Master equations and quasinormal modes of spin-3/2 fields in Schwarzschild (A)dS black hole spacetimes

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In this work, we consider spin-3/2 fields in Schwarzschild (anti-)de Sitter [(A)dS] black hole spacetimes. As this spacetime is different from the Ricci-flat cases, it is necessary to modify the covariant derivative to the supercovariant derivative in order to maintain the gauge symmetry, as noted in our earlier works, and this is done here by including terms related to the cosmological constant. Together with the eigenmodes for spin-3/2 fields on an n-sphere, we derive the master radial equations, which have effective potentials that in general include an explicitly imaginary part and energy dependence. We found that for the asymptotically AdS cases the explicit imaginary dependence automatically disappears because of the negative cosmological constant. We take this case as an example and obtain the quasinormal modes by using the Horowitz-Hubeny method [Phys. Rev. D **62**, 024027 (2000)].

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I. INTRODUCTION

Obtaining the master equations for black hole perturbation theory by considering different fields in general dimensional spherically symmetric black hole backgrounds is a well-studied research topic, except for the spin-3/2 fields, for which only a limited amount of literature exists. This paper is the third part of a systematic study of spin-3/2 fields in higher-dimensional Schwarzschild [1] and Reissner-Nordström black hole spacetimes [2], in which we now seek to confirm the methods used for obtaining the radial equation for cases involving a cosmological constant and to test how the behavior of the spin-3/2 field shall affect the quasinormal modes (QNMs).

As in our previous works, we consider the Rarita-Schwinger equation [3] in the form which describes the behavior of a gravitino in supergravity theories [4,5],

$$\gamma^{\alpha\mu\nu}\nabla_\mu\psi_\nu = 0, \tag{1.1}$$

where $\gamma^{\alpha\mu\nu} = \gamma^{[\alpha}\gamma^{\mu}\gamma^{\nu]}$ is the totally antisymmetric product of Dirac gamma matrices, ∇_μ is the covariant derivative, and ψ_ν is a spinor-vector field. The Rarita-Schwinger equation in this form is convenient for generalizing into higher-dimensional cases; however, the spinor-vector field ψ_ν will in general not be gauge invariant under the transformation

$$\psi'_\nu = \psi_\nu + \nabla_\nu\varphi, \tag{1.2}$$

where φ is a gauge spinor. To maintain the gauge invariance of Eq. (1.1), we have to replace the covariant derivative with a "supercovariant derivative," which we derived previously in the Reissner-Nordström case [2] and which we shall derive for the (anti-)de Sitter [(A)dS] spacetimes here. This becomes the essential difference for the study of spin-3/2 fields when compared with the study of the other spin fields in higher-dimensional spherically symmetric spacetimes.

In this paper, we shall therefore generalize our previous works by including the cosmological constant in the Einstein field equations, which indicates that the black hole will not be located in an asymptotically flat spacetime but an (A)dS one. The master radial equations, and their effective potentials, shall be derived, and it is found that they shall have an explicit imaginary part as well an energy dependence. As such, it shall be more convenient to choose the higher-dimensional asymptotically AdS cases in pursuing our QNM analysis. Note that in this case the effective potentials behave like a confining box, which is a general expectation for perturbing fields in an asymptotically AdS background. This allows us to use the Horowitz-Hubeny method [6] with Dirichlet boundary conditions to study the QNMs.

Recall that the black hole QNMs originally represented the cosmological black hole's ringing, but in the asymptotically

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AdS cases, this may be slightly different, as is suggested by the AdS/CFT correspondence [6–8]. Note that some works have given a more precise comment on the QNMs under such an equivalence, by relating them to the poles of retarded Green's functions of gauge-invariant operators in supersymmetric Yang-Mills theory [9]. Furthermore, many works on QNMs in asymptotically AdS cases have been done with bosonic perturbations [10–14], Dirac perturbations [15,16], and large overtone spin-3/2 perturbations [17]. Our works have attempted to fill the gap in this literature, by developing formulas for spin-3/2 perturbations.

As such, we have structured this paper as follows. In the next section, we define the supercovariant derivative for the Rarita-Schwinger equation, Eq. (1.1), corresponding to the Schwarzschild-(A)dS background. In Sec. III, we obtain the master radial equations both for the "non–transverse and traceless (non-TT) related" and "transverse and traceless (TT) related" modes [1,2]. We will then present some discussion about the properties of the effective potentials in Sec. IV and in Sec. V present the modified Horowitz-Hubeny method to fit the equations under investigation here, obtaining the spin-3/2 QNMs for the asymptotically AdS black hole background. Finally, in Sec. VI, we present our conclusions and some related future directions for this work.

II. SUPERCOVARIANT DERIVATIVE

To maintain gauge invariance of the spinor-vector field Ψ_ν in the Rarita-Schwinger equation, we have to rewrite the derivative in terms of a supercovariant derivative, which incorporates the gauge transformation, Eq. (1.2), into Eq. (1.1). The condition shall therefore be

$$\gamma^{\alpha\mu\nu}[\mathcal{D}_\mu, \mathcal{D}_\nu]\varphi = 0, \tag{2.1}$$

where $\mathcal{D}_\mu = \nabla_\mu + a\sqrt{\Lambda}\gamma_\mu$ and a is a yet-to-be-determined factor, which we are required to solve to satisfy Eq. (2.1). Together with the relation $\nabla_\mu \gamma_\nu = \nabla_\mu e^a_\nu \gamma_a = 0$, Eq. (2.1) becomes

$$\begin{aligned} 0 = \gamma^{\alpha\mu\nu}[\mathcal{D}_\mu, \mathcal{D}_\nu]\psi &= \gamma^{\alpha\mu\nu}[\nabla_\mu, \nabla_\nu]\psi + a^2\Lambda\gamma^{\alpha\mu\nu}[\gamma_\mu, \gamma_\nu]\psi \\ &= \gamma^\mu G_\mu{}^\alpha\psi + a^2\Lambda(-2(D-2)(D-1)\gamma^\alpha)\psi, \end{aligned} \tag{2.2}$$

where $G_\mu{}^\alpha$ is the Einstein tensor. Together with the source free Einstein field equations, we found that if

$$a = \frac{i}{\sqrt{2(D-2)(D-1)}} \tag{2.3}$$

then Eq. (2.1) will always be true. That is, if we rewrite the Rarita-Schwinger equation, Eq. (1.1), with the supercovariant derivative in Schwarzschild (A)dS spacetimes, then

$$\begin{aligned} 0 &= \gamma^{\alpha\mu\nu}\mathcal{D}_\mu\psi_\nu, \\ &= \gamma^{\alpha\mu\nu}\left(\nabla_\mu + i\sqrt{\frac{\Lambda}{2(D-2)(D-1)}}\gamma_\mu\right)\psi_\nu. \end{aligned} \tag{2.4}$$

It is worth recalling that the supercovariant derivative from our previous work, Ref. [2], arose for spin-3/2 fields in general dimensional Reissner-Nordström cases because the spacetime was no longer Ricci flat. Note that we do not need to have the supercovariant derivative in Ricci flat cases as Eq. (2.1) will be automatically satisfied [1]. For the higher-dimensional Reissner-Nordström cases, we had to introduce the supercovariant derivative, as in Eq. (2.9) in Ref. [2], to ensure the gauge invariance of the Rarita-Schwinger equation. This is modified for our current spacetime in Eq. (2.4), to reflect the presence of the cosmological constant. This is the essential difference when studying spin-3/2 perturbations in comparison with other field perturbations in spherically symmetric spacetimes. Note that the supercovariant derivative we have presented in Eq. (2.4), and our previous works, is consistent with the results of Ref. [18].

III. EQUATIONS OF MOTION AND MASTER RADIAL EQUATIONS

To obtain the master radial equations, we begin by defining the line element that we will use. In this case, it is given as

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_N^2, \tag{3.1}$$

where

$$f(r) = 1 - \frac{2M}{r^{D-3}} - \frac{2\Lambda r^2}{(D-2)(D-1)} \tag{3.2}$$

and $d\Omega_N^2$ is the metric of an N-sphere with $D = N + 2$. Note that we will use the overbars to denote terms from the $d\Omega_N^2$ metric. For the Rarita-Schwinger equation, Eq. (2.4), in this spacetime, we may obtain two sets of master radial equations, the non-TT related and TT related modes. This comes from the separation of the angular part of the equations of motion by using the "non-TT eigenmodes" and "TT eigenmodes" on S^N. For details on the spinor-vector eigenmodes, as well as the choice of Dirac gamma matrices, and the calculation of the spin connections, we refer the reader to Refs. [1,2].

A. Radial equations with non-TT eigenmodes

We proceed by taking the spin-3/2 fields in the form

$$\begin{aligned} &\psi_t = \phi_t \otimes \bar{\psi}_{(\lambda)} \quad \text{and} \quad \psi_r = \phi_r \otimes \bar{\psi}_{(\lambda)}, \\ &\psi_{\theta_i} = \phi_\theta^{(1)} \otimes \bar{\nabla}_{\theta_i}\bar{\psi}_{(\lambda)} + \phi_\theta^{(2)} \otimes \bar{\gamma}_{\theta_i}\bar{\psi}_{(\lambda)}, \end{aligned} \tag{3.3}$$

where $\bar{\psi}_{(\lambda)}$ is an eigenspinor on S^N, with eigenvalues $i\bar{\lambda}$, and ϕ_t, ϕ_r, $\phi_\theta^{(1)}$, and $\phi_\theta^{(2)}$ are functions of r and t only and behave as 2-spinors. The eigenvalues $\bar{\lambda}$ are given by $\bar{\lambda} = j + (D-3)/2$, where $j = 3/2, 5/2, 7/2, \ldots$ [1]. We use the Weyl gauge, that is, $\phi_t = 0$, in which case out of the four equations of motion only three are independent. After simplifying, they can be written as

$$\begin{aligned}
0 &= \left(i\bar{\lambda}\partial_r + (D-3)\frac{i\bar{\lambda}}{2r} - \frac{(D-2)(D-3)}{4r\sqrt{f}}i\sigma^3 - \frac{\bar{\lambda}}{\sqrt{f}}\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\sigma^2\right)\phi_\theta^{(1)} \\
&\quad + \left((D-2)\partial_r + (D-3)\frac{i\bar{\lambda}}{r\sqrt{f}}i\sigma^3 + \frac{(D-2)(D-3)}{2r} + (D-2)\frac{i}{\sqrt{f}}\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\sigma^2\right)\phi_\theta^{(2)} \\
&\quad - \left(i\bar{\lambda} + \frac{D-2}{2}\sqrt{f}i\sigma^3 + ir\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\sigma^1\right)\phi_r, \\
0 &= \left(-\frac{i\bar{\lambda}}{\sqrt{f}}\partial_t + (D-3)\frac{i\bar{\lambda}\sqrt{f}}{2r}\sigma^1 + \frac{i\bar{\lambda}f'}{4\sqrt{f}}\sigma^1 - \frac{(D-2)(D-3)}{4r}\sigma^2 - \bar{\lambda}\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}i\sigma^3\right)\phi^{(1)} \\
&\quad + \left(-\frac{D-2}{\sqrt{f}}\partial_t + \frac{(D-3)(D-2)}{2r}\sqrt{f}\sigma^1 + \frac{(D-2)f'}{4\sqrt{f}}\sigma^1 + (D-3)\frac{i\bar{\lambda}}{r}\sigma^2 - (D-2)\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\sigma^3\right)\phi_\theta^{(2)}, \\
0 &= \left(\frac{i}{r\sqrt{f}}\sigma^3\partial_t + \frac{\sqrt{f}}{r}\sigma^2\partial_r + \frac{f'}{4r\sqrt{f}}\sigma^2 + (D-4)\frac{\sqrt{f}}{2r^2}\sigma^2 + \frac{i}{r}\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\right)\phi_\theta^{(1)} - \frac{D-4}{r^2}\sigma^1\phi_\theta^{(2)} - \frac{\sqrt{f}}{r}\sigma^2\phi_r. \qquad (3.4)
\end{aligned}$$

However, the variables ϕ_r, $\phi_\theta^{(1)}$ and $\phi_\theta^{(2)}$ are not gauge-invariant functions. As such, and as we have done previously, we will need to construct gauge-invariant variables. From the gauge transformation in Eq. (1.2),

$$\begin{aligned}
\psi'_{\theta_i} = \psi_{\theta_i} + \nabla_{\theta_i}\varphi &\Rightarrow \phi_\theta'^{(1)} \otimes \bar{\nabla}_{\theta_i}\bar{\psi}_{(\lambda)} + \phi_\theta'^{(2)} \otimes \bar{\gamma}_{\theta_i}\bar{\psi}_{(\lambda)} \\
&= (\phi_\theta^{(1)} + \phi) \otimes \bar{\nabla}_{\theta_i}\bar{\psi}_{(\lambda)} + \left[\phi_\theta^{(2)} + \left(\frac{\sqrt{f}}{2}i\sigma^3 + \frac{ir}{D-2}\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\sigma^1\right)\phi\right] \otimes \bar{\gamma}_{\theta_i}\bar{\psi}_{(\lambda)}. \qquad (3.5)
\end{aligned}$$

Hence, a gauge-invariant variable can be defined as

$$\Phi = -\left(\frac{\sqrt{f}}{2}i\sigma^3 + \frac{ir}{D-2}\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\sigma^1\right)\phi_\theta^{(1)} + \phi_\theta^{(2)}. \qquad (3.6)$$

We now use this gauge-invariant variable in Eq. (3.4) and simplify to obtain the following equation of motion for Φ:

$$\begin{aligned}
&\left(i\bar{\lambda} + \frac{D-2}{2}\sqrt{f}i\sigma^3 - ir\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\sigma^1\right)\left[\sigma^1\partial_t - \frac{(D-3)f}{2r} - \frac{f'}{4} - i\bar{\lambda}\frac{D-3}{D-2}\frac{\sqrt{f}}{r}i\sigma^3 - i\sqrt{\frac{\Lambda f(D-2)}{2(D-1)}}\sigma^2\right]\Phi \\
&= \left(i\bar{\lambda} - \frac{D-2}{2}\sqrt{f}i\sigma^3 - ir\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\sigma^1\right)\left[f\partial_r + i\bar{\lambda}\frac{\sqrt{f}}{(D-2)r}i\sigma^3 + \frac{(2D-7)f}{2r} + \frac{2i}{D-2}\sqrt{\frac{\Lambda f(D-2)}{2(D-1)}}\sigma^2\right]\Phi. \qquad (3.7)
\end{aligned}$$

If we take

$$\Psi = \left(i\bar{\lambda} - \frac{D-2}{2}\sqrt{f}i\sigma^3 - ir\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\sigma^1\right)\Phi,$$

then we can rewrite the above equations as

$$f\partial_r\Psi + (\mathcal{A} + \mathcal{B}i\sigma^3 + \mathcal{D}\sigma^2)\Psi = \sigma^1\partial_t\Psi, \qquad (3.8)$$

where

$$\mathcal{A} = \frac{1}{-\bar{\lambda}^2 + \frac{(D-2)^2}{4} f + r^2 \frac{\Lambda(D-2)}{2(D-1)}} \left[-\bar{\lambda}^2 \left(\frac{f'}{4} + \frac{D-4}{2r} f \right) + \frac{(D-2)^2}{4} f \left(-\frac{3}{4} f' + \frac{D-4}{2r} f \right), \right.$$
$$\left. - r^2 \frac{\Lambda(D-2)}{2(D-1)} \left(-\frac{f'}{4} - \frac{(D-8)f}{2r} \right) \right],$$
$$\mathcal{B} = \frac{i\bar{\lambda}\sqrt{f}}{r} \left[1 + \frac{1}{-\bar{\lambda}^2 + \frac{(D-2)^2}{4} f + r^2 \frac{\Lambda(D-2)}{2(D-1)}} \left(\frac{(D-2)(D-3)M}{r^{D-3}} \right) \right],$$
$$\mathcal{D} = -i\sqrt{\frac{\Lambda f(D-2)}{2(D-1)}} \left[\frac{D-4}{D-2} + \frac{1}{-\bar{\lambda}^2 + \frac{(D-2)^2}{4} f + r^2 \frac{\Lambda(D-2)}{2(D-1)}} \left(\frac{(D-2)(D-3)M}{r^{D-3}} \right) \right]. \tag{3.9}$$

By using a further transformation $\Psi = \mathcal{K}(r)\tilde{\Psi}$, we can remove the $\mathcal{A}$ term, provided $\mathcal{K}(r)$ satisfies the differential equation

$$\frac{f}{\mathcal{K}} \frac{d\mathcal{K}}{dr} + \mathcal{A} = 0. \tag{3.10}$$

We are then left with an equation of the form

$$f\partial_r \tilde{\Psi} + (\mathcal{B} i\sigma^3 + \mathcal{D}\sigma^2)\tilde{\Psi} = \sigma^1 \partial_t \tilde{\Psi}. \tag{3.11}$$

Finally, we can separate the spinor $\tilde{\Psi}$ into its components by making the choice

$$\tilde{\Psi} = \left[\sin\left(\frac{\theta}{2}\right)\sigma^3 + \cos\left(\frac{\theta}{2}\right)\sigma^2 \right] e^{-i\omega t} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{3.12}$$

where the $\phi_{1,2}$ are functions of the radial coordinate only, and

$$\theta = \tan^{-1}\left(\frac{-\mathcal{D}}{i\mathcal{B}} \right). \tag{3.13}$$

Note that a similar transformation was introduced for obtaining the radial equation of a massive Dirac particle in a four-dimensional Schwarzschild black hole spacetime [19]. Substituting the expression in Eq. (3.12) into Eq. (3.11) and simplifying, we get

$$\left(f\partial_r + \frac{f}{2} \left[\frac{\partial}{\partial r} \left(\frac{-\mathcal{D}}{i\mathcal{B}} \right) \right] \left(\frac{\mathcal{B}^2}{\mathcal{B}^2 - \mathcal{D}^2} \right) i\sigma^1 + \sqrt{\mathcal{D}^2 - \mathcal{B}^2}\sigma^3 \right)$$
$$\times \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = i\omega\sigma^1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{3.14}$$

which can be expanded to give

$$(\partial_{r_*} + W)\phi_1 = i\omega\phi_2,$$
$$(\partial_{r_*} - W)\phi_2 = i\omega\phi_1. \tag{3.15}$$

Note that $\partial_{r_*} = \mathcal{F}\partial_r$ is the tortoise coordinate and

$$\mathcal{F} = f\left[1 + \frac{f}{2\omega} \left(\frac{\partial}{\partial r} \frac{\mathcal{D}}{i\mathcal{B}} \right) \left(\frac{\mathcal{B}^2}{\mathcal{B}^2 - \mathcal{D}^2} \right) \right]^{-1},$$
$$W = \sqrt{\mathcal{D}^2 - \mathcal{B}^2}\left[1 + \frac{f}{2\omega} \left(\frac{\partial}{\partial r} \frac{\mathcal{D}}{i\mathcal{B}} \right) \left(\frac{\mathcal{B}^2}{\mathcal{B}^2 - \mathcal{D}^2} \right) \right]^{-1}. \tag{3.16}$$

After decoupling Eqs. (3.15), we obtain

$$(\partial_{r_*}^2 + \omega^2 - V_{\mathrm{eff}})\phi_{1,2} = 0, \tag{3.17}$$

where the effective potential is

$$V_{\mathrm{eff}} = \mp \partial_{r_*} W + W^2. \tag{3.18}$$

There is a notable difference here for the effective potential from previous cases studied in Refs. [1,2], in that the potential has an imaginary component and a dependence on the QNM ω. Note that in Refs. [1,2] the potential relies only on the radial coordinate. The presence of the ω does mean that the potential function, in its current form, is not guaranteed to be a real valued function. The function becomes real by taking the limit $\Lambda \to 0$ (removing the effect of the cosmological constant), in which the potential returns to the Schwarzschild effective potential case. To have a more complete understanding of the effective potential, including how it is affected by the cosmological constant, we will present some of its characteristics in Sec. IV.

B. Radial equations for TT eigenmodes

For the TT related radial equation, we set ψ_r and ψ_t in the same manner as in Eq. (3.3), and the angular component changes to

$$\psi_{\theta_i} = \phi_\theta \otimes \bar{\psi}_{\theta_i}, \tag{3.19}$$

where $\bar{\psi}_{\theta_i}$ is the TT mode eigenspinor vector, with eigenvalue $\bar{\zeta} = j + (D-3)/2$, with $j = 3/2, 5/2, 7/2, \ldots$, as described in Ref. [1]. ϕ_θ depends only on the radial coordinate r, and it behaves like a 2-spinor. We again use the Weyl gauge, in which for the TT related case this means $\phi_t = 0$, and ϕ_r will automatically be zero to satisfy the equations of motion (as in our previous works, Refs. [1,2]). The only nonzero equation of motion is

$$\left(\frac{1}{r\sqrt{f}}i\sigma^3\partial_t+\frac{\sqrt{f}}{r}\sigma^2\partial_r+\frac{f'}{4r\sqrt{f}}\sigma^2+\frac{\sqrt{f}}{2r^2}(D-4)\sigma^2+\frac{i\bar{\zeta}}{r^2}\sigma^1+\frac{i\sqrt{\Lambda}}{r}\sqrt{\frac{D-2}{2(D-1)}}\right)\phi_\theta=0. \tag{3.20}$$

In this case, the function ϕ_θ is already gauge invariant, and we set

$$\phi_\theta = f^{-\frac{1}{4}}r^{-(\frac{D-4}{2})}\sigma^2 e^{-i\omega t}\tilde{\phi}_\theta. \tag{3.21}$$

We then obtain the equation

$$\left(f\partial_r-\frac{\bar{\zeta}\sqrt{f}}{r}\sigma^3+\sqrt{\frac{f\Lambda(D-2)}{2(D-1)}}i\sigma^2\right)\tilde{\phi}_\theta = i\omega\sigma^1\tilde{\phi}_\theta. \tag{3.22}$$

Following a similar method as in the non-TT related case, we take

$$\tilde{\phi}_\theta = \left[\sin\left(\frac{\theta}{2}\right)\sigma^3+\cos\left(\frac{\theta}{2}\right)\sigma^2\right]\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \tag{3.23}$$

where $\varphi_{1,2}$ are functions of r only and

$$\theta = \tan^{-1}\left(\frac{ir}{\bar{\zeta}}\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\right). \tag{3.24}$$

As a result, Eq. (3.22) can be rewritten into two coupled equations

$$\begin{aligned}(\mathbb{F}\partial_r+\mathbb{W})\varphi_1 &= -i\omega\varphi_2,\\ (\mathbb{F}\partial_r-\mathbb{W})\varphi_2 &= -i\omega\varphi_1,\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}\mathbb{F} &= f\left[1-\frac{if}{2\omega}\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\left(\frac{2\bar{\zeta}(D-1)}{2\bar{\zeta}^2(D-1)-r^2\Lambda(D-2)}\right)\right]^{-1},\\ \mathbb{W} &= \left(\frac{\bar{\zeta}^2 f}{r^2}-\frac{f\Lambda(D-2)}{2(D-1)}\right)^{\frac{1}{2}}\left[1-\frac{if}{2\omega}\sqrt{\frac{\Lambda(D-2)}{2(D-1)}}\times\left(\frac{2\bar{\zeta}(D-1)}{2\bar{\zeta}^2(D-1)-r^2\Lambda(D-2)}\right)\right]^{-1}.\end{aligned} \tag{3.26}$$

Using the tortoise coordinate $\partial_{\hat{r}_*} = \mathbb{F}\partial_r$, we obtain the TT related radial equation

$$(\partial^2_{\hat{r}_*}+\omega^2-\mathbb{V}_{\rm eff})\varphi_{1,2} = 0, \tag{3.27}$$

where the effective potential is

$$\mathbb{V}_{\rm eff} = \mp\,\partial_{\hat{r}_*}\mathbb{W}+\mathbb{W}^2. \tag{3.28}$$

We note that the TT related effective potential is basically constructed from the first terms of $\mathcal{B}$ and $\mathcal{D}$ in Eqs. (3.9) of the non-TT related effective potential. For another remark, we would like to point out that there are no TT eigenmodes on S^2, as the first one appears on S^3 [1]. As such, we do not have the TT related radial equation in four-dimensional cases, and Eq. (3.27) can only be true for $D \geq 5$.

IV. PROPERTIES OF THE EFFECTIVE POTENTIALS

The effective potential is the most important physical quantity when working in black hole perturbation theory, as it determines the behaviors of different spin fields in a curved spacetime. We have shown in Sec. III the non-TT related and TT related effective potentials of spin-3/2 fields in a general dimensional Schwarzschild (A)dS spacetime in Eqs. (3.18) and (3.28). Note that both of these potentials represent the behavior of spin-3/2 fields, and these potentials are analogous to both the Regge-Wheeler potential [20] and the Zerilli potential [21], which are related to the spin-2 fields in spherically symmetric black hole backgrounds. It is clear that the basic characteristics for the effective potentials in Eqs. (3.18) and (3.28) are that they both include an imaginary part and an ω dependence. This imaginary part and ω dependence are also present in the Teukolsky equations [22], in which the background spacetime in that case is the Kerr black hole. As such, this is the first time the background spacetime has been spherically symmetric, and the effective potential contains both of these characteristics. Note that until this point the cosmological constant has been left general; that is, the effective potentials in Eqs. (3.18) and (3.28) represent both the asymptotically de Sitter (dS) and AdS cases. We shall analyze our background further by choosing some specific parameters and studying how the effective potentials behave.

A. Schwarzschild de Sitter cases

In this case, the cosmological constant $\Lambda \geq 0$, and in general, the imaginary part and the ω dependence remain for both the non-TT related and the TT related cases. In the earlier studies of the Teukolsky equation, Chandrasekhar and Detweiler [23,24] imposed a transformation on the radial equation to ensure it possessed a purely real and short-range potential; later, Sasaki and Nakamura extended this method for studying the radial equation with a source [25]. However, a direct application of the method in these references may not work for our effective potential, and some further analytic study is needed, which is beyond the scope of the current work.

Nevertheless, there is a special case for the non-TT related potential, when $D = 4$. The fraction of $\mathcal{D}$ over $i\mathcal{B}$ in Eq. (3.16) becomes r independent in this case, and as such, the second term in the square brackets in Eq. (3.16) vanishes. We may conclude that if $\mathcal{D}^2 - \mathcal{B}^2 \geq 0$ the effective potential shall be purely real and without an ω dependence. The potential then behaves like a standard black hole perturbation theory case, which is barrierlike and vanishes at the event horizon as well as the cosmic horizon. In Fig. 1, we present the case of $M = 1$ and $j = 5/2$, in which the corresponding changes in the metric function and the effective potential are shown when the cosmological constant Λ is varied. Note that the $\Lambda = 0$ case is equivalent to the Schwarzschild case studied previously [1,26].

B. Schwarzschild anti-de Sitter cases

For the effective potential in Eqs. (3.18) and (3.28) with cosmological constant $\Lambda \leq 0$, we have found that the imaginary part automatically vanishes, leaving a purely real valued effective potential, and this can be easily deduced from the $\mathcal{D}$ function in Eq. (3.9), the superpotential W in Eq. (3.16) for the non-TT related cases, and the superpotential $\mathbb{W}$ in Eq. (3.26) for TT related cases. However, the ω dependence remains, and because of this dependence, we are not able to deduce the exact behavior of the effective potential until we solve the QNM frequency ω. We can, nevertheless, still estimate the behavior of the potential, in particular its asymptotic behavior, for some given value of ω, as in Fig. 2. Note that the parameter choice in these plots shall be $\Lambda = -(D-1)(D-2)/2$, which represents the "AdS radius" as one unit, with M set to 1 and then 10, and $\omega = 10$, along with a variation of the dimension D for the non-TT related effective potential.

In the AdS spacetimes, the standard expectation is that the potential function will tend to infinity as $r \to \infty$, and the fields will be reflected by this infinite potential. This is true for our cases in which $D \geq 5$, both for non-TT related and TT related modes. However, it is not true for the four-dimensional case, which we shall analyze further in the

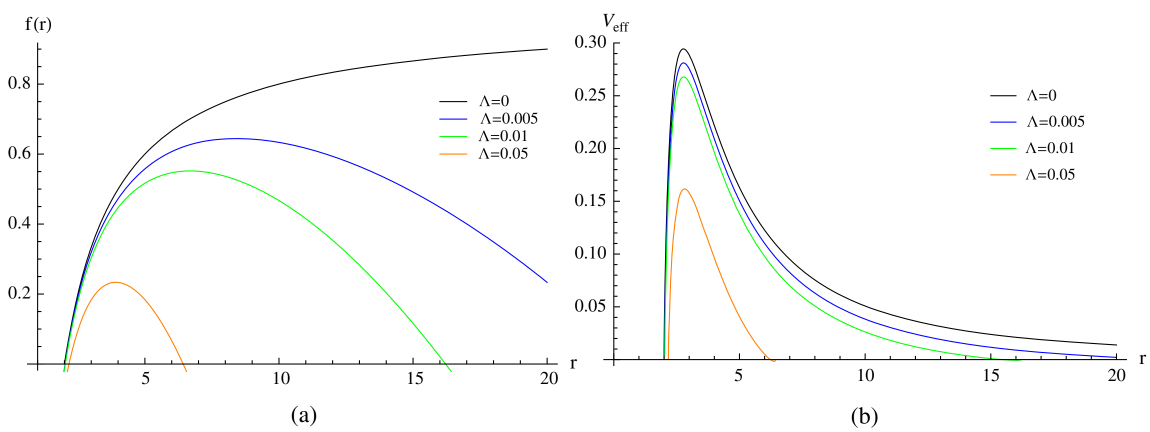

FIG. 1. Plots of the metric function and non-TT related effective potential for $D = 4$, $j = 5/2$ and $\Lambda = 0, 0.005, 0.01, 0.05$. (a) The metric function $f(r)$. (b) The spin-3/2 non-TT related effective potential in a de Sitter black hole spacetime.

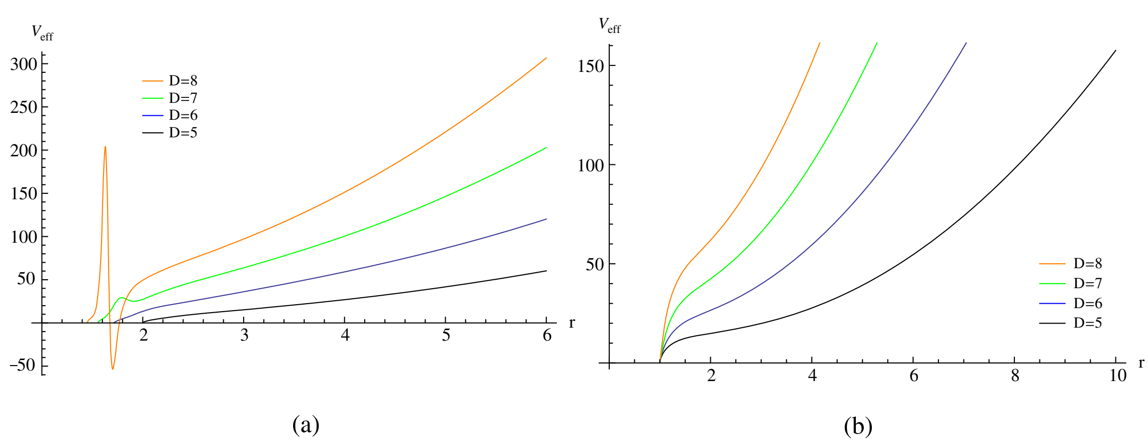

FIG. 2. The spin-3/2 non-TT related effective potential in anti-de Sitter black hole spacetimes for $D = 5, 6, 7, 8$, $j = 5/2$, $\Lambda = -\frac{(D-1)(D-2)}{2}$, and $\omega = 10$. (a) For $M = 1$. (b) For $M = 10$.

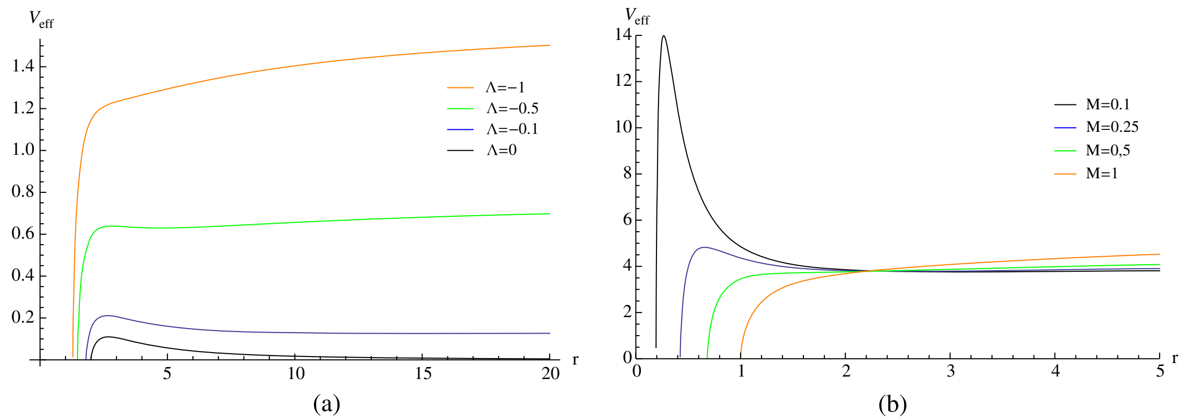

FIG. 3. The spin-3/2 non-TT related effective potential in the four-dimensional Schwarzschild AdS spacetime. (a) For $j = 3/2$ and $M = 1$. (b) For $j = 3/2$ and $\Lambda = -3$ (taking the AdS radius as one).

next subsection. For the higher-dimensional Schwarzschild anti-de Sitter cases, though, it is clearly the most suitable candidate for studying the QNMs, see Sec. V, as we can use a well-known method in obtaining the QNM frequencies without too much modification [6].

C. Four-dimensional Schwarzschild anti-de Sitter case

In the four-dimensional case, the second term in the square brackets of Eq. (3.16) vanishes (similarly in the four-dimensional de Sitter case), and the effective potential becomes ω independent. Furthermore, we also have features similar to the higher-dimensional Schwarzschild anti-de Sitter case, in that it becomes a purely real potential function. From these two points, the effective potential acquires the standard form of other previously studied fields in spherically symmetric black hole spacetimes (see, for example, Ref. [27]). However, the usual infinite potential wall at infinity of the AdS black hole does not, in general, appear. Instead, the potential tends to some finite nonzero value when $r \to \infty$. In Fig. 3, we show two sets of effective potentials with different parameter choices. In Fig. 3(a), we take the black hole mass as 1, where the $\Lambda = 0$ case represents the flat Schwarzschild case. In Fig. 3(b), we take the AdS radius as 1 and plot different values of the mass. Both of these plots present an evolution from a barrierlike potential to a steplike potential, and the potential reaches an asymptotic finite nonzero value that becomes larger for larger values of $|\Lambda|$ or M. From these two plots, we can also see that for small AdS black holes (Λ small or M small) the potential possesses a barrier shape near the event horizon, while for large black holes, this barrier disappears, and the potential becomes steplike.

V. QUASINORMAL MODES FOR ASYMPTOTICALLY ADS CASES

From our discussions in Sec. IV on the effective potentials, we shall use the asymptotically AdS cases as an example for studying the QNMs using the Horowitz-Hubeny method [6]. The Horowitz-Hubeny method works by taking a Taylor series expansion of the potential function around the event horizon and then imposing a purely ingoing boundary condition near the event horizon and Dirichlet boundary conditions on the field solutions at spatial infinity. However, the Taylor series expansion for our potential will include noninteger orders. As such, modifications are necessary for this method to be applied in the current situation.

A. Modified Horowitz-Hubeny method

To investigate the entire region $r_+ < r < \infty$, where r_+ is the event horizon, we map the space to a finite region by changing coordinates to $x = 1/r$. We begin by setting the purely ingoing boundary condition $\phi_2 \sim \Xi(r)e^{-i\omega r_*}$, and the radial equation, Eq. (3.17), becomes

$$\left[\mathcal{F}\frac{d^2}{dr^2} + (\mathcal{F}' - 2i\omega)\frac{d}{dr} - \left(\frac{V_{\rm eff}}{\mathcal{F}}\right)\right]\Xi(r) = 0. \tag{5.1}$$

By redefining $\Lambda = -(D-1)(D-2)/2$, the coefficient of the r^2 term in $f(r)$ becomes -1, which is equivalent to saying that we are setting the AdS radius to 1. Using $r = 1/x$ and

$$M = \frac{1}{2}\left[\frac{1 + x_+^2}{x_+^{(D-1)}}\right],$$

where $x_+ = 1/r_+$ and $f(r_+) = 0$, the radial equation becomes

$$\left[s(x)\frac{d^2}{dx^2} + \frac{t(x)}{(x - x_+)}\frac{d}{dx} - \frac{u(x)}{(x - x_+)^2}\right]\Xi(x) = 0, \tag{5.2}$$

with

$$s(x) = \frac{\mathcal{F}(x)}{x - x_+} x^4,$$
$$t(x) = 2x^3 \mathcal{F}(x) + x^4 (\partial_x \mathcal{F}(x)) + 2x^2 i\omega,$$
$$u(x) = (x - x_+) \left(\frac{V_{\mathrm{eff}}}{\mathcal{F}(x)} \right). \tag{5.3}$$

For $\Xi(x)$ to satisfy the ingoing boundary condition at the horizon, we must write it as

$$\Xi(x) = \sum_{k=0}^{n} a_{\frac{k}{2}} (x - x_+)^{\frac{k}{2}}. \tag{5.4}$$

Note that half-integral powers $k/2$ are necessary, as we require noninteger terms in the Taylor expansion of $u(x)$, where $n/2$ is the order of the expansion, and n shall be an even integer for ease of calculation. Plugging Eq. (5.4) into Eq. (5.2), and expanding Eq. (5.3) when x is close to x_+, we can then compare $(x - x_+)$ order by order to determine the value of $a_{\frac{k}{2}}$. That is,

$$a_{\frac{k}{2}} = -\frac{1}{P_{\frac{k}{2}}} \sum_{q=0}^{k-1} \left[\frac{q}{2} \left(\frac{q}{2} - 1 \right) s_{\frac{k-q}{2}} + \frac{q}{2} t_{\frac{k-q}{2}} - u_{\frac{k-q}{2}} \right] a_{\frac{q}{2}},$$
$$P_{\frac{k}{2}} = \frac{k}{2} \left(\frac{k}{2} - 1 \right) s_0 + \frac{k}{2} t_0, \tag{5.5}$$

where $k = 0, 1, 2, \ldots n$, $s_{\frac{k}{2}}$, $t_{\frac{k}{2}}$, and $u_{\frac{k}{2}}$ represents the corresponding coefficients of the expansion polynomials.

We next consider the Dirichlet boundary condition, which means that the wave function must vanish when $x \to 0$; therefore, we have

$$\Phi = \sum_{k=0}^{n} a_{\frac{k}{2}} (-x_+)^{\frac{k}{2}} = 0. \tag{5.6}$$

Putting all the expressions in Eq. (5.5) into Eq. (5.6) allows us to solve the QNM frequencies with respect to the $\frac{n}{2}$th-order expansion. As a remark, note that the notation shown in this subsection corresponds to the non-TT related cases only. However, for the TT related cases, just a replacement of $\mathcal{F}$ and V_{eff} by $\mathbb{F}$ and $\mathbb{V}_{\mathrm{eff}}$ is needed.

B. Quasinormal modes and some related difficulties

In the numerical calculation of the QNM frequencies in the currently considered cases, there were some intrinsic difficulties, in which for the scalar perturbation case of Horowitz and Hubeny [6] their method gave stable results to three significant figures with an expansion to around 100 orders. These results were reproducible by us after several hours of computing time for one mode. Note that a large number of studies for bosonic perturbations [10–14] and Dirac perturbations [15,16] using this method have been performed, and we suppose that these calculations were also working to a similar order of expansion. However, because of the complexity of our spin-$3/2$ effective potential, an expansion to 100 orders in our current case is much more difficult to achieve.

It is well known that bosonic perturbations in all types of spherically symmetric spacetimes are determined by the Regge-Wheeler equation [20] and the Zerilli equation [21], which are not greatly different with regards to performing numerical calculations. For the other case of Dirac perturbations [27], the superpotential related to the effective potential shall be $\lambda\sqrt{f}/r$, which is just the leading term of our coefficient $\mathcal{B}$ in Eq. (3.9); this is also true for our TT related cases. Together with our earlier discussions, as related to the ω dependence and the noninteger order required when doing the Taylor expansions, we are faced with a far more complicated and nested algorithm that takes a lot of computing time to converge to a reliable QNM frequency.

As such, we present the QNM frequencies from this Horowitz-Hubeny method with expansions up to the 14th order. Note that our modified Horowitz-Hubeny method has an expansion in half-integer powers in the Taylor series expansion, so this compares to approximately 30 order in the original Horowitz-Hubeny method. Achieving results to this order already required more than 10 h per mode, and to go to the next order would cost approximately three times more computing time. In the following tables, Tables I, II, and III, we present a typical set of results for the QNMs of the non-TT related potential for the cases of $D = 5$, 6, 7, 8, $x_+ = 0.001$, 0.01, 0.025, 0.1, 1, 1.2, and $\ell = 0$, 1, 2, where $\ell = j - 3/2$ is related to the spinor-vector eigenmodes on S^N. And in Tables IV, V,

TABLE I. Spin-$3/2$ non-TT related QNMs for $\ell = 0$ in $D = 5, 6, 7, 8$-dimensional Schwarzschild AdS black hole spacetimes.

	QNMs, $\ell = 0$			
x_+	$D = 5$	$D = 6$	$D = 7$	$D = 8$
0.001	2421.34-2768.09i	3345.87-2982.35i	3892.93-3306.22i	4260.42-3581.29i
0.01	242.6-277.064i	335.012-298.455i	389.523-330.907i	425.945-358.546i
0.025	97.3393-110.99i	134.267-119.606i	155.722-132.922i	169.601-144.366i
0.1	24.7773-28.0351i	34.051-30.9378i	38.8121-36.1762i	50.6502-10.5606i
1	3.97854-2.91456i	2.92626-13.028i	4.6083-23.1799i	8.95824-13.0857i
1.2	1.69155-0.02631i	2.41208-6.97679i	1.95646-29.6368i	8.22353-25.1656i

TABLE II. Spin-3/2 non-TT related QNMs for $\ell = 1$ in $D = 5, 6, 7, 8$-dimensional Schwarzschild AdS black hole spacetimes.

	QNMs, $\ell = 1$			
x_+	$D = 5$	$D = 6$	$D = 7$	$D = 8$
0.001	2421.55-2768.2i	3346.03-2982.42i	3893.03-3306.27i	4260.47-3581.34i
0.01	242.779-277.174i	335.142-298.591i	389.416-331.262i	425.4-359.119i
0.1	25.0869-28.3967i	34.6927-32.1749i	57.1463-13.584i	59.2126-9.43011i
1	4.02362-1.92365i	2.43502-6.05751i	1.74403-15.59i	2.01607-28.2032i
1.2	3.91933-0.81368i	0.81334-3.40948i	1.47253-13.8613i	1.70049-25.9925i

TABLE III. Spin-3/2 non-TT related QNMs for $\ell = 2$ in $D = 5, 6, 7, 8$-dimensional Schwarzschild AdS black hole spacetimes.

	QNMs, $\ell = 2$			
x_+	$D = 5$	$D = 6$	$D = 7$	$D = 8$
0.001	2421.76-2768.32i	3346.19-2982.49i	3893.13-3306.32i	4260.5-3581.41i
0.01	242.947-277.283i	335.263-298.753i	389.253-331.712i	424.717-359.844i
0.1	25.224-28.9174i	34.3238-32.525i	40.9097-38.3034i	46.9668-44.8927i
1	1.51351-0.0204i	4.69729-0.676194i	5.985-10.5436i	1.04004-17.6066i
1.2	0.989231-0.0587i	1.25176-0.20764i	4.43287-6.24447i	0.787022-9.37979i

TABLE IV. Spin-3/2 TT related QNMs for $\ell = 0$ in $D = 5, 6, 7, 8$-dimensional Schwarzschild AdS black hole spacetimes.

	QNMs, $\ell = 0$			
x_+	$D = 5$	$D = 6$	$D = 7$	$D = 8$
0.001	2419.23-2766.92i	3344.42-2981.75i	3891.78-3306.04i	4259.37-3581.22i
0.01	240.533-275.868i	333.588-297.804i	388.535-330.502i	425.355-358.088i
0.025	95.3413-109.741i	132.9-118.847i	155.007-132.105i	169.778-143.179i
0.1	23.0666-26.454i	32.7355-29.2607i	38.3477-32.7801i	42.101-35.5639i
1	3.42834-1.25205i	4.11508-1.23027i	5.58248-221.521i	9.79495-373.605i
1.2	3.29817-0.518027i	3.87221-0.360491i	4.74886-1.21018i	4.43772-0.255522i

TABLE V. Spin-3/2 TT related QNMs for $\ell = 1$ in $D = 5, 6, 7, 8$-dimensional Schwarzschild AdS black hole spacetimes.

	QNMs, $\ell = 1$			
x_+	$D = 5$	$D = 6$	$D = 7$	$D = 8$
0.001	2418.6-2766.57i	3344.09-2981.62i	3891.57-3306.02i	4259.2-3581.22i
0.01	239.923-275.48i	333.275-297.66i	388.331-330.47i	425.193-358.08i
0.025	94.7727-109.297i	132.607-118.685i	154.813-132.064i	169.622-143.163i
0.1	22.8024-25.8189i	32.5663-29.0236i	38.201-32.6995i	41.9761-35.5093i
1	4.2281-0.566104i	4.67677-0.520969i	5.19813-0.538036i	5.77799-0.59755i
1.2	2.06008-0.43201i	2.03588-132.369i	4.06663-232.399i	7.43144-393.25i

and VI, we present a similarly typical set of values for the QNMs for the TT related potential for the cases of $D = 5$, 6, 7, 8, $x_+ = 0.001$, 0.01, 0.025, 0.1, 1.2 and $\ell = 0$, 1, 2, where again $\ell = j - 3/2$ is related to the eigenmodes on S^N.

In Fig. 4, we have plotted the real and the imaginary parts of the QNM frequencies separately, for the non-TT related case with $\ell = 0$, vs the position of the event horizon r_+. Both the real parts and the magnitudes of the imaginary parts of the frequencies increase with r_+, as well as with the

TABLE VI. Spin-3/2 TT related QNMs for $\ell = 2$ in $D = 5, 6, 7, 8$-dimensional Schwarzschild AdS black hole spacetimes.

	QNMs, $\ell = 2$			
x_+	$D = 5$	$D = 6$	$D = 7$	$D = 8$
0.001	2417.97-2766.21i	3343.77-2981.49i	3891.36-3305.99i	4259.03-3581.22i
0.01	239.321-275.08i	332.965-297.513i	388.13-330.436i	425.032-358.071i
0.025	94.231-108.824i	132.324-118.513i	154.623-132.02i	169.468-143.145i
0.1	22.6914-25.157i	32.4534-28.7701i	38.0709-32.6065i	45.2679-1765.84i
1	5.1336-0.10657i	1.96347-135.005i	3.99961-235.105i	7.42369-395.964i
1.2	0.6921-73.5791i	1.6762-136.719i	3.45014-239.447i	6.45744-404.69i

spacetime dimension D. Moreover, for large black holes ($r_+ = 100$, 40, 10 or $x_+ = 0.01$, 0.025, 0.1), a linear relation between the QNM frequencies ($\mathrm{Re}(\omega)$ or $|\mathrm{Im}(\omega)|$) and r_+ is apparent. This is consistent with the study of Horowitz and Hubeny [6] in which they found these linear relations between QNM frequencies and Hawking temperatures of large AdS black holes, as the temperature is proportional to r_+ for large AdS black hole cases. Note that for small black holes ($r_+ = 1$, 0.83 or $x_+ = 1, 1.2$), as indicated in Fig. 4, the QNM frequencies do not conform to this relation. Lastly, from Tables I, II, and III, we can see that the values of the frequencies do not vary

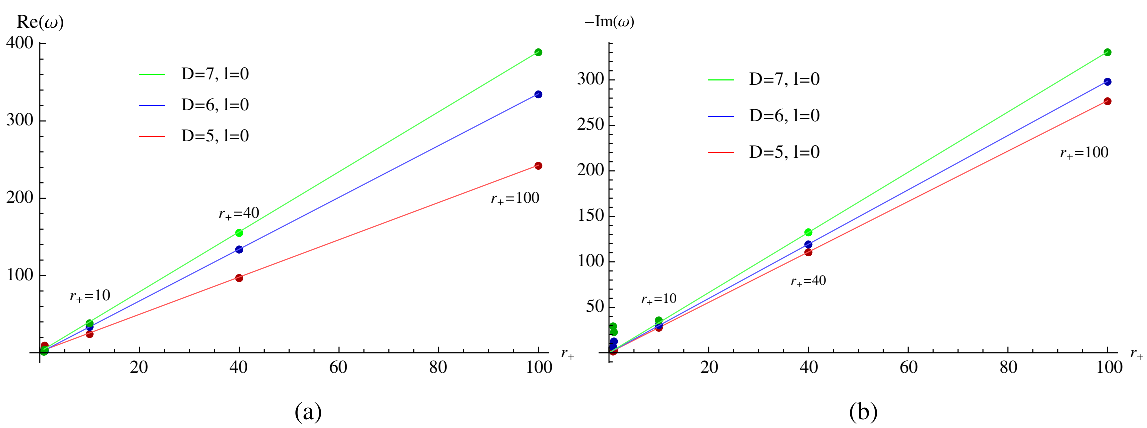

FIG. 4. QNMs, TT related case, with $\ell = 0$, $D = 5, 6, 7$, and $x_+ = 0.01, 0.025, 0.1, 1, 1.2$. (a) Linear relation for the real part up to the 14th iteration. (b) Linear relation for the imaginary part up to the 14th iteration.

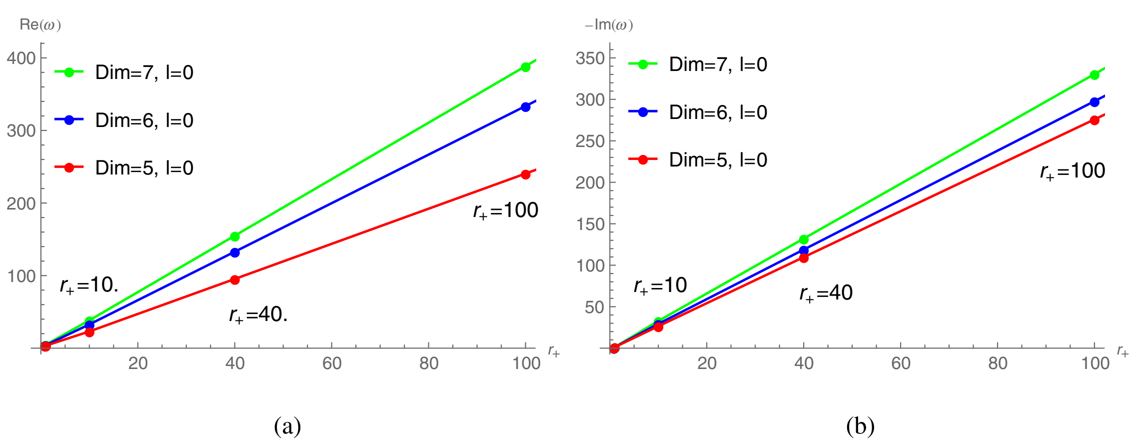

FIG. 5. QNMs, TT related case, with $\ell = 0$, $D = 5, 6, 7$, and $x_+ = 0.01, 0.025, 0.1, 1, 1.2$. (a) Linear relation for the real part up to the 14th iteration. (b) Linear relation for the imaginary part up to the 14th iteration.

with the angular momentum parameter ℓ. Hence, we expect the linear relation to hold also for large black holes in the $\ell = 1, 2$ cases.

In Figs. 5, we have plotted the numerical values of the QNMs related to the TT potential, with $\ell = 0$, $D = 5, 6, 7$, and $x_+ = 0.01, 0.025, 0.1, 1, 1.2$. We have the same trend as seen in the case of the non-TT modes of the spin-3/2 fields. That is, we see that for large values of r_+ we have a linear relationship between r_+ and the real values of the QNMs. Furthermore, the same linear behavior is seen between the r_+ value and the imaginary value of the QNMs. Although we have only considered the $\ell = 0$ modes in the figure, it is clear from the tabulated values in Tables IV, V and VI that the same relationship is seen in the case of $\ell > 0$. In the case of $r_+ \leq 1$, we see that there is some divergence from this linear trend. Therefore, as expected, the TT modes behave very similarly to the non-TT modes.

VI. CONCLUSIONS

The main result of this paper is the obtaining of the master radial equations of spin-3/2 fields in the Schwarzschild (A) dS black hole spacetimes. This has been done for the non-TT related case in Eq. (3.18) and the TT related case in Eq. (3.28). A systematic study of the effective potentials has also been performed. However, because of the varied complexities of these potentials, the issues around calculating the QNMs for asymptotic dS spacetimes and the study of a special case for the four-dimensional Schwarzschild AdS black hole must be left for future works, as greatly different techniques for their calculation need to be employed. This is something beyond the focus of the current investigation of obtaining the master radial equations for spin-3/2 fields in Schwarzschild (A)dS spacetimes. Nevertheless, for an application of our master radial equations, we have presented results on the QNM frequencies in higher-dimensional asymptotic AdS black hole cases. From these results, a linear relationship was successfully obtained between QNM frequencies and the corresponding Hawking temperatures of the large AdS black holes, as previously noted in scalar perturbation studies [6].

Concerning spin-3/2 perturbations in higher-dimensional black hole perturbation spacetimes, there are few references for us to compare our results with, except some for the topologically AdS cases related to the AdS/CFT correspondence. Notably in Ref. [17], a study of the large-overtone QNM frequencies (as well as numerical values of some low-overtone ones) were presented for higher-dimensional Schwarzschild-AdS spacetimes. However, there exist intrinsic differences between this analysis and ours, which prevent us from doing the comparison.

First of all, the authors of Ref. [17] arrived at their equations of motion by following the procedures in Ref. [28]. Basically, they started with the massive Rarita-Schwinger equation,

$$\gamma^{\mu\nu\alpha}\nabla_\nu\Psi_\alpha - m\gamma^{\mu\nu}\Psi_\nu = 0, \tag{6.1}$$

and then by imposing the transverse and the traceless gauge conditions, $\nabla^\mu\Psi_\mu = \gamma^\mu\Psi_\mu = 0$, it was reduced to a Dirac-like equation,

$$\gamma^\mu\nabla_\mu\Psi_\nu - m\Psi_\nu = 0. \tag{6.2}$$

On the contrary, we stick with the massless Rarita-Schwinger equation [Eq. (1.1)]. Gauge invariance is maintained by using the supercovariant derivative as in Eq. (2.4), and the equation of motion is derived for a gauge-invariant variable. Our equation of motion in four dimensions is found to be consistent with the one obtained previously in Ref. [29].

Second, other than the difference in the equations of motion, the setting of the background spacetime in Ref. [17] was also different from ours, even though both correspond to the Schwarzschild AdS one. The line element in Ref. [17] is a $D-1$ brane spacetime with an extra-dimensional bulk coordinate. This setting is strongly linked to that of the

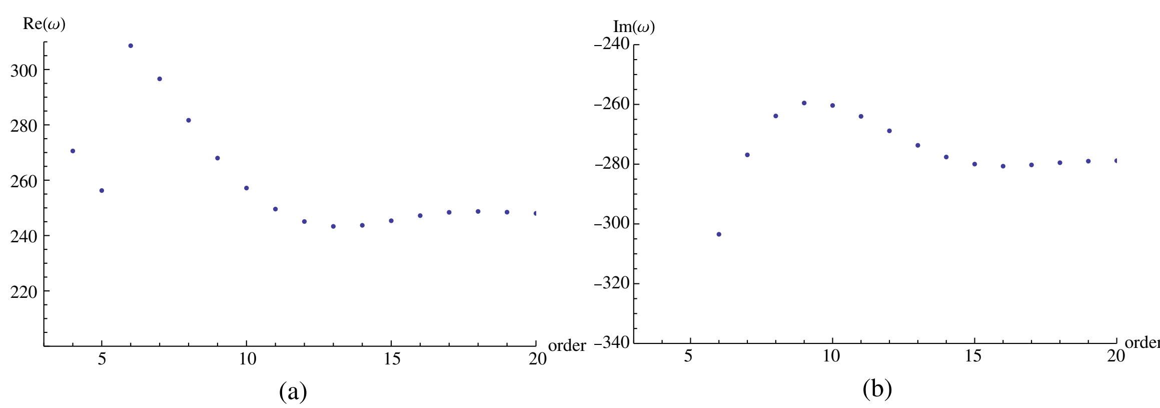

FIG. 6. QNMs, TT related case with 20th order expansion, $D = 5$, $r_+ = 100$, $\ell = 0$. (a) Real part. (b) Imaginary part.

AdS/CFT correspondence. On the other hand, our starting point is the black hole perturbation theory around a spacetime which is spherically symmetric as indicated in Eq. (3.1). As such, plus the discussion above, we are considering different sets of modes, and the comparison of their QNM frequencies would not be appropriate.

Lastly, we would like to return to the discussion of the accuracy of the QNM frequencies in our tables. To have an idea of that, we present in Fig. 6 a plot for the convergence of the QNMs up to 20th order in the expansion of a mode of the TT related potential. As we mentioned earlier, the TT related effective potential is the leading term of the non-TT related effective potential and as such has allowed us to extend these numerics to a few higher orders. The values in Fig. 6 start to stabilize after 15 or 16 orders of expansion. Hence, this indicates that the values of the QNM frequencies in Tables I–VI can be taken, at best, as a good approximation. To obtain more accurate values of the QNM frequencies, as well as those of higher overtone QNMs, the expansion in the Horowitz-Hubeny method would need to be taken to much higher orders. As such, the computing power and time required to do this would not be feasible and would require a new numerical method, for example, the "asymptotic iteration method" [30], to evaluate the QNM frequencies in these cases to a more desirable accuracy. This we will save for future works, and we conclude by saying that the values of the QNM frequencies we are able to calculate demonstrate consistent linear relations with r_+, as observed in Ref. [6].

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