Study on the Quantization of Fields

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This independent study entitled "Study on the Quantization of Fields" submitted in partial fulfillment of the requirements for Bachelor of Science Degree in Physics is hereby approved.

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Abstract

In this article, starting from the problems in the quantum mechanics, which is the formalism based on the Schrödinger equation lacking of the Lorentz invariance and the difficulty in the description for the system with arbitrary number of multi-quantum particles. We review the canonical quantization of fields which can overcome these problems. In the process of this, we start with the four-dimensional space-time with the Lorentz invariance. Then we obtain the scalar, vector, and SL(2,C) spinor fields as one of the possible representations of fields in the Lorentz invariant four dimensional space-time. Finally, we construct the action for each of these fields with imposing the fundamental physical requirements.

Key words : canonical quantization, quantum mechanics and quantum field theory

Chapter 1

Introduction

1.1 Background

It is known that there are following problems :

1. the formalism in the quantum mechanics is not invariant under the Lorentz transformation,

2. the number of particle in the quantum mechanics is basically one partical.

in the quantum mechanics, and these are considered as a motivation to the quantum field theory from the quantum mechanics. This article reviews the quantization of fields according to canonical quantization which cures these problems. In this introduction, we overview these problems and how to cure these.

First we take the problem that the quantum mechanics is not invariant under the Lorentz transformation. The basic equation in the quantum mechanics is the Schrödinger equation:

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi(\vec{x},t),$$

where $\psi(\vec{x},t)$ is a wave function for a state, m is its mass and V is potential terms. Then we can see readily that the Schrödinger equation is not invariant under the Lorentz transformation. As seen from the discussion in section.2, the fundamental equation should be invariant under the Lorentz transformation. To cure this point, we consider the equation for the quantum state with the Lorentz invariance. If the wave function is invariant under the Lorentz transformation, the relativistic relation: $E^2 - \vec{p}^2 = m^2$ should be satisfied, where $(m,\vec{p})$ is the four-momentum vector and E is the energy of the wave function.

Hence, to obtain the equation for the quantum state with the Lorentz invariance, we replace E and p for the operators $-i\hbar/c{\cdot}\partial/\partial t$ and $i\hbar\vec{\nabla}$ which correspond to the momentum and energy, respectively. Toward the relativistic relation in which the four-momentum vector is replaced with the operators, acting on the wave function, we can obtain the following equation:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2}{\hbar^2}\right)\psi(\vec{x},t) = 0.$$

This is the relativistic quantum equation in which the first point is cured.

However the quantum system with the above equation has other problems. It is the difficulty to describe arbitrary number of multi-quantum particle system. Actually, to describe the collision and scattering processes of quantum particles, the formalism which can treat the arbitrarily number of multi-quantum particles is necessary. However the wave function in the quantum mechanics is for one particle and does not correspond to the arbitrary number of multi-quantum particles.

To correspond to this problem, we consider the quantization of the field. To be more specific, instead of the commutation relation for the conjugate coordinates in the phase space $[x_i, p_j] = i\hbar\delta_{ij}$ $(i, j = x, y, z)$, we consider the commutation relation for the conjugate fields as

$$[\psi(\vec{x}, t), \pi(\vec{y}, t)] = i\hbar\delta^3(\vec{x} - \vec{y}),$$

where $\pi(\vec{x}, t)$ is the conjugate momentum of $\psi(\vec{x}, t)$. This is the one way of field quantization which is called the canonical quantization. By this, actually we will show in section.3 in this article, the system can be given by the states discretized by the number of particle, and description of the arbitrarily number of multi-quantum particle system becomes possible. By this we can consider that second part is cured.

We mention the organization of this paper. In Subsection.2.1, starting with the Lorentz invariance in the four-dimensional space-time, we introduce the Lorentz algebra which is the commutator relation in which the generators of the Lorentz group should satisfy. In Subsection.2.2, based on the commutator relation in which the generators of the Lorentz group should satisfy obtained in Subsection.2.1, we discuss what kinds of the fields are possible in the space-time with Lorentz symmetry. As a result, we obtain the scalars, vectors and SL(2,C) spinors as ones of the possible fields. In Subsection.2.3, we discuss the Noether theorem which makes a point that there are always the conserved quantities, if the actions are invariant under the continuous transformations. In Subsection.2.4, showing the physically necessary fundamental conditions which the actions of fields should satisfy, we discuss the forms of actions for the real and complex scalar fields, the forms of the potential terms and dimensionality of fields. Further, based on the Noether theorem, in the given each action, we discuss the preservation of the number of particle. In Subsection.2.5, we construct the actions for the spinor fields with the discussion of the preservation of the number of particle in the actions, and we finally introduce the Dirac spinors and Majorana spinors. In section.3, we perform the quantization of the fields according to the canonical quantization. In Subsection.3.1, we quantize the free scalar fields described by the Klein-Gordon equation, and show that the system of the quantized fields is given by the states discretized by the number of particle. In Subsection.3.2, we quantize the free Dirac fields described by the Dirac equation as well as Subsection.3.1.

In this article, we will not treat the quantization of the vector fields. Because it has a highly involved problems concerning the gauge symmetries, we would need another article to discuss it.

Finally, more detailed and concrete discussion on the problems in quantum mechanics and the necessarily of the field quantization mentioned in this section is given ref.[1]. Further refs.[2, 3] were referred so much in the writing of this article.

1.2 Objective

To formulate the framework of theory that can describe the creation/annihilation quantum process of particles.

1.3 Framework

The canonical quantization in the scalar and spinor fields described by free Klein-Gordon and Dirac equations respectively in the $D = 1 + 3$ Minkowski space-time

1.4 Expected Use

- Description of the quantum processes with the created/annihilated particles
- Application to the semiconductors
- Analysis in the experimental confirmation of the quantum electrodynamics based on the measurement of the fine structure

1.5 Procedure

- Introduction : Motivation to the quantum field theory
  - Problem points in the quantum mechanics
  - Quantization of fields
- Possible representation of the fields in the Minkowski space-time
- Noether theorem
- Quantization of the fields
- Conclusion

1.6 Outcome

The canonical quantization of the fields that enables to describe the quantum processes with the created/annihilated particles in the $D = 1 + 3$ Minkowski space-time

Chapter 2

Representation of Lorentz Group and Fields

2.1 Lorentz Group

Notation: One of the most fundamental properties in the particle physics is the Lorentz invariance. We now start with running through our notation.

Summing up the time coordinate t and spacial coordinate x as the four-dimensional vectors, we write as

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \boldsymbol{x}), \tag{2.1}$$

We call this as the four-dimensional upper index vector or the contravariant vector.

The four-dimensional Minkowski space-time is defined with the following inner products:

$$x \cdot y \equiv xy \equiv g_{\mu\nu} x^\mu y^\nu = x^0 y^0 - \boldsymbol{x} \cdot \boldsymbol{y}. \tag{2.2}$$

where $g_{\mu\nu}$ is the Minkowski metric,

$$g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1) \tag{2.3}$$

and its inverse metric $g^{\mu\nu}$ is same with $g_{\mu\nu}$, which satisfies

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho \quad (\delta^\mu{}_\rho\text{: Kronecker delta}). \tag{2.4}$$

We can raise and lower the vector and tensor indices freely using the metrics $g^{\mu\nu}$ and $g_{\mu\nu}$. For instance, we can have the vector with lower index x_μ by lowering the index of x^μ as

$$x_\mu \equiv g_{\mu\nu} x^\nu = (x^0, -\mathbf{x}). \tag{2.5}$$

We call the vector with lower index the covariant vector. Conversely, raising the index by $g^{\mu\nu}$, it goes back to the original one as

$$g^{\mu\nu} x_\nu = g^{\mu\nu} g_{\nu\rho} x^\rho = \delta^\mu{}_\rho x^\rho = x^\mu. \tag{2.6}$$

In the same way we can raise and lower indices freely as

$$T_{\mu\nu} = g^{\mu\rho} g_{\nu\sigma} T^{\rho\sigma}, \quad T_{\mu}{}^{\nu} = g_{\mu\rho} T^{\rho\nu} \quad \text{and} \quad T^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma}, \tag{2.7}$$

Lorentz transformation: Among the liner transformations of four-dimensional vectors x^μ,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.8}$$

the one which keeps its absolute value x^2,

$$x^2 \equiv g_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu \tag{2.9}$$

invariant is called the Lorentz transformation.

Infinitesimal transformation: In order to understand the Lorenz transformation, we look at the infinitesimal Lorentz transformation. To this purpose we write the Lorentz transformation as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu, \tag{2.10}$$

where $\epsilon^\mu{}_\nu \ll 1$. Expanding ϵ up to the first order, we can obtain the following relation:

$$g_{\rho\sigma} + g_{\mu\rho} \epsilon^\mu{}_\rho + g_{\rho\nu} \epsilon^\nu{}_\rho + \mathcal{O}(\epsilon^2) = g_{\rho\sigma}, \tag{2.11}$$

which leads to

$$g_{\nu\mu} + g_{\mu\nu} = 0. \tag{2.12}$$

We can see that the number of the independent parameters in $g_{\mu\nu}$ is six.

Now let us rewrite the Lorentz transformation given in (2.8) as

$$x'^\mu = x^\mu + \epsilon^\mu{}_\nu x^\nu \equiv \left(1 - \frac{i}{2} \epsilon^{\rho\sigma} M_{\rho\sigma}\right)^\mu{}_\nu x^\nu, \tag{2.13}$$

where

$$(M_{\rho\sigma})^\mu{}_\nu \equiv i(\delta^\mu_\rho g_{\sigma\nu} - \delta^\mu_\rho g_{\rho\nu}). \tag{2.14}$$

We can check that $(M_{\rho\sigma})^\mu{}_\nu$ satisfies the following relation:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\rho} M_{\nu\sigma} - g_{\mu\rho} M_{\nu\sigma} - g_{\mu\rho} M_{\nu\sigma} + g_{\mu\rho} M_{\nu\sigma}). \tag{2.15}$$

Writing of eq.(2.13) corresponds to writing $\hat{\Lambda}$

$$\hat{\Lambda} = \exp\left(-\frac{i}{2} \epsilon^{\rho\sigma} \hat{M}_{\rho\sigma}\right) \approx 1 - \frac{i}{2} \epsilon^{\rho\sigma} \hat{M}_{\rho\sigma}. \tag{2.16}$$

After all, $\hat{\Lambda}$ is elements of $SO(3,1)$ Lie group, and $\hat{M}_{\rho\sigma} = -\hat{M}_{\sigma\rho}$ is the $SO(3,1)$ Lie algebras. Eq.(2.14) is a representation, and the relation given at (2.15) is independent of specific representation. Writing the representation-independent Lie generators $\hat{M}_{\rho\sigma}$, the relation of eq.(2.15) in the representation-independent way can be written as

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] = i(g_{\mu\rho} \hat{M}_{\nu\sigma} - g_{\mu\rho} \hat{M}_{\nu\sigma} - g_{\mu\rho} \hat{M}_{\nu\sigma} + g_{\mu\rho} \hat{M}_{\nu\sigma}). \tag{2.17}$$

2.2 Various Fields: Possible Representations of Lorentz Group

The fundamental quantities in the field theories are the fields.

Lorentz Transformation of Fields: We consider a space-time point P to observer O in an inertial system and O' in an another inertial system. At this time we describe the coordinates of P in O and O' as x^μ and x'^μ, respectively. These are linked to together by the following Lorentz transformation,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{2.18}$$

At this time, since the field $\phi(x)$ for the O and the field $\phi'(x')$ for the O' are the fields at the same point, these should be linked to together. In the case of the scalar field ϕ, the same values should be observed at the same point independent of the inertial system as

$$\phi'(x') = \phi(x). \tag{2.19}$$

On the other hand, the relation in the case of the vector fields A_μ is given as

$$A'_\mu(x') = \Lambda_\mu{}^\nu A'_\nu(x). \tag{2.20}$$

In the case of general fields ϕ_i, writing the representation of Λ as $D(\Lambda)$ which is $N \times N$ matrices,

$$\phi'_i(x') = D(\Lambda)_i{}^j \phi_i(x). \tag{2.21}$$

Hence the question of what kind of fields are possible are equal to the question of what kind of representation are possible, and we can check all kinds of the fields by finding out all the representations of the Lorentz transformation.

Representations of Lorentz Group: We have given the infinitesimal Lorentz transformation at eq.(2.10). We know the representation of the Lorentz transformation if we know the representation of the infinitesimal Lorentz transformation. We write the representation of the infinitesimal Lorentz transformation as

$$D(\Lambda)_i{}^j = \delta_i{}^j - \frac{i}{2}\epsilon^{\mu\nu}(S_{\mu\nu})_i{}^j \equiv \left(1 - \frac{i}{2}\epsilon^{\mu\nu}S_{\mu\nu}.\right)_i{}^j \tag{2.22}$$

The $N \times N$ matrices $S_{\mu\nu}$ give the representations of the generators in the Lorentz transformation $D(\hat{M}_{\mu\nu})$. Hence toward the Lorentz transformations:

$$\hat{\Lambda} = \exp\left(-\frac{i}{2}\epsilon^{\mu\nu}\hat{M}_{\mu\nu}\right), \tag{2.23}$$

the representation of the finite Lorentz transformations can be written as

$$D(\hat{\Lambda}) = \exp\left(-\frac{i}{2}\epsilon^{\mu\nu}D(\hat{M}_{\mu\nu})\right) = \exp\left(-\frac{i}{2}\epsilon^{\mu\nu}S_{\mu\nu}\right). \tag{2.24}$$

The representation matrices $S_{\mu\nu} = D(\hat{M}_{\mu\nu})$ should satisfy the commutator relations of $\hat{M}_{\mu\nu}$ in eq.(2.17).

To obtain all the $S_{\mu\nu}$, first we define the angular momentum operator $\vec{J}$ and the boost operator $\vec{K}$ from $\hat{M}_{\mu\nu}$

$$J_i \equiv \frac{1}{2}\epsilon_{ijk}\hat{M}^{jk} = (\hat{M}_{23}, \hat{M}_{31}, \hat{M}_{12}), \tag{2.25}$$

$$K_i \equiv \hat{M}_{23} = -\hat{M}_{23}, \tag{2.26}$$

we rewrite the commutator relation of $\hat{M}_{\mu\nu}$, where the number of $\vec{J}$ and $\vec{K}$ is totally six.

Then we can see easily that

$$[J_i, J_j] = \epsilon_{ijk}J_k, \tag{2.27}$$

$$[J_i, K_j] = \epsilon_{ijk}J_k, \tag{2.28}$$

$$[K_i, K_j] = \epsilon_{ijk}J_k. \tag{2.29}$$

Further more by defining the operators of **A**-spin and **B**-spin:

$$\mathbf{A} \equiv \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \quad \mathbf{B} \equiv \frac{1}{2}(\mathbf{J} - i\mathbf{K}) \tag{2.30}$$

we can obtain the following relations

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \tag{2.31}$$

$$[B_i, B_j] = i\epsilon_{ijk}B_k, \tag{2.32}$$

$$[B_i, B_j] = 0. \tag{2.33}$$

We can see that each of **A** and **B** satisfies the angular momentum algebra, the Lie algebra of $SU(2)$. Hence the representations of the Lorentz group can be specified by the magnitude of **A**-spin and **B**-spin (A,B) (A,B : integers or half-integers), and the dimension of its representation space is turned out as $(2A + 1)(2B + 1)$. The fields corresponding to this representation is the fields composed of $(2A + 1)(2B + 1)$ component

$$\phi_{a,b}^{(A,B)}(x), \quad \begin{pmatrix} a = -A, \cdots, A \\ b = -B, \cdots, B, \end{pmatrix}, \tag{2.34}$$

where a,b are the eigenvalues of A_3 and B_3. By these, it can be said that we have found out all kind of the irreducible field.

SL(2,C) Spinor: The magnitudes of the spin are integers or half-integers, and the simplest representations are $(A, B) = (0, 1/2)$ or $(A, B) = (1/2, 0)$, which are two-dimensional representations. First let's consider $(0, 1/2)$-representation. The $\phi(x)$ in eq.(2.34) is the two-component. We denote it as $\xi_\alpha(x)$ $(\alpha = 1, 2)$. $(A, B) = (0, 1/2)$ means that **A**-spin and **B**-spin are given as

$$D(\mathbf{A}) = 0, \quad D(\mathbf{B}) = \sigma/2 \quad (\sigma \ : \ \text{Pauli matrices}). \tag{2.35}$$

Hence the representation of **J** and **K** are

$$D(\mathbf{J}) = \frac{\sigma}{2}, \quad D(\mathbf{J}) = i\frac{\sigma}{2}. \tag{2.36}$$

When the parameters are given by

$$(\epsilon_{23}, \epsilon_{31}, \epsilon_{12}) = -\theta, \quad (\epsilon_{10}, \epsilon_{20}, \epsilon_{30}) = \omega, \tag{2.37}$$

the finite Lorentz transformations (2.23) are

$$\hat{\Lambda} = \exp\left(i\theta\mathbf{J} + i\omega\mathbf{K}\right). \tag{2.38}$$

Describing 2×2 matrices $D(\Lambda)$ in the case of $(0, 1/2)$-representation as

$$a_\alpha{}^\beta = \Big[\exp(i\theta \cdot \frac{\sigma}{2} - i\omega \cdot \frac{\sigma}{2})\Big]_\alpha{}^\beta \tag{2.39}$$

where $\xi_\alpha(x)$ are transformed as

$$\xi'_\alpha(x') = a_\alpha{}^\beta \xi_\alpha(x). \tag{2.40}$$

We can see that a with the parameter θ and ω given at eq.(2.39) can run over $SL(2, C)$ complex matrices entirely, and $\xi_\alpha(x)$ are the two-spinors transformed by the $SL(2, C)$ group. By eqs.(2.38) and (2.39), the corresponding relation between the Lorentz group and $SL(2, C)$ group has been given. Indeed it can be seen that this correspondence is one-to-two correspondence. In effect, for example, we take the spacial rotation angular θ to $\theta + 2\pi$. Then the element of the Lorentz group $\hat{\Lambda}$ can be back to what it was. However the elements a in $SL(2, C)$ given in eq.(2.39) becomes $-a$. We have to perform a rotation by 4π in order to back a to what it was. By this meaning, the $SL(2, C)$ spinor representations are the two-valued representation.

Next, we consider the quantities which get the same transformation with the complex conjugate of ξ_α, $(\xi_\alpha)^*$. We describe these as $\eta^{\dot{\alpha}}$ ($\dot{\alpha} = 1, 2$), and name these the dotted spinor. Toward this, we name the first ξ_α the undotted spinor. Say,

$$\eta'_{\dot{\alpha}}(x') = a^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \eta_{\dot{\beta}}(x) \quad (a^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \equiv (a_{\dot{\alpha}}^{\dot{\beta}})^*). \tag{2.41}$$

These are called 2^*-spinor in $SL(2, C)$. Reading out the representation matrices of **A**-spin and **B**-spin from a^*, the followings are turned out:

$$D(A) = -\frac{\sigma^*}{2}, \quad D(B) = 0 \tag{2.42}$$

.

2.3 Noether Theorem

We have discussed about the Lorentz invariance from subsection.2.1. To find out the symmetries and invariance have been the fundamental and one of the central issues in the particle physics. In this subsection we discuss about the Noether theorem. The Noether theorem makes a point that there are always the conserved quantities, if the actions are invariant under the continuous transformations.

We consider the action described as

$$S[\phi] = \int d^4x \mathcal{L}(\phi_i, \partial_\mu \phi_i). \tag{2.43}$$

We consider the infinitesimal transformation,

$$\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \epsilon G_i(\phi(x)), \tag{2.44}$$

where ϵ is the infinitesimal parameter. Under this transformation we assume that the action is invariant: $S[\phi] = S[\phi']$ up to the total deliberation. Namely,

$$\delta\mathcal{L} = \mathcal{L}(\phi'_i(x), \partial_\mu \phi'_i(x)) - \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) = \epsilon \partial_\mu X^\mu(\phi(x)). \tag{2.45}$$

On the other hand, the variation of the action under the transformation in eq.(2.44) is given as

$$\begin{aligned} \delta\mathcal{L} &= (\partial\mathcal{L}/\partial\phi_i)\epsilon G_i(\phi) + (\partial\mathcal{L}/\partial(\partial_\mu\phi_i))\epsilon\partial_\mu G_i(\phi) && (2.46) \\ &= \epsilon[\partial_\mu(\partial\mathcal{L}/\partial(\partial_\mu\phi_i)) \cdot G_i(\phi) + (\partial\mathcal{L}/\partial\phi_i)\partial_\mu G_i(\phi)]. && (2.47) \end{aligned}$$

Here on the first line we have used the property that the transformation parameter ϵ is independent of x (the global transformation), and at the second line we have used the Euler-Lagrange equation $\frac{\delta S}{\delta \phi_i(x)} \equiv \frac{\delta \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))} \right) = 0$. By equating equations (2.45) and (2.46) , we obtain the conserved law of the current as

$$\partial_\mu j^\mu(x) = 0 \tag{2.48}$$

with the current,

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i(x))} G_i(\phi(x)) - X^\mu(\phi(x)). \tag{2.49}$$

There is an important point in the Noether theorem to be noticed. The conserved charge obtained from the Noether theorem,

$$Q = \int d^3x j^0(x) and \quad dQ/dt = 0 \tag{2.50}$$

play the role of the transformation's generators. Therefore the following equations are valid:

$$\text{In the classical theory: } \{\phi_i(x), Q\}_{\mathrm{P}} = G_i(\phi(x)), \tag{2.51}$$

$$\text{In the quantum theory: } [iQ, \phi_i(x)]_{\mathrm{P}} = G_i(\phi(x)), \tag{2.52}$$

where $\{\ ,\ \}_{\mathrm{P}}$ is the Poisson bracket, and $[\ ,\ ]_{\mathrm{P}}$ means the commutator relation.

A group composed of all of the translations and Lorentz transformations are called the Poincaré group. The conserved charges playing the role of the generators for these are

$$P_\rho = \int d^3x T_\rho{}^0, \quad and M_{\rho\sigma} = \int d^3x \mathcal{M}^0_{\rho\sigma}. \tag{2.53}$$

2.4 The Action of Scalar Fields

In the previous subsection we have written the action of $\phi_i(x)$ as in eq.(2.43). We have noticed that the following conditions are imposed generally when actions are given.

- The locality: The Lagrangian density is given on a coordinate point x as in eq.(2.43), which means the assumption that the interaction works only at the vicinity of x.
- $\mathcal{L}$ is real numbers or Hermitian: This corresponds to what the energy is real in classical theories, and time evolution is unitary in the quantum theories, which means that the probability can be conserved in the quantum theories.
- $\mathcal{L}$ is a function of ϕ and $\partial\phi$, and the quadratic order of $\partial\phi$ at most. This is the assumption that the derivative of the time is no more than the second order in the equations of motion. In the quantum theory, if the equations involve the higher derivative terms, there always appears the following two difficulties: i) The appearance of the negative definite particles leading to the negative probabilities and ii) The appearance of the tachyons.
- The Poincare invariance and inner symmetries etc : The Lorentz invariance requires that $\mathcal{L}$ is invariant under the Lorentz transformations, and translational symmetries require that $\mathcal{L}$ does not depend on x^μ explicitly. Furthermore, some inner symmetries such as the isospin symmetries and some discrete symmetries such as the time reflection $\mathcal{T}$ and space reflection $\mathcal{P}$ are required,
- Renormalizability: If the renormalizability is required, the dimensionality of the Lagrangian density $\mathcal{L}$ should be lower than 4.

Putting the above assumptions, let us consider the Lagrangian density $\mathcal{L}$ for the scalar fields.

The natural Scalar Fields: Let us consider only one kind of real-scalar field $\phi(x)$. In this case, the general Lagrangian densities satisfying the above requirements are given as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{2.54}$$

We call the first and second terms the kinetic and potential terms, respectively. In the form of $V(\phi)$, since the constant part plays the role only to shift the origin of the energy, we can ignore it. Further more, since the liner order term of ϕ can be canceled by the redefinition: $\phi(x) \to \psi(x)+c$ (constant), we can also ignore it. As a result, we can consider that the potential can start from the quadratic order as

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{3!}g\phi^3 + \frac{1}{4!}\lambda\phi^4 + \cdots . \tag{2.55}$$

Relating to the requirement of the renormalizability 5), now we have to discuss the dimensionality. First, we employ the natural units: $c = \hbar = 1$, which means

$$c = [L/T] = 1, \quad \hbar = [E \cdot T] = 1. \tag{2.56}$$

Hence the length L and time T are in the same dimensionality, and the dimensionality of energy E is the dimensionality of T^{-1}. The dimensionality of M is equivalent to the dimensionality of energy by the Einstein relation: $E = Mc^2 = M$. Hence we can write down the dimensionality of any quantities using M:

$$[T] = [L] = [M^{-1}], \quad [E] = [M]. \tag{2.57}$$

We call this the mass dimension.

Since the dimensionality of the action is 0, the dimensionality of the Lagrangian density $\mathcal{L}$ is given by the inverse of the dimensionality of volume element $d^n x$, namely $[\mathcal{L}] = n$. Hence from counting the kinetic term's dimensionality, we can see that

$$\dim[\phi] = (n-2)/2. \tag{2.58}$$

If we require the renormalizability, since $\dim[\mathcal{L}] \leq n$, the potential term should be given as the expansion to the quartet order at $n = 4$ and the triplet-order at $n = 6$. On the other hand, the case of $n = 2$ is particular. Because we can prove that $[\psi] = 0$ at that time. As a result, there is no limitation in the expansion order of the potential term.

In the case of four-dimensional, requiring the invariance under the discrete transformation: $\phi \to -\phi$, it is quite often that we take the potential term as

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4!}\lambda\phi^4. \tag{2.59}$$

This is called the ϕ-forth model.

The Euler-Lagrange equation in this system can be obtained as $(\Box + \mu^2)\phi = -\dfrac{\lambda\phi^3}{3!}$ $(\Box \equiv \partial^\mu\partial_\mu)$. As a result, we can see that μ plays the role of mass, and we call the first and second terms in eq.(2.59) the mass and interaction terms, respectively.

The Complex Scalar Fields We can consider the case of the complex scalar field similarly. The ϕ-forth model in the the complex scalar field can be written as

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - \mu^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2. \tag{2.60}$$

The different point from the real field is that the model is invariant under the following $U(1)$ global phase transformation

$$\begin{cases} \phi(x) \to \phi'(x) = e^{i\theta}\phi(x), \\ \phi^*(x) \to \phi^{*\prime}(x) = e^{-i\theta}\phi^*(x). \end{cases} \tag{2.61}$$

Since this is a continuous transformation, there is the conserved current according to the Noether theorem. In eq.(2.61), the infinitesimal transformation in eq.(2.61) can be written as

$$\delta\phi(x) \equiv \psi'(x) - \phi(x) = i\theta\phi(x) and \quad \delta\phi(x)^* = \phi'(x) - \phi(x) = -i\theta\phi^*(x). \tag{2.62}$$

Then from $\delta\mathcal{L} = 0$, the Noether current given in eq.(2.48) in subsection 2.3 can be obtained as

$$\begin{aligned} j^\mu(x) &= (\partial\mathcal{L}/\partial(\partial_\mu\phi^*))i\phi + (\partial\mathcal{L}/\partial(\partial_\mu\phi^*))(-i\phi^*) \quad (2.63)\\ &= -i\phi^*\overleftrightarrow{\partial}\phi \quad (f\overleftrightarrow{\partial}g \equiv f\partial_\mu g - \partial_\mu f\cdot g). \quad (2.64)\end{aligned}$$

The Noether charge corresponding to this can be also obtained as

$$Q = \int d^3x j^0(x) = \int d^3x i(\dot{\psi^*}\psi - \psi^*\dot{\psi}). \quad (2.65)$$

As mentioned in the previous subsection, we can have the following relations in the quantum theory:

$$[iQ, \phi(x)] = i\psi(x), \quad (2.66)$$

$$[iQ, \phi^*(x)] = -\, i\phi^*(x). \quad (2.67)$$

We can confirm that these can be the generators of the transformation in eq.(2.61) from the process of the canonical quantization performed later.

It can be said from eq.(2.66) that the complex scalar field has the charge. Actually taking $|*\rangle$ as an eigenstates of the operator Q with the charge q, namely $q|*\rangle = Q|*\rangle$. Then

$$Q\phi(x)|*\rangle = ([Q, \phi(x)] + \phi(x)Q)|*\rangle = (q+1)\phi(x)|*\rangle. \quad (2.68)$$

Hence we can see that $\phi(x)$ and $\phi^*(x)$ are the operators creating a quantum with $Q = +1$ and $Q = -1$, respectively. Toward this, the real scalar field has no phase transformation, and it describes the neutral particles.

2.5 The Action of Spinor Fields

In what follows, we use the following notation as the 2 and 2^*-representations of $SL(2, C)$ spinors:

$$\xi_\alpha(x) \quad \eta^{\dot{\alpha}}(x). \quad (2.69)$$

The Kinetic Term: Let us first consider the kinetic term of the spinors. It can be seen that the most simple Lorentz scalar composed of the spinors including the derivative ∂_μ can be obtained by contracting the spinor indices as

$$\eta^{*\alpha}(\sigma^\mu_{\alpha\dot{\beta}})\partial_\mu\eta^{\dot{\beta}} \equiv \eta^\dagger\sigma^\mu\partial_\mu\eta. \quad (2.70)$$

Here from the definition of the spinors with a dotted and without dots, we can see that the complex conjugate of the dotted spinors have the translation property as the non-dotted spinors (or the inverse of these), say

$$(\eta^{\dot{\alpha}})^* = (\eta^*)^\alpha and \quad (\xi_\alpha)^* = (\xi^*)_{\dot{\alpha}}. \quad (2.71)$$

To make eq.(2.70) real, we need the complex conjugate of these. However adding it becomes the total derivative $\partial_\mu(\eta^\dagger\sigma^\mu\eta)$. Hence we perform subtraction:

$$\mathcal{L}_{\mathrm{L}}^k = \frac{i}{2}(\eta^\dagger\sigma^\mu\partial_\mu\eta - \partial_\mu\eta^\dagger\cdot\sigma^\mu\eta) = \frac{i}{2}\eta^\dagger\sigma^\mu\partial_\mu\eta. \tag{2.72}$$

The above is the kinetic term of η. In a similar fashion, the kinetic term of ξ is invariant under the transformations with θ and θ' as

$$\mathcal{L}_{\mathrm{R}}^k = \frac{i}{2}\xi^*{}_{\dot\beta}(\bar\sigma^\mu)^{\dot\beta\alpha}\partial_\mu\xi_\alpha = \frac{i}{2}\xi^\dagger(\bar\sigma^\mu)\partial_\mu\xi. \tag{2.73}$$

If ξ and η are independent of each other, the system of $\mathcal{L}_{\mathrm{L}}^k + \mathcal{L}_{\mathrm{R}}^k$ is invariant under the transformations

$$\xi'(x) = e^{i\theta}\xi(x), \quad \xi^{*\prime}(x) = e^{-i\theta}\xi^*(x), \tag{2.74}$$
$$\eta'(x) = e^{i\theta}\eta(x), \quad \eta^{*\prime}(x) = e^{-i\theta}\eta^*(x). \tag{2.75}$$

Writing $U(1)_{\mathrm{R}}$ and $U(1)_{\mathrm{L}}$ in the transformation of ξ and η respectively, we call the above transformations the chiral $U(1)_{\mathrm{L}}\times U(1)_{\mathrm{R}}$ transformation. If there is this invariance, two Noether currents exist, and the following are conserved individually:

$$\text{The charge of } \xi \equiv \text{the number of particle of } \xi, \tag{2.76}$$
$$\text{The charge of } \eta \equiv \text{the number of particle of } \eta. \tag{2.77}$$

By using the derivative twice, as well as the scalar fields, we can make the kinetic term of the Lorentz scalar $\partial_\mu\xi^\dagger\cdot\partial^\mu\eta + \text{h.c.}$ (h.c. is the complex conjugate). However such a kinetic term obtained by mixing ξ and η is known to lead to negative definite particle. Therefore, we do not employ this.

The Mass Term: It can be seen that the Lorentz scalar without the derivative is given as

$$\eta^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta} = \xi_\alpha\epsilon^{\alpha\beta}\xi_\beta = \xi^T\epsilon\xi. \tag{2.78}$$

However this seems to vanish. Actually, since

$$\eta^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta} = \eta^1\eta^2 - \eta^2\eta^1 \tag{2.79}$$

this vanishes if η and ξ are normal numbers. However the spinors follow the Fermi-distribution in the quantum theories, and correspondingly in the classical theory, spinors are treated as the Grassmann numbers. Hence the minus sign appears as $\eta^2\eta^1 = -\eta^1\eta^2$, eq.(2.79) becomes $2\eta^1\eta^2$ and does not vanish. In what follows, we treat the spinors as the Grassmann numbers.

So, as the real quantities with eq.(2.78), we can obtain the Lagrangian density as

$$\mathcal{L}_{\mathrm{L}}^k = -\frac{1}{2}(m_{\mathcal{L}}\eta^T\epsilon\eta - m_{\mathcal{L}}^*\eta^\dagger\epsilon\eta^*), \tag{2.80}$$
$$\mathcal{L}_{\mathrm{R}}^k = -\frac{1}{2}(m_{\mathcal{R}}\xi^\dagger\epsilon\xi - m_{\mathcal{R}}^*\xi^T\xi). \tag{2.81}$$

These are named the Majorana mass term. We should notice that eq.(2.80) violates the chiral $U_{\rm L}(1) \times U_{\rm R}(1)$ symmetry. For this reason, if there is the Majorana mass term, the number of particle is not conserved.

On the other hand if we allow the transition between ξ and η, we can have another Lorentz scalar $\xi^*{}_{\dot\alpha}\eta^{\dot\alpha} = \xi^\dagger\eta$. So the following mass term is possible:

$$\mathcal{L}_{\rm D}^m = -(m\xi^\dagger\eta + m^*\eta^\dagger\xi). \tag{2.82}$$

We call this the Dirac mass term. The Dirac mass term also breaks the chiral $U(1)_{\rm L} \times U(1)_{\rm R}$ symmetry, however it does not break completely unlike the Majorana mass term. In fact we can see that the transformation with $\theta = \theta'$ at eq.(2.74):

$$\begin{pmatrix} \xi'(x) \\ \eta'(x) \end{pmatrix} = e^{i\theta} \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix} \quad \begin{pmatrix} \xi^{*\prime}(x) \\ \eta^{*\prime}(x) \end{pmatrix} = e^{i\theta} \begin{pmatrix} \xi^{*\prime}(x) \\ \eta^{*\prime}(x) \end{pmatrix}. \tag{2.83}$$

We call this $U(1)_V$ subgroup. Hence in the case with the Dirac mass term, there is the conserved quantity Q corresponding to eq.(2.83), and ξ and η have the charge $Q = +1$, and ξ^* and η^* have the charge $Q = -1$.

We have treated $m_{\rm R}$ $m_{\rm L}$ and m as the complex quantities at eqs.(2.80) and (2.82). However by the redefinition of the phase of ξ and η, we can take arbitrary two of the three to real positives.

Dirac Four-Spinor: In the case of the massless or the case that the mass term is given by the Majorana mass term given at eq.(2.62), the spinor fields ξ_α and $\eta^{\dot\alpha}$ do not mix up each other. As a result, the theory can be described by the either of ξ_α or $\eta^{\dot\alpha}$. However the particles with the Dirac mass term is described by ξ_α and $\eta^{\dot\alpha}$, and we consider a state composed of these as

$$\psi(x) = \begin{pmatrix} \xi_\alpha(x) \\ \eta^{\dot\alpha}(x) \end{pmatrix}. \tag{2.84}$$

We call this the Dirac spinor.

Correspondingly, we combine the four-dimensional Pauli matrices into the two-story matrix form and define the following 4×4 matrix:

$$\gamma^\mu \equiv \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\beta} \\ (\sigma^\mu)_{\alpha\dot\beta} & 0 \end{pmatrix} \tag{2.85}$$

with

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{2.86}$$

The most fundamental property of the γ-matrices is

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.87}$$

Further more

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} (\sigma^\mu)_\alpha{}^\beta & 0 \\ 0 & (\sigma^\mu)^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}. \tag{2.88}$$

Let us define the following matrix:

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.89}$$

The representation of the 2×2-matrices from eq.(2.85) to eq.(2.89) can be obtained by combining ξ and η to the one such as eq.(2.66), which is called the spinor-representation or the chiral representation. When discussing the non-relativistic limit, it is convenient to take $(\xi + \eta)/2$ and $(\xi - \eta)/2$ as the upper and lower two-components.

In this representation, the kinetic term can be written by eq.(2.72) + eq.(2.73)

$$\mathcal{L}^{\mathrm{k}} = \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi. \tag{2.90}$$

The Dirac mass term can be written (now $m = m^*$)

$$\mathcal{L}^m_{\mathrm{D}} = -m\bar{\psi}\psi. \tag{2.91}$$

Here $\bar{\psi}$ means $\psi^\dagger\gamma_0$, which is called the Dirac conjugate. As can be seen in eq.(2.89), the two-component $\xi_\alpha(x)$ and $\eta^{\dot{\alpha}}(x)$ correspond to the eigenstates with the eigen value $+1$ and -1 of γ_5. The eigenvalue of γ_5 is usually called the chirality. The decomposition of the Dirac spinor into the components with the chirality ± 1 can be performed using the operators $(1 \pm \gamma_5)/2$

$$\psi_R = \left(\frac{1+\gamma_5}{2}\right)\psi = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix} \quad \psi_L = \left(\frac{1+\gamma_5}{2}\right)\psi = \begin{pmatrix} 0 \\ \eta_\alpha \end{pmatrix} \tag{2.92}$$

ψ_R and ψ_L are called the right-handed and left-handed components, respectively.

Majorana Spinor: When only the two-component $\eta^{\dot{\alpha}}$ exists, and it has the Majorana mass term, the Lagrangian density is given from eqs.(2.57) and (2.62) as

$$\mathcal{L} = \frac{i}{2}\eta^\dagger\sigma\partial_\mu\eta - \frac{m}{2}(\eta^T\epsilon\eta - \eta^\dagger\epsilon\eta^*). \tag{2.93}$$

Although this field is the two-component, as well as the Dirac spinors, we can combine into the four-component representation. Namely if we consider the following combination

$$\psi_{\mathrm{M}} \equiv \begin{pmatrix} \eta^{*\beta}\epsilon_{\beta\alpha} \\ \eta^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} -i\sigma_2\eta^* \\ \eta \end{pmatrix} \tag{2.94}$$

as well as the Dirac spinors, the upper two component has the transformation property as the undotted spinors, and the lower two component has the transformation property as the dotted spinors. However the independent component is only the two-component complex η. As a result, this four-component ψ_M is self-conjugate under the charge conjugate transformation $\mathcal{C}$:

$$\psi^C_M = \psi_M. \tag{2.95}$$

Namely, ψ_M is the real Dirac field by the sense of the charge conjugate. We call such a self-conjugate fields Majorana field. The Lagrangian density given in eq.(2.93) can be rewritten by using ψ_M as

$$\mathcal{L} = \frac{i}{4}\bar{\psi_M}\gamma^\mu\partial\psi_M - \frac{1}{2}\bar{\psi}_M\psi_M. \tag{2.96}$$

This is the formally same with the Dirac field.

We can also produce the Majorana field only by ξ_α as

$$\psi_M = \begin{pmatrix} \xi_\alpha \\ \epsilon^{\dot{\alpha}\dot{\beta}}\xi^*_{\dot{\beta}} \end{pmatrix} \tag{2.97}$$

Chapter 3

Quantization of Fields

3.1 Free Scalar Fields

First we consider the real scalar field given by the $\lambda\phi^4$ theory. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2) - \frac{\lambda}{4!}\phi^4 \tag{3.1}$$

We perform the quantization according to the canonical quantization. The Lagrangian L is given, by fixing the time $x^0 = t$ fixed, as

$$L = \int d^3x\mathcal{L}(\phi(t,x), \dot{\phi}(t,x)), \quad \left(\dot{\phi}(t,\phi) = \frac{\partial\phi(t,x)}{\partial t}\right) \tag{3.2}$$

Then we can obtain the canonical momentum $\pi(t,x)$ for $\phi(t,x)$ as

$$\pi(t,x) \equiv \frac{\delta L}{\delta\dot{\phi}(t,x)} = \frac{\delta\mathcal{L}}{\delta\dot{\phi}(x)} = \dot{\phi}(x) \tag{3.3}$$

the canonical quantization is performed by taking π and ϕ as the operators and requiring the following equal-time commutation relations

$$[\phi(t,x), \pi(t,y)] = i\delta^3(x-y) \tag{3.4}$$
$$[\phi(t,x), \phi(t,y) = [\pi(t,x), \pi(t,x)] = 0 \tag{3.5}$$

The Hamiltonian can be obtained as

$$H = \int d^3(\pi\dot{\phi} - \mathcal{L}) = \int d^3\left(\frac{1}{2}(\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2) + \frac{\lambda}{4!}\phi^4\right). \tag{3.6}$$

Hence the Heisenberg equation of motion can be given

$$i\dot{\phi} = [\phi, H] = i\pi, \tag{3.7}$$
$$i\dot{\pi} = [\pi, H] = -i\left\{(-\nabla^2 + \mu^2)\phi + \frac{\lambda}{3!}\phi^3\right\} \tag{3.8}$$

This can agree to the Eular-Lagrange equation obtained from eq.(3.1):

$$(\Box + \mu^2)\phi = -\frac{\lambda}{3!}\phi^3 \tag{3.9}$$

In what follows we consider no interaction system ($\lambda = 0$). The equation of motion at $\lambda = 0$ is called the Klein-Gordon equation. To see that the commutation relation given at eq.(3.4) gives the particle-picture, first let us perform the Fourier transformation as

$$\phi(t,x) = \int \frac{d^3k}{(2\pi)^{3/2}} q_k(t) e^{ikx} \tag{3.10}$$

At this time the Klein-Gordon equation at eq.(3.9) can be written as

$$\ddot{q}_k + (k^2 + \mu^2) q_k = 0 \tag{3.11}$$

we can see that $q_k(t)$ is Harmonic oscillator, and the general solution is

$$q_k(t) = q_1(k) e^{-ik_0 t} + q_2(k) e^{ik_0 t} \tag{3.12}$$

$$k_0 \equiv \sqrt{k^2 + \mu^2} > 0 \tag{3.13}$$

From the fact that eq.(3.10) is real (Hermite operator), what $q_2(k) = q_1^\dagger(-k)$ is following, $q_1(k) = a(k)/\sqrt{2k_0}$, from eqs.(3.10) and (3.12),

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k_0}} \{a(k)e^{-ikx} + a^\dagger(k)e^{ikx}\} \tag{3.14}$$

We can see that $\pi(x) = \dot{\phi(x)}$

$$\pi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k_0}} \{-ik_0 a(k)e^{-ikx} + ik_0 a^\dagger(k)e^{ikx}\} \tag{3.15}$$

Furthe more we can see easily that

$$a(k) = (f_k, \phi) \equiv i \int d^3x f_k^*(x) \partial_0 \phi(x) \tag{3.16}$$

$$f_k(x) \equiv e^{-ikx}/\sqrt{(2\pi)^3 2k_0} \tag{3.17}$$

Using these, we can see that the commutator relations given at eq.(3.4) are equivalent to the commutator relations of $a(k)$ and $a^\dagger(k)$ as

$$[a(k), a(q)^\dagger] = \delta(k-q) \tag{3.18}$$

$$[a(k), a(q)] = [a^\dagger(k), a^\dagger(q)] = 0 \tag{3.19}$$

The energy momentum P^μ can be obtained by using the general representation of the energy-momentum tensor $T_\rho{}^\mu$ as

$$P^\mu = \int d^3x (\dot{\phi}\partial^\mu\phi - g^{\mu 0}\mathcal{L}) \tag{3.20}$$

$$= \int d^3x \frac{1}{2} k^\mu [a^\dagger(k)a(k) + a(k)a^\dagger(k)] \tag{3.21}$$

Here using eq.(3.18) with $k = q$

$$[a(k), a^\dagger(k)] = \delta^3(p = 0) = \int \frac{d^3x}{(2\pi)^3} e^{ipx}|_{p=0} = (2\pi)^{-3} \int d^3x \tag{3.22}$$

we define the number operator as

$$n(k) = a^\dagger(k)a(k). \tag{3.23}$$

Then using this we rewrite eq.(3.20)

$$P^\mu = \int d^3k k^\mu n(k) + \int \frac{d^3k d^3x}{(2\pi)^3} \frac{1}{2} k^\mu \tag{3.24}$$

The second term is just a numbers (called c-number). We can see that this second term vanishes in the symmetric-integration of d^3k at $\mu = 1, 2, 3$, and remains at the case of $\mu = 0$. $\mu = 0$ corresponds to the case of the energy. We can interpret the contribution at $\mu = 0$ as there is the quantum degree of freedom in a unit phase-space of $(2\pi\hbar)$ and the contribution at $\mu = 0$ is the integration of $(1/2)\hbar\sqrt{k^2 + \mu^2}$ over all the degree of freedom. Hence we can get the interpret that the vacuum is the energy of the ground state. This term diverges by the two causes: 1) due to the infinity of space-time volume, 2) the violet divergence. To make P^μ well-defined. By displacing the origin of the energy we can disregard the c-number term, and finally we can write

$$P^\mu = \int d^3k k^\mu n(k) \tag{3.25}$$

As known in the case of quantum harmonic oscillator, the ground state of this system is the $|0\rangle$ which satisfies

$$a(k)|0\rangle = 0(\text{For arbitrary } k) \tag{3.26}$$

We can see from $P^\mu|0\rangle = 0$ that the energy and momentum in $|0\rangle$ are zero. The general states in this system can be written by the liner combination of

$$|k_1, k_2, \cdots, k_n\rangle = a^\dagger(k_1)a^\dagger(k_2)\cdots a^\dagger(k_n)|0\rangle \tag{3.27}$$

From the commutator relation at eq.(3.26),

$$[P^\mu, a^\dagger(k)] = k^\mu a^\dagger(k) \tag{3.28}$$

we can have

$$P^\mu|k_1, k_2, \cdots, k_n\rangle = ({k_1}^\mu + {k_2}^\mu + \cdots + {k_n}^\mu)|k_1, k_2, \cdots, k_n\rangle \tag{3.29}$$

So we can see that the states at eq.(3.27) are eigenstates with eigenvalues ${k_1}^\mu + {k_2}^\mu + \cdots + {k_n}^\mu$. From this and the $k^\mu = (\sqrt{k^2 + \mu^2}, k)$, we can interpret that $a^\dagger(k)$ is the operator creating a particle with mass μ momentum k.

We can do in the same fashion with the case of the complex scalar field, The expansion of the field corresponding to eq.(3.14)

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2k_0}}\{a_-(k)e^{-ikx} + a_+^\dagger(k)e^{ikx}\}, \tag{3.30}$$

and the expansion of $\phi^*(x)$ can be obtained by the Hermite conjugate of the above. The canonical commutation relations are

$$[a_+(k), a_+(q)^\dagger] = [a_-(k), a_-(q)] = \delta(k-q), \tag{3.31}$$

and all the other combinations are commute. The number operator for $+$ and $-$-quantums are

$$n_+(k) = a^\dagger(k)a(k), \quad n_-(k) = a^\dagger(k)a(k), \tag{3.32}$$

and the energy-momentum is

$$P^\mu = \int d^3x \frac{1}{2} k^\mu [n_+(k) + n_-(k)], \tag{3.33}$$

where we have ignored the zero-point energy. the conserved charge is

$$Q = \int d^3k[n_+(k) - n_-(k)] \tag{3.34}$$

3.2 Free Dirac Fields

The Lagrangian density for the free Dirac field is known to be given as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \tag{3.35}$$

We again fix the time $x^0 = t$. The momentum $\pi^\alpha(x,t)$ (α is the indices for the four-component spinor) conjugate to the coordinate variables $\phi_\alpha(x,t)$ is

$$\pi_\psi^\alpha = \partial\mathcal{L}/\partial\dot{\psi}_\alpha = i\psi^\beta(x)(\gamma^0)_\beta{}^\alpha = i\psi(x). \tag{3.36}$$

Here in the derivative of $\dot{\psi}$, we have followed the right-derivative rule:

$$\begin{aligned} \text{left} - \text{derivative}: & \quad [(\partial/\partial\theta)A]B + (-)^{|A|}A[(\partial/\partial\theta)B], & (3.37)\\ \text{Right} - \text{derivative}: & \quad \partial(AB)/\partial\theta = A(\partial B/\partial\theta) + (-)^{|B|}(\partial A/\partial\theta)B & (3.38) \end{aligned}$$

Generally in the derivative of the Grassmann number θ, the left- and right-handed derivatives can be defined, and these are defined to satisfy the above Leipnitz rule. $|A|$ in the index-parts are 1 when $|A|$ is the Grassmann-odd, and 0 when $|A|$ is the Grassmann-even.

Since it has been turned out from eq.(3.36) that the complex conjugate ψ^* is the conjugate momentum variable π_ψ for ψ, we do not derive the conjugate momentum of

ψ^* anymore. To set the spinors to the fermion fields, we impose the following equal-time commutation relation:

$$\{\psi_\alpha(x,t), \pi_\psi^\alpha(y,t)\} = i\delta_\alpha^\beta\delta^3(x-y), \tag{3.39}$$

$$\{\psi_\alpha(x,t), \psi_\beta(x,t)\} = \{\psi_\alpha^{*\alpha}(x,t), \psi_\beta^{*\beta}(y,t)\} = 0 \tag{3.40}$$

We may write the first equation in the above using eq.(3.36) as

$$\{\psi(x,t), \bar{\psi}(y,t)\} = \gamma^0\delta^3(x-y) \tag{3.41}$$

Here ψ is the quantity in which the four-component are put horizontally and $\bar{\psi}$ is the quantity in which the four-component are put longitudinally. Hamiltonian can be obtained as

$$H = \int d^3x(\pi_\psi\dot{\psi} - \mathcal{L}) \tag{3.42}$$

$$= \int d^3x\bar{\psi}(-i\gamma^k\partial_k + m\beta)\psi \tag{3.43}$$

$$= \int d^3x\bar{\psi}(-i\alpha\partial_k \cdot \nabla + m\beta)\psi, \tag{3.44}$$

where we have used $\alpha \equiv \gamma^0\gamma$,$\beta \equiv \gamma^0$. From the Heisenberg equation: $i\dot{\psi} = [\psi, H]$ with using eq.(3.39), we can obtain

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{3.45}$$

This can agree to the Euler-Lagrange equation obtained from eq.(3.45), and this is called the Dirac equation.

Chapter 4

Summary and Remark

In this article, first we have put the problems in the quantum mechanics as in the beginning of the introduction. Then, starting with the four-dimensional space-time with the Lorentz invariance, we have finally performed the canonical quantization of fields. In the process of this, we have obtained the scalar, vector and SL(2,C) spinor fields as ones of the possible representation of fields in the four-dimensional space-time with the Lorentz invariance, and we have constructed the action for each these fields with imposing the fundamental physical requirements. The result obtained by this can overcome these problems.

The quantum field theory works as the formalism in the particle physics, which has achieved great progress since G 'tHooft had proved the renormalizability in the non-Abelian gauge theory at the beginning of 1970. Concretely, while it has been the Weinberg-Salam model that unifies the weak and electro-magnetic forces and could give the brilliant prediction for the neutral current and W and Z bosons, it has been the asymptotic free, which has been the significant finding leading to the quantum chromodynamics that is the the non-Abelian gauge theory to describe the strong force. At the moment, the quantum chromodynamics plays an indispensable role in the description of the strongly coupled system in not only particle physics of hadron but also condensed matter physics. Furthermore, it has been also extremely important achievement that the interactions of all the four forces in the nature can be describe uniformly by the universal form in which the gauge fields mediate. These lead to the ground unified theory using larger gauge groups such as SU(5),SO(10) and E_8, and theories in the higher dimensional space-time and its compactifications, supersymmetric theory and superstring theory to get the ultimate description and understanding for all the four forces, the origin of the space-time and the universe. What has been reviewed in this article forms the fundamental of these theories.

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