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Thermodynamics and phase transition of spherically symmetric black hole in de Sitter space from Rényi statistics

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Abstract Schwarzschild black holes in a de Sitter background are studied in terms of their thermodynamics based on the Rényi statistics. This leads to thermodynamically stable black hole configurations for some certain range of black hole radii; namely, within this range the corresponding black holes have positive heat capacity. Moreover, for a certain background temperature there can exist at most three configurations of black hole, one among which is thermodynamically stable. These configurations are investigated in terms of their free energies, resulting in the moderate-sized stable black hole configuration being the most preferred configuration. Furthermore, a specific condition on the Rényi non-extensive parameter is required if a given hot spacetime were to evolve thermally into the moderate-sized stable black hole.

1 Introduction and motivations

The cosmic accelerating expansion of our universe has been proved to exist through numerous observations [1–3]. To model this expansion, theorists have come up with various theories, one of which is the cosmological constant model [4]. The key feature of the cosmological constant model is that the cosmic expansion is driven by the constant, Λ, which is treated as an additional matter/energy spreading throughout our universe. This model has been known as one of the most successful, yet simplest, cosmological models because it fits nicely with observations [5].

With the presence of the cosmological constant, a mathematical spherically symmetric black hole solution can be considered as a generalization of the standard Schwarzschild solution. Depending on the sign of Λ, the corresponding black hole solution can represent either a Schwarzschild black hole on a de Sitter background for positive value of Λ, formally known

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as the Schwarzschild–de Sitter (Sch–dS) solution, or that on an anti-de Sitter background for negative value of Λ, known as the Schwarzschild–anti-de Sitter (Sch–AdS) solution. Since our late-time universe is found observationally to correspond to positive cosmological constant [5], then if black holes were to exist in our universe, they should correspond to the Sch–dS black holes. An interesting point of the Sch–dS spacetime is that it possesses two event horizons, *i.e.* the black hole horizon and the cosmological horizon. A thermodynamic similarity between these two horizons has been established since the work of Gibbons and Hawking [6]. Despite the fact that our universe prefers the positive cosmological constant, there have been several theoretical works which indicate that black holes tends to be unstable in de Sitter spacetime background [7,8].

If a black hole is treated as a thermodynamic system, the black hole can be considered to have its own temperature due to the presence of the corresponding Hawking radiation [9]. Taking the Schwarzschild black hole as an example, the Hawking radiation produced around the event horizon of the black hole happens in such a way that the corresponding Hawking temperature decreases as it gains energy, indicating that the Schwarzschild black hole is a thermal object with negative heat capacity [9] (see also [10]). In particular, for a Schwarzschild black hole with its temperature above the background temperature, it will evaporate away as it radiates, while the temperature increases, whereas for that with temperature below the background temperature, it will keep increasing in its size, while the temperature decreases. As a consequence, the black hole will never be in thermodynamic equilibrium with the background. For the Sch–dS black hole in four-dimensional spacetime, the situation is even worse; the presence of the positive cosmological constant makes the heat capacity even more negative given that black hole's thermodynamic quantities are evaluated at the black hole's horizon, which is evident in the following articles [6,11] as the black hole temperature is a decreasing function of its mass.

On the other hand, although it appears to go against with observations, the Sch–AdS black hole exhibits very nice thermal behaviours; there exist configurations with positive heat capacity which makes the black hole a thermodynamically stable object [12]. In other words, the Sch–AdS black holes can be in thermal equilibrium with the background in a stable way. These features of the Sch–AdS are very interesting theoretically, and they have been studied in a variety of ways, especially in the application to address thermal behaviours of gauge theories through AdS/CFT correspondence.

Those previous predictions are based on one common idea which is that all of the thermodynamic behaviours are described using the Gibbs–Boltzmann statistics. In particular, the so-called Bekenstein–Hawking entropy of a black hole of interest follows the standard thermodynamics based on the Gibbs–Boltzmann statistics. However, the entropy being proportional to the area, usually known as the area law, suggests that a black hole is not an extensive system, i.e. its entropy is not proportional to its volume. This fact leads to various studies on thermodynamics of black holes using more general statistics which applies to non-extensive systems. One of simple approaches is to treat the Bekenstein–Hawking entropy as the Tsallis entropy [13]. However, the Tsallis entropy appears to have a difficulty in defining thermodynamic temperature because of its incompatibility with the zeroth law of thermodynamics [14]. To fix such a problem, an appropriate formulation of entropy which utilizes the formal logarithm of the Tsallis entropy is formally known as the Rényi entropy [15].[1]

Rényi entropy exhibits very interesting features in the black hole thermodynamics as it effectively modifies the thermodynamics as if the black hole is in the AdS background

[1] Another way to fix the problem of incompatibility is to define an effective temperature as a conjugate variable to the Tsallis entropy (see, for example, [16]). Through this approach, a relation analogous to the standard zeroth law can be obtained.

without the necessity of introducing the negative cosmological constant. For example, there is an interesting study on a Schwarzschild black hole in an asymptotically flat spacetime using the Rényi entropy [10]. Although, in the Gibbs–Boltzmann context, the black hole is said to have a negative heat capacity, the Rényi statistics suggests that the Schwarzschild black hole can possibly have a positive heat capacity within a certain range of parameters, which resembles the thermal property of the Sch–AdS black hole according to Gibbs–Boltzmann statistics. In addition, the Rényi statistic treatment has been proved to be very interesting in the context of a Reissner–Nordström black hole in which the thermodynamic stability of the hole exists, and in this language, the Reissner–Nordström black hole exhibits intriguing thermal behaviours as if it is a van der Waal liquid–gas system [17].

As motivated by the observational data, since it could represent a black hole in our late-time universe, the Sch–dS black hole is considered in the present work. It is well known that the Sch–dS black hole generally possesses two horizons: event horizon and cosmic horizon. As a result, the Sch–dS spacetime provides two distinct temperatures leading to the non-equilibrium state. This is one of the main obstructions to investigate thermodynamics of the Sch–dS black hole. There have been several attempts to overcome this obstruction. Here, we briefly classify them into three approaches. First, one may treat the thermodynamic system of each horizons independently. Therefore, each system can be characterized by their own temperatures and thermodynamic behaviours. In this approach, one may think that the process of the heat flow between two system happens at a timescale which is much longer than the timescale of the thermodynamic phase transition [11]. In other words, an equivalent view for this approach is treating one of two horizons as a boundary [8,18,19]. Furthermore, one can consider the black hole which is enclosed in an isothermal cavity at fixed temperature where the cavity can play the role of the reservoir [20–22]. Second, the thermodynamic system can be considered in a global view to construct a globally effective temperature and other effective thermodynamic quantities. In this approach, the observer could imagine oneself being in the region between the event horizon and the cosmic horizon [23]. According to this idea, several extensions to a more general black hole have been investigated [24–29]. Note that in order to avoid unphysical behaviours of the effective temperature, one may identify the effective entropy as the difference of cosmological entropy and black hole entropy instead of the sum of entropies [30,31]. Third, the thermal equilibrium can be achieved by requiring the additional degree of freedom to set both temperature to be equal, for example scalar hairy black hole [32]. Alternatively, it is possible to combine two latter approaches by requiring the effective temperature be equal to both the black hole horizon temperature and the cosmic horizon temperature. This can be performed by identifying the additional terms for the total entropy which can be viewed as a part of correlation of both systems [33–36]. In the present work, we choose to consider the thermodynamics of Sch–dS black hole horizon separately from the thermodynamics of cosmic horizon, following the first approach. We will discuss about this approach in more detail below.

As mentioned above, even though the two horizons cannot be in thermodynamic equilibrium due to the difference of their temperatures, the black hole horizon and cosmic horizon can be classically viewed as two separate thermodynamic systems. Following the Euclidean black hole method proposed in [8,18], one can discuss either one of the two horizons as a thermodynamic object by treating the other as a boundary. When the cosmic horizon is chosen as the boundary, the Sch—dS space is reduced to be a thermodynamic system of the black hole horizon contained in a space of given cosmic horizon. Manifestly, there is no notion of spatial infinity for this reduced Sch–dS system, because its space is ended at the cosmic horizon; the spatial infinity exists beyond the cosmic horizon. Lacking the asymptotic region in our consideration, it seems to be ambiguous in defining the black hole mass. It is well known that the black hole mass can be defined in the asymptotically flat black hole spacetime since the mass–energy conservation law can be well defined by an observer in asymptotic region. In other words, through the language of Euclidean black hole method, one may fix the mass parameter in the Schwarzschild metric, for example, at spatial infinity when the path integral in the semi-classical approximation is evaluated using an extremum of the action. However, in defining the black hole mass in the Sch–dS with setting the boundary at the cosmic horizon, one can do in a similar way by fixing the field variables at the cosmic horizon, instead of at the spatial infinity. Then, in the present work, the black hole mass can be defined by an observer at the cosmic horizon, analogous to the observer in asymptotic flat region in the case of Schwarzschild spacetime. As discussed in [19], the conserved quantity of this reduced Sch–dS can be obtained through this approach, where the black hole mass serves in the same way as the ADM mass. It is important to remark here that it is also possible to define the black hole mass in this set-up through the Abbott–Deser–Tekin (ADT) approach [37–39], in which the conserved charges are in full agreement with the charges reported in [8,18] for non-rotating cases.

Moreover, it is important to emphasize that we study the black hole thermodynamics in dS space in an equilibrium sense. To achieve this, we use the assumptions as in [11] which are the following.

i Although this system of two horizons is not in equilibrium in general, the temperatures are proportional to $\hbar$ in their dimensionful forms. This results in a small difference between the two temperatures and also a large timescale of the thermal transfer between the horizons. We then assume hereafter that the timescale of this thermal transfer is much longer than those of the following thermodynamic processes so that the thermodynamic system of interest can be assumed to be in "quasi-equilibrium". This then allows us to consider thermodynamic behaviours arising from the black hole horizon separately, as mentioned above.

ii We assume that there is an infinite reservoir to transfer thermal energy into the black hole system in order to describe dynamics of black hole. Furthermore, the characteristic time of the thermal transfer is also assumed, in this work, to be small compared to the timescale of the thermal transfer between the black hole event horizon and the cosmic horizon.

With these assumptions, our discussions on the black hole phase transitions can be acceptable.

Although the Sch–dS black holes are thermodynamically unstable in the context of the Gibbs–Boltzmann statistics, it is interesting to study the thermal behaviours of the Sch–dS black holes using Rényi statistics. In Sect. 2, the black hole thermodynamics according to Rényi statistics are investigated and it turns out that a Sch–dS black hole can be thermodynamically stable; namely, it can have a positive heat capacity for some certain parameter set-up. Additionally, for an appropriate background temperature, there are up to three possible states of the Sch–dS black hole which can be in thermal equilibrium to the background and only one of them is thermally stable. The free energies corresponding to the Rényi statistics are also considered in Sect. 3 in order to find the preferred state among the three possible ones. Interestingly, it turns out that the one with positive heat capacity corresponds to the lowest free energy which renders it a most preferred state. Finally, we also investigate the situation where a given hot space can actually evolve into a Sch–dS black hole with positive heat capacity through the free energy analysis and we report the condition of the parameter set-up for such a situation. We finish this article by giving concluding remarks in Sect. 4.

2 Black hole in de Sitter space

A spherically symmetric black hole solution to the Einstein's field equation in the presence of the positive cosmological constant Λ, frequently known as the Schwarzschild–de Sitter (Sch–dS) solution, is given by the following line element,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2, \tag{1}$$

where

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \tag{2}$$

given that M is an integration constant which corresponds to a mass of the black hole. At the black hole event horizon defined as r_+, we have $f(r_+) = 0$ and we can find the black hole mass as follows,

$$M = \frac{r_+}{2}\left(1 - \frac{\Lambda r_+^2}{3}\right). \tag{3}$$

Note that M is positive when $r_+^2 < \frac{3}{\Lambda}$. Interestingly, the Hawking temperature of the black hole becomes lowered in the de Sitter space. This can be expressed through the formula $T = \frac{f'(r_+)}{4\pi}$. Then, we obtain

$$\begin{aligned} T &= \frac{1}{4\pi}\left(\frac{2M}{r_+^2} - \frac{2\Lambda r_+}{3}\right), \\ &= \frac{1}{4\pi}\left(\frac{1}{r_+} - \frac{\Lambda r_+}{3} - \frac{2\Lambda r_+}{3}\right), \\ &= \frac{1}{4\pi r_+}\left(1 - \Lambda r_+^2\right). \end{aligned} \tag{4}$$

Since this formula holds for any horizon specified by a root of $f(r) = 0$, the largest positive root, i.e. the cosmic horizon r_c, is also associated with a temperature which is generally different from the black hole temperature [6,11]. This difference in temperatures causes thermal energy transfer between the two horizons which implies a non-equilibrium situation.[2] Nevertheless, as discussed above, the difference in temperatures is very small such that we can use the aforementioned assumptions allowing us to study the thermodynamics of black hole in the equilibrium sense.

The Bekenstein–Hawking entropy, which follows the area law, is in the following form,

$$S_{\mathrm{BH}} = \frac{A}{4} = \pi r_+^2. \tag{5}$$

From the entropy formula, the area law states that the entropy is proportional to the area of the thermal system. This is one of the main reasons as to why black holes have been argued to be non-extensive systems, which does not coincide with the usual thermodynamic concepts. As previously mentioned in the introduction, in order to understand such systems, there are several kinds of thermal statistics which incorporates non-extensivity into thermal systems. To this end, the Tsallis entropy is one of the possible candidates. However, using Tsallis

[2] In Rényi statistics, it is possible to have a thermodynamic system in which the two horizons are in thermal equilibrium. Such treatment on this possibility is described in 1.

entropy leads to the problem of the incompatibility with the zeroth law of thermodynamics [14] (see also in 1). There are a suggestion to mathematically transform it to Rényi entropy. By assuming that the black hole entropy follows the Tsallis statistics, the Rényi entropy is expressed in the following form [15],

$$\begin{aligned} S_{\rm R} &= \frac{1}{\lambda} \ln\left(1 + \lambda S_{\rm BH}\right) \\ &= \frac{1}{\lambda} \ln\left(1 + \lambda \pi r_+^2\right), \end{aligned} \tag{6}$$

where $-\infty < \lambda < 1$ is the non-extensive parameter. Note that in the limit $\lambda \to 0$, this formula reduces to the Bekenstein–Hawking entropy S_{BH}.

Using Rényi statistics, the black hole stability is investigated through its temperature profile. With an appropriate definition of the thermodynamic internal energy, the corresponding conjugate Rényi temperature can be found. As mentioned in Sect. 1, the mass parameter M can appropriately serve as a conserved quantity and as a thermodynamic internal energy of the system [8,18]. The Rényi temperature can then be determined through $T_{\rm R} = \frac{dM}{dS_{\rm R}}$. Using Eq. (3), the differentiation of the black hole mass is

$$dM = \frac{1}{2}\left(1 - \Lambda r_+^2\right) dr_+. \tag{7}$$

It is obvious that the extremum of the mass profile is at $r_+^2 = \frac{1}{\Lambda}$ and since the second derivative of M is always negative for positive values of r_+, this extremum is a maximum. By knowing that $M = 0$ when $r_+ = 0$, the mass M monotonically increases in r_+ for $0 < r_+^2 < \frac{1}{\Lambda}$. By using Eq. (6), the differentiation of the Rényi entropy is

$$\begin{aligned} dS_{\rm R} &= \frac{1}{\lambda} \frac{2\lambda \pi r_+}{\left(1 + \lambda \pi r_+^2\right)} dr_+, \\ &= \frac{2\pi r_+}{1 + \lambda \pi r_+^2} dr_+. \end{aligned} \tag{8}$$

Using Eqs. (7) and (8), we obtain the Rényi temperature $T_{\rm R}$ in the form of the Hawking temperature T as follows

$$\begin{aligned} T_{\rm R} &= \frac{1}{4\pi r_+}\left(1 - \Lambda r_+^2\right)\left(1 + \lambda \pi r_+^2\right) \\ &= \left(1 + \lambda \pi r_+^2\right) T. \end{aligned} \tag{9}$$

Note that the black hole of size as parametrically large as $r_+^2 = \frac{1}{\Lambda}$ corresponds to a zero temperature, which is related to the Nariai limit [40,41]. The above expression can be rewritten in the following form,

$$T_{\rm R} = \frac{1}{4\pi r_+}\left(1 + (\lambda\pi - \Lambda) r_+^2 - \Lambda \lambda \pi r_+^4\right). \tag{10}$$

The Rényi temperature profile is expressed in Fig. 1. The Rényi temperature of the Sch–dS black hole has a very interesting feature. Particularly, the non-extensive parameter in the Rényi statistics provides the thermal effect as if the system is in a background with negative cosmological constant [10]. As a result, the Schwarzschild black hole thermodynamics according to the Rényi statistics is similar to the Bekenstein–Hawking thermodynamics of a Schwarzschild–anti-de Sitter black hole [10]. In this case where the Sch–dS black hole is considered, the non-extensive parameter changes the temperature profile of the Sch–dS

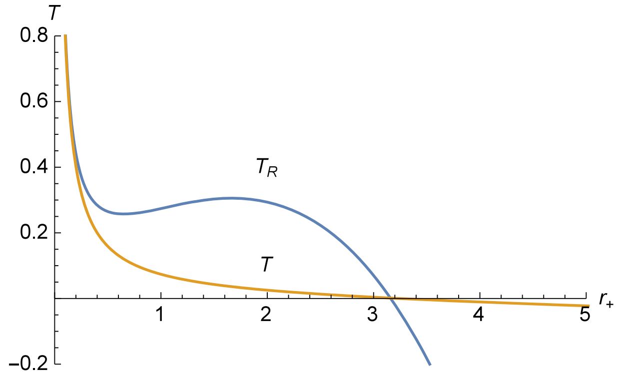

Fig. 1 Temperature profile corresponding to that of the Rényi entropy (blue) and the Bekenstein–Hawking temperature (yellow) of the Sch–dS black hole are shown here. The corresponding parameters are set as follows, $\lambda = 0.9$, $\Lambda = 0.02$

black hole in an interesting way. In particular, the Rényi temperature is modified from the Bekenstein–Hawking temperature such that there exist two extrema: one minimum and one maximum. To this end, let us consider the following,

$$\begin{aligned} dT_{\mathrm{R}} &= \left(-\frac{1}{4\pi r_+^2} + \frac{\lambda\pi - \Lambda}{4\pi} - \frac{3\Lambda\lambda}{4} r_+^2\right) dr_+ \\ &= -\frac{1}{4\pi r_+^2}\left(1 - (\lambda\pi - \lambda) r_+^2 + 3\Lambda\lambda\pi r_+^4\right) dr_+. \end{aligned} \tag{11}$$

By considering $\frac{dT_{\mathrm{R}}}{dr_+} = 0$, we know that T_{R} becomes extremal at r_+ satisfying

$$3\Lambda\lambda\pi r_+^4 - (\lambda\pi - \Lambda) r_+^2 + 1 = 0. \tag{12}$$

The roots of this equation follows

$$r_{+1,2}^2 = \frac{\lambda\pi - \Lambda \pm \sqrt{(\lambda\pi - \Lambda)^2 - 12\Lambda\lambda\pi}}{6\Lambda\lambda\pi}. \tag{13}$$

From Eq. (13), to have both roots to be positive, we need $\lambda\pi - \Lambda > 0$ from which, according to the definition of $\Lambda > 0$ corresponding to the de Sitter background, it can be implied that $\lambda > 0$. Moreover, we also need $(\lambda\pi - \Lambda)^2 > 12\Lambda\lambda\pi$ so that the roots are not complex numbers. By requiring $(\lambda\pi - \Lambda)^2 - 12\Lambda\lambda\pi > 0$, first we recall that

$$\begin{aligned} (\lambda\pi - \Lambda)^2 - 12\Lambda\lambda\pi = &\left(\lambda\pi - \Lambda\left(7 + \sqrt{48}\right)\right) \\ &\left(\lambda\pi - \Lambda\left(7 - \sqrt{48}\right)\right), \end{aligned} \tag{14}$$

meaning that for $(\lambda\pi - \Lambda)^2 - 12\Lambda\lambda\pi > 0$ we require either $\lambda\pi > \Lambda(7 + \sqrt{48}) \approx 13.93\Lambda$ or $\lambda\pi < \Lambda(7 - \sqrt{48}) \approx 0.07180\Lambda$. Since the latter case means that $\lambda\pi < \Lambda$, then the only applicable case is that $\lambda\pi > \Lambda(7 + \sqrt{48})$. This marks the lower bound for the value of λ.

Since in this scenario there is a local minimum in temperature, it can be proved that the minimum temperature is always positive for any allowed set of parameters, i.e. it cannot be as low as zero. Since at the minimum temperature, it also satisfies Eq. (12). Then we can find the minimum temperature from simplifying Eq. (9) using Eq. (12) as follows,

$$T_{\mathrm{R},min} = \frac{1}{6\pi r_{+min}}\left(2+(\lambda\pi-\Lambda)\,r_{+min}^2\right),$$
$$= \frac{1}{6\pi r_{+min}}\Bigg(2+\frac{\lambda\pi-\Lambda}{6\Lambda\lambda\pi}$$
$$\left[(\lambda\pi-\Lambda)-\sqrt{(\lambda\pi-\Lambda)^2-12\Lambda\lambda\pi}\right]\Bigg). \tag{15}$$

Since from requiring that $\lambda\pi-\Lambda>0$ and $\Lambda>0$, the square bracket in the second line is always positive, rendering the minimum temperature always positive.

The corresponding heat capacity can be evaluated as

$$C=\frac{dM}{dT_{\mathrm{R}}}=\frac{-2\pi r_+^2\left(1-\Lambda r_+^2\right)}{\left(1-(\lambda\pi-\Lambda)r_+^2+3\Lambda\lambda\pi r_+^4\right)}. \tag{16}$$

Let us consider the case where both of the $r_{+1,2}^2$ are positive, in other words the case which coincides with Fig. 1. Note that the denominator in Eq. (16) is exactly the same function as in Eq. (12) whose roots are those in Eq. (13). Since $\lambda>0$, $\Lambda>0$ (which renders the denominator a convex parabola in an argument r_+^2), it can then be implied that the denominator in Eq. (16) is negative when $r_{+1}^2<r_+^2<r_{+2}^2$. For the numerator in Eq. (16), it can be shown that both the roots, $r_{+1,2}^2$, are less than $\frac{1}{\Lambda}$ by the following. From the requirement that $\lambda\pi-\Lambda>0$, we can express the roots as follows,

$$r_{+1,2}^2=\frac{\lambda\pi\left(1-\frac{\Lambda}{\lambda\pi}\right)\pm\lambda\pi\sqrt{1-\frac{14\Lambda}{\lambda\pi}+\frac{\Lambda^2}{(\lambda\pi)^2}}}{6\Lambda\lambda\pi},$$
$$\approx\frac{1}{3\Lambda}\left(1-\frac{4\Lambda}{\lambda\pi}+\mathcal{O}(2)\right),\ \frac{1}{\lambda\pi}\left(1-\frac{\Lambda}{12\,(\lambda\pi)}+\mathcal{O}(2)\right). \tag{17}$$

From Eq. (17), it is obvious that as $\lambda\pi-\Lambda>0$, $r_{+1}^2<\frac{1}{3\Lambda}<\frac{1}{\Lambda}$ and $r_{+2}^2<\frac{1}{\lambda\pi}<\frac{1}{\Lambda}$. For a Sch–dS black hole of a size corresponding to $r_{+1}^2<r_+^2<r_{+2}^2$, the numerator in Eq. (16) is thus always negative. Note that these bounds are in consistency with the positive black hole mass bound, $\Lambda r_+^2\le 3$. Taking these bounds into account, we can conclude that for the range of interest, $r_{+1}^2<r_+^2<r_{+2}^2$, the corresponding heat capacity is always positive, meaning that the holes in this range are all locally thermodynamically stable, whereas for those with sizes smaller than r_{+1}^2 or larger than r_{+2}^2. The corresponding heat capacities for various sizes of black holes are shown in Fig. 2.

The Rényi temperature profile with two-extrema nature also exhibits very interesting thermal behaviours. For example, if a system is in a temperature which lies between the local minimum temperature and the local maximum temperature, there can exist three kinds of black holes (the constant temperature line cuts through three points in the temperature profile as in Fig. 3). As previously mentioned that the mass M monotonically increases in r_+, then the r_+ axis can be also treated as an increasing M axis. The unstable small holes of higher temperature will evaporate away, while those of lower temperature will evolve into stable moderate-sized black holes as they receive thermal radiation from the system and gain mass, resulting in an increase in r_+. The unstable large black holes of higher temperature will shrink as they lose masses through their own thermal radiation and eventually evolve into the stable moderate-sized black holes. On the other hand, those of lower temperature will keep receiving thermal radiation from the system and finally evolve into the zero-temperature

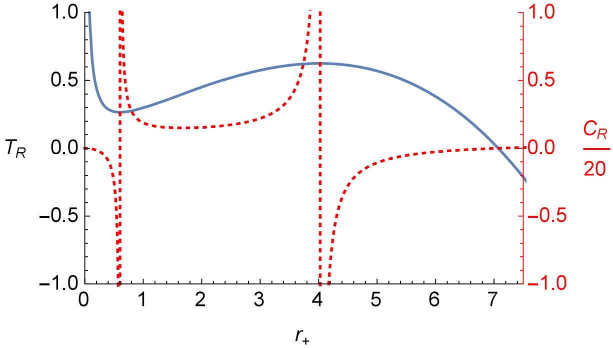

Fig. 2 Heat capacities corresponding to black holes of different radii (red dotted line) is shown in comparison with the temperature profile (blue line). The corresponding parameters are set as follows, $\lambda = 0.9$, $\Lambda = 0.02$

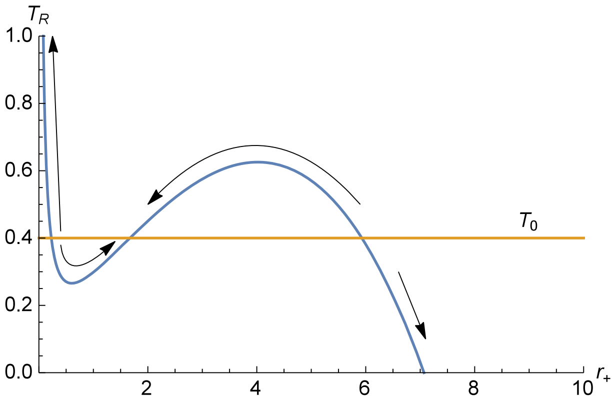

Fig. 3 Temperature profile corresponding to that of the Rényi entropy is shown here with the constant temperature T_0, indicating that there are three possible black holes. The arrows denote directions of evolution for each kind of black holes when surrounded by a system of temperature T_0. The corresponding parameters are set as follows, $\lambda = 0.9$, $\Lambda = 0.02$, $T_0 = 0.4$

black holes which are the largest possible holes. Note that these behaviours arise because the black holes in consideration have negative heat capacities.

The mentioned features arise because of the effect of the non-extensive parameter, λ. This gives rise to the two extrema in the Rényi temperature profile. There also exists a specific parameter set-up where the two extrema merge and become only one extremum; particularly, it satisfies the vanishing discriminant, or $(\lambda\pi - \Lambda)^2 - 12\Lambda\lambda\pi = 0$. In this case, $r_{+1}^2 = r_{+2}^2$ and the thermodynamically stable black hole does not exist because the corresponding heat capacity is always negative. This specific set of parameters can be viewed as a point of phase transition in a parameter space; namely, it defines a boundary in the parameter space separating the region in which there exists a locally thermodynamically stable black hole and the region of no possible thermodynamically stable black hole. This phase transition is shown graphically in Fig. 4 where various parameter set-ups are shown in comparison. Moreover, the allowed region corresponding to the existence of thermodynamically stable black holes can be illustrated in a $\lambda - r_+$ phase space as shown in Fig. 5.

Figure 5 shows that for a given parameter set-up, there is a minimum value of λ below which no thermodynamically stable black hole can be found. The minimum λ can be found

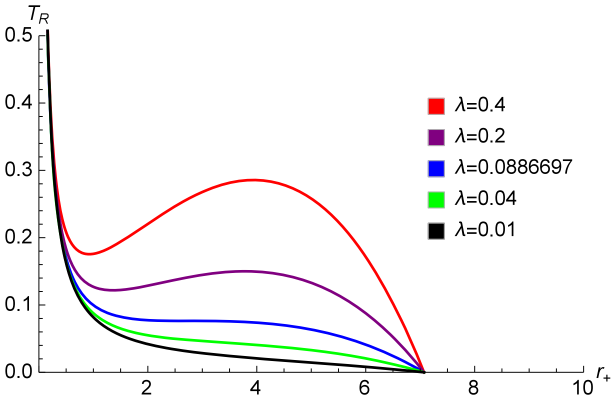

Fig. 4 Various Rényi temperature profiles corresponding to various λ as $\Lambda = 0.02$ are shown here. The blue graph corresponds to a parameter set which exhibits the phase transition between a phase of no possible thermally stable black hole into that of possible ones

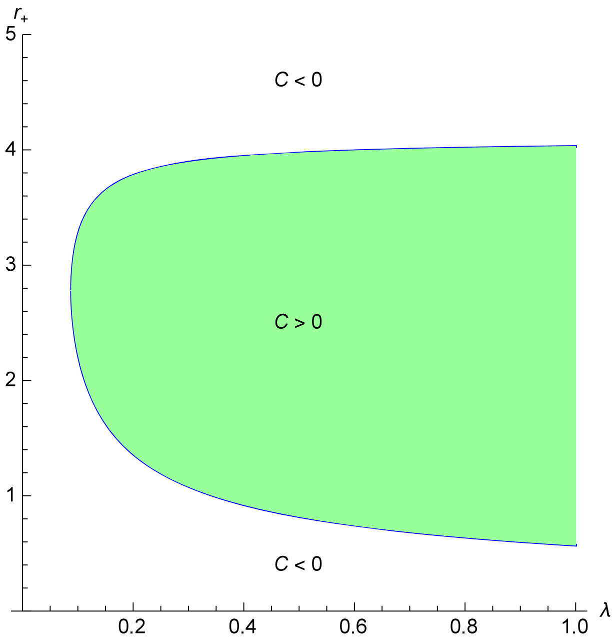

Fig. 5 Various configurations of (λ, r_+) are shown here given that $\Lambda = 0.02$. The lines denote boundaries between phases of different behaviours of the heat capacities, thus different thermodynamic stabilities, for each configuration of (λ, r_+). The green shaded region denotes the configurations corresponding to positive heat capacity, while the unshaded region corresponds to negative heat capacity configurations

analytically through Eq. (14). Since we require $\lambda\pi > \Lambda$ for the temperature to have the extrema on the positive side of r_+, then the condition for the phase transition point is that $\lambda\pi = \Lambda(7 + \sqrt{48})$ for a particular value of Λ.

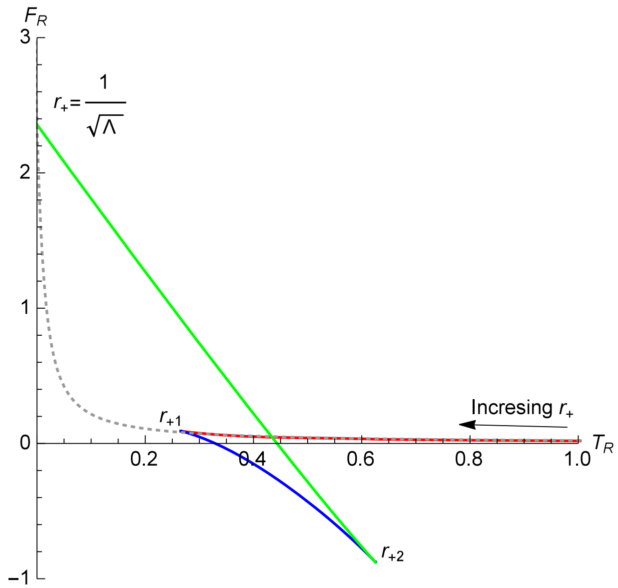

Fig. 6 Free energy profile according to each temperature is shown here. The parameter set-up is the following, $\lambda = 0.9, \Lambda = 0.02$. The graph shows the various values of free energy according to different possible configurations for a given temperature; the red line, blue line and green line correspond to configurations in which $0 < r_+ < r_{+1},\ r_{+1} < r_+ < r_{+2},$ and $r_{+2} < r_+ < \frac{1}{\sqrt{\Lambda}}$, respectively. The dotted grey line indicates the free energy profile when $\lambda = 0, \Lambda = 0.02$ according to which the Rényi free energy reduces into the Gibbs–Boltzmann free energy

3 Free energies

According to the unique feature of the Rényi temperature of the Sch–dS black hole, if the background temperature lies between the minimum and the maximum temperature, then there are at most three configurations of black hole, one of which is thermodynamically stable where the other two are not. This could lead one to think that there are three possible states of black hole which can be formed from hot gas in the spacetime of given temperature. These three states can be shown that they are not equally likely to happen by considering each of their corresponding free energies. By formulating thermodynamic quantities following the Rényi statistics, we can define the corresponding Helmholtz free energy as a Legendre transformation of the mass,

$$F_R = M - T_R S_R, \tag{18}$$

$$= \frac{r_+}{2}\left(1 - \frac{\Lambda r_+^2}{3}\right) - \frac{\left(1 + (\lambda\pi - \Lambda)\, r_+^2 - \Lambda\lambda\pi r_+^4\right)}{4\pi\lambda r_+} \ln\left(1 + \lambda\pi r_+^2\right). \tag{19}$$

A plot of the free energy against the temperature is plotted in Fig. 6. It can be seen that for a given temperature the free energy corresponding to the positive heat capacity black hole is the lowest, indicating that the thermodynamically stable black hole is preferred. This behaviour arises from the fact that the slope of the $F_R - T_R$ graph is negative Rényi entropy, specifically

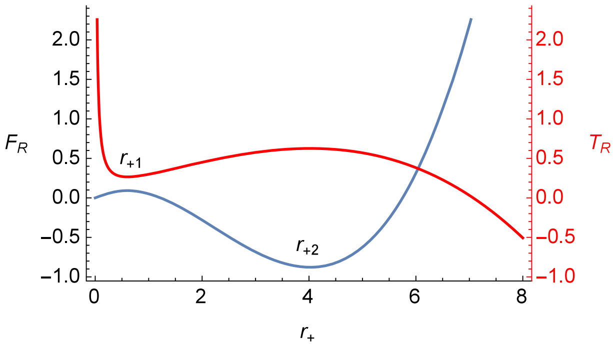

Fig. 7 Free energy profile according to each temperature is shown here. The parameter set-up is the following, $\lambda = 0.9$, $\Lambda = 0.02$. The graph shows the various values of free energy according to different possible configurations for a given temperature

$\frac{dF_R}{dT_R} = -S_R$. Since the entropy is an increasing function in r_+ then as r_+ increases, the entropy increases which also makes the slope more negative. If we trace the $F_R - T_R$ graph from $r_+ = 0$ to $r_+ = r_{+1}$ (the red line in Fig. 6), we will see that the graph between this range of r_+ results in increasing F_R but decreasing T_R and the slope will be more negative as r_+ is increasing. However, when we trace the graph from $r_+ = r_{+1}$ to $r_+ = r_{+2}$ (the blue line in Fig. 6), the graph is such that F_R decreases as T_R increases, but the slope still keeps being negatively steeper. Lastly, as we trace the graph from $r_+ = r_{+2}$ to $r_+ = \frac{1}{\sqrt{\Lambda}}$ (the green line in Fig. 6), F_R begins to increase again as T_R decreases and the slope is still negatively steeper. These behaviours result in the lowest values of free energy corresponding to the section of graph in the range between $r_+ = r_{+1}$ to $r_+ = r_{+2}$ and suggest that the black holes of sizes $r_{+1} < r_+ < r_{+2}$, the thermally stable ones, correspond to the lowest free energy at a given temperature and are more likely to form than the other configurations.

Given that a spacetime of consideration is at some certain temperature, the free energy of the hot spacetime itself is zero, meaning that in order for a black hole to form itself from the hot spacetime, the free energy of the hole must be negative. This can be realized by considering the local minimum of the free energy (because the maximum and the minimum of possible r_+ result in the free energies not less than zero, meaning that the global extrema of the free energy cannot be global minima.). From $dF_R = -S_R dT_R$, we know that the local maximum of the temperature is automatically the local minimum of the free energy, making r_{+2} corresponds to the local minimum of F_R as shown in Fig. 7.

Since r_{+2} satisfies Eq. (12), then we can use Eq. (12) to simplify Eq. (19) as follows,

$$F_R(r_{+2}) = \frac{1 + 8r_{+2}^2 + \epsilon r_{+2}^2 - \left(6 + 3(1-\epsilon) r_{+2}^2\right) \ln\left(1 + \lambda\pi r_{+2}^2\right)}{18 r_{+2}}, \tag{20}$$

where we have defined $\epsilon \equiv \frac{\Lambda}{\lambda\pi}$. By requiring that $F_R < 0$, then the numerator in Eq. (20) must be negative.

After substituting in the expression for r_{+2}^2, we have the following condition,

$$8 - \epsilon - \epsilon^2 + (\epsilon + 8)\sqrt{1 - 14\epsilon + \epsilon^2} < 3\left(1 + 10\epsilon + \epsilon^2 + (1-\epsilon)\sqrt{1 - 14\epsilon + \epsilon^2}\right) \ln\left(\frac{1 + 5\epsilon + \sqrt{1 - 14\epsilon + \epsilon^2}}{6\epsilon}\right). \tag{21}$$

In order to satisfy Eq. (21), it turns out numerically that we must have $\epsilon < 0.0328$. This bound is even stronger than the previous bound, *i.e.* $\epsilon \equiv \frac{\Lambda}{\lambda\pi} < \frac{1}{7+\sqrt{48}} \approx 0.0718$, required to have both minimum and maximum in the temperature profile. This suggests that for a given temperature, which lies between the minimum and the maximum temperature, the bound $\epsilon < 0.0328$ must be additionally satisfied in order to have a thermodynamically stable black hole as a preferred state in a hot spacetime.

4 Conclusions

In this article, we have studied thermodynamics of a Schwarzschild black hole in a de Sitter background in the approach of Rényi statistics. When ones study thermodynamic behaviours of black holes in the dS space, usually there are obstructions in realizing the system as a thermal one because generally there are two horizons, dubbed the black hole event horizon and the cosmic horizon. These two horizons both have their associated surface gravities and then their associated temperatures which generally are different. This results in thermal energy transfer between the two horizons and then the equilibrium thermodynamic treatment is not suitable for such a system. In this work, the thermodynamic system in consideration is assumed as follows. Firstly, since the temperatures of the two horizons are different from one another by an order of $\hbar$ which results in a large timescale of thermal energy transfer, the considered thermodynamic processes are assumed to occur with their timescale considerably much smaller than the timescale for the thermal energy transfer between the two horizons, i.e. the system is treated to be in "quasi-equilibrium". Secondly, in order to study thermodynamics of the black hole and its phase transition, the black hole system is then assumed to be in contact with an infinite reservoir which provides required energy for the black hole to be at a certain temperature and the timescale of this process is assumed to be much smaller than the timescale of the thermal energy transfer between the two horizons. These assumptions then allow us to treat a space with two horizons as a thermodynamic system in an equilibrium sense. The Sch–dS black hole is known to have a negative heat capacity when its thermodynamic quantities are evaluated on the hole's horizon based on the Gibbs–Boltzmann statistics. The possible reason behind this may be that the Gibbs–Boltzmann statistics is not applicable to non-extensive systems such as black holes. One of the approaches for treating non-extensive systems is to consider thermodynamics based on Rényi statistics. When the black hole is considered in the language of Rényi statistics, compared with that in Gibbs–Boltzmann statistics, the corresponding temperature profile shows an increasing temperature for a specific range of black hole radii, indicating an existence of positive heat capacity which can be seen as an example in Figs. 1 and 2. This interesting behaviour only appears when the non-extensive parameter λ and the cosmological constant Λ obey the following bound: $\epsilon \equiv \frac{\Lambda}{\lambda\pi} < \frac{1}{7+\sqrt{48}} \approx 0.0718$. For a certain value of background temperature, there exists a temperature such that it lies between the local minimum and the local maximum temperatures, resulting in three possible configurations of thermal black hole as shown in Fig. 3. In particular, the configuration with the moderate black hole radius is thermodynamically stable, while the other two, namely those with the smaller and bigger radii, are unstable. According to our findings, the Rényi statistics of the Sch–dS black hole suggests that the smaller holes can either evaporate away if the holes have higher temperature than that of the background or evolve into the moderate-sized stable black holes if they have lower temperature than that of the background. For the bigger holes, they can either evolve into the moderate-sized stable ones if their temperatures are higher than that of the background or evolve into zero-temperature black holes which are the largest possible holes given that black holes of negative temperature do not exist.

From the free energy analysis, these three possible configurations correspond to different values of free energies, suggesting that these three configurations are preferred differently for a thermal background of some certain temperature. From the previous analysis on their free energies, the moderate-sized stable black hole which is in thermal equilibrium with the background appears to be the preferred configuration compared with the other two (unstable) configurations as shown in Fig. 6. Furthermore, in order for a hot spacetime to evolve itself into a spacetime with a thermal Sch–dS black hole, the free energy of the preferred configuration of the black hole must be lower than that of the hot spacetime, which is zero. Requiring that the free energy of the preferred black hole is less than zero, a stronger bound of the parameter set-up has been found to satisfy Eq. (21) or numerically $\epsilon < 0.0328$. This bound might suggest that if the real black holes were to be explained by Rényi statistics, then the Rényi non-extensive parameter, λ, would be possibly determined by the cosmological constant, Λ.

According to the previously mentioned analyses, we found that based on Rényi statistics, it is possible to have thermodynamically stable black holes in dS background if non-extensivity is taken into account. Moreover, the non-extensive parameter λ behaves as if it produces thermal effects usually seen on black holes in AdS background even though the actual background is not AdS. In particular, it can be seen from the second term in Eq. (10) that the non-extensive parameter λ seems to compete with the cosmological constant Λ since they are of opposite signs. Thanks to these features, this study may give us possibilities in describing black holes in our universe in the context of their thermodynamic stabilities or even broadening the research field towards AdS/CFT correspondence.

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Appendix

Zeroth law compatibility

In Sect. 2, the Rényi entropy is considered as thermodynamic entropy of the black hole system instead of the Tsallis entropy. The reason behind this is that the Tsallis entropy which obeys the following composition,

$$S_{12} = S_1 + S_2 + \lambda S_1 S_2, \tag{22}$$

is not compatible with the zeroth law of thermodynamics which is crucial in defining temperature of a thermal system. In order to see this, let us consider a thermal isolated system with constant entropy, S. We can always consider this system as being composed of two weakly interacting subsystems, each of which has entropy of S_1 and S_2. Moreover, let us assume that the entropy composition rule is that of Gibbs–Boltzmann statistics as follows,

$$S = S_1 + S_2. \tag{23}$$

If the subsystems are allowed to exchange only thermal energy to one another, since the whole system is isolated, then according to the conservation of energy, the thermal energy gained by the subsystem 1 must be from the subsystem 2,

$$T_1 dS_1 = -T_2 dS_2. \tag{24}$$

If the system enter a thermal equilibrium, $T_1 = T_2$, then we obtain $dS_1 + dS_2 = d\left(S_1 + S_2\right) = dS = 0$ which implies that there is no change in entropy which is reasonable for an isolate system. This statement is true only in the context of the Gibbs–Boltzmann entropy which obeys $S = S_1 + S_2$. On the other hand, when the Tsallis entropy is considered instead, the above analysis will imply a change in total entropy even when an isolated system is assumed, showing a contradiction between the Tsallis entropy and the zeroth law of thermodynamics. The more detailed investigation is reported in Ref. [14]. It is also worth mentioning that there is an approach which

On the thermal equilibrium between the two horizons

Although in the article the thermodynamic behaviour of the black hole event horizon is treated separately with those of the cosmic horizon, there is a possibility for the two horizons to be in thermal equilibrium in the language of Rényi statistics. The Gibbs–Boltzmann temperatures of both horizons can be found through their respective surface gravities, κ as follows,

$$T_+ = \frac{\kappa_+}{2\pi}, \quad T_c = \frac{\kappa_c}{2\pi}, \tag{25}$$

where the subscripts $+$ and c denote the quantity being evaluated at the black hole event horizon and at the cosmic horizon, respectively. Their corresponding Rényi temperatures are, after substituting the surface gravities,

$$T_{\mathrm{R},+} = \frac{1}{4\pi r_+}\left(1 - \Lambda r_+^2\right)\left(1 + \lambda\pi r_+^2\right), \tag{26}$$

$$T_{\mathrm{R},c} = \frac{1}{4\pi r_c}\left(\Lambda r_c^2 - 1\right)\left(1 + \lambda\pi r_c^2\right). \tag{27}$$

Note that the sign of $T_{\mathrm{R},c}$ is flipped because only the magnitude of the cosmic surface gravity is considered. In the situation where the two horizons are present, we must have $9M^2\Lambda < 1$ (where $9M^2\Lambda = 1$ implies an extremal case) and the black hole event horizon and the cosmic horizon can be expressed explicitly as follows,

$$r_+ = \frac{2}{\sqrt{\Lambda}}\cos\left[\frac{\arctan\left(\sqrt{\frac{1}{9M^2\Lambda} - 1}\right) + \pi}{3}\right], \tag{28}$$

$$r_c = \frac{2}{\sqrt{\Lambda}}\cos\left[\frac{\arctan\left(\sqrt{\frac{1}{9M^2\Lambda} - 1}\right) - \pi}{3}\right], \tag{29}$$

which are functions of M and Λ. Equating Eq. (26) with Eq. (27) allows us to find a suitable value of λ as a function of Λ and M so that the two horizons are in thermal equilibrium. Some of the possible configuration are given in Fig. 8 for specific values of Λ.

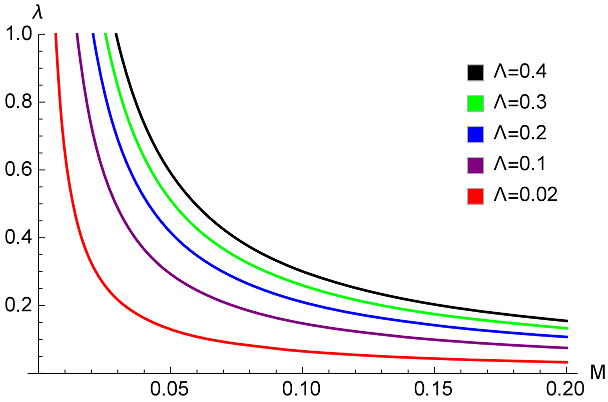

Fig. 8 Possible (λ, M) configurations when thermal equilibrium between the black hole event horizon and the cosmic horizon is assumed are given for specific values of Λ

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