, the surface term in Eq. (42) in fact comes from

$$\hat{\nabla}_\mu\left(\frac{e}{\kappa^2}e^{a\mu}e^{b\nu}\delta\omega_{\nu ab}\right)d^4x = \partial_\mu\left(\frac{e}{\kappa^2}e^{a\mu}e^{b\nu}\delta\omega_{\nu ab}\right)d^4x = d\left(\frac{1}{3!}\frac{1}{\kappa^2}\epsilon_{\mu\beta\gamma\delta}e^{a\mu}e^{b\nu}\delta\omega_{\nu ab}dx^{\beta\gamma\delta}\right). \tag{45}$$

The other terms in Eq. (42) will be combined with nonsurface terms in $\delta\mathbf{L}_M$ into field equations.

3. Variation of the fermion action

Now, let us consider the variation of the fermion action. This is quite similar to the usual variation of the Dirac fermion action in curved space, except that now the fermion also couples to torsion. However, it is straightforward to see that the torsion does not contribute to the surface term through this variation. After some tedious calculations, we obtain the result as follows:

$$\begin{aligned}\delta\mathbf{L}_M &= \delta(\epsilon\mathcal{L}_M) = \frac{1}{2}g^{\mu\nu}\delta(e^a_\mu\eta_{ab}e^b_\nu)\epsilon\mathcal{L}_M + \epsilon\delta\mathcal{L}_M\\ &= \epsilon\mathcal{L}_M e^\mu_a\delta e^a_\mu + d\left[\frac{1}{3!}\frac{1}{2}\epsilon_{\mu\beta\gamma\delta}(\delta\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta\psi)dx^{\beta\gamma\delta}\right] + \epsilon[(\overset{*}{\nabla}_\alpha\bar{\psi}\gamma^\alpha - m\bar{\psi} + S_{\alpha\beta}{}^\beta\bar{\psi}\gamma^\alpha)\delta\psi - \delta\bar{\psi}(\gamma^\alpha\overset{*}{\nabla}_\alpha\psi + m\psi + S_{\alpha\beta}{}^\beta\gamma^\alpha\psi)]\\ &\quad -\frac{1}{2}\epsilon\bar{\psi}\gamma^a\overset{\leftrightarrow}{\overset{*}{\nabla}}_\mu\psi\delta e^\mu_a - \frac{1}{4}\bar{\psi}\gamma^{\mu\nu\rho}\psi(\nabla_\rho\delta e^a_\mu e_{a\nu} + S_{\rho\mu}{}^\sigma\delta e^a_\sigma e_{a\nu} - \eta_t e_{a\nu}\delta S_{\rho\mu}{}^a)\epsilon + \frac{\eta_t - 1}{4}\epsilon(\bar{\psi}\gamma^{b\nu\rho}\psi)S_{\mu\nu\rho}\delta e^\mu_b,\end{aligned} \tag{46}$$

where $\mathbf{L}_M \coloneqq \mathcal{L}_M\epsilon$. In arriving at the above, we have used Eq. (40).

The $\delta\psi$ and $\delta\bar{\psi}$ terms in Eq. (46) give the field equation (28). On the other hand, by combining the nonsurface terms associated with $\delta S_{\rho\mu}{}^a$ in both Eqs. (42) and (46), we obtain the field equation for the torsion field:

$$\left[-2S^{(\nu|\rho|\mu)} - S^{\mu\nu\rho} + 2S^\nu{}_\sigma{}^\sigma g^{\rho\mu} - 2S^\rho{}_\sigma{}^\sigma g^{\mu\nu} + 2S^\mu{}_\sigma{}^\sigma g^{\nu\rho} + \eta_t\frac{\kappa^2}{4}\bar{\psi}\gamma^{\mu\nu\rho}\psi\right]\epsilon e_{a\nu} = 0. \tag{47}$$

It is straightforward to solve it and arrive at Eq. (27).

Next, we combine the terms involving $\nabla_\rho\delta e^a_\mu$ in Eqs. (42) and (46) with the help of Eq. (27), and then integrate by parts to obtain the additional surface term. Explicitly, the combined result gives

$$\begin{aligned}&\frac{\eta_t - 1}{4}(\bar{\psi}\gamma^{\mu\nu\rho}\psi)e_{a\nu}\nabla_\rho\delta e^a_\mu\epsilon\\ &\quad = d\left(\frac{1}{3!}\epsilon_{\rho\alpha_2\alpha_3\alpha_4}\frac{\eta_t - 1}{4}(\bar{\psi}\gamma^{\mu\nu\rho}\psi)e_{a\nu}\delta e^a_\mu dx^{\alpha_2\alpha_3\alpha_4}\right)\\ &\qquad - (\eta_t - 1)\Sigma^{[\mu\nu]}e_{a\nu}\delta e^a_\mu\epsilon,\end{aligned} \tag{48}$$

where

$$\Sigma_{\mu\nu} \coloneqq \frac{1}{2}[\bar{\psi}\gamma_\nu\nabla_\mu\psi - (\nabla_\mu\bar{\psi})\gamma_\nu\psi]. \tag{49}$$

Combining the last term on the rhs of Eq. (48) with the other terms involving δe^a_μ in Eqs. (42) and (46), we arrive at the field equation (29) after using the on-shell relation

$$\Sigma_{[\mu\nu]} = \bar{\Sigma}_{[\mu\nu]}, \tag{50}$$

upon using Eq. (27).

4. Combining into the symplectic potential

Finally, collecting the surface terms in Eqs. (42), (46), and (48), we obtain the symplectic potential:

$$\begin{aligned}\mathbf{\Theta} &= \frac{1}{3!}\frac{1}{\kappa^2}\epsilon_{\mu\beta\gamma\delta}e^{a\mu}e^{b\nu}\delta\omega_{\nu ab}dx^{\beta\gamma\delta}\\ &\quad + \frac{1}{3!}\frac{1}{2}\epsilon_{\mu\beta\gamma\delta}(\delta\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta\psi)dx^{\beta\gamma\delta}\\ &\quad + \frac{\eta_t - 1}{4}\frac{1}{3!}\epsilon_{\mu\beta\gamma\delta}(\bar{\psi}\gamma^{\mu\nu\rho}\psi)e_{a\rho}\delta e^a_\nu dx^{\beta\gamma\delta}.\end{aligned} \tag{51}$$

Note that this is not yet the final form of Eq. (38). To achieve this, we need to use the on-shell relation (27) and Eq. (40) which can be further simplified, by the antisymmetry of the torsion tensor $S^{\alpha\beta\gamma} = S^{[\alpha\beta\gamma]}$, into

$$e^{a\mu}e^{b\nu}\delta\omega_{\nu ab} = g^{\alpha\mu}g^{\beta\nu}\triangle^{\tau\sigma\lambda}_{\beta\nu\alpha}\nabla_{[\tau}\delta e^a_{\sigma]}e_{a\lambda}. \tag{52}$$

Then, Eq. (51) will be turned into Eq. (38) by throwing away an exact form,

$$d\left(\frac{1}{4\kappa^2}\epsilon_{\mu\beta\rho_1\rho_2}g^{\mu\alpha}g^{\beta\nu}\delta e^a_{[\alpha}e_{\nu]a}dx^{\rho_1\rho_2}\right). \tag{53}$$

This ambiguity is allowed as can be seen from the defining equation (1) of the symplectic potential because Eq. (1) still holds under

$$\boldsymbol{\Theta} \to \boldsymbol{\Theta} + \delta\mu + d\mathbf{Y}(\phi, \delta\phi), \tag{54}$$

as explained in Ref. [7]. Note that μ is due to the shift of $\mathbf{L}$: $\mathbf{L} \to \mathbf{L} + d\mu$.

5. Obtaining the Noether charge

We will now derive the Noether current and hence the Noether charge associated with the vector field ξ, and finally the explicit expression of the quasilocal energy for the minimally coupled Einstein-Cartan-fermion system.

According to the prescription of Refs. [6,7], the Noether current 3-form is given by

$$\mathbf{J}_\xi = \boldsymbol{\Theta}(\phi, \mathcal{L}_\xi \phi) - \xi \cdot \mathbf{L}. \tag{55}$$

Our goal is to extract the Noether charge by rewriting $\mathbf{J}_\xi$ as

$$\mathbf{J}_\xi = d\mathbf{Q}_\xi + (\text{on-shell}), \tag{56}$$

where the terms in (on-shell) vanish when imposing the on-shell condition.

To explicitly carry out the evaluation, we need the Lie derivatives of the vielbein and fermion field, i.e.,

$$\mathcal{L}_\xi e^a_\alpha = \nabla_\alpha \xi^\beta e^a_\beta - \xi^\beta \omega_\beta{}^a{}_c e^c_\alpha - 2S_{\alpha\beta}{}^a \xi^\beta, \tag{57}$$

$$\begin{aligned}\mathcal{L}_\xi \psi &= \xi^\mu \partial_\mu \psi \\ &= \xi^\mu \left(\overset{*}{\nabla}_\mu \bar{\psi} - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\psi - \frac{\eta_t - 1}{4} K_{\mu\nu\rho}\gamma^{\nu\rho}\psi \right).\end{aligned} \tag{58}$$

Here we adopt the convention of Ref. [40] for the Lie derivative of the fermion field by treating it like a scalar. Besides, the following identities are useful in the process of derivation:

$$[\nabla_\alpha, \nabla_\beta] v^\gamma = R_{\alpha\beta\mu}{}^\gamma v^\mu - 2S_{\alpha\beta}{}^\nu \nabla_\nu v^\gamma, \tag{59}$$

$$(\mathcal{L}_\xi g)_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} + 4\xi^\gamma S_{\gamma(\mu}{}^\beta g_{\nu)\beta} \approx 2\nabla_{(\mu}\xi_{\nu)}, \tag{60}$$

$$\nabla_\gamma S^{\alpha\beta\gamma} \approx -\eta_t \frac{\kappa^2}{2} (\nabla^{[\alpha}\bar{\psi}\gamma^{\beta]}\psi + \bar{\psi}\gamma^{[\alpha}\nabla^{\beta]}\psi), \tag{61}$$

$$\xi \cdot \mathbf{L}_M \approx 0, \tag{62}$$

where $\approx$ denotes the weak equality that holds on shell.

We break down the derivation into steps. We first deal with the first term of Eq. (38), called $\boldsymbol{\Theta}^{(1)}(\phi, \mathcal{L}_\xi \phi)$. It yields

$$\begin{aligned}2\kappa^2 \Theta^{(1)} &= \frac{1}{3!}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3} g^{\mu\alpha} g^{\beta\gamma} (\nabla_\beta \nabla_\alpha \xi_\gamma + \nabla_\beta \nabla_\gamma \xi_\alpha - \nabla_\alpha \nabla_\beta \xi_\gamma - \nabla_\alpha \nabla_\gamma \xi_\beta) dx^{\rho_1\rho_2\rho_3} \\ &= \frac{1}{3!}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3} [(2R^\mu{}_\sigma \xi^\sigma + 4S^{\mu\beta\sigma}\nabla_\sigma \xi_\beta) + 2\nabla_\beta \nabla^{[\beta}\xi^{\mu]}] dx^{\rho_1\rho_2\rho_3}.\end{aligned} \tag{63}$$

In arriving at the above, the identities (59) and (60) were used. Moreover, the last term of the last line can be used to make up a total derivative term:

$$\begin{aligned}&\frac{1}{3!}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3} 2\nabla_\beta \nabla^{[\beta}\xi^{\mu]} dx^{\rho_1\rho_2\rho_3} \\ &\quad = \frac{1}{2!}\nabla_{\rho_1}(\boldsymbol{\epsilon}_{\mu\beta\rho_2\rho_3}\nabla^{[\beta}\xi^{\mu]}) dx^{\rho_1\rho_2\rho_3} \\ &\quad = d\left(\frac{1}{2!}\boldsymbol{\epsilon}_{\mu\beta\rho_2\rho_3}\nabla^\beta \xi^\mu dx^{\rho_2\rho_3}\right) + \frac{1}{3} S^{\beta\mu\sigma}\boldsymbol{\epsilon}_{\sigma\rho_1\rho_2\rho_3}\nabla_\beta \xi_\mu dx^{\rho_1\rho_2\rho_3}.\end{aligned} \tag{64}$$

We have used the total antisymmetry of the torsion tensor $S^{\alpha\beta\gamma} \approx S^{[\alpha\beta\gamma]}$ to arrive at the last equality.

Now comes the second term of Eq. (38), denoted as $\boldsymbol{\Theta}^{(2)}(\phi, \mathcal{L}_\xi \phi)$, which by using Eq. (57) can be further simplified as follows:

$$\boldsymbol{\Theta}^{(2)} = -\frac{1}{4!}\bar{\psi}\gamma^{\alpha\nu\sigma}\psi \boldsymbol{\epsilon}_{\sigma\rho_1\rho_2\rho_3} (\nabla_\alpha \xi_\nu - \xi^\beta \omega_{\beta\nu\alpha} - 2S_{\alpha\beta\nu}\xi^\beta) dx^{\rho_1\rho_2\rho_3}. \tag{65}$$

Finally, we now deal with the last term of Eq. (38), called $\boldsymbol{\Theta}^{(3)}(\phi, \mathcal{L}_\xi \phi)$. Using Eq. (58), we can arrive at

$$\begin{aligned}\boldsymbol{\Theta}^{(3)} = &-\frac{1}{3!}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3}\xi_\alpha \bar{\Sigma}^{\alpha\mu} dx^{\rho_1\rho_2\rho_3} \\ &+ \frac{1}{4!}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3}\bar{\psi}\gamma^{\alpha\beta\mu}\psi \xi^\nu \omega_{\nu\alpha\beta} dx^{\rho_1\rho_2\rho_3} \\ &- \frac{1}{3!}\frac{\eta_t - 1}{4}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3}\xi^\alpha S_{\alpha\beta\gamma}\bar{\psi}\gamma^{\beta\gamma\mu}\psi dx^{\rho_1\rho_2\rho_3}.\end{aligned} \tag{66}$$

Then, combining all of the above with

$$\xi \cdot \mathbf{L}_R = \xi \cdot \boldsymbol{\epsilon}(R - 2\Lambda) = \frac{1}{3!}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3}\xi^\nu \delta^\mu_\nu (R - 2\Lambda) dx^{\rho_1\rho_2\rho_3} \tag{67}$$

and using Eq. (62), we can rewrite the Noether current (55) as follows:

$$\mathbf{J}_\xi \approx \frac{1}{3!\kappa^2}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3}\left((G^{\mu\nu}+\Lambda g^{\mu\nu})\xi_\nu - \kappa^2\xi_\nu\bar{\Sigma}^{\nu\mu} + \frac{\eta_t+1}{\eta_t}S^{\mu\beta\gamma}\nabla_\gamma\xi_\beta\right)dx^{\rho_1\rho_2\rho_3}$$
$$+ d\left(\frac{1}{2\kappa^2}\frac{1}{2!}\boldsymbol{\epsilon}_{\mu\beta\rho_2\rho_3}\nabla^\beta\xi^\mu dx^{\rho_2\rho_3}\right) - \frac{1}{3!}\frac{\eta_t+1}{4}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3}\xi^\alpha S_{\alpha\beta\gamma}\bar{\psi}\gamma^{\beta\gamma\mu}\psi dx^{\rho_1\rho_2\rho_3}. \tag{68}$$

Let us "integrate by parts" the last term in the big round bracket in the first line of Eq. (68), and using Eqs. (27) and (61) we can arrive at

$$\frac{1}{3!\kappa^2}\frac{\eta_t+1}{\eta_t}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3}S^{\mu\beta\gamma}\nabla_\gamma\xi_\beta = d\left(-\frac{\eta_t+1}{2\eta_t}\frac{1}{2\kappa^2}\boldsymbol{\epsilon}_{\mu\beta\rho_2\rho_3}S^{\mu\beta\gamma}\xi_\gamma dx^{\rho_2\rho_3}\right) - \frac{\eta_t+1}{2\eta_t}\frac{1}{3\kappa^2}S_{\beta\mu}{}^{\sigma}\boldsymbol{\epsilon}_{\sigma\rho_1\rho_2\rho_3}S^{\mu\beta\gamma}\xi_\gamma dx^{\rho_1\rho_2\rho_3}$$
$$-\frac{\eta_t+1}{2\eta_t}\frac{1}{3}\boldsymbol{\epsilon}_{\mu\rho_1\rho_2\rho_3}\Sigma^{[\mu\nu]}\xi_\nu dx^{\rho_1\rho_2\rho_3}, \tag{69}$$

where we have omitted the term proportional to $S_{\beta\sigma}{}^{\sigma}$ which vanishes on shell. Putting this back into the Noether current expression (68), we see that the quadratic torsion terms cancel each other, and the stress tensor part combined together well, i.e.,

$$-\kappa^2\xi_\nu\bar{\Sigma}^{\nu\mu} - \kappa^2(\eta_t+1)\bar{\Sigma}^{[\mu\nu]}\xi_\nu = -\kappa^2(\xi_\nu\bar{\Sigma}^{(\mu\nu)} + \eta_t\xi_\nu\bar{\Sigma}^{[\mu\nu]}). \tag{70}$$

Thus, we indeed obtain the graviton field equation in the big round bracket in the first line of Eq. (68) after "integrating by parts." Moreover, by imposing the on-shell conditions, the η_t dependence is dropped and the Noether current is put into the desired form of Eq. (39).

In summary, we have derived the explicit form of the symplectic potential (38) and the Noether charge (39) associated with a vector field ξ for the Einstein-Cartan-fermion system. We can then use them to evaluate the corresponding quasilocal energy (12) or its variation (9).

In the remainder of this paper we will apply the above results to some specific examples. One example is to evaluate the quasilocal energy for the perturbative solutions of AdS space due to the fermion field up to second order of the Newton constant: we find that it agrees with the holographic relative entropy. The other example is to evaluate the ADM mass as well as the angular momentum for the asymptotically flat and AdS backgrounds.

B. Comparison with the holographic relative entropy

In this section we would like to explicitly evaluate the quasilocal energy (9) for the perturbative solution, i.e., Eqs. (33) and (36), of the Einstein-Cartan-fermion system. We will find that our result coincides with the holographic relative entropy calculated in Ref. [21].

We will consider the quasilocal energy associated with the so-called entanglement wedge Σ, which is a spatial region bounded by a boundary disk B and the Ryu-Takayanagi minimal surface $\tilde{B}$, whose area gives the holographic entanglement entropy associated with region B for the dual CFT. In Refs. [13,14] it was argued that the quasilocal energy associated with a vector field satisfying the boundary conditions (14)–(15) with $\kappa_s = 2\pi$ [Eqs. (16) and (17)[3]] is nothing but the holographic relative entropy of the dual CFT.[4] Although the argument is very general, only some cases of Einstein gravity were explicitly checked. In particular, in Ref. [13] it was calculated in the Hollands-Wald gauge [10] which requires the gauge transformation to fix the boundary conditions for the associated Killing vector field and the position of $\tilde{B}$.

On the other hand, in Ref. [21] we evaluated the holographic relative entropy for the solution (33) and (36) of the Einstein-Cartan-fermion system, and find that the resultant holographic relative entropy can be negative. If the equivalence between the quasilocal energy and holographic relative entropy also holds for the Einstein-Cartan-Fermion system, this implies that the positive energy condition can be violated. This is intriguing as there is no obvious pathology for the underlying theory and its solutions. Moreover, in this calculation we do not need to specify any vector field as in the definition of the quasilocal energy. Therefore, it is not so trivial to check if the equivalence still holds for the gravity theory with torsion and fermions by direct evaluation of the quasilocal energy (12) for some vector field ξ satisfying the aforementioned boundary conditions.

1. Effect of torsion and fermions

From the results for the symplectic potential (38) and the Noether charge (39), it is obvious that there will be no

[3] In Ref. [14] this condition was put in an asymptotical form: $\mathcal{L}_\xi g_{\mu\nu}|_{z\to 0} = \mathcal{O}(z^{d-2})$, where z is the radial coordinate of AdS_{d+1}. This rapid falling behavior ensures that the modular Hamiltonian only receives a contribution from the leading-order perturbation.

[4] The relative entropy for comparing reduced density matrices ρ_A and σ_A on region A can be evaluated as follows: $S(\rho_A||\sigma_A) = \Delta\langle H_A\rangle - \Delta S_A$, where H_A is the modular Hamiltonian and S_A is the entanglement. Here Δ means taking the difference with respect to two different states. Holographically, $\langle H_A\rangle$ can be obtained via the holographic stress tensor if region A is a disk, and S_A via the Ryu-Takayanagi formula.

direct torsion contribution to the physical charges. However, we emphasize that torsion can still contribute to the physical charges indirectly by sourcing the graviton. For example, the value of the quasilocal energy (which will be computed shortly) depends explicitly on η_t.

Regarding the evaluation of the quasilocal energy based on the symplectic potential (38) and the Noether charge (39), the first question is whether or not the additional terms related to the fermion will contribute. We first consider the integral of Eq. (9) over $\tilde{B}$: in this case the vector field ξ vanishes [i.e., Eq. (16)], so that the $\xi \cdot \mathbf{\Theta}$ and the second term in Eq. (39) vanish as well. Thus, only the usual term in Einstein gravity contributes to the integral of Eq. (9) over $\tilde{B}$.

On the other hand, the integral of Eq. (9) over B is more subtle: the second term in Eq. (39) will not contribute because its pullback vanishes on B due to the fact that ξ is time-like there. Thus, only the usual term of Einstein gravity in Eq. (39) contributes. For $\xi \cdot \mathbf{\Theta}$ we notice that the second and third terms are subleading terms of κ^2 order in comparison with the first term, i.e., the usual term in Einstein gravity. This means that these terms could be suppressed by positive powers of z when approaching the boundary. As we have the explicit solution (33) and (36), we can then perform the power counting of z for those terms associated with the fermion. After doing this, it turns out that both terms vanish on B,[5] e.g., power counting of the third term of Eq. (38):

$$\sqrt{-g}(\delta\bar{\psi}\gamma^{\mu}\psi - \bar{\psi}\gamma^{\mu}\delta\psi) = \frac{\ell^2}{z^4}\delta^{\mu}_{z}\frac{i\alpha^2\beta^2 {r_L}^2 z^7 (3{\eta_t}^2 + 2\mu_0 m^2\ell^2)}{3\ell^{12}}\mathrm{K} \to 0 \quad \text{as} \quad z \to 0. \tag{71}$$

From the above analysis, although the symplectic potential and Noether charge contain the terms associated with the fermion, they will not contribute to the quasilocal energy (9). Therefore, the effect of torsion and fermions comes into play only through the solution of the field equations. We should say that this conclusion is quite general because the power counting is controlled by the fall-off behavior of the on-shell solution, which is completely determined by the metric of AdS space.

2. Fix the vector field

The next step in evaluating the quasilocal energy is to choose an appropriate vector field ξ that satisfies the required boundary conditions. As mentioned, in Ref. [13] this was done by choosing the Hollands-Wald gauge so that ξ and the positions of B and $\tilde{B}$ are fixed. This will save some time when finding the new ξ in the deformed background, but it requires solving the gauge conditions. The latter turns out to be a tedious procedure, as demonstrated in Ref. [13]. Instead, we will directly find the deformed vector field in the perturbative AdS space up to order K^2, and then use it to evaluate the quasilocal energy. This should be equivalent to the evaluation in the Hollands-Wald gauge.

We start with the Killing vector of AdS space,

$$\xi_{\mathrm{AdS}} = \frac{\pi}{R_A}(R_A^2 - z^2 - r^2 - t^2)\partial_t - \frac{2\pi}{R_A}t(z\partial_z + r\partial_r), \tag{72}$$

which satisfies the required conditions, and is the vector field used in the Hollands-Wald gauge. However, after the perturbation away from the pure AdS space there can be no Killing vector field. Despite this, for our purpose of evaluating the quasilocal energy it is sufficient to find a vector field that satisfies these boundary conditions on $B, \tilde{B}$. In general, there is more than one solution for ξ that satisfies the above boundary conditions. However, the details of ξ in the bulk of Σ are not relevant, as the quasilocal energy (9) is given by the integral over B and $\tilde{B}$. Therefore, all of the ξ's satisfying the same boundary conditions will yield the same quasilocal energy.

Since we solve the field equation perturbatively up to order K^2, we also only need to solve ξ perturbatively up to the same order of K. Here, we will solve the solution using the following ansatz:

$$\xi = \frac{\pi}{R}(r(z)^2 - r^2 - t^2)\partial_t - \frac{2\pi}{R}t[(f_0(z) + \mathrm{K}f_1(z) + \mathrm{K}^2 f_2(z))\partial_z + g(r)\partial_r], \tag{73}$$

where $f_0 = z$, and

$$g(r) = \sqrt{R_A^2 - z^2} + \mathrm{K}g_1(z) + \mathrm{K}^2 g_2(z) + O(\mathrm{K}^3) \quad \text{on} \quad \tilde{B}. \tag{74}$$

Note also that $r(z)$ is the Ryu-Takayanagi minimal surface solved with respect to the perturbed metric (36), and the details can be found in Eqs. (105)–(110) of Ref. [21].

One finds that the real challenge of this ansatz is to solve the condition (15), and this will fix the unknown functions f_1, f_2 and g_1, g_2. The explicit forms of f_1, f_2 and g_1, g_2 are shown in the Appendix. Note that the second-order functions f_2 and g_2 are quite complicated in comparison with f_1 and g_1. The former even contains log pieces, which are necessary to cancel the log pieces arising from the integral of the terms involving f_1 and g_1. In this sense, it is quite nontrivial to have the quasilocal energy evaluated here agree with the holographic relative entropy obtained in Ref. [21], as shown below.

[5] Explicitly, we can also see that the second term in $\xi \cdot \mathbf{\Theta}$ vanishes by the tensor structure itself even before taking $z \to 0$.

3. Evaluation of the quasilocal energy

We are now ready to explicitly evaluate the quasilocal energy (12) perturbatively up to the second order in κ based on the above discussions. In particular, all of the terms involving the fermion in the symplectic potential and Noether charge will not contribute. Moreover, asymptotically, both the Noether charge and $-\xi\cdot\mathbf{K}$ do not contribute beyond the linear order because of the asymptotic fall-off behavior of fields.

We first consider the integral over $\tilde{B}$, for which only the first term in Eq. (39) contributes. Using the vector field (73) with the solutions (A5)–(A7) found in the Appendix, and the minimal surface $r(z)$ solved in Ref. [21], we obtain

$$\Delta\int_{\tilde{B}}\mathbf{Q}_\xi = S_1 + S_2, \tag{75}$$

where

$$S_1 = \frac{\pi^2\alpha\beta m\mu_0 {R_A}^3}{2\ell^2},$$
$$S_2 = \frac{4\pi^3\alpha^2\beta^2 G_N R_A^6(2\eta_t^2 - \mu_0^2 m^2\ell^2)}{35\ell^8}. \tag{76}$$

Here S_1 is the first-order result in κ and S_2 is the second-order one. These are exactly Eqs. (113) and (114) of Ref. [21]. Although the functions $r(z)$, f_2, and g_2 are very complicated and contain log pieces, it is amazing that the log pieces all cancel out to yield a simple final result.

Now we consider the integral over the boundary disk B, on which the vector ξ reduces to the conformal Killing vector ζ,

$$\xi|_B = \frac{\pi}{R_A}(R_A^2 - r^2)\partial_t. \tag{77}$$

Unlike $\tilde{B}$, the shape of B is independent of the perturbation of the metric. Thus, the integral of the Noether charge turns out to be

$$\Delta\int_B\mathbf{Q}_\xi = \int_B\mathbf{Q}_\xi(\phi) - \int_B\mathbf{Q}_\xi(\phi_0) = \frac{\pi^2\alpha\beta(3\mu_0 - 4)m{R_A}^3}{12\ell^2}, \tag{78}$$

while the integral of the symplectic potential term gives

$$-\int_B\xi\cdot\mathbf{\Theta}^{(\mathrm{R})} = \frac{\pi^2\alpha\beta(3\mu_0 + 4)m{R_A}^3}{12\ell^2}. \tag{79}$$

The use of the symplectic potential is sufficient because $\mathbf{K}$ does not contribute beyond linear order.

Summing up Eqs. (78) and (79), we get

$$\Delta\int_B(\mathbf{Q}_\xi - \xi\cdot\mathbf{K}) = \frac{\pi^2\alpha\beta\mu_0 m{R_A}^3}{2\ell^2}, \tag{80}$$

which agrees with the change of the expectation values of the modular Hamiltonian previously found in Ref. [21]. Note that the above result is only first order in κ because of the fall-off behavior in z of the perturbative solution for the field equations. This kind of fall-off behavior also ensures the integrability condition (4) and thus the existence of the full H_ξ, as discussed earlier.

In summary, by subtracting Eq. (75) from Eq. (80)[6] we obtain the quasilocal energy for the perturbative solution (33) and (36):

$$\Delta H_\xi = \frac{4\pi^3\alpha^2\beta^2 G_N R_A^6(\mu_0^2 m^2\ell^2 - 2\eta_t^2)}{35\ell^8}. \tag{81}$$

This happens to be the same as the holographic relative entropy obtained in Ref. [21]. The positive energy condition can be violated if

$$\mu_0^2 m^2\ell^2 < 2\eta_t^2. \tag{82}$$

C. Fermion and torsion effect on the ADM mass and angular momentum

Having the expression for quasilocal energy at hand, one is able to obtain the ADM mass or other global charges by simply replacing the spatial region Σ with the 2-sphere at infinity. We have seen from Eqs. (38) and (39) that there are no direct contributions from torsion to the physical charges, as they are formally independent of η_t. However, we emphasize again that a torsion effect can show up through sourcing the graviton field. With this observation, it is then natural to ask if the presence of the fermion deforms the global charges. As far as we know, the inclusion of fermions is rarely discussed in the literature. We will answer this question for the ADM mass and angular momentum in both asymptotically flat spacetime and the AdS background.

1. Review of ADM quantities in Einstein gravity

We first review the derivation of the ADM mass in asymptotically flat spacetime for pure Einstein gravity (as done in Ref. [7]) to demonstrate how the quasilocal energy is linked to the ADM mass. The variation of the ADM mass is given by

[6]The relative sign reflects the opposite normal directions of B and $\tilde{B}$.

$$\delta H_\xi = \int_{S^2_\infty} (\delta \mathbf{Q}_\xi - \xi \cdot \mathbf{\Theta}), \tag{83}$$

where S^2_∞ is the 2-sphere at infinity and ξ is the asymptotic Killing vector $\xi := \hat{t} = \partial_t$. To find the ADM mass, we need to find $\mathbf{K}$ defined in Eq. (10) and set the absolute value of $H_{\hat{t}}$ for Minkowski space to zero. An asymptotically flat spacetime has the following boundary condition for the metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + O(1/r), \tag{84}$$

where $\eta_{\mu\nu}$ is the metric of Minkowski space, and r is the radial component in polar coordinates,

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 + O(1/r). \tag{85}$$

To find $\mathbf{K}$, we first evaluate $-\hat{t}\cdot\mathbf{\Theta}$ and try to pull out δ in accordance with Eq. (84). The result is

$$\int_{S^2_\infty} \hat{t}\cdot\mathbf{\Theta} = -\frac{1}{2\kappa^2}\delta \int_{S^2_\infty} dS((\partial_r g_{00} - \partial_0 g_{r0}) + r^k h^{ij}(\partial_i h_{kj} - \partial_k h_{ij})), \tag{86}$$

where $i, j, k = 1, 2, 3$, $r^k = \delta^k_r$, h_{ij} is the spatial metric, dS is the area element of a 2-sphere of radius r, and we have repeatedly used the asymptotic boundary condition (84). $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol and $\tilde{\epsilon}_{0r\theta\varphi} = 1$. Therefore, $\mathbf{K}$ is given by

$$(\hat{t}\cdot\mathbf{K})_{\alpha\beta} = -\frac{1}{2\kappa^2}\boldsymbol{\epsilon}_{\alpha\beta}((\partial_r g_{00} - \partial_0 g_{r0}) + r^k h^{ij}(\partial_i h_{kj} - \partial_k h_{ij})). \tag{87}$$

Similarly, the Noether-charge part can be calculated straightforwardly,

$$\int_{S^2_\infty} \mathbf{Q}_{\hat{t}} = -\frac{1}{2\kappa^2}\int_{S^2_\infty} dS(\partial_r g_{00} - \partial_0 g_{r0}), \tag{88}$$

where the asymptotic boundary condition (84) is used in the last equality. Notice that the higher subleading terms in the asymptotic expansion of the metric do not contribute.

Summing up Eq. (88) and the integration of Eq. (87), we get precisely the well-known ADM mass formula,

$$H_{\hat{t}} = \int_{S^2_\infty} (\mathbf{Q}_{\hat{t}} - \hat{t}\cdot\mathbf{K}) = \frac{1}{2\kappa^2}\int_\infty dS r^k h^{ij}(\partial_i h_{kj} - \partial_k h_{ij}) := M_{\mathrm{ADM}}. \tag{89}$$

If the system admits an asymptotic rotational Killing vector, e.g., $\hat{\varphi} := \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$, there is an associated angular momentum defined by[7]

$$J_{\hat{\varphi}} := -\int_{S^2_\infty} \mathbf{Q}_{\hat{\varphi}}. \tag{90}$$

Notice that the $-\hat{\varphi}\cdot\mathbf{K}$ term drops because it is pulled back to zero on S^2_∞. For Einstein gravity, this form of the angular momentum is exactly in the form of the Komar formula [41],[8]

$$J^{(\mathrm{Komar})}_{\hat{\varphi}} = \frac{1}{2!}\frac{1}{2\kappa^2}\int_{S^2_\infty} \boldsymbol{\epsilon}_{\alpha\beta\rho_1\rho_2}\tilde{\nabla}^\alpha \xi^\beta dx^{\rho_1\rho_2}. \tag{91}$$

2. ADM quantities in the Einstein-Cartan-fermion system

Now we are in a position to go beyond Einstein gravity and check if the fermion contributes to the ADM mass or angular momentum. We first consider again the asymptotically flat spacetime. As we will not explicitly determine $\mathbf{K}$ in the torsion gravity case, we need to confirm its existence by the integrability condition (4). To this end, we compute the symplectic current,

$$\begin{aligned}\boldsymbol{\omega} = {} & \frac{1}{2\kappa^2}\boldsymbol{\epsilon}_\alpha P^{\alpha\beta\mu\nu\rho\sigma}(\delta_2 g_{\beta\mu}\tilde{\nabla}_\nu \delta_1 g_{\rho\sigma} - \delta_1 g_{\beta\mu}\tilde{\nabla}_\nu \delta_2 g_{\rho\sigma}) + \boldsymbol{\epsilon}_\mu\Big[-\frac{1}{4}(\bar{\psi}\gamma^{\alpha\beta\mu}\psi)e^\gamma_b e_{\beta a}\delta_1 e^b_\gamma \delta_2 e^a_\alpha - \frac{1}{4}\delta_1(\bar{\psi}\gamma^{\alpha\beta\mu}\psi)\delta_2 e^a_\alpha e_{\beta a} \\ & -\frac{1}{4}(\bar{\psi}\gamma^{\alpha\beta\mu}\psi)\delta_2 e^a_\alpha \delta_1 e_{\beta a} + \frac{1}{2}(\delta_2\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\delta_2\psi)e^\alpha_a \delta_1 e^a_\alpha + \frac{1}{2}(\delta_2\bar{\psi}\gamma^\mu\delta_1\psi - \delta_1\bar{\psi}\gamma^\mu\delta_2\psi) \\ & -\frac{1}{2}(\delta_2\bar{\psi}\gamma^\alpha\psi - \bar{\psi}\gamma^\alpha\delta_2\psi)e^\mu_b \delta_1 e^b_\alpha\Big] - (\delta_1 \leftrightarrow \delta_2),\end{aligned} \tag{92}$$

[7]The minus sign comes from the fact that the Killing vector $\hat{\varphi}$ is space-like.

[8]It is worth noting that the Komar anomalous factor of 2 between the Komar mass and angular momentum formulas is naturally resolved in Wald's formalism [7].

where $-(\delta_1 \leftrightarrow \delta_2)$ is only applied to the second term above, and

$$P^{\alpha\beta\mu\nu\rho\sigma} = g^{\alpha\rho}g^{\sigma\beta}g^{\mu\nu} - \frac{1}{2}g^{\alpha\nu}g^{\beta\rho}g^{\sigma\mu} - \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}g^{\rho\sigma} - \frac{1}{2}g^{\beta\mu}g^{\alpha\rho}g^{\sigma\nu} + \frac{1}{2}g^{\beta\mu}g^{\alpha\nu}g^{\rho\sigma}. \tag{93}$$

Recall that $\tilde{\nabla}_\mu$ is the Riemannian covariant derivative without including torsion.

To satisfy Eq. (4), with Eq. (84), we need the following asymptotic boundary condition for the fermion:

$$\delta\psi \sim \frac{1}{r^n}, \qquad n > 1, \qquad \psi \sim \frac{1}{r^m}, \qquad m > 0. \tag{94}$$

In fact, we can do better. As $\delta\phi$ satisfies the linearized field equations in Wald's formalism, their asymptotic boundary conditions are constrained by on-shell relations. Hence, we may start from the marginal value of the boundary condition for the metric—i.e., the value that gives a finite ADM mass, $g = \eta + O(1/r)$—and look for the boundary condition for the fermions via the consistency of the field equation.

The Minkowski background version of the perturbative Einstein field equation for the Einstein-Cartan-fermion system is given by [21]

$$\tilde{G}^{(1)}_{\mu\nu} = \tilde{\sigma}^{(0)}_{\mu\nu}, \tag{95}$$

whose linearized version is, schematically,[9]

$$\Box\delta g_{\mu\nu} = \tilde{\nabla}^{(0)}_{(\mu}\bar{\psi}\gamma^{(0)}_{\nu)}\psi^{(0)} - \bar{\psi}\gamma^{(0)}_{(\nu}\tilde{\nabla}^{(0)}_{\mu)}\psi^{(0)}, \tag{96}$$

where $\Box$ denotes a second-order derivative operator that includes the Laplacian. As $\delta g \sim 1/r$, by simple power counting of Eq. (96) we conclude that

$$\psi^{(0)} \sim 1/r. \tag{97}$$

Notice that Eq. (97) is compatible with Eq. (94) as $\delta\psi$ should be subleading in the $1/r$ expansion. Thus, $\delta\psi$ should obey Eq. (94) so that the integrability condition is indeed satisfied.

Similarly, further power counting can tell us whether the non-Einstein parts of the Noether charge (39) and the symplectic potential (38) will contribute to the ADM mass and angular momentum. A straightforward power counting then shows that all of the fermion terms in $-\xi \cdot \mathbf{\Theta}$ do not contribute to the ADM mass. For the fermion part in the Noether charge, notice that it is pulled back to zero on S^2_∞ due to the tensorial structure and $\xi = \hat{t}$. Therefore, we find that the fermion does not contribute to the ADM mass for our Einstein-Cartan-fermion system in asymptotically flat space.

However, it is a different story for the angular momentum (90), because the fermion term in the Noether charge no longer has a vanishing pullback to S^2_∞. This is because the Killing vector is now $\hat{\varphi}$, not $\hat{t}$. Moreover, the power counting shows that the fermion part has a finite contribution. Therefore, we expect that the presence of the spin-1/2 fermion will deform the angular momentum, and this may be an observable physical effect in, for example, compact binary inspirals [32–34]. Let us perform the power counting explicitly for this finite contribution. Use the metric (85), we construct vielbeins connecting the polar coordinates with the orthonormal frame:

$$\mathbf{e}^{(t)} = dt, \qquad \mathbf{e}^{(r)} = dr, \qquad \mathbf{e}^{(\theta)} = rd\theta, \qquad \mathbf{e}^{(\varphi)} = r\sin\theta d\varphi, \tag{98}$$

where the boldface letter $\mathbf{e}$ again represents differential forms, and the indices in parentheses label the orthonormal frame indices. Notice that, in particular,

$$\mathbf{e}^{(\varphi)}_\varphi = r\sin\theta,$$
$$\Rightarrow \gamma^\varphi = g^{\nu\varphi}e^a{}_\nu\gamma_a = g^{\varphi\varphi}e^{(\varphi)}_\varphi\gamma_{(\varphi)} = \frac{1}{r\sin\theta}\gamma_{(\varphi)}. \tag{99}$$

Now, let us check the contribution of the fermion term in Eq. (90),

$$\boldsymbol{\epsilon}_{\alpha\beta\rho_1\rho_2}\bar{\psi}\gamma^{\alpha\beta\gamma}\psi g_{\gamma\delta}\xi^\delta dx^{\rho_1\rho_2} = 4\bar{\psi}\gamma^{0r\varphi}\psi(r^2\sin^2\theta)\frac{1}{r\sin\theta}(r^2\sin\theta d\theta d\phi), \tag{100}$$

which is finite because $\bar{\psi} \sim \psi \sim 1/r$ and $\gamma^\varphi \sim 1/r$ according to Eq. (99). Therefore, when the fermion is present, we find that the angular momentum is the following extension of the Komar formula:

$$J_{\hat{\varphi}} = J^{(\mathrm{Komar})}_{\hat{\varphi}} - \frac{1}{2!}\frac{1}{2}\int_{S^2_\infty}\boldsymbol{\epsilon}_{\alpha\beta\rho_1\rho_2}\frac{1}{4}\bar{\psi}\gamma^{\alpha\beta\gamma}\psi\xi_\gamma dx^{\rho_1\rho_2},$$
$$= J^{(\mathrm{Komar})}_{\hat{\varphi}} - \frac{1}{4}\int_{S^2_\infty}\bar{\psi}\gamma_{\rho_2}\gamma_5\psi\xi_{\rho_1}dx^{\rho_1\rho_2}, \tag{101}$$

where $J^{(\mathrm{Komar})}_{\hat{\varphi}}$ is the Komar angular momentum and we follow the gamma matrix convention from Ref. [21]. This means that the axial current generates angular momentum for the Einstein-Cartan-fermion system in asymptotically flat spacetime.

For the case of the AdS background, we have the explicit solutions (33) and (36). It is then straightforward to perform a power counting and conclude that the existence of

[9] On the rhs of Eqs. (95) and (96) appearing in Ref. [21] there is an overall IR factor r_L^2. For simplicity, we just set it to one here.

fermions does not deform the ADM mass or angular momentum formulas.

IV. IMPLICATIONS FOR GRAVITATIONAL-WAVE PHYSICS

LIGO's discovery of gravitational waves from binary black hole inspirals and mergers has opened a new era of gravitational-wave astronomy. Three events [28–30] have already been confirmed as binary black hole mergers, and we may expect dozens or hundreds more in the near future. As more and more gravitational-wave data is collected, more precise tests of Einstein gravity can be expected. Therefore, this is the right time to push the modified gravity theories into the regime of precision tests. As Einstein gravity has passed many precision tests on the scale of the Solar System (see, e.g., Refs. [42,43]), we would not expect detectable deviations from Einstein gravity in the weak-field regime. On the other hand, the black hole mergers discovered by LIGO are in the strong-field regime, for which we may like to test the deviation from Einstein gravity in a more precise way. Some constraints on modified gravities derived from LIGO's discoveries have been studied, for example, in Ref. [31], which however did not include torsion gravity.

There are many proposals for modified gravity theories: the most common types are the scalar-tensor gravities, such as Brans-Dicke theory or the more general Horndeski theory. They can be treated formally in the usual second-order metric formulation of Einstein gravity. On the other hand, by adding torsion (as shown in this work), one needs to adopt the first-order formulation involving vielbeins and spin connections.

There are many physical reasons to introduce torsion, in particular in the new gravitational-wave physics. The key point is that the torsion is naturally sourced by some high-energy coherent states of fermionic matter, which is the main constituent of all astronomical objects, including black holes and neutron stars. We can imagine that for massive fermion stars (say, hundreds of solar masses), the torsion coupling could affect the pattern of gravitational radiation originating from the inspirals and mergers of such stars. In particular, it is possible that some fermions are dark matter candidates, which form the dark fermion stars which could only couple to gravity. In this case the LIGO observations can serve as a window onto dark matters and torsion gravity. Moreover, in order to find the waveform of these dark stars' merger, we need to implement numerical relativity calculations. Numerical relativity is formulated as a $3+1$ Cauchy problem—e.g., the famous BSSNOK formulation [37–39]—which, however, is formulated in terms of the metric and extrinsic curvature, not the vielbein and spin connection. To develop a similar $3+1$ Cauchy problem for torsion gravity coupled to dark fermion matter, we instead need to adopt the first-order formulation in terms of the vielbein and spin connection, and the formulation developed in this work should be quite useful for this purpose. We should emphasize that the dynamics of dark stars should be clearer, simpler, and more massive than that of neutron stars without the complication of nuclear interactions and the Tolman-Oppenheimer-Volkoff limit on the mass of neutron stars. This should be helpful for testing modified gravity theories such as torsion gravity using gravitational radiation in the strong-field regime.

On the other hand, even without introducing the fermion matter, the torsion could also be induced at low energies by either the high-energy fermion matter or the nonlinear self-interaction of gravitons, e.g., the Routhian for the low-energy effective dynamics of the spin bodies of mass m contains the terms

$$\frac{1}{2}\omega_\mu^{ab}S_{ab}u^\mu+\frac{1}{2m}R_{\alpha\beta\gamma\delta}S^{\gamma\delta}u^\alpha S^{\beta\sigma}u_\sigma, \tag{102}$$

where S^{ab} is the spin tensor in the local flat frame characterized by the tetrad e_μ^a, $S^{ab}\coloneqq S^{\mu\nu}e_\mu^a e_\nu^b$, and u^μ is the 4-velocity of the spinning body. In the above, the Riemann tensor $R_{\alpha\beta\gamma\delta}$ and the spin connection ω_μ^{ab} are the ones for torsion gravity defined in Eqs. (21) and (22), respectively. By using this Routhian with the help of the tricks developed in this work, we believe that one can derive the gravitational waveform of the inspirals of binary black holes by following the effective approach to post-Newtonian (PN) gravity developed in Refs. [44–46]. By incorporating torsion in this way, its effect on the gravitational waveform should be comparable with the pure Riemannian gravity effect at higher PN order, e.g., 4PN. Furthermore, this torsion effect could be significant for the self-force of a spin body around a supermassive Kerr black hole, which can also be studied by generalizing the standard self-force problem (see, e.g., Ref. [47]). The result should be relevant to the quasinormal modes in the merger phase of a binary black hole due to the torsion effect. We remark that the 1PN analysis in the context of Poincaré gauge theory [48] was already done more than three decades ago [49], and no difference from the Einstein gravity was found. However, this is expected as the torsion effect should manifest as a finite-size effect at the higher PN order. In summary, we are optimistic that we will see the torsion effect when the precision of detectors is improved in the future.

V. CONCLUSION

The positive energy condition is important for the stability of solutions of the theory of gravity. However, in the past it was mostly done for Einstein gravity. In this paper we adopted Wald's formalism of the covariant Noether charge to construct an explicit expression of the quasilocal energy for the Einstein-Cartan-fermion system. We evaluated the quasilocal energy of the entanglement wedge for some particular solution, and found that it is not always positive definite for an arbitrary fermion mass and torsion-axial current coupling. This result implies a violation of the positive energy theorem for the Einstein-Cartan-fermion system, and provides a nontrivial example of a swampland beyond the symmetry principle and possibly the weak gravity conjecture. Though we cannot see the whole scope of the swampland from a simple example, it can be a stepping stone for further exploration of the issue. Moreover, the on-shell value of the quasilocal energy agrees with the holographic relative entropy evaluated in Ref. [21]. This generalizes their equivalence from Einstein gravity to torsion gravity.

Besides, we found that the quasilocal energy is *formally* independent of the torsion-axial current coupling, although the dependence will come in through the solutions of the field equations when evaluating it on shell. Despite that, this "pseudotopological" nature is unexpected and thus intriguing, and may have some deep implication for AdS/MERA duality along similar lines as the bulk/edge correspondence for topological insulators.

Torsion is a very natural addition to Einstein gravity in the presence of fermion sources, and thus torsion gravity should be called for the tests of the observation data. In view of this, we have also discussed the implications of our results for the tests of torsion gravity in the context of gravitational-wave astronomy. We elaborated that the techniques developed here should be useful in generating the gravitational waveform templates of torsion gravity from either the effective-one-body formalism or from numerical relativity simulations, because there has not been much discussion in the literature on the first-order formulation in terms of the vielbein and spin connection in the aforementioned approaches. We are currently working in this direction, and hope to report the results in the near future.

ACKNOWLEDGMENTS

This work is partially supported by National Natural Science Foundation of China (NCTS). F. L. L. is supported by the Taiwan Ministry of Science and Technology through Grants No. 103-2112-M-003 -001 -MY3, No. 103-2811-M-003 -024, No. 106-2112-M-003 -004 -MY3, No. 106-2918-I-003 -008, and No. 06-2811-M-003 -026. B. N. is supported in part by NSFC under Grant No. 11505119. We thank Carlos Cardona, Chong-Sun Chu, Dimitrios Giataganas, Kyung Kiu Kim, Rong-Xin Miao, Pichet Vanichchapongjaroen, and Dong-han Yeom for their discussions. This project was initiated at the IF-YITP GR + HEP + Cosmo International Symposium VI, 3–5 August, 2016. F. L. L. thanks members of IF for the hospitality during his visit. S.-L. K. is grateful to NCTS and NTNU for their kind hospitality during his visits while working on this project.

APPENDIX: SOLVING THE VECTOR FIELD ξ UP TO $O(\kappa^4)$

From the minimal surface equation

$$r = r(z), \tag{A1}$$

which was also obtained in Ref. [21], one obtains the two unit normal vectors as

$$n^{\mu}_{(1)} = -\frac{g^{0\mu}}{\sqrt{-g^{00}}},$$
$$n^{\mu}_{(2)} = \frac{1}{\sqrt{g^{rr} - 2g^{zr}r'(z) + g^{zz}r'(z)^2}}(g^{\mu r} - g^{\mu z}r'(z)). \tag{A2}$$

The binormal vector is then

$$\begin{aligned} n^{\mu\nu} &= n^{\mu}_{(1)}n^{\nu}_{(2)} - n^{\mu}_{(2)}n^{\nu}_{(1)} \\ &= \frac{1}{\sqrt{-g^{00}(g^{rr} - 2g^{zr}r' + g^{zz}r'^2)}} \\ &\quad \times [-g^{0\mu}g^{r\nu} + g^{0\mu}g^{z\nu}r'(z) - (\mu \leftrightarrow \nu)], \end{aligned} \tag{A3}$$

which has only nonzero (independent) components (tr) and (tz). On the other hand,

$$\tilde{\nabla}^{[\mu}\xi^{\nu]} = g^{\mu[\rho}g^{\sigma]\nu}\partial_{\rho}(\xi^{\tau}g_{\tau\sigma}). \tag{A4}$$

The boundary condition (15) then gives us two equations for the unknown functions at each order. One then obtains the following solutions:

$$g_1 = \frac{\alpha\beta\mu_0 m r_L^2 z^3(3z^2 - 5R_A^2)}{8\ell^4\sqrt{R_A^2 - z^2}},$$
$$f_1 = -\frac{\alpha\beta\mu_0 m r_L^2 z^2(7z^2 - 3R_A^2)}{8\ell^4}, \tag{A5}$$

$$\begin{aligned} g_2(z) = {}& \frac{1}{40320R_A^2\ell^{10}(R_A^2 - z^2)^{3/2}}\alpha^2\beta^2 r_L^4[7776\mu_0^2 m^2 R_A^8\ell^2(R_A^4 - 3R_A^2 z^2 + 2z^4)\log(R_A) \\ & - 7776\mu_0^2 m^2 R_A^8\ell^2(R_A^4 - 3R_A^2 z^2 + 2z^4)\log(R_A + z) + z(7776\mu_0^2 m^2 R_A^{11}\ell^2 - 3888\mu_0^2 m^2 R_A^{10} z\ell^2 \\ & - 7776\mu_0^2 m^2 R_A^9 z^2\ell^2 + 4050\mu_0^2 m^2 R_A^8 z^3\ell^2 + 1485\mu_0^2 m^2 R_A^6 z^5\ell^2 + 6720\mu_0 m^2 R_A^6 z^5\ell^2 - 4480 m^2 R_A^6 z^5\ell^2 \\ & + 33993\mu_0^2 m^2 R_A^4 z^7\ell^2 - 13440\mu_0 m^2 R_A^4 z^7\ell^2 + 8960 m^2 R_A^4 z^7\ell^2 - 77220\mu_0^2 m^2 R_A^2 z^9\ell^2 + 6720\mu_0 m^2 R_A^2 z^9\ell^2 \\ & - 4480 m^2 R_A^2 z^9\ell^2 + 40320\mu_0^2 m^2 z^{11}\ell^2 + 10080\eta_t^2 R_A^4 z^7 - 20160\eta_t^2 R_A^2 z^9 + 10080\eta_t^2 z^{11})], \end{aligned} \tag{A6}$$

$$f_2(z) = \frac{1}{20160R_A^2\ell^{10}(R_A+z)}\alpha^2\beta^2 r_L^4 z[-7776\mu_0^2 m^2 R_A^9\ell^2\log(R_A+z) + 7776\mu_0^2 m^2 R_A^9\ell^2\log(R_A) + 3888\mu_0^2 m^2 R_A^8 z\ell^2 + 7776\mu_0^2 m^2 R_A^8 z\ell^2\log(R_A) - 7776\mu_0^2 m^2 R_A^8 z\ell^2\log(R_A+z) - 4671\mu_0^2 m^2 R_A^5 z^4\ell^2 - 4671\mu_0^2 m^2 R_A^4 z^5\ell^2 - 7110\mu_0^2 m^2 R_A^3 z^6\ell^2 + 3360\mu_0 m^2 R_A^3 z^6\ell^2 - 2240 m^2 R_A^3 z^6\ell^2 - 7110\mu_0^2 m^2 R_A^2 z^7\ell^2 + 3360\mu_0 m^2 R_A^2 z^7\ell^2 - 2240 m^2 R_A^2 z^7\ell^2 + 20160\mu_0^2 m^2 R_A z^8\ell^2 + 20160\mu_0^2 m^2 z^9\ell^2 - 5040\eta_t^2 R_A^3 z^6 - 5040\eta_t^2 R_A^2 z^7 + 5040\eta_t^2 R_A z^8 + 5040\eta_t^2 z^9]. \tag{A7}$$

[1] R. Arnowitt, S. Deser, and C. W. Misner, Energy and the criteria for radiation in general relativity, Phys. Rev. **118**, 1100 (1960).

[2] J. M. Bardeen, B. Carter, and S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. **31**, 161 (1973).

[3] G. W. Gibbons and S. W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D **15**, 2752 (1977).

[4] J. D. Brown and J. W. York, Jr., Quasilocal energy and conserved charges derived from the gravitational action, Phys. Rev. D **47**, 1407 (1993).

[5] L. B. Szabados, Quasi-local energy-momentum and angular momentum in GR: A review article, Living Rev. Relativ. **7**, 4 (2004).

[6] R. M. Wald, Black hole entropy is the Noether charge, Phys. Rev. D **48**, R3427 (1993).

[7] V. Iyer and R. M. Wald, Some properties of Noether charge and a proposal for dynamical black hole entropy, Phys. Rev. D **50**, 846 (1994).

[8] V. Iyer and R. M. Wald, A comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes, Phys. Rev. D **52**, 4430 (1995).

[9] R. M. Wald and A. Zoupas, A general definition of "conserved quantities" in general relativity and other theories of gravity, Phys. Rev. D **61**, 084027 (2000).

[10] S. Hollands and R. M. Wald, Stability of black holes and black branes, Commun. Math. Phys. **321**, 629 (2013).

[11] S. Ryu and T. Takayanagi, Holographic Derivation of Entanglement Entropy from AdS/CFT, Phys. Rev. Lett. **96**, 181602 (2006).

[12] S. Ryu and T. Takayanagi, Aspects of holographic entanglement entropy, J. High Energy Phys. 08 (2006) 045.

[13] N. Lashkari and M. Van Raamsdonk, Canonical energy is quantum Fisher information, J. High Energy Phys. 04 (2016) 153.

[14] N. Lashkari, J. Lin, H. Ooguri, B. Stoica, and M. Van Raamsdonk, Gravitational positive energy theorems from information inequalities, Prog. Theor. Exp. Phys. **2016**, 12C109 (2016).

[15] R. Schon and S.-T. Yau, On the proof of the positive mass conjecture in general relativity, Commun. Math. Phys. **65**, 45 (1979).

[16] R. Schon and S.-T. Yau, Proof of the positive mass theorem. 2., Commun. Math. Phys. **79**, 231 (1981).

[17] E. Witten, A simple proof of the positive energy theorem, Commun. Math. Phys. **80**, 381 (1981).

[18] G. W. Gibbons, S. W. Hawking, G. T. Horowitz, and M. J. Perry, Positive mass theorems for black holes, Commun. Math. Phys. **88**, 295 (1983).

[19] C.-C. M. Liu and S.-T. Yau, Positivity of Quasilocal Mass, Phys. Rev. Lett. **90**, 231102 (2003).

[20] L. F. Abbott and S. Deser, Stability of gravity with a cosmological constant, Nucl. Phys. **B195**, 76 (1982).

[21] B. Ning and F. L. Lin, Relative Entropy and Torsion Coupling, Phys. Rev. D **94**, 126007 (2016).

[22] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick, and J. M. Nester, General relativity with spin and torsion: Foundations and prospects, Rev. Mod. Phys. **48**, 393 (1976).

[23] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, The string landscape, black holes and gravity as the weakest force, J. High Energy Phys. 06 (2007) 060.

[24] X.-G. Wen, Topological orders and edge excitations in fractional quantum Hall states, Adv. Phys. **44**, 405 (1995).

[25] M. Z. Hasan and C. L. Kane, Topological insulators, Rev. Mod. Phys. **82**, 3045 (2010).

[26] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Symmetry-protected topological orders in interacting bosonic systems, Science **338**, 1604 (2012).

[27] Y. F. Cai, S. Capozziello, M. De Laurentis, and E. N. Saridakis, $f(T)$ teleparallel gravity and cosmology, Rep. Prog. Phys. **79**, 106901 (2016).

[28] B. P. Abbott *et al.* (LIGO Scientific and VIRGO Collaborations), GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Phys. Rev. Lett. **118**, 221101 (2017).

[29] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Phys. Rev. Lett. **116**, 241103 (2016).

[30] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Tests of General Relativity with GW150914, Phys. Rev. Lett. **116**, 221101 (2016).

[31] N. Yunes, K. Yagi, and F. Pretorius, Theoretical physics implications of the binary black-hole mergers GW150914 and GW151226, Phys. Rev. D **94**, 084002 (2016).

[32] J. Steinhoff, G. Schaefer, and S. Hergt, ADM canonical formalism for gravitating spinning objects, Phys. Rev. D **77**, 104018 (2008).

[33] J. Steinhoff, S. Hergt, and G. Schaefer, Spin-squared Hamiltonian of next-to-leading order gravitational interaction, Phys. Rev. D **78**, 101503 (2008).

[34] J. Steinhoff, Spin and quadrupole contributions to the motion of astrophysical binaries, Fundam. Theor. Phys. **179**, 615 (2015).

[35] A. Buonanno and T. Damour, Effective one-body approach to general relativistic two-body dynamics, Phys. Rev. D **59**, 084006 (1999).

[36] T. Damour and A. Nagar, The effective one body description of the two-body problem, Fundam. Theor. Phys. **162**, 211 (2011).

[37] M. Shibata and T. Nakamura, Evolution of three-dimensional gravitational waves: Harmonic slicing case, Phys. Rev. D **52**, 5428 (1995).

[38] T. Nakamura, K. Oohara, and Y. Kojima, General relativistic collapse to black holes and gravitational waves from black holes, Prog. Theor. Phys. Suppl. **90**, 1 (1987).

[39] T. W. Baumgarte and S. L. Shapiro, On the numerical integration of Einstein's field equations, Phys. Rev. D **59**, 024007 (1998).

[40] D. Z. Freedman and A. Van Proeyen, *Supergravity* (Cambridge University Press, Cambridge, England, 2012).

[41] A. Komar, Covariant conservation laws in general relativity, Phys. Rev. **113**, 934 (1959).

[42] I. I. Shapiro, Fourth Test of General Relativity, Phys. Rev. Lett. **13**, 789 (1964).

[43] B. Bertotti, L. Iess, and P. Tortora, A test of general relativity using radio links with the Cassini spacecraft, Nature (London) **425**, 374 (2003).

[44] W. D. Goldberger and I. Z. Rothstein, An effective field theory of gravity for extended objects, Phys. Rev. D **73**, 104029 (2006).

[45] R. A. Porto, Post-Newtonian corrections to the motion of spinning bodies in NRGR, Phys. Rev. D **73**, 104031 (2006).

[46] M. Levi and J. Steinhoff, Equivalence of ADM Hamiltonian and effective field theory approaches at next-to-next-to-leading order spin1-spin2 coupling of binary inspirals, J. Cosmol. Astropart. Phys. 12 (2014) 003.

[47] L. Barack and C. O. Lousto, Computing the gravitational self-force on a compact object plunging into a Schwarzschild black hole, Phys. Rev. D **66**, 061502 (2002).

[48] F. W. Hehl, Four lectures on Poincaré gauge field theory, in *Cosmology and Gravitation: Spin, Torsion, Rotation and Supergravity*, edited by P. G. Bergmann and V. de Sabbata (Plenum, New York, 1980), p. 5.

[49] M. Schweizer, N. Straumann, and A. Wipf, Postnewtonian generation of gravitational waves in a theory of gravity with torsion, Gen. Relativ. Gravit. **12**, 951 (1980).