Red Light, Blue Light, Feynman and a Layer of Glass: The Causal Propagator

E. B. Manoukian
The Institute for Fundamental Study,
Naresuan University, Phitsanulok 65000, Thailand
Email Address: manoukian_eb@hotmail.com

To be published in: Journal of Electromagnetic Waves and Applications

Abstract

A propagator-description approach of the propagation of EM waves is provided, in complex situations with underlying boundary conditions. To this end, the so-called Schwinger-Feynman causal *propagator* is explicitly derived, and spelled out, for the first time, for transmission of a photon through, and reflection off, a layer of glass of flat parallel surfaces. In a very general context, the corresponding transition amplitudes of a *polarized or unpolarized* photon are obtained, and the conservation of probabilities, and the underlying consistency conditions of the results, are established. Inspired by Feynman's intuitive and well publicized non-technical treatment of such a problem with red and blue light, one, in a quantum view-point, is interested in the *probabilities* of the events just described in determining the fate of a photon. This analysis provides an *alternative* way of looking at the problem in question, through propagation of photons, as a physically more appealing approach, in Feynman's spirit, than by matching plane waves at boundaries as we often do. It also shows how a propagator may be derived, in more complex situations with specific boundary conditions, than just simply in infinite extended spaces. Finally, it is hoped that this work will be stimulating and of interest to practitioners with different emphases on EM wave propagation.

Keywords: Propagator theory as a means of describing propagation of electromagnetic waves, photons in a medium, computations in electrodynamics and boundary conditions, quantum probability.

1 Introduction

With the aim of providing a physically appealing approach to the propagation of electromagnetic waves in complex situations with appropriate boundary conditions, we develop a propagator-description approach in spacetime, in the presence of obstacles, as a time evolution process thus emphasizing the dynamical aspects of the formalism. This provides an alternative description of propagation, for example, to the standard approach, in terms of oscillating electromagnetic fields. To this end, the so-called Schwinger-Feynman causal *propagator* is derived, for the first time, and spelled out, to describe the fate of photons impinging on a layer of glass of flat parallel surfaces. The derivation, surprisingly, turned up to be far from trivial even for such a seemingly simple situation, where a slab is taken with constant dielectric permitivity (independent of frequency) but is nevertheless highly illuminating physically and opens the way for further generalizations. The underlying probabilistic interpretation that emerges for describing the fate of such photons from an actual quantum field theory analysis, is much in the spirit of a fascinating, intuitive and highly non-technical well publicized treatment by Feynman.[1] We explicitly derive the causal propagator for transmission through, and reflection off, such a layer of glass. The corresponding transition amplitudes of a polarized or unpolarized photon are then obtained, in a quantum field analysis, as the photons travel from an emission source to detection sources. Inspired by Feynman's study of the fate of red and blue photons, one, in a quantum-view point, is interested in the explicit *probabilities*, associated with quantum amplitudes, for determining the fate of a photon confronted with the situation just described. The Schwinger-Feynman propagator for the photon at hand, is also useful in its own right to be used in the present configuration of space, as the photon propagator, in infinite extension of space, in vacuum, may be used. In this respect, this analysis also shows how propagators may be derived in more complex situations with underlying boundary conditions, than just in infinite extended spaces. Also this alternative analysis, is perhaps physically more appealing with photons propagating, in a time evolution process, in Feynman's spirits, than the more standard method of matching plane waves at boundaries as we often do. Earlier studies (see, e.g., [2-5]) on photon dynamics were to define wave functions for the photon, as is done in quantum mechanics, and are different from the propagator approach developed here, which is much in the spirit of field theory in describing photon dynamics, and answering underlying probabilistic questions as well, in general environments, with underlying boundary conditions. Several interesting other studies have been also carried out in the literature (see, e.g., [7-15]), in general, in describing quantum particles, as the photon, in general classical situations. We should also mention an interesting work [16] (see also [17]) based on linear response theory which is not, however, developed as a time evolution process. This method, being linear, however, does not allow any non-linearities to be introduced into the theory that may be generated by any charges that may be present in the theory as encountered in quantum electrodynamics. Needless to say, our formalism as a time evolution process, in terms of a photon propagator, is just what is required for applications in quantum field theories with built in non-linearities embodied in them which certainly go beyond of applying only Maxwell's equations as such. The present method is also expected to have applications for the propagation of EM waves in plasmas (see, e.g., [18]), with possible non-linearities present in the formalism. For additional clarity, we have spelled out many details in the presentation, thus providing many steps in the derivations and hopefully making it *easier* for the reader to follow through. Sources are conveniently introduced in our treatment as emitters and detectors. They are localized in various regions and are not point-like as one might expect. They are simply introduced as a simple way of generating amplitudes, in a quantum mechanical setting, and are withdrawn or switched off once they create or introduce photons into the system, to be analyzed, and are finally detected. They do not participate in defining a medium. This method of generating amplitudes by introducing sources was developed and applied for years by Julian Schwinger culminating in his monumental work described in one of his books.[17] We hope that this work will be stimulating and of interest to practitioners with different emphases on electromagnetic waves propagation and also to witness the physically interesting approach of describing propagation of electromagnetic waves in terms of propagators.

In Sect.2, we set up the equations and the boundary conditions to be satisfied by the propagator. Sect.3, deals with the general structure of the propagator as described in the various regions in reference to the region of emission of photons. The boundary conditions are then applied in Sect.4, to provide the explicit solution of the propagator relating the transmission and reflection region to the emission one. In Sects.5, and 6, the transition amplitudes for reflection and transmission are, respectively, derived for unpolarized photons. The corresponding amplitudes for *polarized* photons are obtained in Sect.7. In the final section (Sect.8), the conservation of probabilities, and the underlying consistency conditions, are established and applications to Feynman's red and blue photons are given.

2 Setting-Up Equations and Boundary Conditions

We work in the celebrated temporal gauge for the vector potential $A^0 = 0$. The Lagrangian density of electrodynamics in a medium of constant permittivity ε, in the presence of an external source J^μ is, in this gauge, given by

$$\mathcal{L} = -\frac{1}{4}\,F_{ij}\,F^{ij} - \frac{\varepsilon}{2}\,F_{0i}F^{0i} + J^i A_i,\ i,j = 1,2,3, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{1}$$

with a summation over repeated indices understood.

Variation of the corresponding action with respect to the field components A^i gives

$$\Big[\big(-\partial^2 + \varepsilon\partial_0^2\big)\delta^{ik} + \partial^i\partial^k\Big]A^k = J^i. \tag{2}$$

This equation may be equivalently rewritten as

$$\Big(-\partial^2 + \varepsilon\partial_0^2\Big)A^j(x) = \Big[\delta^{jk} - \frac{\partial^j\partial^k}{\varepsilon\partial_0^2}\Big]J^k(x). \tag{3}$$

We may then introduce the propagator $D^{jk}(x',x)$ satisfying the equation

$$\Big(-\partial'^2 + \varepsilon\partial_0'^2\Big)D^{jk}(x',x) = \Big[\delta^{jk} - \frac{\partial'^j\partial'^k}{\varepsilon\partial_0'^2}\Big]\delta^{(4)}(x',x). \tag{4}$$

Because translational invariance is broken along the z-axis, due to the presence of the layer of glass, with the upper surface at $z = 0$, and the lower one at $z = -D$, as shown below, we have written the arguments of D^{jk} and $\delta^{(4)}$ as (x',x) rather than as $(x'-x)$:

$$\begin{array}{c} z > 0 \\ - - - - - - - - \; z = 0 \\ \varepsilon \\ - - - - - - - - \; z = -D \end{array}$$

Now we set up the boundary conditions at each of the two interfaces to obtain the expression for the propagator. To this end, the boundary conditions on the components of the electric and magnetic fields (see, e.g., the classic book [20]) may be now expressed in terms of the propagator as follows:

$$\begin{array}{l} D^{ak}(x',x), \qquad \varepsilon(z')D^{3k}(x',x), \qquad \text{and} \quad [\partial'^3 D^{ak}(x',x) - \partial'^a D^{3k}(x',x)] \\ \text{are continuous at} \quad z' = 0 \quad \text{and at} \quad z' = -D, \quad a = 1,2; \;\; k = 1,2,3, \end{array} \tag{5}$$

where

$$\varepsilon(z') = \varepsilon \quad \text{for} \quad -D < z' < 0, \quad \text{and} \quad \varepsilon(z') = 1, \quad \text{otherwise}. \tag{6}$$

Finally we note that with the external sources *localized* in time (and space), we also have

$$\partial'_j D^{jk}(x',x)=0, \quad \partial_k D^{jk}(x',x)=0, \quad \text{for} \quad x^{'0}\neq x^0. \tag{7}$$

With the source for the emission of photons situated at $z>0$, we consider the three different regions defined by $z'>0$, $z'<-D$, and $-D<z'<0$.

Region $z'>0$:

With the causal arrangement $x'^0>x^0$, we look for a particular as well as for a homogeneous solution of (4). To this end, a particular solution, for $a,b=1,2$, is elementary and, by complex integration in the energy plane, is given by

$$D^{ab}_{\rm p}(x',x)=\mathrm{i}\int\frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\frac{\mathrm{d}q}{2\pi}\,\mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\rm T}-\mathbf{x}_{\rm T})}\mathrm{e}^{\mathrm{i}q(z'-z)}\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x^{'0}-x^0)}\Big(\delta^{ab}-\frac{K^aK^b}{\mathbf{k}^2}\Big). \tag{8}$$

On the other hand a solution of the homogeneous equation

$$\Big(-\partial^{'2}+\partial_0^{'2}\Big)D^{jk}_{\rm h}(x',x)=0, \quad \Big(-\partial^2+\partial_0^2\Big)D^{jk}_{\rm h}(x',x)=0, \tag{9}$$

is of the form

$$D^{ab}_{\rm h}(x',x)=\mathrm{i}\int\frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\frac{\mathrm{d}q'\mathrm{d}q}{2\pi}\,\mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\rm T}-\mathbf{x}_{\rm T})}\mathrm{e}^{\mathrm{i}q'z'}\mathrm{e}^{-\mathrm{i}qz}\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x^{'0}-x^0)}\tilde{D}^{ab}_{\rm h}, \tag{10}$$

where the i factor is chosen for convenience, and $\mathbf{K}^2+q^{'2}=\mathbf{k}^2$, $\mathbf{K}^2+q^2=\mathbf{k}^2$, where the second equation follows from the second equation in (9). From these last two equalities, we *derive that* $q'=\pm q$. In the reflection region $z'>0$, we must choose $q'=-q$, $q<0$, with the latter negativity condition achieved by the restriction set by an emission source $J^i(x')$ in region $z'>0$, when carrying the integration $\int(\mathrm{d}x')J^i(x')D^{jk}(x',x)$, as given in (43), expressed in terms of the momenta of a photon. Eqn(10) becomes

$$D^{ab}_{\rm h}(x',x)=\mathrm{i}\int\frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\int\frac{\mathrm{d}q}{2\pi}\,\mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\rm T}-\mathbf{x}_{\rm T})}\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x^{'0}-x^0)}\mathrm{e}^{-\mathrm{i}q(z'+z)}A^{ab}_{>}, \tag{11}$$

where $A^{ab}_{>}$ will be determined from the boundary conditions.

We may thus write the solution in question for this region as

$$D^{ab}_{>}(x',x)=\mathrm{i}\int\frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\int\frac{\mathrm{d}q}{2\pi}\,\mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\rm T}-\mathbf{x}_{\rm T})}\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x^{'0}-x^0)}\Big[\mathrm{e}^{\mathrm{i}q(z'-z)}\Big(\delta^{ab}-\frac{K^aK^b}{\mathbf{k}^2}\Big)+\mathrm{e}^{-\mathrm{i}q(z'+z)}A^{ab}_{>}\Big], \tag{12}$$

From (7), we may readily obtain the corresponding expressions for $D^{3b}_{>}$, $D^{a3}_{>}$, and $D^{33}_{>}$. Accordingly, for this region, we have

$$D^{jk}_{>}(x',x)=\mathrm{i}\int\frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\int\frac{\mathrm{d}q}{2\pi}\,\mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\rm T}-\mathbf{x}_{\rm T})}\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x^{'0}-x^0)}\tilde{D}^{jk}_{>}, \tag{13}$$

and, with $a=1,2$,

$$\tilde{D}^{a3}_{>}=\mathrm{e}^{\mathrm{i}q(z'-z)}\Big(-\frac{K^aq}{\mathbf{k}^2}\Big)+\mathrm{e}^{-\mathrm{i}q(z'+z)}\Big(-\frac{A^{ab}_{>}K^b}{q}\Big), \tag{14}$$

$$\tilde{D}^{33}_{>}=\mathrm{e}^{\mathrm{i}q(z'-z)}\Big(\frac{\mathbf{K}^2}{\mathbf{k}^2}\Big)+\mathrm{e}^{-\mathrm{i}q(z'+z)}\Big(-\frac{K^aA^{ab}_{>}K^b}{q^2}\Big), \tag{15}$$

$$\tilde{D}^{3a}_{>}=\mathrm{e}^{\mathrm{i}q(z'-z)}\Big(-\frac{K^aq}{\mathbf{k}^2}\Big)+\mathrm{e}^{-\mathrm{i}q(z'+z)}\Big(\frac{K^bA^{ba}_{>}}{q}\Big). \tag{16}$$

Region $z' < -D$:

The structure of D^{jk} may be essentially written down from the considerations of the above region, except now the question of a particular solution does not arise ($z' \neq z$). Also, we have $q' = q$, $q < 0$. Accordingly

$$D^{jk}_{<}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \int \frac{\mathrm{d}q}{2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\mathrm{T}} - \mathbf{x}_{\mathrm{T}})} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0 - x^0)} \tilde{D}^{jk}_{<}, \tag{17}$$

$$\tilde{D}^{ab}_{<} = \mathrm{e}^{\mathrm{i}q(z'-z)} A^{ab}_{<}, \qquad \tilde{D}^{a3}_{<} = \mathrm{e}^{\mathrm{i}q(z'-z)} \Big(- \frac{A^{ab}_{<} K^b}{q} \Big), \tag{18}$$

$$\tilde{D}^{33}_{<} = \mathrm{e}^{\mathrm{i}q(z'-z)} \Big(\frac{K^a A^{ab}_{<} K^b}{q^2} \Big), \quad \tilde{D}^{3a}_{<} = \mathrm{e}^{\mathrm{i}q(z'-z)} \Big(- \frac{K^b A^{ba}_{<}}{q} \Big). \tag{19}$$

The unknown $A^{ba}_{<}$ will be also determined from the boundary conditions.

Region $-D < z' < 0$:

Here we have the homogeneous equation:

$$\Big(- \partial'^2 + \varepsilon \partial_0'^2 \Big) D^{jk}(x', x) = 0, \quad \Big(- \partial^2 + \partial_0^2 \Big) D^{jk}(x', x) = 0. \tag{20}$$

Thus upon writing

$$D^{ab}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \frac{\mathrm{d}q'\mathrm{d}q}{2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\mathrm{T}} - \mathbf{x}_{\mathrm{T}})} \mathrm{e}^{\mathrm{i}q'z'} \mathrm{e}^{-\mathrm{i}qz} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0 - x^0)} \tilde{D}^{ab}, \tag{21}$$

we have from (20), $\mathbf{K}^2 + q'^2 = \varepsilon \mathbf{k}^2$, $\mathbf{K}^2 + q^2 = \mathbf{k}^2$. From these last two equalities we obtain $q' = \pm Q$, $Q = \sqrt{(\varepsilon - 1)\mathbf{K}^2 + \varepsilon q^2}$. Hence the general solution for $a, b = 1, 2$ is given by

$$D^{ab}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \frac{\mathrm{d}q}{2\pi} \mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\mathrm{T}} - \mathbf{x}_{\mathrm{T}})} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0 - x^0)} [\mathrm{e}^{\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} M_1^{ab} + \mathrm{e}^{-\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} M_2^{ab}], \tag{22}$$

From (7), we then have, for the general solution for this region

$$D^{jk}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \frac{\mathrm{d}q}{2\pi} \mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\mathrm{T}} - \mathbf{x}_{\mathrm{T}})} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0 - x^0)} \underline{D}^{jk}, \tag{23}$$

with

$$\underline{D}^{a3} = \mathrm{e}^{\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \Big(- \frac{M_1^{ab} K^b}{q} \Big) + \mathrm{e}^{-\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \Big(- \frac{M_2^{ab} K^b}{q} \Big), \tag{24}$$

$$\underline{D}^{33} = \mathrm{e}^{\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \Big(\frac{K^a M_1^{ab} K^b}{qQ} \Big) + \mathrm{e}^{-\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \Big(- \frac{K^a M_2^{ab} K^b}{qQ} \Big), \tag{25}$$

$$\underline{D}^{3a} = \mathrm{e}^{\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \Big(- \frac{K^b M_1^{ba}}{Q} \Big) + \mathrm{e}^{-\mathrm{i}Qz'} \mathrm{e}^{-\mathrm{i}qz} \Big(\frac{K^b M_2^{ba}}{Q} \Big). \tag{26}$$

3 Satisfying the Boundary Conditions

Boundary Conditions at $z' = 0$:

The boundary conditions may be read directly from the previous section, and according to the continuity conditions spelled out in (5), respectively, are given, respectively, by the constraints

$$\Big(\delta^{ab} - \frac{K^a K^b}{\mathbf{k}^2} \Big) + A^{ab}_{>} = M_1^{ab} + M_2^{ab}, \; - \frac{K^b q}{\mathbf{k}^2} + \frac{K^a A^{ab}_{>}}{q} = \varepsilon \Big(- \frac{K^a M_1^{ab}}{Q} + \frac{K^a M_2^{ab}}{Q} \Big), \tag{27}$$

$$q\delta^{ab} - qA^{ab}_{>} - K^a \frac{K^c A^{cb}_{>}}{q} = Q\big(M_1^{ab} - M_2^{ab}\big) + K^a \Big(\frac{K^c M_1^{cb}}{Q} - \frac{K^c M_2^{cb}}{Q} \Big). \tag{28}$$

We then have

$$2M_1^{ab} = \Big(\frac{Q+q}{Q}\Big)\delta^{ab} - \Big(1+\frac{q}{\varepsilon\,Q}\Big)\frac{K^a k^b}{\mathbf{k}^2} + \Big(\frac{Q-q}{Q}\Big)A_>^{ab} + \Big(\frac{1}{\varepsilon}-1\Big)K^a\frac{K^c A_>^{cb}}{Qq}, \tag{29}$$

$$2M_2^{ab} = \Big(\frac{Q-q}{Q}\Big)\delta^{ab} - \Big(1-\frac{q}{\varepsilon\,Q}\Big)\frac{K^a k^b}{\mathbf{k}^2} + \Big(\frac{Q+q}{Q}\Big)A_>^{ab} - \Big(\frac{1}{\varepsilon}-1\Big)K^a\frac{K^c A_>^{cb}}{Qq}. \tag{30}$$

Boundary Conditions at $z' = -D$:

According to the the expressions in the previous section and the continuity conditions spelled out in (7) we have, respectively,

$$\mathrm{e}^{-\mathrm{i}QD}M_1^{ab} + \mathrm{e}^{\mathrm{i}QD}M_2^{ab} = \mathrm{e}^{-\mathrm{i}qD}A_<^{ab}, \tag{31}$$

$$\mathrm{e}^{-\mathrm{i}qD}\Big(-\frac{K^a A_<^{ab}}{q}\Big) = \varepsilon\Big[\mathrm{e}^{-\mathrm{i}QD}\Big(-\frac{K^a M_1^{ab}}{Q}\Big) + \mathrm{e}^{\mathrm{i}QD}\Big(\frac{K^a M_2^{ab}}{Q}\Big)\Big], \tag{32}$$

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}qD}\Big(qA_<^{ab} + K^a\frac{K^c A_<^{ab}}{q}\Big) &= Q\Big(\mathrm{e}^{-\mathrm{i}QD}M_1^{ab} - \mathrm{e}^{\mathrm{i}QD}M_2^{ab}\Big) \\ &\quad - K^a\Big(-\mathrm{e}^{-\mathrm{i}QD}\frac{K^c M_1^{cb}}{Q} + \mathrm{e}^{\mathrm{i}QD}\frac{K^c M_2^{cb}}{Q}\Big).\end{aligned} \tag{33}$$

We then obtain

$$2M_1^{ab} = \mathrm{e}^{+\mathrm{i}(Q-q)D}\Big[\Big(\frac{Q+q}{Q}\Big)A_<^{ab} + \Big(1-\frac{1}{\varepsilon}\Big)K^a\frac{K^c A_<^{cb}}{qQ}\Big], \tag{34}$$

$$2M_2^{ab} = \mathrm{e}^{-\mathrm{i}(Q+q)D}\Big[\Big(\frac{Q-q}{Q}\Big)A_<^{ab} - \Big(1-\frac{1}{\varepsilon}\Big)K^a\frac{K^c A_<^{cb}}{qQ}\Big]. \tag{35}$$

Upon comparing these two expressions with the corresponding ones in (29), (30), gives

$$a_> = \frac{(Q^2-q^2)\big(\mathrm{e}^{\mathrm{i}QD}-\mathrm{e}^{-\mathrm{i}QD}\big)}{(Q-q)^2\mathrm{e}^{-\mathrm{i}QD}-(Q+q)^2\mathrm{e}^{\mathrm{i}QD}}, \quad a_< = \frac{4\,qQ\,\mathrm{e}^{\mathrm{i}qD}}{(Q+q)^2\mathrm{e}^{\mathrm{i}QD}-(Q-q)^2\mathrm{e}^{-\mathrm{i}QD}}. \tag{36}$$

$$b_> = \Big[\frac{q^2}{\mathbf{k}^2}\frac{(Q^2-\varepsilon^2q^2)}{(Q+\varepsilon q)^2\mathrm{e}^{\mathrm{i}QD}-(Q-\varepsilon q)^2\mathrm{e}^{-\mathrm{i}QD}} + \frac{(Q^2-q^2)}{(Q+q)^2\mathrm{e}^{\mathrm{i}QD}-(Q-q)^2\mathrm{e}^{-\mathrm{i}QD}}\Big]\big(\mathrm{e}^{\mathrm{i}QD}-\mathrm{e}^{-\mathrm{i}QD}\big), \tag{37}$$

$$b_< = 4\,qQ\,\Big[\frac{\varepsilon(q^2/\mathbf{k}^2)}{(Q+\varepsilon q)^2\mathrm{e}^{\mathrm{i}QD}-(Q-\varepsilon q)^2\mathrm{e}^{-\mathrm{i}QD}} - \frac{1}{(Q+q)^2\mathrm{e}^{\mathrm{i}QD}-(Q-q)^2\mathrm{e}^{-\mathrm{i}QD}}\Big]\mathrm{e}^{\mathrm{i}qD}. \tag{38}$$

4 Explicit Solution of Propagator for Reflection and Transmission

For the propagator describing the reflection process, we have from (13)

$$D_>^{ij}(x',x) = \mathrm{i}\int\frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\int_{q<0}\frac{\mathrm{d}q}{2\pi}\,\mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_\mathrm{T}-\mathbf{x}_\mathrm{T})}\mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0-x^0)}\Big[\mathrm{e}^{\mathrm{i}q(z'-z)}A_0^{ij} + \mathrm{e}^{-\mathrm{i}q(z'+z)}A_>^{ij}\Big]. \tag{39}$$

where the explicit expression for A_0^{ij} follows from (19) - (21), to be given by $A_0^{ij} = \delta^{ij} - k^i k^j/\mathbf{k}^2$, $\mathbf{k} = (\mathbf{K}, q)$. A tedious analysis shows that the expressions for $A_>^{ab}$, $A_>^{3a}$, $A_>^{a3}$, $A_>^{33}$, in (22) - (24), may be expressed in a unified manner as

$$A_>^{ij} = \Big(\delta^{ij} a_> + \frac{k'^i k^j}{\mathbf{K}^2} b_>\Big) - \frac{1}{q^2}\Big[(\mathbf{k}^2 + 2\,q^2) a_> + \frac{\mathbf{k}^4}{\mathbf{K}^2} b_>\Big]\delta^{i3}\delta^{j3} - \frac{1}{q}\big[\,k'^i\delta^{j3} - \delta^{i3}k^j\,\big]\Big(a_> + \frac{\mathbf{k}^2}{\mathbf{K}^2} b_>\Big), \qquad \mathbf{k}' = (\mathbf{K}, -q), \tag{40}$$

The following important transversality conditions are readily verified $k'^i\, A_>^{ij} = 0$, $A_>^{ij}\, k^j = 0$.

For the propagator describing the transmission process, we have from (17)

$$D_<^{ij}(x', x) = \mathrm{i} \int \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2} \int_{q<0} \frac{\mathrm{d}q}{2\pi}\, \mathrm{e}^{\mathrm{i}\mathbf{K}.(\mathbf{x}'_{\mathrm{T}} - \mathbf{x}_{\mathrm{T}})} \mathrm{e}^{-\mathrm{i}|\mathbf{k}|(x'^0 - x^0)} \mathrm{e}^{\mathrm{i}q(z'-z)} A_<^{ij}, \tag{41}$$

and a tedious analysis shows that the expressions for $A_<^{ab}$, $A_<^{3a}$, $A_<^{a3}$, $A_<^{33}$, in (22) - (24), may be expressed in a unified manner as

$$A_<^{ij} = \Big(\delta^{ij} a_< + \frac{k^i k^j}{\mathbf{K}^2} b_<\Big) + \Big(a_< + \frac{\mathbf{k}^2}{\mathbf{K}^2} b_<\Big)\Big[\,\frac{\mathbf{k}^2}{q^2}\,\delta^{i3}\delta^{j3} - \frac{k^i\delta^{j3} + \delta^{i3}k^j}{q}\,\Big], \quad \mathbf{k} = (\mathbf{K}, q), \tag{42}$$

The following important transversality conditions are also readily verified $k^i\, A_<^{ij} = 0$, $A_<^{ij}\, k^j = 0$.

5 Transition Amplitude for Reflection

The transition amplitude for reflection is obtained from the expression

$$\mathrm{i}J_2^{i^*}(k')\; A_>^{ij}\; \mathrm{i}J_1^j(k) = \sum_{\alpha,\beta=1,2} \big(\mathrm{i}\mathbf{J}_2^*(k') \cdot \mathbf{e}'_\beta\big)\big[\,\mathrm{e}'^{i^*}_\beta\; A_>^{ij}\; \mathrm{e}^j_\alpha\,\big]\big(\mathbf{e}^*_\alpha \cdot \mathrm{i}\mathbf{J}_1(k)\big) \tag{43}$$

where $J_1^i(k), J_2^j(k')$ are Fourier transforms of an emission source and detection source, respectively, set causally in the $z > 0$ region, and $A_>^{ij}$ is given in (40). Here e'_α, e_α, $(\alpha, \beta = 1, 2)$, are polarization vectors, and we have the completeness relations:

$$\delta^{ij} = \frac{k'^i k'^j}{\mathbf{k}^2} + \sum_{\alpha=1,2} \mathrm{e}'^{i^*}_\alpha\, \mathrm{e}'^j_\alpha = \frac{k'^i k'^j}{\mathbf{k}^2} + \sum_{\alpha=1,2} \mathrm{e}'^i_\alpha \mathrm{e}'^{j^*}_\alpha = \frac{k^i\, k^j}{\mathbf{k}^2} + \sum_{\alpha=1,2} \mathrm{e}^{i^*}_\alpha\; \mathrm{e}^j_\alpha = \frac{k^i\, k^j}{\mathbf{k}^2} + \sum_{\alpha=1,2} \mathrm{e}^i_\alpha\; \mathrm{e}^{j^*}_\alpha, \tag{44}$$

$$\mathbf{k}' \cdot \mathbf{e}'_\alpha = 0, \quad \mathbf{k}' \cdot \mathbf{e}'^*_\alpha = 0, \quad \mathbf{e}'_\alpha \cdot \mathbf{e}'^*_\beta = \delta_{\alpha\beta}, \quad \mathbf{k} \cdot \mathbf{e}_\alpha = 0, \quad \mathbf{k} \cdot \mathbf{e}^*_\alpha = 0, \quad \mathbf{e}_\alpha \cdot \mathbf{e}^*_\beta = \delta_{\alpha\beta}. \tag{45}$$

The amplitude of a photon with momentum and polarization $\mathbf{k}$, $\mathbf{e}_\alpha$ being reflected by the layer of glass and ending up with momentum and polarization $\mathbf{k}'$, $\mathbf{e}'_\beta$, is then given by

$$(\mathbf{k}', \beta|\mathbf{k}, \alpha) = \mathrm{e}'^{i^*}_\beta\; A_>^{ij}\, \mathrm{e}^j_\alpha, \quad \mathbf{k} = (\mathbf{K}, q), \;\; \mathbf{k}' = (\mathbf{K}, -q), \quad q < 0, \tag{46}$$

with $\mathbf{e}^*_\alpha \cdot \mathrm{i}\mathbf{J}_1(k)$, $\mathrm{i}\mathbf{J}_2(k') \cdot \mathbf{e}'_\beta$, denoting, respectively, amplitudes for emitting and detecting a photon with such attributes.

For unpolarized photons, we average over the initial polarization states and sum over the final polarization states, using the completeness relations in (44). A tedious analysis gives for the probability of reflection of an unpolarized photon the result

$$\text{Prob}\Big|_{\text{refl.}} = \sum_{\beta=1,2}\Big(\frac{1}{2}\sum_{\alpha=1,2}|(\mathbf{k}',\beta|\mathbf{k},\alpha)|^2\Big) = \frac{1}{2}\Big[|a_>|^2 + \frac{\mathbf{k}^4}{q^4}\,|a_> + b_>|^2\Big], \tag{47}$$

$$|a_>|^2 = \frac{(Q^2-q^2)^2\,\sin^2 QD}{[4\,q^2Q^2 + (Q^2-q^2)^2\,\sin^2 QD]}, \quad \frac{\mathbf{k}^4}{q^4}|a_> + b_>|^2 = \frac{(Q^2-\varepsilon^2 q^2)^2\,\sin^2 QD}{4\,\varepsilon^2 q^2 Q^2 + (Q^2 - \varepsilon^2 Q^2)^2\,\sin^2 QD}. \tag{48}$$

where $Q = \sqrt{(\varepsilon-1)\mathbf{K}^2 + \varepsilon q^2}$. It is interesting to *compare these probabilities*, given here for special cases, as obtained from our very general analysis, with the so-called standard *reflection coefficients* given, e.g., in [20]. The probability of reflection of a *polarized* photon is worked out in § 7.

6 Transition Amplitude for Transmission

With $A^{ij}_<$ given in (42), the expression of the amplitude of transmission is extracted from

$$\mathrm{i}J_3^{i^*}(k)\;A^{ij}_<\;\mathrm{i}J_1^j(k), \quad \text{to be } (\mathbf{k},\beta|\mathbf{k},\alpha) = \mathrm{e}^{i^*}_\beta\,A^{ij}_<\,\mathrm{e}^j_\alpha, \tag{49}$$

where $J_3^i(k)$ is the Fourier transform of a detection source set causally in the $z < -D$ region, and $(\mathbf{k},\beta|\mathbf{k},\alpha)$ is the amplitude of *transmission* of the photon with polarization specified by the parameter β, where we have used the completeness relation in (44) to write the corresponding expression for the latter.

For unpolarized photons, we average over the initial polarization states and sum over the final polarization states as before. Using the completeness relations in (44), the following expression emerges for the probability of transmission of an unpolarized photon

$$\text{Prob}\Big|_{\text{trans.}} = \sum_{\beta=1,2}\Big(\frac{1}{2}\sum_{\alpha=1,2}|(\mathbf{k},\beta|\mathbf{k},\alpha)|^2\Big) = \frac{1}{2}\Big[|a_<|^2 + \frac{\mathbf{k}^4}{q^4}\,|a_< + b_<|^2\Big], \tag{50}$$

$$|a_<|^2 = \frac{4\,q^2Q^2}{[4\,q^2Q^2 + (Q^2-q^2)^2\,\sin^2 QD]}, \quad \frac{\mathbf{k}^4}{q^4}|a_< + b_<|^2 = \frac{4\,\varepsilon^2 q^2 Q^2}{4\,\varepsilon^2 q^2 Q^2 + (Q^2 - \varepsilon^2 Q^2)^2\,\sin^2 QD}. \tag{51}$$

where $Q = \sqrt{(\varepsilon-1)\mathbf{K}^2 + \varepsilon q^2}$. Again, it is interesting to *compare these probabilities*, which are given here for special cases from our general analysis, with the so-called standard *transmission coefficients* given, e.g., in [20]. The probability of transmission of a *polarized* photon is worked out in the next section.

7 Polarized Photons

In this section, we work out the probabilities of reflection and transmission for a polarized photon. To this end and for definiteness, we choose the momenta and real polarization vectors as ($q < 0$,

$K_2 \equiv K$),

$$\mathbf{k} = (0, K, q), \mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = \frac{1}{|\mathbf{k}|}(0, q, -K), \mathbf{k}' = (0, K, -q), \mathbf{e}'_1 = (1,0,0), \mathbf{e}'_2 = \frac{1}{|\mathbf{k}|}(0, -q, -K). \tag{52}$$

The following key relations should be noted $\mathbf{e}'_\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha 1}\delta_{\beta 1} + \delta_{\alpha 2}\delta_{\beta 2}(\mathbf{K}^2 - q^2)/\mathbf{k}^2$, $\mathrm{e}'^3_\alpha \mathrm{e}^3_\beta = \delta_{\alpha 2}\delta_{\beta 2}\mathbf{K}^2/\mathbf{k}^2$. From Eq.(40), (42) the explicit expression for the probabilities of reflection or transmission of a photon with initial and final polarizations specified by the parameters α, β, then readily emerge to be, respectively,

$$\mathrm{Prob}\big|_{\mathrm{refl.}}[\mathbf{k}, \alpha \to \mathbf{k}', \beta] = |a_>|^2\, \delta_{\alpha 1}\delta_{\beta 1} + \frac{\mathbf{k}^4}{q^4}|a_> + b_>|^2\, \delta_{\alpha 2}\, \delta_{\beta 2}, \tag{53}$$

$$\mathrm{Prob}\big|_{\mathrm{trans.}}[\mathbf{k}, \alpha \to \mathbf{k}, \beta] = |a_<|^2\, \delta_{\alpha 1}\delta_{\beta 1} + \frac{\mathbf{k}^4}{q^4}|a_< + b_<|^2\, \delta_{\alpha 2}\, \delta_{\beta 2}. \tag{54}$$

8 Conservation of Probability, Consistency and the Fate of Feynman's Red and Blue Photons

The probabilities of reflection and transmission of an unpolarized or polarized photon are given, respectively, in (47)/(50), (53)/(54), and on account of the equalities in (36) - (38), we obtain the consistency check $\mathrm{Prob}\Big|_{\mathrm{refl.}} + \mathrm{Prob}\Big|_{\mathrm{trans.}} = 1$, for both cases. Thus the conservation laws of probability pass the test with flying colors.

It is impossible not to be tempted to investigate the fate of, say, a red and blue photon in the spirit of Feynman. For more generality, consider a photon with, say, polarization vector $\mathbf{e}_2$. Then the probabilities of reflection and transmission of such a photon are from (53), (54), and the second relations in (48), (51):

$$\mathrm{Prob}\big|_{\mathrm{refl.}} = \frac{(Q^2 - \varepsilon^2 q^2)^2 \sin^2 QD}{4\,\varepsilon^2 q^2 Q^2 + (Q^2 - \varepsilon^2 Q^2)^2 \sin^2 QD}, \mathrm{Prob}\big|_{\mathrm{trans.}} = \frac{4\,\varepsilon^2 q^2 Q^2}{4\,\varepsilon^2 q^2 Q^2 + (Q^2 - \varepsilon^2 Q^2)^2 \sin^2 QD}, \tag{55}$$

with $Q = \sqrt{(\varepsilon - 1)\mathbf{K}^2 + \varepsilon q^2}$. Typically for glass we may take $\varepsilon = (3 + \sqrt{3})/2$ for the permittivity. For $|K| = |q|$, corresponding to an angle of incidence of 45^o, this gives $Q^2 = 2\varepsilon^2 q^2/3$, $Q^2 - \varepsilon^2 q^2)^2/(4\varepsilon^2 q^2 Q^2) = 1/24$. For such values, the transmission probability oscillates between 0.96 and one, while the reflection probability oscillates between zero and 0.04. We note that since $|\mathbf{k}| = \sqrt{2}|q|$, and $|\mathbf{k}| = 2\pi/\lambda$, where λ is the wavelength, we have $Q = 2\,\pi\varepsilon/\sqrt{3}\lambda$. Suppose red and blue light are prepared and emitted, in turn, by an emitting source, with wavelengths $\lambda_{\mathrm{red}} = (3 + \sqrt{3})(11/10)(\sqrt{\pi/2})\, 10^{-5}$ cm, $\lambda_{\mathrm{blue}} = (3 + \sqrt{3})(11/10)(\sqrt{\pi}/2)\, 10^{-5}$ cm. For glass of thickness $D = (\sqrt{3\pi}11)/10$ cm, the probability of reflection of the blue photon is zero since $Q|_{\mathrm{blue}}D = 2\pi(10^5)$, and $\sin Q|_{\mathrm{blue}}D = 0$, while for the red photon we have $Q|_{\mathrm{red}}D = (2\pi/\sqrt{2})\, 10^5 = 2\pi(70710 + .6781)$, with 70710 full cycles plus an angle of 38.85^o. This gives a reflection probability for the red photon to be 0.016. On the other hand, for a thickness of glass $D = (\sqrt{3\pi/2})(11/10)$ cm, the situation is exactly reversed with the red photon not being reflected and there is a probability of 0.016 for the blue photon to be reflected.

Clearly, other cases may be similarly treated. Needless to say, and as mentioned in the Introduction, this propagator is also useful in its own right, in such a complex situation, in the same way a propagator may be used in infinite extension of space, in vacuum. In particular, it shows how propagators are derived in more complex situations, with underlying boundary conditions, and emphasizes the physical aspect of photons propagating in spacetime in an alternative way, and, perhaps, in a physically more appealing way, than the one provided by matching plane waves at boundaries, as we often do. Hopefully, this work will be of interest to practitioners working on different aspects, and with different ways, with electromagnetic waves, and realize how the propagator approach, not only being physically appealing, but also very rich for other applications and various generalizations of the present development such as in introducing non-linearities into a theory, as one encounters in quantum electrodynamics, which go beyond of just making use of Maxwell's equations alone.

References

[1] Feynman R P. QED: The Strange Theory of Light and Matter, Princeton University Press: Princeton, New Jersey; 1985. p. 17-35, p. 64-72.

[2] Bialynicki-Birula I, Progress in Optics. Photon wave function. 1996; 36:245-294.

[3] Sipe J E. Photon wave functions. Phys Rev A. 1995; 52:1875-1883.

[4] Inagaki T. Quantum mechanical approach to a free photon. Phys Rev A. 1994; 49:2839-2843.

[5] Cook R J. Lorentz covariance of photon dynamics. Phys Rev A. 1982; 26:2754-2760.

[6] Cook R J. Photon dynamics. Phys Rev A. 1982; 25:2164-2167.

[7] Manoukian E B. Reflection-off a reflecting surface in quantum mechanics: Where do the reflections occur? I. Nuovo Cimento B. 1987; 99:133-143.

[8] Manoukian E B. Reflection-off a reflecting surface in quantum mechanics: Where do the reflections occur? II: the law of Reflection. Nuovo Cimento B. 1987; 100:185-193.

[9] Manoukian E B. Reflection off a reflecting surface in quantum mechanics. Nuovo Cimento B. 1990; 105:745-748.

[10] Manoukian E B. Quantum field theory of reflection and refraction of light. Hadronic J. 1997; 20:251-265.

[11] Kennedy G, Critchley R, Dowker J. Finite temperature field theory with boundaries: Stress tensor and surface action renormalization. Ann Phys (NY). 1980; 125:346-400.

[12] Deutsch D, Candelas P. Boundary effects in quantum field theory. Phys Rev D. 1979; 20:3063-3080.

[13] Schwinger J, deRaad L L Jr, Milton K A. Casimir effect in dielectrics. Ann Phys (NY). 1978; 115:1-23.

[14] Balian R, Duplantier B. Electromagnetic waves near perfect conductors.I. Multiple scattering. Ann Phys (NY). 1977; 104:303-335.

[15] Brown L S, Maclay G J. Vacuum stress between conducting plates: An image solution. Phys Rev. 1969; 184:1272-1270.

[16] Gottam M G, Maradudin A A. Surface Excitations: Agranovich V M, Loudon R, editors. North-Holland: Amsterdam; 1984. p.1-194.

[17] Lahlaouti M L. Electromagnetic waves in finite superlattices with buffer and cap layers. J Opt Soc Am A. 1999; 16:1703-1714.

[18] Melrose D B, McPhedran R C. Electromagnetic Processes in Dispersive Media, Cambridge University Press: Cambridge (UK); 1991.

[19] Schwinger J. Particles, Sources and Fields, Addison-Wesley: Reading, Massachusetts; 1970.

[20] Born M, Wolf E. Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light. 7th ed. Cambridge University Press: Cambridge (UK); 1999.