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Some Generalised Fixed Point Theorems Applied to Quantum Operations

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Abstract: In this paper, we consider an order-preserving mapping T on a complete partial b-metric space satisfying some contractive condition. We were able to show the existence and uniqueness of the fixed point of T. In the application aspect, the fidelity of quantum states was used to establish the existence of a fixed quantum state associated to an order-preserving quantum operation. The method we presented is an alternative in showing the existence of a fixed quantum state associated to quantum operations. Our method does not capitalise on the commutativity of the quantum effects with the fixed quantum state(s) (Luders's compatibility criteria). The Luders's compatibility criteria in higher finite dimensional spaces is rather difficult to check for any prospective fixed quantum state. Some part of our results cover the famous contractive fixed point results of Banach, Kannan and Chatterjea.

Keywords: partial b-metric space; order-preserving; fixed point; quantum operation; fidelity of quantum state; quantum state

1. Introduction

The early research motivations in the area of fixed point theory were for solving problems in differential equations [1–3]. In 1883, Poincaré [4] established a theorem that was later proved as an equivalence to the Brouwer's [5] fixed point theorem. It was in 1912 that Brouwer [5] published his fixed point theorem of self-continuous mappings on a closed ball, while in the same year (1912), Poincaré [6] published his fixed point theorem for area-preserving mappings of an annulus, see [7,8]. No doubt, Poincaré understood the early fixed point theorems and was using them as a tool in finding solutions of some differential equations see [3,4,6,9]).

On the other hand, another research motivation can be linked to the work of Picard [2]; he was utilising systematic application of successive approximations method for finding solutions of different differential equation problems, see [10]. As a consequence, the famous Banach contraction principle [11] emerged in 1922, see [7]. Moreover, it was the same year that boundary value problems of nonlinear ordinary differential equations prompt Birkhoff–Kellogg [1] to lead the struggle for extending Brouwer's fixed point theorem to function space, see [7].

Another angle of fixed point research emerged with the advent of the Knaster–Tarski Fixed point theorem [12,13]. The idea was first initiated from both authors (Knaster and Tarski) in 1927 [12], and later Tarski found some improvement of the work in 1939, which he discussed in some public lectures between 1939 and 1942 [13,14]. Finally, in 1955, Tarski [13] published the comprehensive results together with some applications. A distinctive property of this theorem is that it involves an order relation defined on the space of consideration. Indeed, the order relation serve as an alternative to the continuity and contraction of the mappings as found in Brouwer [5] and Banach [11] fixed point theorems, respectively, see [13].

After the advent of the Brouwer [5], Banach [11] and Knaster–Tarski [13] fixed point theorems, many researchers engage in extension [15–17], generalisation [15,18–21] and improvements [22–26] of the theorems using different spaces and functions. Along the direction of generalising spaces was Bourbaki–Bakhtin–Cezerwik's b-metric space [27–29], Matthews's partial metric space [30] and Shukla's Partial b-metric space [31].

Looking into the direction of quantum operations, many researchers are interested in finding the condition(s) that guarantees the existence of fixed points/states of quantum operations and the properties attached to the fixed point sets of the quantum operations, see [32–36].

In the area of quantum information theory, qubit is seen as a quantum system, whereas quantum operation can be viewed as measurement of quantum system; it describes the evolution of the system through the quantum states. Measurements use to have some errors which can be corrected through quantum error correction codes. The quantum error correction codes are easily developed through the information-preserving structures with the help of the fixed points set of the associated quantum operation. Therefore, the study of quantum operations is vital in the field of quantum information theory, at least in developing the error correction codes, knowing the state of the system (qubit) and the description of energy dissipation effects due to loss of energy from a quantum system [37].

In 1951, Lüders [38] discussed the compatibility of quantum states in measurements (quantum operations). He also showed that the compatibility of quantum states in measurements is equivalent to commutativity of the states with each quantum effects in the measurement.

In 1998, Busch et al. [33] proved a proposition that generalises the Lüder's theorem and shows that a state is invariant under a quantum operation if the state commutes with every quantum effect that described the quantum operation.

In 2002, Arias et al. [32] studied the fixed point sets of a quantum operation and gave some conditions to which the set is equal to a commutant set of the quantum effects that described the quantum operation.

In 2011, Long and Zhang [35] studied the fixed point set of quantum operations, they gave some necessary and sufficient conditions for the existence of a non-trivial fixed point set. Similarly, in 2012, Zhang and Ji [34] studied the existence of a non-trivial fixed point set of a generalised quantum operation.

In 2016, Zhang and Si [39] investigated the conditions for which the fixed point set of a quantum operation ($\phi_{\mathcal{A}}$) with respect to a row contraction $\mathcal{A}$ equals to the fixed point set of the power of the quantum operation $\phi^{j}_{\mathcal{A}}$ for some $1 \leq j < \infty$.

Remark 1. *It is worth noting that the existence of fixed point(s) of a quantum operation in a finite dimensional Hilbert space depends on the compatibility criteria as provided by Lüders [38]; fixed quantum states must commute with all quantum effects. Therefore, it is difficult to test the compatibility criteria in higher dimensional spaces; testing commutativity of the state with many quantum effects. Thus, the need for other alternatives arises.*

In this paper, motivated by Batsari et al. [18], Du et al. [21] and Dung et al. [40], we establish some fixed point results in partial b-metric spaces with a contraction condition that is different from that of

Banach [11], Kannan [26] and Chatterjea [23]. As an application of our result(s), we consider using some contractive conditions in establishing the existence of fixed point of a depolarising and generalised amplitude damping quantum operations. For, the depolarising quantum operation is an important source of noise/error in quantum communication that can be found in finite dimensional cases when the quantum system interact with the environment, whereas the generalised amplitude damping is used in the description of energy dissipation effects due to loss of energy from a quantum system.

Moreover, the technique we adopted in establishing the existence of fixed point of quantum operation is entirely different to that of Arias et al. [32], Busch et al. [33] and Lüders [38]. We do not utilise the properties of quantum effects, rather we utilise the properties of the Bloch vectors associated to the quantum states in consideration. Thus, it is an alternative to the existing methods in the literature. Our results generalise and improve some existing results in the literature.

2. Preliminaries

Let X be a nonempty set, $\mathbb{R}_+$ denotes the set of non negative real numbers, $\mathbb{R}$ denotes the set of real numbers, $(X, \preceq)$ denotes the partially ordered set on X and (X, d) is a metric space.

A **b-metric** on X is a function $d_s : X \times X \to \mathbb{R}_+$ such that,

(D_s1) $\forall x, y \in X,\ \ d_s(x, y) \geq 0.$

(D_s2) $\forall x, y \in X,\ \ d_s(x, y) = 0 \iff x = y.$

(D_s3) $\forall x, y \in X,\ \ d_s(x, y) = d_s(y, x).$

(D_s4) There exists a real number $s \geq 1$, for which $d_s(x, y) \leq s\,[d_s(x, z) + d_s(z, y)]\,,\ \ \forall x, y, z \in X.$

(X, d_s) denotes the $b-$metric space. It is clear to see that, every metric is a b-metric with $s = 1$ (see [27–29]).

The converse is not true in general. For example, taking $d_s : X \times X \to \mathbb{R}_+$, if $d_s(x, y) = |y - x|^2,\ x, y \in \mathbb{R}$, then d_s is a $b-$metric with $s = 2$. However, it is not a metric for $x = 5, y = 3$ and $z = 4$, condition $d_s(x, y) \leq d_s(x, z) + d_s(z, y)$ fails [18].

Example 1. *[18] Let $X = \mathbb{R}$, $n \in 2\mathbb{N}$. Define $d_b : X \times X \to \mathbb{R}_+$ by $d_s = (x - y)^n$, $\forall x, y \in X$. Then, d_s is a b-metric with $s = 2^{n-1}$ and d_s is not a metric.*

A **partial metric or** *pmetric* on X is a function $p : X \times X \to \mathbb{R}_+$ such that,

(P1) $\forall x, y \in X,\ \ x = y \iff p(x, x) = p(x, y) = p(y, y).$

(P2) $\forall x, y \in X,\ \ p(x, x) \leq p(x, y).$

(P3) $\forall x, y \in X,\ \ p(x, y) = p(y, x).$

(P4) $\forall x, y, z \in X,\ \ p(x, z) \leq p(x, y) + p(y, z) - p(y, y).$

(X, p) denotes the partial metric space. Observe that every metric is a partial metric with $p(x, x) = 0, \forall x \in X$ (see [30]). However, the converse is not necessary true.

Example 2. *[41] Define a mapping $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ by*

$$\Psi(x_1, x_2) = \max\{x_1, x_2\},$$

where $x_1, x_2 \in \mathbb{R}$. Therefore, Ψ is not a metric but, a partial metric.

A **partial b-metric** on the set X is a function $p_s : X \times X \to \mathbb{R}_+$ such that,

(P_b1) $\forall x, y \in X,\ \ x = y \iff p_s(x, x) = p_s(x, y) = p_s(y, y).$

(P_b2) $\forall x, y \in X,\ \ p_s(x, x) \leq p_s(x, y).$

(P_b3) $\forall x, y \in X,\ \ p_s(x, y) = p_s(y, x).$

(P_b4) There exist a real number $s \geq 1$ such that, $\forall x, y, z \in X\ \ p_s(x, z) \leq s\,[p_s(x, y) + p_s(y, z)] - p_s(y, y).$

(X, p_s) denotes the partial $b-$metric space. Note that, every partial metric is a partial b-metric with $s = 1$. Also, every b-metric is a partial b-metric with $p_s(x, x) = 0,\ \ \forall\, x \in X$ (see [31]).

Example 3. *[42] Let $X = [0, 100]$. Define $p_s : X \times X \to \mathbb{R}_+$ by $p_s(x, y) = e^{|x-y|}$ for all $x, y \in X$. Then, p_s is a partial b-metric on X, which is neither a b-metric nor a partial metric on X.*

An open b-ball for a partial b-metric $p_s : X \times X \to \mathbb{R}_+$ is a set of the form $B_b(x, \epsilon) := \{y \in X : p_s(x, y) < \epsilon + p_s(x, x)\}$ for $\epsilon > 0$, and $x \in X$ [31].

Every partial b-metric defined on a nonempty set X generates a topology τ_b on X, whose base is the family of the open b-balls, where $\tau_b = \{B_b(x, \epsilon) : x \in X \text{ and } \epsilon > 0\}$. Moreover, the topological space (X, τ_b) is T_0 but not necessary T_1 [31].

A sequence $\{x_n\}$ in the space (X, p_s) converges with respect to the topology τ_b to a point $x \in X$, if and only if

$$\lim_{n\to\infty} p_s(x_n, x) = p_s(x, x) \tag{1}$$

(see [31]). The sequence $\{x_n\}$ is Cauchy in (X, p_s) if the below limit exists and is finite

$$\lim_{n,m\to\infty} p_s(x_n, x_m) < \infty \tag{2}$$

(see [31]). A partial b-metric space (X, p_s) is complete, if every Cauchy sequence $\{x_n\}$ in (X, p_s) converges to a point $x \in X$ such that,

$$\lim_{n,m\to\infty} p_s(x_n, x_m) = p_s(x, x)$$

(see [31]).

A mapping T is said to be order-preserving on X, whenever $x \preceq y$ implies $T(x) \preceq T(y)$ $\forall x, y \in X$.

3. Results

Theorem 1. *Let (X, p_s) be a complete partial b-metric space with $s \geq 1$, and associated with a partial order $\preceq$. Suppose an order preserving mapping $T : X \to X$ satisfies*

$$p_s(T(x), T(y)) \leq \frac{\beta}{2}\Big[\max\{p_s(x, y), p_s(x, T(y)), p_s(y, T(x))\} + \min\{p_s(x, T(x)), p_s(y, T(y))\}\Big], \tag{3}$$

for all comparable $x, y \in X$, where $\beta \in [0, \alpha)$ and $\alpha = \min\{\frac{1}{s^2}, \frac{2}{2s+1}\}$. If there exist $x_0 \in X$ such that $x_0 \preceq T(x_0)$, then T has a unique fixed point $\hat{x} \in X$ such that $p_s(\hat{x}, \hat{x}) = 0$.

Proof of Theorem 1. First, we will prove the uniqueness of the fixed point assuming it exists. Let $x_1, x_2 \in X$ be two distinct comparable fixed points of T. Then,

$$\begin{aligned}
& p_s(x_1, x_2) \\
= \; & p_s(T(x_1), T(x_2)) \\
\leq \; & \frac{\beta}{2}\Big[\max\{p_s(x_1, x_2), p_s(x_1, T(x_2)), p_s(x_2, T(x_1))\} + \min\{p_s(x_1, T(x_1)), p_s(x_2, T(x_2))\}\Big] \\
= \; & \frac{\beta}{2}\left[\max\{p_s(x_1, x_2), p_s(x_1, x_2), p_s(x_2, x_1)\} + \min\{p_s(x_1, x_1), p_s(x_2, x_2)\}\right] \\
< \; & \frac{\alpha}{2}\left[p_s(x_1, x_2) + p_s(x_1, x_2)\right] \\
= \; & \frac{\alpha}{2}\left[2p_s(x_1, x_2)\right] \\
= \; & \alpha p_s(x_1, x_2) \\
< \; & p_s(x_1, x_2).
\end{aligned}$$

Thus, is a contradiction. Therefore, the fixed point is unique if it exist, for $x_1 = x_2$.

Next we prove that if $\hat{x} \in X$ is a fixed point of T, then $p_s(\hat{x}, \hat{x}) = 0$. Suppose $p_s(\hat{x}, \hat{x}) \neq 0$. Then,

$$\begin{aligned}
p_s(\hat{x}, \hat{x}) &= p_s(T(\hat{x}), T(\hat{x})) \\
&\leq \frac{\beta}{2}\Big[\max\{p_s(\hat{x}, \hat{x}), p_s(\hat{x}, T(\hat{x})), p_s(\hat{x}, T(\hat{x}))\} + \min\{p_s(\hat{x}, T(\hat{x})), p_s(\hat{x}, T(\hat{x}))\}\Big] \\
&= \frac{\beta}{2}\Big[\max\{p_s(\hat{x}, T(\hat{x})), p_s(\hat{x}, T(\hat{x})), p_s(\hat{x}, T(\hat{x}))\} + \min\{p_s(\hat{x}, T(\hat{x})), p_s(\hat{x}, T(\hat{x}))\}\Big] \\
&= \frac{\beta}{2}\left[p_s(\hat{x}, T(\hat{x})) + p_s(\hat{x}, T(\hat{x}))\right] \\
&= \frac{\beta}{2} 2p_s(\hat{x}, \hat{x}) \\
&< \frac{\alpha}{2} 2p_s(\hat{x}, \hat{x}) \\
&= \alpha p_s(\hat{x}, \hat{x}) \\
&< p_s(\hat{x}, \hat{x}).
\end{aligned}$$

Thus contradicting the fact that $p_s(\hat{x}, \hat{x}) \neq 0$. Therefore, $p_s(\hat{x}, \hat{x}) = 0$.

Now, we proceed to prove the existence of the fixed point of T satisfying (3). Let $x_0 \in X$ be such that $x_0 \preceq T(x_0)$. If $T(x_0) = x_0$ then, x_0 is a fixed point of T. Recall that, T is order-preserving and $x_0 \preceq T(x_0)$ then, we have $x_0 \preceq T(x_0) = x_1$, $x_1 \preceq T(x_1) = x_2$, $x_2 \preceq T(x_2) = x_3, \cdots,$ $x_n \preceq T(x_n) = x_{n+1}$. By transitivity of $\preceq$, we have $x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$.

Suppose $x_0 \neq T(x_0)$, define a sequence $\{x_n\} \subseteq X$ by $x_n = T^n(x_0)$ and let $q_n = p_s(x_n, x_{n+1})$. It is clear that if $x_n = x_{n+1}$ for some natural number n, then x_n is a fixed point of T, i.e., $x_{n+1} = T(x_n) = x_n$. Let $x_{n+1} \neq x_n \;\; \forall n \in \mathbb{N}$. Then, we proceed as follows,

$$\begin{aligned}
& q_n \\
= \; & p_s(x_n, x_{n+1}) \\
= \; & p_s(T(x_{n-1}), T(x_n)) \\
\leq \; & \frac{\beta}{2}\Big[\max\{p_s(x_{n-1}, x_n), p_s(x_{n-1}, T(x_n)), p_s(x_n, T(x_{n-1}))\} + \min\{p_s(x_{n-1}, T(x_{n-1})), p_s(x_n, T(x_n))\}\Big] \\
= \; & \frac{\beta}{2}\Big[\max\{p_s(x_{n-1}, x_n), p_s(x_{n-1}, x_{n+1}), p_s(x_n, x_n)\} + \min\{p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1})\}\Big] \\
\leq \; & \frac{\beta}{2}\Big[\max\{p_s(x_{n-1}, x_n), s(p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})) - p_s(x_n, x_n), p_s(x_n, x_n)\}\Big] \\
& + \frac{\beta}{2}\left[\min\{p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1})\}\right] \\
\leq \; & \frac{\beta}{2}\left[\max\{p_s(x_{n-1}, x_n), s(p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})), p_s(x_n, x_n)\}\right] \\
& + \frac{\beta}{2}\left[\min\{p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1})\}\right] \\
= \; & \frac{\beta}{2}\left[s(p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})) + \min\{p_s(x_{n-1}, x_n), p_s(x_n, x_{n+1})\}\right] \\
\leq \; & \frac{\beta}{2}\left[s(p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})) + \left(\frac{p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1})}{2}\right)\right] \\
= \; & \frac{\beta}{2}\left[\frac{(2s+1)(p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1}))}{2}\right] \\
= \; & \beta\left[\frac{(2s+1)(p_s(x_{n-1}, x_n) + p_s(x_n, x_{n+1}))}{4}\right] \\
= \; & \beta\left[\frac{(2s+1)(q_{n-1} + q_n)}{4}\right].
\end{aligned}$$

Thus, we have

$$q_n \quad \leq \quad \beta(2s+1)\left(\frac{q_{n-1}+q_n}{4}\right),$$

which implies

$$\left(\frac{4-(2\beta s+\beta)}{4}\right)q_n \leq \left(\frac{2\beta s+\beta}{4}\right)q_{n-1}. \tag{4}$$

By simplifying (4), we have

$$q_n \leq \left(\frac{2\beta s+\beta}{4-2\beta s-\beta}\right)q_{n-1}. \tag{5}$$

For $\beta \in [0,\alpha)$, we deduce that

$$0 \leq \frac{2\beta s+\beta}{4-2\beta s-\beta} \leq 1.$$

Therefore, from (5), we conclude that $p_s(x_n, x_{n+1}) = q_n \leq q_{n-1} = p_s(x_{n-1}, x_n)$. Thus, $\{q_n\}_{n=1}^{\infty}$ is a monotone non-increasing sequence of real numbers, and bounded below by 0. Therefore, $\lim_{n\to\infty} q_n = 0$, see Chidume et al. [43].

Next, we show $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Let $x_n, x_m \in X$, $\forall n,m \in \mathbb{N}$. Then,

$$\begin{aligned}
& p_s(x_n, x_m) = p_s(T^n(x_0), T^m x_0) \\
= \;& p_s(T(x_{n-1}), T(x_{m-1})) \\
\leq \;& \beta\left(\max\{p_s(x_{n-1}, x_{m-1}), p_s(x_{n-1}, T(x_{m-1})), p_s(x_{m-1}, T(x_{n-1}))\}\right) \\
& +\beta\left(\min\{p_s(x_{n-1}, T(x_{n-1})), p_s(x_{m-1}, T(x_{m-1}))\}\right) \\
= \;& \beta\Big(\max\{p_s(x_{n-1}, x_{m-1}), p_s(x_{n-1}, x_m), p_s(x_{m-1}, x_n)\} + \min\{p_s(x_{n-1}, x_n), p_s(x_{m-1}, x_m)\}\Big) \\
\leq \;& \beta\left(\max\{A, s[p_s(x_{n-1}, x_n) + p_s(x_n, x_m)], s[p_s(x_{m-1}, x_m) + p_s(x_m, x_n)]\}\right) \\
& +\beta\left(\min\{p_s(x_{n-1}, x_n), p_s(x_{m-1}, x_m)\}\right) \\
= \;& \beta\Big(s(p_s(x_{n-1}, x_n) + s(p_s(x_n, x_m) + p_s(x_m, x_{m-1}))) + \min\{p_s(x_{n-1}, x_n), p_s(x_{m-1}, x_m)\}\Big),
\end{aligned}$$

where $A = s(p_s(x_{n-1}, x_n) + s(p_s(x_n, x_m) + p_s(x_m, x_{m-1}))$. By further simplifying we have

$$(1-\beta s^2)(p_s(x_n, x_m)) \quad \leq \quad \beta s p_s(x_{n-1}, x_n) + \beta s^2 p_s(x_m, x_{m-1}) + \beta\Big[\min\{p_s(x_{n-1}, x_n), p_s(x_{m-1}, x_m)\}\Big],$$

which implies

$$\begin{aligned}
& p_s(x_n, x_m) \\
\leq \;& \frac{\beta}{(1-\beta s^2)}\Big[s p_s(x_{n-1}, x_n) + s^2 p_s(x_{m-1}, x_m) + \min\{p_s(x_{n-1}, x_n), p_s(x_{m-1}, x_m)\}\Big] \\
\leq \;& \frac{\beta}{1-\beta s^2}\left(s p_s(x_{n-1}, x_n) + s^2 p_s(x_{m-1}, x_m) + \frac{p_s(x_{n-1}, x_n) + p_s(x_{m-1}, x_m)}{2}\right) \\
= \;& \frac{\beta}{1-\beta s^2}\left(\frac{(2s+1)p_s(x_{n-1}, x_n) + (2s^2+1)p_s(x_{m-1}, x_m)}{2}\right).
\end{aligned} \tag{6}$$

Now, taking the limit as $n, m \to \infty$ in (6), we have

$$\lim_{n,m\to\infty} p_s(x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. For X being complete, there exists $\hat{x} \in X$ such that

$$\lim_{n\to\infty} p_s(x_n, \hat{x}) = \lim_{n,m\to\infty} p_s(x_n, x_m) = p_s(\hat{x}, \hat{x}) = 0.$$

For showing $\hat{x} \in X$ is a fixed point of T, we proceed as follows,

$$\begin{aligned} p_s(\hat{x}, T(\hat{x})) &\leq s\left[p_s(\hat{x}, x_{n+1}) + p_s(x_{n+1}, T(\hat{x}))\right] - p_s(x_{n+1}, x_{n+1}) \\ &\leq s\left[p_s(\hat{x}, x_{n+1}) + p_s(T(x_n), T(\hat{x}))\right] \\ &\leq s\left[p_s(\hat{x}, x_{n+1}) + \frac{\beta}{2}(\max\{p_s(x_n, \hat{x}), p_s(x_n, T(\hat{x})), p_s(\hat{x}, T(x_n))\})\right] \\ &\quad + s\frac{\beta}{2}\Big[\min\{p_s(x_n, x_{n+1}), p_s(\hat{x}, T(\hat{x}))\}\Big]. \end{aligned} \tag{7}$$

Case I: Suppose $\max\{p_s(x_n, \hat{x}), p_s(x_n, T(\hat{x})), p_s(\hat{x}, T(x_n))\} = p_s(x_n, \hat{x})$. Then, from inequality (7), we have

$$\begin{aligned} & p_s(\hat{x}, T(\hat{x})) \\ \leq\ & s\left[p_s(\hat{x}, x_{n+1}) + \frac{\beta}{2}p_s(x_n, \hat{x}) + \frac{\beta}{2}\min\{p_s(x_n, x_{n+1}), p_s(\hat{x}, T(\hat{x}))\}\right] \\ \leq\ & s\left[p_s(\hat{x}, x_{n+1}) + \frac{\beta}{2}\left(p_s(x_n, \hat{x}) + \frac{p_s(x_n, x_{n+1}) + p_s(\hat{x}, T(\hat{x}))}{2}\right)\right] \\ =\ & s\left[p_s(\hat{x}, x_{n+1}) + \beta\left(\frac{2p_s(x_n, \hat{x}) + p_s(x_n, x_{n+1}) + p_s(\hat{x}, T(\hat{x}))}{4}\right)\right]. \end{aligned} \tag{8}$$

From inequality (8), we have

$$\left(\frac{4 - s\beta}{4}\right) p_s(\hat{x}, T(\hat{x})) \leq \frac{s}{4}\left[4p_s(\hat{x}, x_{n+1}) + \beta\left(2p_s(x_n, \hat{x}) + p_s(x_n, x_{n+1})\right)\right],$$

which implies

$$p_s(\hat{x}, T(\hat{x})) \leq \frac{s}{4 - s\beta}\left[4p_s(\hat{x}, x_{n+1}) + \beta\left(2p_s(x_n, \hat{x}) + p_s(x_n, x_{n+1})\right)\right]. \tag{9}$$

We can observe that, for $\beta \in [0, \alpha)$, we have

$$4 - s\beta > 4 - s\alpha. \tag{10}$$

If $\alpha = \frac{1}{s^2}$, then from inequality (10) we have

$$\begin{aligned} 4 - s\beta &> 4 - s\alpha \\ &= 4 - s\frac{1}{s^2} \\ &= 4 - \frac{1}{s} \\ &> 0, \ \ \forall s \geq 1. \end{aligned} \tag{11}$$

Similarly, if $\alpha = \frac{2}{2s+1}$, inequality (10) implies

$$\begin{aligned} 4 - s\beta &> 4 - s\alpha \\ &= 4 - \frac{2s}{2s+1} \\ &> 0, \ \ \forall s \geq 1. \end{aligned} \tag{12}$$

From the inequalities (11) and (12), we conclude that, the right hand side of (9) is non-negative.

Case II: Suppose $\max\{p_s(x_n,\hat{x}), p_s(x_n,T(\hat{x})), p_s(\hat{x},T(x_n))\} = p_s(x_n,T(\hat{x}))$. Then, from inequality (7), we have

$$\begin{aligned}
& p_s(\hat{x},T(\hat{x})) \\
\leq \quad & s\left[p_s(\hat{x},x_{n+1}) + \frac{\beta}{2}p_s(x_n,T(\hat{x})) + \frac{\beta}{2}\min\{p_s(x_n,x_{n+1}), p_s(\hat{x},T(\hat{x}))\}\right] \\
\leq \quad & s\left[p_s(\hat{x},x_{n+1}) + \frac{\beta}{2}\left(p_s(x_n,T(\hat{x})) + \frac{p_s(x_n,x_{n+1}) + p_s(\hat{x},T(\hat{x}))}{2}\right)\right] \\
\leq \quad & s\Big[p_s(\hat{x},x_{n+1}) + \frac{\beta}{2}\Big(s(p_s(x_n,\hat{x}) + p_s(\hat{x},T(\hat{x}))) - p_s(\hat{x},\hat{x}) + \frac{p_s(x_n,x_{n+1}) + p_s(\hat{x},T(\hat{x}))}{2}\Big)\Big] \\
\leq \quad & s\Big[p_s(\hat{x},x_{n+1}) + \frac{\beta}{2}\Big(s(p_s(x_n,\hat{x}) + p_s(\hat{x},T(\hat{x}))) + \frac{p_s(x_n,x_{n+1}) + p_s(\hat{x},T(\hat{x}))}{2}\Big)\Big].
\end{aligned} \tag{13}$$

From (13), we have

$$\left(1 - \frac{s^2\beta}{2} - \frac{s\beta}{4}\right)p_s(\hat{x},T(\hat{x})) \leq s\left[p_s(\hat{x},x_{n+1}) + \frac{\beta}{2}\left(sp_s(x_n,\hat{x}) + \frac{p_s(x_n,x_{n+1})}{2}\right)\right],$$

so that

$$p_s(\hat{x},T(\hat{x})) \leq \frac{4s}{4 - 2s^2\beta - s\beta}\left[p_s(\hat{x},x_{n+1}) + \frac{\beta}{2}\left(sp_s(x_n,\hat{x}) + \frac{p_s(x_n,x_{n+1})}{2}\right)\right]. \tag{14}$$

From the fact that, $\beta \in [0,\alpha)$ we have

$$4 - 2s^2\beta - s\beta > 4 - 2s^2\alpha - s\alpha. \tag{15}$$

If $\alpha = \frac{1}{s^2}$, then from inequality (15) we have

$$\begin{aligned}
4 - 2s^2\beta - s\beta &> 4 - 2s^2\alpha - s\alpha \\
&= 4 - 2s^2\frac{1}{s^2} - s\frac{1}{s^2} \\
&= 4 - 2 - \frac{1}{s} \\
&> 0, \quad \forall s \geq 1.
\end{aligned} \tag{16}$$

Similarly, if $\alpha = \frac{2}{2s+1}$, inequality (15) implies

$$\begin{aligned}
4 - 2s^2\beta - s\beta &> 4 - 2s^2\alpha - s\alpha \\
&= 4 - 2s^2\frac{2}{2s+1} - s\frac{2}{2s+1} \\
&\leq 4 - 2s^2\frac{1}{s^2} - s\frac{1}{s^2} \\
&= 4 - 2 - \frac{1}{s} \\
&> 0, \forall s \geq 1.
\end{aligned} \tag{17}$$

From the inequalities (16) and (17), we conclude that, the right hand side of (14) is non-negative.

Case III: Suppose $\max\{p_s(x_n,\hat{x}), p_s(x_n,T(\hat{x})), p_s(\hat{x},T(x_n))\} = p_s(\hat{x},T(x_n))$. Then, from inequality (7), we have

$$\begin{aligned} & p_s(\hat{x},T(\hat{x})) \\ \leq \; & s\left[p_s(\hat{x},x_{n+1}) + \frac{\beta}{2}p_s(\hat{x},T(x_n)) + \frac{\beta}{2}\min\{p_s(x_n,x_{n+1}), p_s(\hat{x},T(\hat{x}))\}\right] \\ \leq \; & s\left[p_s(\hat{x},x_{n+1}) + \frac{\beta}{2}\left(p_s(\hat{x},T(x_n)) + \frac{p_s(x_n,x_{n+1}) + p_s(\hat{x},T(\hat{x}))}{2}\right)\right] \\ = \; & \left[p_s(\hat{x},x_{n+1}) + \frac{\beta}{2}\left(p_s(\hat{x},x_{n+1}) + \frac{p_s(x_n,x_{n+1}) + p_s(\hat{x},T(\hat{x}))}{2}\right)\right]. \end{aligned} \tag{18}$$

By simplifying the inequality (18), we have

$$p_s(\hat{x},T(\hat{x})) \leq \frac{4s}{4-s\beta}\left[p_s(\hat{x},x_{n+1}) + \frac{\beta}{2}\left(p_s(\hat{x},x_{n+1}) + \frac{p_s(x_n,x_{n+1})}{2}\right)\right]. \tag{19}$$

Note that, for any value of $\beta \in [0,\alpha)$ and $s \geq 1$, $4 - s\beta > 0$. Thus, the right hand side of (19) is non-negative.

Taking the limit as $n \to \infty$ of both sides in the respective inequalities (9), (14) and (19), we generally conclude that

$$\begin{aligned} p_s(\hat{x},T(\hat{x})) &= \lim_{n\to\infty} p_s(\hat{x},T(\hat{x})) \\ &= 0. \end{aligned}$$

Thus, $T(\hat{x}) = \hat{x}$. □

Corollary 1. *Let (X,d_s) be a complete b-metric space with $s \geq 1$, and associated with a partial order $\preceq$. Suppose an order-preserving mapping $T : X \to X$ satisfies*

$$d_s(T(x),T(y)) \leq \frac{\beta}{2}\Big[\max\{d_s(x,y), d_s(x,T(y)), d_s(y,T(x))\} + \min\{d_s(x,T(x)), d_s(y,T(y))\}\Big], \tag{20}$$

for all comparable $x,y \in X$, where $\beta \in [0,\alpha)$ and $\alpha = \min\{\frac{1}{s^2}, \frac{2}{2s+1}\}$. If there exist $x_0 \in X$ such that $x_0 \preceq T(x_0)$ then, T has a unique fixed point $\hat{x} \in X$.

Corollary 2. *Let (X,p) be a complete partial metric space associated with a partial order $\preceq$. Suppose an order-preserving mapping $T : X \to X$ satisfies*

$$p(T(x),T(y)) \leq \frac{\beta}{2}\Big[\max\{p(x,y), p(x,T(y)), p(y,T(x))\} + \min\{p(x,T(x)), p(y,T(y))\}\Big], \tag{21}$$

for all comparable $x,y \in X$, where $\beta \in [0,\frac{2}{3})$. If there exist $x_0 \in X$ such that $x_0 \preceq T(x_0)$, then T has a unique fixed point $\hat{x} \in X$ and $p(\hat{x},\hat{x}) = 0$.

Theorem 2. *Let (X,p_s) be a complete partial b-metric space associated with a partial order $\preceq$, and $s \geq 1$. Let an order-preserving mapping $T : X \to X$ comply with*

$$p_s(T(x),T(y)) \leq \frac{\beta}{2}\Big[\max\{p_s(x,y), p_s(x,T(y)), p_s(x,T(x))\} + \min\{p_s(y,T(y)), p_s(y,T(x))\}\Big], \tag{22}$$

for comparable elements $x,y \in X$, where $\beta \in [0,\alpha)$ and $\alpha = \min\{\frac{2}{3s}, \frac{4}{2s^2+s}\}$. If there exist $x_0 \in X$ such that $x_0 \preceq T(x_0)$ then, T has a unique fixed point $\hat{x} \in X$ such that, $p_s(\hat{x},\hat{x}) = 0$.

Proof of Theorem 2. The proof is similar to that of Theorem 1. □

Remark 2. *We can view the difference between Theorems 1 and 2 in the positions that the terms $p_s(x, T(x))$ and $p_s(y, T(x))$ took in conditions (3) and (22).*

Corollary 3. *Let (X, d_s) be a complete b-metric space associated with a partial order $\preceq$ and $s \geq 1$. Let the order-preserving mapping $T : X \to X$ comply with*

$$d_s(T(x), T(y)) \quad \leq \quad \frac{\beta}{2}\Big[\max\{d_s(x,y), d_s(x, T(y)), d_s(x, T(x))\} + \min\{d_s(y, T(y)), d_s(y, T(x))\}\Big], \tag{23}$$

for comparable elements $x, y \in X$, where $\beta \in [0, \alpha)$ and $\alpha = \min\{\frac{2}{3s}, \frac{4}{2s^2+s}\}$. If there exist $x_0 \in X$ such that $x_0 \preceq T(x_0)$ then, T has a unique fixed point $\hat{x} \in X$.

Corollary 4. *Let (X,p) be a complete partial metric space associated with a partial order $\preceq$. Let an order-preserving mapping $T : X \to X$ comply with*

$$p(Tx, Ty) \leq \frac{\beta}{2}\left[\max\{p(x,y), p(x, Ty), p(x, Tx)\} + \min\{p(y, Ty), p(y, Tx)\}\right], \tag{24}$$

for comparable elements $x, y \in X$, where $\beta \in [0, \frac{2}{3})$. If there exist $x_0 \in X$ such that $x_0 \preceq T(x_0)$ then, T has a unique fixed point $\hat{x} \in X$ such that, $p(\hat{x}, \hat{x}) = 0$.

4. Application to Quantum Operations

In quantum systems, measurements can be seen as quantum operations [44]. Quantum operations are very important in describing quantum systems that interact with the environment.

Let $\mathcal{B}(H)$ be the set of bounded linear operators on the separable complex Hilbert space H; $\mathcal{B}(H)$ is the state space of consideration. Suppose $\mathcal{A} = \{A_i, A_i^* : i = 1, 2, 3 \cdots\}$ is a collection of operators A_i's $\in \mathcal{B}(H)$ satisfying $\sum A_i A_i^* \leq I$. A map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ of the form $\phi_{\mathcal{A}}(B) = \sum A_i B A_i^*$ is called a quantum operation [32], quantum operations can be used in quantum measurements of states. If the A_i's are self adjoint then, $\phi_{\mathcal{A}}$ is self-adjoint.

General quantum measurements that have more than two values are described by effect-valued measures [32]. Denote the set of *quantum effects* by $\mathcal{E}(H) = \{A \in \mathcal{B}(H) : 0 \leq A \leq I\}$. Consider the discrete effect-valued measures described by a sequence of $E_i \in \mathcal{E}(H), i = 1, 2, \cdots$ satisfying $\sum E_i = I$ where the sum converges in the strong operator topology. Therefore, the probability that outcome i occurs in the state ρ is $P_\rho(E_i)$ and the post-measurement state given that i occurs is $\frac{E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}}}{tr(\rho E_i)}$ [32]. Furthermore, the resulting state after the execution of measurement without making any observation is given by

$$\phi(\rho) = \sum E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}}. \tag{25}$$

If the measurement does not disturb the state ρ, then we have $\phi(\rho) = \rho$ (fixed point equation).

Furthermore, the probability that an effect A occurs in the state ρ given that, the measurement was performed is

$$P_{\phi(\rho)}(A) = tr\left[A \sum E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}}\right] = tr\left(\sum E_i^{\frac{1}{2}} A E_i^{\frac{1}{2}} \rho\right). \tag{26}$$

If A is not disturbed by the measurement in any state we have

$$\sum E_i^{\frac{1}{2}} A E_i^{\frac{1}{2}} = A,$$

and by defining $\phi(A) = \sum E_i^{\frac{1}{2}} A E_i^{\frac{1}{2}}$, we end up with $\phi(A) = A$.

More measurements are frequently used in quantum dynamics, quantum computation and quantum information theory [37,45,46].

Henceforth we will be dealing with a two-level ($|0\rangle, |1\rangle$) single qubit quantum system. Where a quantum state $|\Psi\rangle$ can be described as

$$|\Psi\rangle = a|0\rangle + b|1\rangle, \;\; with \;\; a, b \in \mathbb{C} \;\; and \;\; |a|^2 + |b|^2 = 1,$$

(see [37]). Considering the representation of a two-level quantum system by the Bloch sphere (Figure 1) above, a quantum state ($|\Psi\rangle$) can be represented with the below density matrix (ρ),

$$|\Psi\rangle = \rho = \frac{1}{2}\begin{pmatrix} 1+\eta\cos\theta & \eta e^{-i\varphi}\sin\theta \\ \eta e^{i\varphi}\sin\theta & 1-\eta\cos\theta \end{pmatrix}, \;\; \eta \in [0,1], \;\; 0 \le \theta \le \pi, \; and \; 0 \le \varphi \le 2\pi. \tag{27}$$

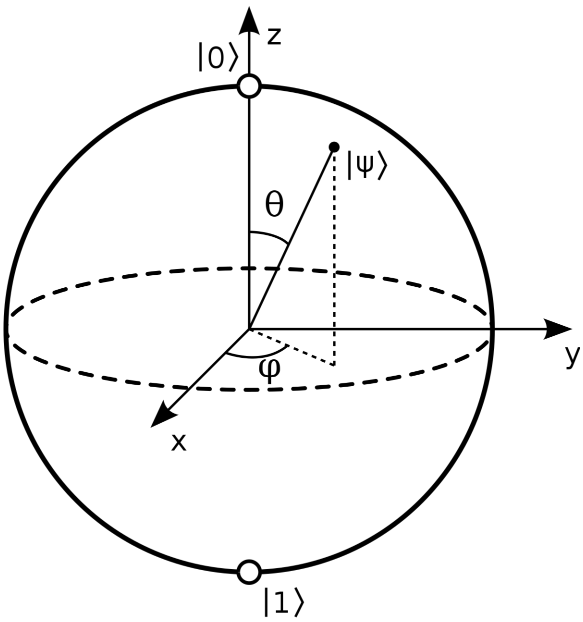

Figure 1. Bloch sphere.

Furthermore, the density (ρ) matrix can also take below representation [37],

$$\rho = \frac{1}{2}[I + \overline{r}_\rho \cdot \overline{\sigma}] = \frac{1}{2}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix} \tag{28}$$

where $\overline{r}_\rho = [r_x, r_y, r_z]$ is the Bloch vector with $\|\overline{r}_\rho\| \le 1$, and $\overline{\sigma} = [\sigma_x, \sigma_y, \sigma_z]$ for $\sigma_x, \sigma_y, \sigma_z$ being the Pauli matrices.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that the Bloch vectors with norm less than one are associated to the mixed quantum states, whereas Bloch vectors with norm equals one are associated to the pure quantum states.

Let ρ, σ be two quantum states in a two level quantum system. Then, the Bures fidelity [47] between the quantum states ρ and σ is defined as

$$F(\rho, \sigma) = [tr\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}]^2,$$

(see [47]). The Bures fidelity satisfies $0 \le F(\rho, \sigma) \le 1$, it is 1 if $\rho = \sigma$ and 0 if ρ and σ have an orthogonal support (perfectly distinguishable) [37].

Now consider a two-level quantum system X represented with the collection of density matrices $\{\rho : \rho\ is\ as\ defined\ in\ Equation\ (28)\}$. Define the function $p_s : X \times X \to \mathbb{R}_+$ by

$$p_s(\rho, \delta) = \begin{cases} 0, & \rho = \delta \\ \max\{\|\bar{r}_\rho\|, \|\bar{r}_\delta\|\} e^{\frac{1}{5}(1-F(\rho,\delta))}, & \rho \neq \delta. \end{cases}$$

It is easy to show that p_s is a b-metric on X (partial b-metric) with $s = e^{\frac{1}{1000}} \approx 1$. Define an order relation $\preceq$ on X by

$$\rho \preceq \delta\ iff\ the\ line\ from\ origin\ joining\ the\ point\ \bar{r}_\delta\ passes\ through\ \bar{r}_\rho. \tag{29}$$

It is obvious that, the order relation defined above (29) is a partial order.

Corollary 5. *Let (p_s, X) be a complete partial b-metric space associated with the above order $\preceq$ (29). Suppose an order-preserving quantum operation $T : X \to X$ that satisfies either conditions in Theorems 1 or 2. Then, T has a fixed point.*

Below example covers both Theorems 1 and 2. However, we precisely execute the solution procedure in favour of Theorem 1.

Example 4. *Consider the depolarising quantum operation T on the Bloch sphere X; $T(\rho) = \frac{I}{2}p + (1-p)\rho$ with the depolarising parameter $p \in [0, 1]$. Let the comparable quantum states satisfy (29).*

We will check that, $T : X \to X$ satisfy all the conditions of our theorem(s), as such, it has a unique fixed point.

Now, let $\rho, \delta \in X$. If the order $\preceq$ is as defined in (29), we will start by showing T is order-preserving. Note that, T is order-preserving if the angle of rotation describing any two comparable quantum states is invariant under T, and the distance from origin to $T(\rho)$ is less than or equal to the distance from origin to $T(\delta)$, i.e., if $\rho \preceq \delta$ then $T\rho \preceq T\delta$.

Therefore, using the Bloch sphere representation of states in a two-level quantum system below

$$\rho = \frac{1}{2}\begin{pmatrix} 1 + \varrho\cos\theta & \varrho e^{-i\varphi}\sin\theta \\ \varrho e^{i\varphi}\sin\theta & 1 - \varrho\cos\theta \end{pmatrix}, \quad \varrho \in [0, 1],\ \ 0 \leq \theta \leq \pi,\ \ and\ \ 0 \leq \varphi \leq 2\pi,$$

we proceed as follows,

$$\begin{aligned} T(\rho) &= \frac{1}{2}\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} + \frac{1-p}{2}\begin{pmatrix} 1 + \varrho\cos\theta & \varrho e^{-i\phi}\sin\theta \\ \varrho e^{i\phi}\sin\theta & 1 - \varrho\cos\theta \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 + \varrho\cos\theta & \varrho e^{-i\phi}\sin\theta \\ \varrho e^{i\phi}\sin\theta & 1 - \varrho\cos\theta \end{pmatrix} - \frac{1}{2}\begin{pmatrix} p + p\varrho\cos\theta & p\varrho e^{-i\phi}\sin\theta \\ p\varrho e^{i\phi}\sin\theta & p - p\varrho\cos\theta \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} p + 1 + \varrho\cos\theta - p - p\varrho\cos\theta & (1-p)\varrho e^{-i\phi}\sin\theta \\ (1-p)\varrho e^{i\phi}\sin\theta & p + 1 - \varrho\cos\theta - p + p\varrho\cos\theta \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 + (1-p)\varrho\cos\theta & (1-p)\varrho e^{-i\phi}\sin\theta \\ (1-p)\varrho e^{i\phi}\sin\theta & 1 - (1-p)\varrho\cos\theta \end{pmatrix}. \end{aligned}$$

Clearly, the angles θ and ϕ are not affected by the depolarising quantum operation T. Furthermore, we can deduce that the distance of the quantum state ρ from origin given by ϱ is greater than or equal to the distance of the new quantum state $T(\rho)$ from origin given by $(1-p)\varrho$, $p \in [0, 1]$. Therefore, for any two comparable quantum states $\rho, \delta \in X$ $(\rho \preceq \delta)$, with respective distances from origin ϱ_ρ and ϱ_δ such that, $\varrho_\rho \leq \varrho_\delta$, the depolarising quantum operation T produces two quantum states

$T(\rho), T(\delta) \in X$, with respective distances from origin $(1-p)\varrho_\rho$ and $(1-p)\varrho_\delta$ for $p \in [0,1]$. As $\varrho_\rho \leq \varrho_\delta$, then $(1-p)\varrho_\rho \leq (1-p)\varrho_\delta$, $\forall p \in [0,1]$. Thus, $T(\rho) \preceq T(\delta)$; T is order-preserving.

The fidelity of any two quantum states $\rho = \frac{1}{2}(I_2 + \vec{r}_\rho \cdot \vec{\sigma})$ and $\delta = \frac{1}{2}(I_2 + \vec{r}_\delta \cdot \vec{\sigma})$ can take the form

$$F(\rho,\delta) = \frac{1}{2}[1 + \vec{r}_\rho \cdot \vec{r}_\delta + \sqrt{1 - \|\vec{r}_\rho\|^2}\sqrt{1 - \|\vec{r}_\delta\|^2}], \tag{30}$$

(see, [48]), where $\vec{r}_\rho \cdot \vec{r}_\delta$ is the inner/dot product between the vectors $\vec{r}_\rho$ and $\vec{r}_\delta$. So, for any comparable quantum states $\rho = \frac{1}{2}(I_2 + \vec{r}_\rho \cdot \vec{\sigma})$ and $\delta = \frac{1}{2}(I_2 + \vec{r}_\delta \cdot \vec{\sigma})$, $\vec{r}_\rho \cdot \vec{r}_\delta = \|\vec{r}_\rho\|\|\vec{r}_\delta\| \cos\vartheta$ for ϑ being the angle between $\vec{r}_\rho$ and $\vec{r}_\delta$. Using Equation (30), one can show that,

1. $F(\rho,\rho) = 1$.
2. $F(\rho,o) = \frac{1}{2}$; for ρ a pure state and o the completely mixed state(origin/center).
3. $F(\rho,\rho_-) = 0$; for ρ_- a pure state that is 180^0 separated from ρ.

Thus, for $\rho, \delta \in X$, $1.000 \leq e^{\frac{1}{5}(1-F(\rho,\delta))} \leq 1.105$.

Furthermore, using $s = 1$ the condition $\beta \in [0, \frac{2}{3})$ is imposed on both Theorems 1 and 2. From the known facts and definitions, we proceed as

$$\begin{aligned}
p_s(T\rho, T\delta) &= \max\{\|T\rho\|, \|T\delta\|\}e^{\frac{1}{5}(1-F(T\rho,T\delta))} \\
&= \frac{1}{4}\|\delta\|e^{\frac{1}{5}(1-F(T\rho,T\delta))} \\
&\leq \frac{1}{4}(\|\delta\|e^{\frac{1}{5}(1-F(T\rho,\delta))} + \|\rho\|e^{\frac{1}{5}(1-F(T\rho,\rho))}) \\
&= \frac{1}{2}\left(\frac{1}{2}(p_s(T\rho,\delta) + p_s(T\rho,\rho))\right) \\
&= \frac{1}{2}\left(\frac{1}{2}[\max\{p_s(\rho,\delta), p_s(\rho,T\delta), p_s(\delta,T\rho)\} + \min\{p_s(\rho,T\rho), p_s(\delta,T\delta)\}]\right).
\end{aligned}$$

Taking $\beta = \frac{1}{2}$, condition (3) in Theorem 1 is satisfied. A similar procedure can be used to prove the compliance of condition (22) in Theorem 2. Finally, in reference to Theorem 1, we conclude that T has a unique fixed point $\frac{I}{2} \in X$ (centre). A similar conclusion can be attained using Theorem 2.

Example 5. *Consider the quantum operation (T) known as the generalised amplitude damping on the Bloch sphere X defined as*

$$T\left(\frac{1}{2}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1+\gamma(2p-1)+r_z\sqrt{1-\gamma} & r_x\sqrt{1-\gamma} - ir_y\sqrt{1-\gamma} \\ r_x\sqrt{1-\gamma} + ir_y\sqrt{1-\gamma} & 1-[\gamma(2p-1)+r_z\sqrt{1-\gamma}] \end{bmatrix}, \tag{31}$$

with damping parameter $\gamma \in [0,1]$ and $p \in [0,1]$. Let the comparable quantum states satisfies (29). Then, T has a fixed point.

In a similar way as we demonstrated in Example 4, one can show the existence of the invariant state $\hat{\rho} = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}$ for the generalised amplitude damping T as presented in Equation (31). The effect of the generalised amplitude damping is like a flow of states on the Bloch sphere (Unit ball) towards the fixed state $\hat{\rho}$. The generalised amplitude damping can be used in description of energy dissipation effects due to loss of energy from a quantum system. Note that, the invariant state is unique for every $p \in [0,1]$.

5. Conclusions

The results in this paper cover some part of the famous contractive fixed point results of Banach [11], Kannan [26] and Chatterjea [23]. The contractive conditions (3) and (22) presented can be seen as an improvement to the work of Batsari et al. [18], Du et al. [21] and Dung et al. [40]; as the conditions contain both maximum and minimum functions. Moreover, our results are generalisations of many other existing results in terms of the space in consideration (partial-b metric space).

On the other hand, although the fidelity function is not a metric, we have shown how it can be utilised in studying fixed points of some quantum operations. Moreover, the existence of fixed points of some quantum operations can be studied without given much attention to the quantum effects as seen from Examples 4 and 5. Thus, the criteria and procedure we presented can serve as an alternative in guaranteeing the existence and finding the fixed points of some quantum operations respectively if compared with the existing ones provided by Lüders [38] and Busch et al. [45]. Our choice for using depolarising and generalised amplitude damping quantum operations was related to their importance as source of quantum error and in description of energy dissipation effect respectively.

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