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1 Gravitational Perturbation Theory

Small Fluctuations

As an essential feature of the analysis presented here, we assume that during most of the history of the universe all departures from homogeneity and isotropy have been small, so that they can be treated as first-order perturbations. Because the observable universe is nearly homogeneous, and its spatial curvature either vanishes or is negligible until very near the present, we will take the unperturbed metric to have the Robertson-Walker form with curvature constant $K = 0$,

$$\bar{g}_{00} = -1, \quad \bar{g}_{i0} = \bar{g}_{0i} = 0, \qquad \bar{g}_{ij} = a^2(t)\,\delta_{ij}, \tag{1.1}$$

where $i, j = 1, 2, 3$ for spatial dimensions and δ_{ij} is the Kronecker delta. The total perturbed metric is then

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{1.2}$$

where $\bar{g}_{\mu\nu}$ is the unperturbed $K = 0$ Robertson-Walker metric and $h_{\mu\nu} = h_{\nu\mu}$ is a small torsionless perturbation. We also have the inverse metric as

$$\bar{g}^{\,00} = -1, \quad \bar{g}^{\,i0} = \bar{g}^{\,0i} = 0, \qquad \bar{g}^{\,ij} = a^{-2}(t)\,\delta^{ij}. \tag{1.3}$$

The inverse of the metric is perturbed by

$$h^{\mu\nu} \equiv g^{\mu\nu} - \bar{g}^{\,\mu\nu} = -\bar{g}^{\,\mu\rho}\,\bar{g}^{\,\nu\sigma}\,h_{\rho\sigma}, \tag{1.4}$$

with components

$$h^{ij} = -a^{-4}h_{ij}, \quad h^{i0} = a^{-2}h_{i0}, \quad h^{00} = -h_{00}. \tag{1.5}$$

Note the $-$ sign in Eq. (1.4) (Exercise); in our notation, the perturbation $\delta g^{\mu\nu}$ to $g^{\mu\nu}$ is *not* given by simply using the unperturbed metric to raise the indices on $\delta g_{\mu\nu}$.

Field Equations

The metric perturbation produces a perturbation to the affine connection

$$\delta\,\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}\,\bar{g}^{\,\mu\rho}\left[-2h_{\rho\sigma}\bar{\Gamma}^{\,\sigma}_{\,\nu\lambda} + \partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\lambda\nu}\right]. \tag{1.6}$$

For $K = 0$, the only non-vanishing components of the unperturbed affine connection are given by

$$\bar{\Gamma}^{\,i}_{\,j0} = \bar{\Gamma}^{\,i}_{\,0j} = \frac{\dot{a}}{a}\,\delta_{ij}, \qquad \bar{\Gamma}^{\,0}_{\,ij} = a\dot{a}\,\delta_{ij}. \tag{1.7}$$

Thus Eq. (1.6) gives the components of the perturbed affine connection as

$$\delta\,\Gamma^{i}_{jk} = \frac{1}{2a^2}\left(-2a\dot{a}\,h_{i0}\,\delta_{jk} + \partial_k h_{ij} + \partial_j h_{ik} - \partial_i h_{jk}\right) \tag{1.8}$$

$$\delta\,\Gamma^{i}_{j0} = \frac{1}{2a^2}\left(-\frac{2\dot{a}}{a}\,h_{ij} + \dot{h}_{ij} + \partial_j h_{i0} - \partial_i h_{j0}\right) \tag{1.9}$$

$$\delta\,\Gamma^{0}_{ij} = \frac{1}{2}\left(2a\dot{a}\,\delta_{ij}\,h_{00} - \partial_j h_{i0} - \partial_i h_{j0} + \dot{h}_{ij}\right) \tag{1.10}$$

$$\delta\Gamma^{i}_{00} = \frac{1}{2a^2}\left(2\dot{h}_{i0} - \partial_i h_{00}\right) \tag{1.11}$$

$$\delta\Gamma^{0}_{i0} = \frac{\dot{a}}{a}h_{i0} - \frac{1}{2}\partial_i h_{00} \tag{1.12}$$

$$\delta\Gamma^{0}_{00} = -\frac{1}{2}\dot{h}_{00} \tag{1.13}$$

In particular, we will need

$$\delta\Gamma^{\lambda}_{\lambda\mu} = \delta_\mu\left[\frac{1}{2a^2}h_{ii} - \frac{1}{2}h_{00}\right]. \tag{1.14}$$

To write the Einstein equations, we need the perturbation to the Ricci tensor

$$\delta R_{\mu\kappa} = \frac{\partial\delta\Gamma^{\lambda}_{\mu\kappa}}{\partial x^\lambda} - \frac{\partial\delta\Gamma^{\lambda}_{\mu\lambda}}{\partial x^\kappa} + \delta\Gamma^{\eta}_{\mu\kappa}\bar{\Gamma}^{\nu}_{\nu\eta} + \delta\Gamma^{\nu}_{\nu\eta}\bar{\Gamma}^{\eta}_{\mu\kappa} - \delta\Gamma^{\eta}_{\mu\nu}\bar{\Gamma}^{\nu}_{\kappa\eta} - \delta\Gamma^{\nu}_{\kappa\eta}\bar{\Gamma}^{\eta}_{\mu\nu}, \tag{1.15}$$

with components

$$\begin{aligned}\delta R_{jk} = {} & \frac{1}{2}\partial_j\partial_k h_{00} + \left(2\dot{a}^2 + a\ddot{a}\right)\delta_{jk}h_{00} + \frac{1}{2}a\dot{a}\,\delta_{jk}\dot{h}_{00} - \frac{1}{2}\left(\partial_j\dot{h}_{k0} + \partial_k\dot{h}_{j0}\right) \\ & -\frac{1}{2a^2}\left(\nabla^2 h_{jk} - \partial_i\partial_j h_{ik} - \partial_i\partial_k h_{ij} + \partial_j\partial_k h_{ii}\right) - \frac{\dot{a}}{2a}\left(\partial_j h_{k0} + \partial_k h_{j0}\right) \\ & +\frac{1}{2}\ddot{h}_{jk} - \frac{\dot{a}}{2a}\left(\dot{h}_{jk} - \delta_{jk}\dot{h}_{ii}\right) - \frac{\dot{a}^2}{a^2}\left(-2h_{jk} + \delta_{jk}h_{ii}\right) - \frac{\dot{a}}{a}\delta_{jk}\partial_i h_{i0},\end{aligned} \tag{1.16}$$

$$\begin{aligned}\delta R_{0j} = {} & \delta R_{j0} = -\frac{\dot{a}}{a}\partial_j h_{00} - \frac{1}{2a^2}\left(\nabla^2 h_{j0} - \partial_j\partial_i h_{i0}\right) + \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right)h_{j0} \\ & -\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{1}{a^2}\left(\partial_j h_{kk} - \partial_k h_{kj}\right)\right],\end{aligned} \tag{1.17}$$

$$\delta R_{00} = -\frac{1}{2a^2}\nabla^2 h_{00} - \frac{3\dot{a}}{2a}\dot{h}_{00} + \frac{1}{a^2}\partial_i\dot{h}_{i0} - \frac{1}{2a^2}\left[\ddot{h}_{ii} - \frac{2\dot{a}}{a}\dot{h}_{ii} + 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right)h_{ii}\right]. \tag{1.18}$$

In general, we can put the Einstein field equations in the form

$$R_{\mu\nu} = 8\pi G\,S_{\mu\nu}, \tag{1.19}$$

where

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}T_{\rho\sigma}. \tag{1.20}$$

the perturbation to the energy-momentum tensor and metric produces a perturbation to the source tensor $S_{\mu\nu}$:

$$\delta S_{\mu\nu} = \delta T_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\delta T^{\lambda}_{\ \lambda} - \frac{1}{2}h_{\mu\nu}\bar{T}^{\lambda}_{\ \lambda}. \tag{1.21}$$

We are not assuming that the contents of the universe form a perfect fluid, but the rotational and translational invariance of the *unperturbed* energy-momentum tensor $\bar{T}^{\mu\nu}$ require that it takes the perfect fluid form:

$$\bar{T}_{\mu\nu} = \bar{p}\,\bar{g}_{\mu\nu} + (\bar{p} + \bar{\rho})\,\bar{u}_\mu\bar{u}_\nu\,, \tag{1.22}$$

where $\bar{\rho}(t)$, $\bar{p}(t)$, and $\bar{u}^{\mu}$ are the unperturbed energy density, pressure, and velocity four-vector, respectively, with $\bar{u}^{0} = 1$ and $\bar{u}^{i} = 0$. Also, we use the unperturbed Einstein equation to write $\bar{\rho}$ and $\bar{p}$ in terms of the Robertson-Walker scale factor and its derivatives

$$\bar{\rho} = \frac{3}{8\pi G}\left(\frac{\dot{a}^2}{a^2}\right), \qquad \bar{p} = -\frac{1}{8\pi G}\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{1.23}$$

It follows in particular that the unperturbed energy-momentum tensor has the trace

$$\bar{T}^{\lambda}{}_{\lambda} = 3\bar{p} - \bar{\rho} = -\frac{3}{4\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{1.24}$$

Hence,

$$\delta S_{jk} = \delta T_{jk} - \frac{a^2}{2}\delta_{jk}T^{\lambda}_{\lambda} + \frac{3}{8\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{jk}, \tag{1.25}$$

$$\delta S_{j0} = \delta T_{j0} + \frac{3}{8\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{j0}, \tag{1.26}$$

$$\delta S_{00} = \delta T_{00} + \frac{1}{2}\delta T^{\lambda}_{\lambda} + \frac{3}{8\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{00}. \tag{1.27}$$

The Einstein equations thus take the form

$$\begin{aligned} 8\pi G\left(\delta T_{jk} - \frac{a^2}{2}\delta_{jk}\,\delta T^{\lambda}_{\lambda}\right) &= \frac{1}{2}\partial_j\partial_k h_{00} + \left(2\dot{a}^2 + a\ddot{a}\right)\delta_{jk}h_{00} + \frac{1}{2}a\dot{a}\,\delta_{jk}\dot{h}_{00} + \frac{1}{2}\ddot{h}_{jk} - \frac{\dot{a}}{2a}\left(\dot{h}_{jk} - \delta_{jk}\dot{h}_{ii}\right) \\ &\quad -\frac{1}{2a^2}\left(\nabla^2 h_{jk} - \partial_i\partial_j h_{ik} - \partial_i\partial_k h_{ij} + \partial_j\partial_k h_{ii}\right) - \frac{\dot{a}}{2a}\left(\partial_j h_{k0} + \partial_k h_{j0}\right) \\ &\quad -\left(\frac{\dot{a}^2}{a^2}\right)\delta_{jk}h_{ii} - \frac{\dot{a}}{a}\delta_{jk}\partial_i h_{i0} - \frac{1}{2}\left(\partial_j\dot{h}_{k0} + \partial_k\dot{h}_{j0}\right) - \left(\frac{\dot{a}^2}{a^2} + \frac{3\ddot{a}}{a}\right)h_{jk}, \end{aligned} \tag{1.28}$$

$$\begin{aligned} 8\pi G\delta T_{j0} &= -\frac{\dot{a}}{a}\partial_j h_{00} - \frac{1}{2a^2}\left(\nabla^2 h_{j0} - \partial_j\partial_i h_{i0}\right) - \left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)h_{j0} \\ &\quad -\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{1}{a^2}\left(\partial_j h_{kk} - \partial_k h_{kj}\right)\right], \end{aligned} \tag{1.29}$$

$$\begin{aligned} 8\pi G\left(\delta T_{00} + \frac{1}{2}\delta T^{\lambda}_{\lambda}\right) &= -\frac{1}{2a^2}\nabla^2 h_{00} - \frac{3\dot{a}}{2a}\dot{h}_{00} + \frac{1}{a^2}\partial_i\dot{h}_{i0} \\ &\quad -\frac{1}{2a^2}\left[\ddot{h}_{ii} - \frac{2\dot{a}}{a}\dot{h}_{ii} + 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right)h_{ii}\right] - 3\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)h_{00}. \end{aligned} \tag{1.30}$$

The component of the energy momentum tensor are subject to the conservation condition that $T^{\mu}_{\nu;\mu} = 0$, which to first order in perturbations gives

$$\partial_\mu\delta T^{\mu}_{\nu} + \bar{\Gamma}^{\mu}_{\mu\lambda}\delta T^{\lambda}_{\nu} - \bar{\Gamma}^{\lambda}_{\mu\nu}\delta T^{\mu}_{\lambda} + \delta\Gamma^{\mu}_{\mu\lambda}\bar{T}^{\lambda}{}_{\nu} - \delta\Gamma^{\lambda}_{\mu\nu}\bar{T}^{\mu}{}_{\lambda} = 0, \tag{1.31}$$

in which the perturbations to the energy-momentum tensor δT^{μ}_{ν} with mixed indices can be calculated from

$$\delta T^{\mu}_{\nu} = \bar{g}^{\mu\lambda}\left[\delta T_{\lambda\nu} - h_{\lambda\kappa}\overline{T}^{\kappa}{}_{\nu}\right]. \tag{1.32}$$

Setting ν equal to a spatial coordinate index j gives the equation of momentum conservation

$$\partial_0 \delta T^0_j + \partial_i \delta T^i{}_j + \frac{2\dot{a}}{a}\delta T^0_j - a\dot{a}\delta T^j{}_0 - (\bar{\rho} + \bar{p})\left(\frac{1}{2}\partial_j h_{00} - \frac{\dot{a}}{a}h_{j0}\right) = 0, \tag{1.33}$$

while setting ν equal to the time coordinate index 0 gives the equation of energy conservation

$$\partial_0 \delta T^0_0 + \partial_i \delta T^i_0 + \frac{3\dot{a}}{a}\delta T^0_j - \frac{\dot{a}}{a}\delta T^i{}_i - \left(\frac{\bar{\rho} + \bar{p}}{2a^2}\right)\left(-\frac{2\dot{a}}{a}h_{ii} + \dot{h}_{ii}\right) = 0. \tag{1.34}$$

These conversion equations are not independent conditions, but may be derived from the Einstein field equations. However, it is often convenient to use either or both in place of one or two of the field equations. Also, in the frequently encountered case where the constituents of the universe are non-interacting fluids (such as one fluid consisting of cold dark matter and another consisting of ordinary matter and radiation) these conservation equations are satisfied *separately* by each fluid, information that could not be derived from the field equations.

The results obtained so far are repulsively complicated. Fortunately, the spatial isotropy and homogeneity of the unperturbed metric and energy-momentum tensor allow us to simplify these results by decomposing the perturbations into scalars, divergenceless vectors, and divergenceless traceless symmetric tensors, which are not coupled to each other by the field equations or conservation equations. The perturbation to the metric can always be out in the form

$$h_{00} = -E, \tag{1.35}$$

$$h_{i0} = a\left[\frac{\partial F}{\partial x^i} + G_i\right], \tag{1.36}$$

$$h_{ij} = a^2\left[A\delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j} + \frac{\partial C_i}{\partial x^j} + \frac{\partial C_j}{\partial x^i} + D_{ij}\right], \tag{1.37}$$

where the perturbations A, B, C_i, $D_{ij} = D_{ji}$, E, F, and G_i are functions of $\mathbf{x}$ and t, satisfying the conditions

$$\frac{\partial C_i}{\partial x^i} = \frac{\partial G_i}{\partial x^i} = 0, \quad \frac{\partial D_{ij}}{\partial x^i} = 0, \quad D_{ii} = 0. \tag{1.38}$$

To carry out a similar decomposition of the energy-momentum tensor, we first note that for a perfect fluid we would have

$$T_{\mu\nu} = p g_{\mu\nu} + (\rho + p)\, u_\mu u_\nu, \tag{1.39}$$

with

$$g^{\mu\nu} u_\mu u_\nu = -1, \tag{1.40}$$

Recalling that $\bar{u}_i = 0$ and $\bar{u}_0 = -1$, we find that the normalization condition Eq. (1.40) gives

$$\delta u^0 = \delta u_0 = \frac{h_{00}}{2}, \tag{1.41}$$

while δu_i is an independent dynamical variable. (Note that $\delta u^\mu \equiv \delta\left(g^{\mu\nu}u_\nu\right)$ is not given by $\bar{g}^{\mu\nu}\delta u_\nu$.) Then the first-order perturbation to the energy-momentum tensor for a perfect fluid is

$$\delta T_{ij} = \bar{p}\,h_{ij} + a^2\delta_{ij}\delta p, \quad \delta T_{i0} = \bar{p}h_{i0} - (\bar{\rho} + \bar{p})\,\delta u_i, \quad \delta T_{00} = -\bar{\rho}h_{00} + \delta\rho. \tag{1.42}$$

More generally, we can always put the perturbed energy-momentum tensor in a form like that of the perturbed metric. In general, we define $\delta\rho$ just as for a perfect fluid, as the difference between δT_{00} and $-\bar{\rho}h_{00}$, but $\delta\rho$ is *not* necessarily given by varying the temperature and chemical potentials in the formula for ρ that applies in thermal equilibrium. Also, in general we define the velocity perturbation δu_i times $\bar{p} + \bar{\rho}$ as for a perfect fluid, as the difference between $-\delta T_{i0}$ and $\bar{p}h_{i0}$, and we decompose δu_i into the gradient of a scalar velocity potential δu and a divergenceless vector δu_i^V. Finally, ew define $a^2\delta p$ as the coefficient of δ_{ij} in the difference between δT_{ij} and $\bar{p}\delta_{ij}$, again without assuming that δp is given by varying the temperature and chemical potentials in the formula for p that applies in thermal equilibrium. The other terms in δT_{ij}, denoted $\partial_i\partial_j\pi^S$, $\partial_i\pi_j^V + \partial_j\pi_i^V$, and π_{ij}^T, represent dissipative corrections to the inertia tensor. That is, we write

$$\delta T_{ij} = \bar{p}\,h_{ij} + a^2\left[\delta_{ij}\delta p + \partial_i\partial_j\pi^S + \partial_i\pi_j^V + \partial_j\pi_i^V + \pi_{ij}^T\right], \tag{1.43}$$

$$\delta T_{i0} = \bar{p}\,h_{i0} - (\bar{\rho} + \bar{p})\left(\partial_i\delta u + \delta u_i^V\right), \tag{1.44}$$

$$\delta T_{00} = -\bar{\rho}\,h_{00} + \delta\rho, \tag{1.45}$$

where π_i^V, π_{ij}^T, and δu_i^V satisfy conditions analogous to the conditions Eq. (1.38) satisfied by C_i, D_{ij}, and G_i:

$$\partial_i\pi_i^V = \partial_i\delta u_i^V = 0, \quad \partial_i\pi_{ij}^T = 0, \quad \pi_{ii}^T = 0. \tag{1.46}$$

To repeat, these formulas can be taken as a definition of the quantities $\delta\rho$, δp, and $\delta u_i \equiv \partial_i\delta u + \delta u_i^V$, as well as of the *anisotropic inertia* terms π^S, π^V, and π^T, which characterize departures from the perfect fluid form of the energy-momentum tensor. The perturbed mixed components (1.32) of energy-momentum tensor, which are needed in the conservation laws,

$$\delta T^i{}_j = \delta_{ij}\,\delta p + \partial_i\partial_j\pi^S + \partial_i\pi_j^V + \partial_j\pi_i^V + \partial\pi_{ij}^T, \tag{1.47}$$

$$\delta T^i_0 = a^{-2}\,(\bar{\rho} + \bar{p})\left(a\partial_i F + aG_i - \partial_i\delta u - \delta u_i^V\right) \tag{1.48}$$

$$\delta T^0_i = (\bar{\rho} + \bar{p})\left(\partial_i\delta u + \delta u_i^V\right), \quad \delta T^0_0 = -\delta\rho, \tag{1.49}$$

$$\delta T^\lambda_\lambda = 3\delta p - \delta\rho + \nabla^2\pi^S. \tag{1.50}$$

With these decompositions, and again using Eq. (1.23), the Einstein field equations (1.28)–(1.29) and conservation equations (1.33) and (1.34) fall into three classes of coupled equations:

Scalar (Compressional) Modes

These are the most complicated, involving the eight scalars E, F, A, B, $\delta\rho$, δP, π^S, and δu. The part of Eq. (1.28) proportional to δ_{jk} gives

$$-4\pi G a^2\left[\delta\rho - \delta p - \nabla^2\pi^S\right] = \frac{1}{2}a\dot{a}\dot{E} + \left(2\dot{a}^2 + a\ddot{a}\right)E + \frac{1}{2}\nabla^2 A - \frac{1}{2}a^2\ddot{A}$$
$$-3a\dot{a}\dot{A} - \frac{1}{2}a\dot{a}\nabla^2\dot{B} + \dot{a}\nabla^2 F. \tag{1.51}$$

The part of Eq. (1.28) of the form $\partial_j\partial_k S$ (where S is any scalar) gives

$$\partial_j\partial_k\left[16\pi G a^2\pi^S + E + A - a^2\ddot{B} - 3a\dot{a}\dot{B} + 2a\dot{F} + 4\dot{a}F\right] = 0. \tag{1.52}$$

The part of Eq. (1.29) of the form $\partial_j S$ (where S is again any scalar) gives

$$8\pi G a\left(\bar{\rho} + \bar{p}\right)\partial_j\delta u = -\dot{a}\partial_j E + a\partial_j\dot{A}. \tag{1.53}$$

Eq. (1.30) gives

$$-4\pi G\left(\delta\rho + 3\delta p + \nabla^2\pi^S\right) = -\frac{1}{2a^2}\nabla^2 E - \frac{3\dot{a}}{2a}\dot{E} - \frac{1}{a}\nabla^2\dot{F} - \frac{\dot{a}}{a^2}\nabla^2 F + \frac{3}{2}\ddot{A}$$
$$+\frac{3\dot{a}}{a}\dot{A} - \frac{3\ddot{a}}{a}E - \frac{1}{2}\nabla^2\ddot{B} + \frac{\dot{a}}{a}\nabla^2\dot{B}. \tag{1.54}$$

The part of the momentum conservation condition (1.33) that is a derivative of ∂_j is

$$\partial_j\left[\delta p + \nabla^2\pi^S + \partial_0\left[(\bar{\rho} + \bar{p})\delta u\right] + \frac{3\dot{a}}{a}\left(\bar{\rho} + \bar{p}\right)\delta u + \frac{1}{2}\left(\bar{\rho} + \bar{p}\right)E\right] = 0, \tag{1.55}$$

and the energy-conservation condition (1.34) is

$$\delta\dot{\rho} + \frac{3\dot{a}}{a}\left(\delta\rho + \delta p\right) + \nabla^2\left[-a^{-1}\left(\bar{\rho} + \bar{p}\right)F + a^{-2}\left(\bar{\rho} + \bar{p}\right)\delta u + \frac{\dot{a}}{a}\pi^S\right]$$
$$+ \frac{1}{2}\left(\bar{\rho} + \bar{p}\right)\partial_0\left[3A + \nabla^2 B\right] = 0. \tag{1.56}$$

In Eqs. (1.55) and (1.56), $\delta\rho$, δp, and π^S are elements of the perturbation to the total energy-momentum tensor, but the same equations apply to each constituent of the universe that does not exchange energy and momentum with other constituents.

Vector (Vortical) Modes

These involve the four divergenceless vectors G_i, C_i, δu_i^V, and π_i^V. The part of Eq. (1.28) of the form $\partial_k V_j$ (where V_j is any vector satisfying $\partial_j V_j = 0$) gives

$$\partial_k\left[16\pi G a^2\pi_j^V - a^2\ddot{C}_j - 3a\dot{a}\dot{C}_j + a\dot{G}_j + 2\dot{a}G_j\right] = 0, \tag{1.57}$$

while the part of Eq. (1.29) of the form V_j (where V_j is again any vector satisfying $\partial_j V_j = 0$) gives

$$8\pi G\left(\bar{\rho} + \bar{p}\right)a\delta u_j^V = \frac{1}{2}\nabla^2 G_j - \frac{a}{2}\nabla^2\dot{C}_j. \tag{1.58}$$

The part of the momentum conservation equation (1.33) that takes the form of a divergenceless vector is

$$\nabla^2 \pi_j^V + \partial_0 \left[(\bar{\rho} + \bar{p}) \, \delta u_j^V \right] + \frac{3\dot{a}}{a} (\bar{\rho} + \bar{p}) \, \delta u_j^V = 0, \tag{1.59}$$

In particular, for a perfect fluid $\pi_i^V = 0$, and Eq. (1.59) tells us that $(\bar{\rho} + \bar{p}) \, \delta u_j^V$ decays as $1/a^3$. In this case, both Eqs. (1.57) and (1.58) imply that the quantity $G_j - a\dot{C}_j$ decays as $1/a^2$. Because they decay, vector modes have not played a large role in cosmology.

Tensor (Rediative) Modes

These involve only the two traceless divergenceless symmetric tensor D_{ij} and π_{ij}^T. There is only one field equation here: the part of Eq. (1.28) of the form of a divergenceless traceless tensor is the wave equation for gravitational radiation

$$-16\pi G a^2 \pi_{ij}^T = \nabla^2 D_{ij} - a^2 \ddot{D}_{ij} - 3a\dot{a}\dot{D}_{ij}. \tag{1.60}$$

The above equations for scalar, vector, and tensor perturbations do not form a complete set. This is in part because we still have the freedom to make changes in the coordinate system, of the same order as the physical perturbations. In the next section, we will see how to remove this freedom by a choice of "gauge."

But even after the gauge has been fixed, the equations for the scalar modes will still not form a complete set, unless the pressure p and anisotropic inertia π^S can be expressed as functions of the energy density ρ. The pressure in thermal equilibrium can usually be expressed as a function of ρ and one or more number densities n that satisfy the condition that the current n^μ is conserved

$$(nu^\mu)_{;\mu} = 0. \tag{1.61}$$

This condition tells us that the unperturbed number density satisfied $\bar{n} \propto a^{-3}$, while the perturbation satisfies

$$\frac{\partial}{\partial t} \left(\frac{\delta n}{\bar{n}} \right) + \frac{1}{a^2} \nabla^2 \delta u + \frac{1}{2} \left(3\dot{A} + \nabla^2 \dot{B} \right) - \frac{1}{a} \nabla^2 F = 0. \tag{1.62}$$

With p given as a function of ρ and n, after gauge fixing the field equations and conservation equations (1.55), (1.56), and (1.62) form a complete set of equations for the scalar modes. Similarly, even after gauge fixing, the equations for vector and tensor modes do not form a complete set unless we have formulas for π_i^V and π_{ij}^T, respectively. This is no problem for perfect fluid, for which $\pi_i^V = \pi_{ij}^T = 0$. In the general case local thermal equilibrium is not maintained, and we must calculate $\delta\rho$, δp, π^S, π_i^V and π_{ij}^T by following changes in the distribution of individual particle positions and momenta, which are governed by Boltzmann equations.

2 Gravitational Gauge Transformation

The equations derived in Section 1 have two unsatisfactory features. First, even with the simplifications introduced by decomposing the equations into scalar, vector, and tensor modes, the equations for the scalar modes are still fearsomely complicated. Second, among the solutions of these equations are unphysical scalar and vector modes, corresponding to a mere coordinate transformation of unperturbed Robertson-Walker metric and energy-momentum tensor. We can eliminate the second problem and ameliorate the first by fixing the coordinate system, adopting suitable conditions on the full perturbed metric or energy-momentum tensor. Consider a spacetime coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{2.1}$$

with $\epsilon^\mu(x)$ small in the same sense that $h_{\mu\nu}$, $\delta\rho$, and other pertubations are small. Under this transformation, the metric tensor will be transformed to

$$g'_{\mu\nu}(x') = g_{\lambda\kappa}(x)\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\kappa}{\partial x'^\nu}. \tag{2.2}$$

Instead of working with such transformations, which affect the coordinates and unperturbed fields as well as the perturbations to the fields, it is more convenient to work with so-called *gauge transformations*, which affect only the field perturbations. For this purpose, after making the coordinate transformation (2.1), we relabel coordinates by dropping the prime on the coordinate argument, and we attribute the whole change in $g_{\mu\nu}(x)$ to a change in the perturbation $h_{\mu\nu}(x)$. The field equations should thus be invariant under the gauge transformation $h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \Delta h_{\mu\nu}(x)$, where

$$\Delta h_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x), \tag{2.3}$$

with the unperturbed Robertson-Walker metric $\bar{g}_{\mu\nu}(x)$ left unchanged, and corresponding gauge transformations of other perturbations. To first order in $\epsilon(x)$ and $h_{\mu\nu}(x)$, Eq. (2.3) is

$$\begin{aligned}\Delta h_{\mu\nu}(x) &= g'_{\mu\nu}(x') - \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda}\epsilon^\lambda(x) - g_{\mu\nu}(x)\\ &= -\bar{g}_{\lambda\mu}(x)\frac{\partial \epsilon^\lambda(x)}{\partial x^\nu} - \bar{g}_{\lambda\nu}(x)\frac{\partial \epsilon^\lambda(x)}{\partial x^\mu} - \frac{\partial \bar{g}_{\mu\nu}(x)}{\partial x^\lambda}\epsilon^\lambda(x),\end{aligned} \tag{2.4}$$

or in more detail

$$\Delta h_{ij} = -\frac{\partial \epsilon_i}{\partial x^j} - \frac{\partial \epsilon_j}{\partial x^i} + 2a\dot{a}\delta_{ij}\epsilon_0, \tag{2.5}$$

$$\Delta h_{i0} = -\frac{\partial \epsilon_i}{\partial t} - \frac{\partial \epsilon_0}{\partial x^i} + 2\frac{\dot{a}}{a}\epsilon_i, \tag{2.6}$$

$$\Delta h_{00} = -2\frac{\partial \epsilon_0}{\partial t}, \tag{2.7}$$

with all quantities evaluated at the same spacetime *coordinate* point, and indices now raised and lowered with the Robertson-Walker matric, so that $\epsilon_0 = -\epsilon^0$ and $\epsilon_i = a^2\epsilon^i$. The field equation will be invariant only if the same gauge transformation is applied to all tensors, and in particular to the energy-momentum tensor, so that we must transform $\delta T_{\mu\nu}(x) \to \delta T_{\mu\nu}(x) + \Delta\delta T_{\mu\nu}(x)$, where $\Delta\delta T_{\mu\nu}$ is given by a formula[1] analogous to Eq. (2.4):

$$\Delta\delta T_{\mu\nu} = -\overline{T}_{\lambda\mu}(x)\frac{\partial\epsilon^\lambda(x)}{\partial x^\nu} - \overline{T}_{\lambda\nu}(x)\frac{\partial\epsilon^\lambda(x)}{\partial x^\mu} - \frac{\partial\overline{T}_{\mu\nu}(x)}{\partial x^\lambda}\epsilon^\lambda(x), \tag{2.8}$$

or more detail

$$\Delta\delta T_{ij} = -\overline{p}\left(\frac{\partial\epsilon_i}{\partial x^j} + \frac{\partial\epsilon_j}{\partial x^i}\right) + \frac{\partial}{\partial t}\left(a^2\overline{p}\right)\delta_{ij}\epsilon_0, \tag{2.9}$$

$$\Delta\delta T_{i0} = -\overline{p}\frac{\partial\epsilon_i}{\partial t} + \overline{\rho}\frac{\partial\epsilon_0}{\partial x^i} + 2\overline{p}\frac{\dot{a}}{a}\epsilon_i, \tag{2.10}$$

$$\Delta\delta T_{00} = 2\overline{\rho}\,\frac{\partial\epsilon_0}{\partial t} + \dot{\overline{\rho}}\epsilon_0. \tag{2.11}$$

Note that we use δ to signify a perturbation, while Δ here denotes the change in a perturbation associated with a gauge transformation.

In order to write these gauge transformations in terms of the scalar, vector, and tensor components introduced in Section 1, it is necessary to decompose the spatial part of ϵ^μ into the gradient of a spatial scalar plus a divergenceless vector:

$$\epsilon_i = \partial_i\epsilon^S + \epsilon_i^V, \quad \partial_i\epsilon_i^V = 0. \tag{2.12}$$

Then the transformations (2.5)–(2.7) and (2.9)–(2.11) give the gauge transformations of the metric perturbation components defined by Eqs. (1.35)–(1.37):

$$\begin{aligned}
\Delta A &= \frac{2\dot{a}}{a}\epsilon_0, \quad \Delta B = -\frac{2}{a^2}\epsilon^S, \\
\Delta C_i &= -\frac{1}{a^2}\epsilon_i^V, \quad \Delta D_{ij} = 0, \quad \Delta E = 2\dot{\epsilon}_0, \\
\Delta F &= \frac{1}{a}\left(-\epsilon_0 - \dot{\epsilon}^S + \frac{2\dot{a}}{a}\epsilon^S\right), \quad \Delta G_i = \frac{1}{a}\left(-\dot{\epsilon}_i^V + \frac{2\dot{a}}{a}\epsilon_i^V\right),
\end{aligned} \tag{2.13}$$

and of the perturbations (1.43)–(1.45) to the pressure, energy density, and velocity potential

$$\Delta\delta p = \dot{\overline{p}}\epsilon_0, \quad \Delta\delta\rho = \dot{\overline{\rho}}\epsilon_0, \quad \Delta\delta u = -\epsilon_0. \tag{2.14}$$

The other ingredients of the energy-momentum tensor are gauge invariant:

$$\Delta\pi^S = \Delta\pi_i^V = \Delta\pi_{ij}^T = \Delta\delta u_i^V = 0. \tag{2.15}$$

Note in particular that the conditions $\pi^S = \pi_i^V = \pi_{ij}^T = 0$ for a perfect fluid and the condition $\delta u_i^V = 0$ for potential (*i.e.* irrotational) flow are gauge invariant.

[1]The right-hand sides of Eqs. (2.4) and (2.8) are known as the *Lie derivatives* of the metric and energy-momentum tensor, respectively.

For the field equations to be gauge-invariant, similar transformations must of course be made on any other ingredients in these equations. For instance, any four-scalar $s(x)$ for which $s'(x') = s(x)$ under arbitrary four-dimensional coordinate transformations would undergo the change $\Delta\delta s(x) \equiv s'(x) - s(x) = s'(x) - s'(x')$, which to first order in perturbations is

$$\Delta\delta s(x) = s(x) - s(x') = -\frac{\partial\bar{s}(t)}{\partial x^\mu}\epsilon^\mu(x) = \dot{\bar{s}}(t)\epsilon_0. \tag{2.16}$$

This applies for instance to the number density n or a scalar field φ. For a perfect fluid both p and ρ are defined as scalars, and the gauge transformations in Eq. (2.14) of δp and $\delta\rho$ are other special cases of Eq. (2.16). Likewise, for a perfect fluid the gauge transformation in Eq. (2.14) of δu can be derived from the vector transformation law of u_μ. Because the gauge transformation properties of $\delta\rho$, δp, δu, etc. do not depend on the conservation laws, Eqs. (2.14) and (2.15) apply to each individual constituent of the universe in any case in which the energy-momentum tensor is a sum of terms for different constituents of the universe, even if these individual terms are not separately conserved.

We can eliminate the gauge degrees of freedom either by working only with gauge-invariant quantities, or by choosing a gauge. The tensor quantities π^T_{ij} and D_{ij} appears in Eq. (1.60) are already gauge invariant, and no gauge-fixing is necessary or possible. For the vector quantities π^V_i, δu^V_i, C_i and G_i, we can write Eqs. (1.57)–(1.59) in terms of the gauge invariant quantities π^V_i, δu^V_i and $\tilde{G}_i \equiv G_i - a\dot{C}_i$, or we can fix a gauge for these quantities by choosing ϵ^V_i so that either C_i or G_i vanishes. For the scalar perturbations it is somewhat more convenient to fix a gauge. There are several frequently considered possibilities.

Newtonian Gauge

Here we choose ϵ^S so that $B = 0$, and then choose ϵ_0 so that $F = 0$. Both choices are unique, so that after choosing Newtonian gauge, there no remaining freedom to make gauge transformations. It is conventional to write E and A in this gauge as

$$E \equiv 2\Phi, \qquad A \equiv -2\Psi, \tag{2.17}$$

so that (now considering only scalar perturbations) the perturbed metric has components

$$g_{00} = -1 - 2\Phi, \quad g_{0i} = 0, \quad g_{ij} = a^2(t)\,(1 - 2\Psi)\,\delta_{ij} \tag{2.18}$$

The gravitational field equations (1.51)–(1.54) then take the form

$$\begin{aligned}
-4\pi G a^2\left[\delta\rho - \delta p - \nabla^2\pi^S\right] &= a\dot{a}\dot{\Phi} + \left(4\dot{a}^2 + 2a\ddot{a}\right)\Phi - \nabla^2\Psi + a^2\ddot{\Psi} + 6a\dot{a}\dot{\Psi}, &(2.19)\\
-8\pi G a^2\partial_i\partial_j\pi^S &= \partial_i\partial_j\left[\Phi - \Psi\right], &(2.20)\\
4\pi G\left(\bar{\rho} + \bar{p}\right)\partial_i\delta u &= -\dot{a}\partial_i\Phi - a\partial_i\dot{\Psi}, &(2.21)\\
4\pi G\left(\delta\rho + 3\delta p + \nabla^2\pi^S\right) &= \frac{1}{a^2}\nabla^2\Phi + \frac{3\dot{a}}{a}\dot{\Phi} + 3\ddot{\Psi} + \frac{6\dot{a}}{a}\dot{\Psi} + \frac{6\ddot{a}}{a}\Phi, &(2.22)
\end{aligned}$$

and the equations (1.55)-(1.56) of momentum and energy conservation become (aside from modes of zero wave number)

$$\begin{aligned}
\delta p + & \nabla^2\pi^S + \partial_0\left[(\bar{\rho}+\bar{p})\,\delta u\right] + \frac{3\dot{a}}{a}(\bar{\rho}+\bar{p})\,\delta u + (\bar{\rho}+\bar{p})\,\Phi = 0, & (2.23)\\
\delta\rho + & \frac{3\dot{a}}{a}(\delta\rho+\delta p) + \nabla^2\left[a^{-2}(\bar{\rho}+\bar{p})\,\delta u + \frac{\dot{a}}{a}\pi^S\right] - 3(\bar{\rho}+\bar{p})\,\dot{\Psi} = 0. & (2.24)
\end{aligned}$$

In particular, Eq. (2.20) shows that Φ and Ψ are not physically independent fields; they differ only by a term arising from the anisotropic part of the stress tensor, and in particular they are equal for a perfect fluid, for which $\pi^S = 0$. The perturbation to the number density of a species of particle whose total number is conserved will satisfy relation (1.62), which in Newtonian gauge reads

$$\frac{\partial}{\partial t}\left(\frac{\delta n}{\bar{n}}\right) + \frac{1}{a^2}\nabla^2\delta u - 3\dot{\Psi} = 0. \tag{2.25}$$

Given an equation of state for p as a function of ρ (or, if p depends also on other quantities like n, then given also field equations for those quantities) and given also a formula for π^S as a linear combination of the other perturbations (such as for instance the formula $\pi^S = 0$ for a perfect fluid) we can regard Eqs. (2.21), (2.23), and (2.23) (and, where needed, Eq. (2.25)) as equations of motion for Ψ, δu, and $\delta\rho$, respectively, with Φ given in terms of Ψ by Eq. (2.20). The remaining equations provide a constraint on the solution of this coupled system of equations. By substracting $3/a^2$ times Eq. (2.19) from Eq. (2.22) and then using Eqs. (2.20) and (2.21) to eliminate π^S and Φ, we find that

$$a^2\delta\rho - 3Ha^3(\bar{\rho}+\bar{p})\,\delta u - \left(\frac{a}{4\pi G}\right)\nabla^2\Psi = 0. \tag{2.26}$$

This is a constraint rather than an equation of motion, because the equations of motion (2.19), (2.21), and (2.22) imply that the left-hand side of Eq. (2.26) is time-independent, so that Eq. (2.26) only has to be imposed as an initial condition.

Synchronous Gauge

Here we choose ϵ_0 so that $E = 0$, and then choose ϵ^S so that again $F = 0$. Considering only scalar perturbations, the complete perturbed metric is then

$$g_{00} = -1, \quad g_{0i} = 0, \quad g_{ij} = a^2\left[(1+A)\,\delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j}\right]. \tag{2.27}$$

In this gauge, the Einstein field equations (1.51)–(1.54) take the form

$$\begin{aligned}
-4\pi G a^2\left[\delta\rho - \delta p - \nabla^2\pi^S\right] &= \frac{1}{2}\nabla^2 A - \frac{1}{2}a^2\ddot{A} - 3a\dot{a}\dot{A} - \frac{1}{2}a\dot{a}\nabla^2\dot{B}, & (2.28)\\
-16\pi G a^2\pi^S &= A - a^2\ddot{B} - 3a\dot{a}\dot{B}, & (2.29)\\
8\pi G\, a(\bar{\rho}+\bar{p})\,\delta u &= a\dot{A}, & (2.30)\\
-4\pi G\left(\delta\rho + 3\delta p + \nabla^2\pi^S\right) &= \frac{3}{2}\ddot{A} + \frac{3\dot{a}}{a}\dot{A} + \frac{1}{2}\nabla^2\ddot{B} + \frac{\dot{a}}{a}\nabla^2\dot{B}, & (2.31)
\end{aligned}$$

and the equations (2.48) and (2.49) of momentum and energy conservation read

$$\delta p+\nabla^2\pi^S+\partial_0\left[(\bar\rho+\bar p)\,\delta u\right]+\frac{3\dot a}{a}(\bar\rho+\bar p)\,\delta u = 0, \tag{2.32}$$

$$\delta\dot\rho+\frac{3\dot a}{a}(\delta\rho+\delta p)+\nabla\left[a^{-2}(\bar\rho+\bar p)\,\delta u+\frac{\dot a}{a}\pi^S\right]+\frac{1}{2}(\bar\rho+\bar p)\,\partial_0\left[3A+\nabla^2 B\right] = 0. \tag{2.33}$$

Note that in this gauge the equation of momentum conservation, which furnished the equation of motion (the Navier–Stokes equation) for an imperfect fluid, does not depend at all on the pertubed metric, while the eqution of energy conservation may be written as

$$\delta\dot\rho+\frac{3\dot a}{a}(\delta\rho+\delta p)+\nabla^2\left[a^{-2}(\bar\rho+\bar p)\,\delta u+\frac{\dot a}{a}\pi^S\right]+(\bar\rho+\bar p)\,\psi=0 \tag{2.34}$$

where

$$\psi\equiv\frac{1}{2}\left[3\dot A+\nabla^2\dot B\right]=\frac{\partial}{\partial t}\left(\frac{h_{ii}}{2a^2}\right). \tag{2.35}$$

We need A and B separately to calculate the motion of individual particles, but the effect of gravitation on a perfect or imperfect fluid is entirely governed by the quantity ψ. Inspection of the field equation (2.31) shows that it provides a differential equation for just this combination of scalar fields:

$$-4\pi Ga^2\left(\delta\rho+3\delta p+\nabla^2\pi^S\right)=\frac{\partial}{\partial t}\left(a^2\psi\right) \tag{2.36}$$

Also, in synchronous gauge the equation (1.62) for particle conservation takes the form

$$\frac{\partial}{\partial t}\left(\frac{\delta n}{\bar n}\right)+a^{-2}\nabla^2\delta u+\psi=0. \tag{2.37}$$

Given an equation of state for p as a function of ρ (and perhaps n) and a formula expressing π^S as a linear combination of the other scalar perturbations we can use Eqs. (2.32), (2.34), (2.36) (and perhaps Eq. (2.37)) to find solutions for the three independent perturbations δu, $\delta\rho$, and ψ, respectively. The left-over equations (2.28)–(2.30) are not needed, for a reason given in Section 1: the full set of equations (2.28)–(2.33) are not independent, because the equations of energy and momentum conservation can be derived from the Einstein field equations.

If we need to know A and B separately we can find them from ψ and $\delta\rho$. By adding 3 times Eq. (2.28), plus $1/2$ the Laplacian of Eq. (2.29), plus a^2 times Eq. (2.31), we obtain the simple relation

$$\nabla^2 A=-8\pi Ga^2\delta\rho+2Ha^2\psi, \tag{2.38}$$

where as usual $H\equiv\dot a/a$. After A is found in this way, we can find B form A and ψ by solving Eq. (2.35).

We can see from Eq. (2.13) that E and F are not affected by a gauge transformation with

$$\epsilon_0(\mathbf{x},t)=-\tau(\mathbf{x}),\quad \epsilon^S(\mathbf{x},t)=a^2(t)\tau(\mathbf{x})\int a^{-2}(t)\mathrm{d}t, \tag{2.39}$$

where $\tau(\mathbf{x})$ is an arbitrary function of $\mathbf{x}$, but not of t. But under this transformation A and B do change

$$\Delta A=-\frac{2\dot a\tau}{a},\quad \Delta B=-2\tau\int a^{-2}(t)\mathrm{d}t. \tag{2.40}$$

In particular, the combination (2.35) undergoes the change

$$\Delta\psi = -3\tau\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{a}\right) - a^{-2}\nabla^2\tau. \tag{2.41}$$

Also, the changes in the perturbations to the energy density, pressure, and velocity potential are given by Eqs. (eq:gaugeall) and (2.39) as

$$\Delta\delta p = -\dot{\bar{p}}\tau, \quad \Delta\delta\rho = -\dot{\bar{\rho}}\tau, \quad \Delta\delta u = \tau, \tag{2.42}$$

while π^S is invariant. The same transformation rules apply for any one of the individual constituents of the universe. Any scalar perturbation δs such as the number density perturbation δn or a scalar field perturbation $\delta\varphi$ undergoes a change like that of the pressure and density perturbations:

$$\Delta\delta s = -\dot{\bar{s}}\tau. \tag{2.43}$$

We can check that all of the equations (2.28)–(2.34) and (2.37) are invariant under these residual gauge transformations. This being the case, for any solution ψ, δp, δu, δn, etc. of these equations there will be another solution $\psi + \Delta\psi$, $\delta p + \Delta\delta p$, $\delta\rho + \Delta\delta\rho$, $\delta u + \Delta\delta u$, $n + \Delta\delta n$, etc. and since the field equations are linear, this means that $\Delta\psi$, $\Delta\delta p$, $\Delta\delta\rho$, $\Delta\delta u$, $\Delta\delta n$, etc. is also a solution. (For this solution there is no scalar anisotropic inertia, because π^S is gauge invariant.)

Newtonian–Synchronous Gauge Conversion

We will find it convenient to do calculations using Newtonian gauge in some eras, and synchronous gauge in others. To connect results for different eras, we need to be able to convert them from one gauge to another.

Suppose first that we begin in Newtonian gauge, and make an infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$, with $\epsilon_i = \partial_i\epsilon^S$ so as not to induce vector perturbations. According to Eqs. (2.13) and (2.17), in order to give E the synchronous gauge value $E = 0$, we need

$$\dot{\epsilon}_0 = -\Phi \tag{2.44}$$

Then according to Eq. (2.13), to keep $F = 0$ we need

$$\frac{\partial}{\partial t}\left(\frac{\epsilon^S}{a^2}\right) = -\frac{\epsilon_0}{a^2}. \tag{2.45}$$

The A and B components of the spatial metric in synchronous gauge are given by Eqs. (2.13) and (2.17):

$$A = -2\Psi + 2H\epsilon_0, \quad B = -\frac{2}{a^2}\epsilon^S, \tag{2.46}$$

where $H \equiv \dot{a}/a$. In particular, the field of ψ is

$$\psi = -3\dot{\Psi} + 3\frac{\partial}{\partial t}(H\epsilon_0) + \frac{\nabla^2\epsilon_0}{a^2}. \tag{2.47}$$

Also, Eq. (2.14) allows us to calculate the synchronous gauge pressure perturbation δp^s, energy density perturbation $\delta\rho^s$, and velocity potential δu^s from the corresponding quantities δp, $\delta\rho$, and δu in Newtonian gauge:

$$\delta p^s = \delta p + \epsilon\dot{\bar{p}}, \quad \delta\rho^s = \delta\rho + \epsilon_0\dot{\bar{\rho}}, \quad \delta u^s = \delta u - \epsilon_0. \tag{2.48}$$

Given Φ we can calculate ϵ_0 from Eq. (2.44), and then given Ψ we can obtain ψ from Eq. (2.47) and the synchronous gauge pressure, energy density, and velocity potential perturbations from Eq. (2.48). The quantity ϵ_0 is determined by Eq. (2.44) only up to a time-dependent function of position, so the values of the synchronous gauge quantities A, B, ψ, $\tilde{p}$, $\tilde{\rho}$, and $\delta\tilde{u}$ are only determined up to a residual gauge transformation (2.40)–(2.42).

Next suppose that we begin in synchronous gauge, with metric fields A and B, and want to convert to Newtonian gauge. According to Eq. (2.13), to make g_{ij} proportional to δ_{ij} we need to take

$$\epsilon^S = a^2 B/2. \tag{2.49}$$

Then to keep $g_{i0} = 0$, Eq. (2.13) tells us that we must take

$$\epsilon_0 = -a^2\dot{B}/2. \tag{2.50}$$

Using Eq. (2.13) again together with the definitions (2.18), we have then

$$\Phi = \dot{\epsilon}_0 = -\frac{1}{2}\frac{\partial}{\partial t}\left(a^2\dot{B}\right), \tag{2.51}$$

$$\Psi = -\frac{1}{2}\left(A + \frac{2\dot{a}}{a}\epsilon_0\right) = -\frac{1}{2}A + \frac{a\dot{a}}{2}\dot{B}. \tag{2.52}$$

In contrast to the previous case, Eqs. (2.51) and (2.52) give Φ and Ψ uniquely. Not only that – it is easy to see that the results for Φ and Ψ are unaffected if the A and B with which we start are subjected to the residual gauge transformation (2.40).

Other Gauges

In choosing a gauge, it is not necessary to impose conditions only on the scalar fields appearing in the metric tensor. Instead, some of the gauge conditions can impose constraints on the scalars appearing in the energy-momentum tensor. For instance, in *co-moving gauge* we choose ϵ_0 so that $\delta u = 0$ (which for scalar perturbations makes the velocity perturbation δu^i vanish). Where the only "matter" is a single scalar field, as in popular theories of inflation, this mean that the time coordinate is defined so that at any given time the scalar field equals its unperturbed value, with all perturbations relegated to components of the metric. In the *constant density gauge* we choose ϵ_0 so that $\delta\rho = 0$. In either case, after fixing ϵ_0 we can make F vanish with a suitable choice of ϵ^S, so that the scalar perturbations still have $g_{i0} = 0$. Note that although this procedure fixes ϵ_0, it only fixes ϵ^S up to terms of the form $a^2(t)\tau(\mathbf{x})$, so these gauges share the drawback of synchronous gauge, of leaving a residual gauge symmetry.

3 Random Cosmic Fields and their Statistical Description

In this section we succinctly recall current ideas about the physical origin of stochasticity in cosmic fields in different cosmological scenarios. We then present the statistical tools that are commonly used to describe random cosmic fields such as power spectra, probability distribution functions, moments and cumulants, and give some mathematical properties of interest.

The Need for a Statistical Approach

The current explanation of the large-scale structure of the universe is that the present distribution of matter on cosmological scales results from the growth of primordial, small, seed fluctuations on an otherwise homogeneous universe amplified by gravitational instability. Tests of cosmological theories which characterize these primordial seeds are not deterministic in nature but rather statistical, for the following reasons. First, we do not have direct observational access to primordial fluctuations (which would provide definite initial conditions for the deterministic evolution equations). In addition, the time-scale for cosmological evolution is so much longer than that over which we can make observations, that is not possible to follow the evolution of single systems. In other words, what we observe through our the past light cone is different objects at different times of their evolution, therefore testing the evolution of structure must be done statistically.

The observable universe is thus modeled as a stochastic realization of a statistical ensemble of possibilities. The goal is to make statistical predictions, which in turn depend on the statistical properties of the primordial perturbations leading to the formation of large-scale structures. Among the two classes of models that have emerged to explain the large-scale structure of the universe, the physical origin of stochasticity can be quite different and thus give rise to very different predictions.

Correlation Functions and Power Spectra

From now on, we consider a cosmic scalar field whose statistical properties we want to describe. This field can either be the cosmic density field, $\delta(\mathbf{x})$, the cosmic gravitational potential, the velocity divergence field, or any other field of interest.

A random field is called *statistically homogeneous*[2] if all the joint multipoint *probability distribution functions* $p(\delta_1, \delta_2, \dots)$ or its *moments*, ensemble averages of local density products, remain the same under translation of the coordinates $\mathbf{x}_1, \mathbf{x}_2, \dots$ in space(here $\delta_i \equiv \delta(\mathbf{x}_i)$). Thus the probabilities depend only on the relative positions. A stochastic field is called *statistically isotropic* if $p(\delta_1, \delta_2, \dots)$ is invariant under spatial rotations. We will assume that cosmic fields are statistically homogeneous and isotropic, as predicted by most cosmological theories. The validity of this assumption can and should be tested against the observational data.

The two-point correlation function is defined as the joint ensemble average of the density at two different locations,

$$\xi(r) = \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r}) \rangle, \tag{3.1}$$

which depends only on the norm of $\mathbf{r}$ due to statistical homogeneity and isotropy. The density contrast

[2] This is in contrast with a *homogeneous* field, which takes the same value everywhere in space.

$\delta(\mathbf{x})$ is usually written in terms of its Fourier components,

$$\delta(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\delta(\mathbf{k})\,\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}). \tag{3.2}$$

The quantities $\delta(\mathbf{k})$ are then complex random variables. As $\delta(\mathbf{x})$ is real, it follows that

$$\delta(\mathbf{k}) = \delta^*(-\mathbf{k}). \tag{3.3}$$

The density field is therefore determined entirely by the statistical properties of the random variable $\delta(\mathbf{k})$. We can compute the correlators in Fourier space,

$$\langle\delta(\mathbf{k})\delta(\mathbf{k}')\rangle = \int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{r}\,\langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle\,\exp[-\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x} - \mathrm{i}\mathbf{k}'\cdot\mathbf{r}] \tag{3.4}$$

which gives,

$$\begin{aligned}\langle\delta(\mathbf{k})\delta(\mathbf{k}')\rangle &= \int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{r}\,\xi(r)\,\exp[-\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x} - \mathrm{i}\mathbf{k}'\cdot\mathbf{r}] \\ &= (2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}')\int \mathrm{d}^3\mathbf{r}\,\xi(r)\,\exp(-\mathrm{i}\mathbf{k}'\cdot\mathbf{r}) \\ &\equiv (2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}')\,P(k),\end{aligned} \tag{3.5}$$

where $P(k)$ is by definition the density *power spectrum*. The inverse relation between two-point correlation function and power spectrum thus reads

$$\xi(r) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,P(k)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}). \tag{3.6}$$

There are basically two conventions in the literature regarding the definition of the power spectrum, which differ by a factor of $(2\pi)^3$. In this lecture we use the convention

$$f(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\tilde{f}(\mathbf{k})\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}). \tag{3.7}$$

and

$$\tilde{f}(\mathbf{k}) = \int \mathrm{d}^3\mathbf{x}\,f(\mathbf{x})\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}). \tag{3.8}$$

Another popular choice is to reverse the role of $(2\pi)^3$ factors in the Fourier transforms, i.e. $\delta(\mathbf{k}) \equiv \int \mathrm{d}^3\mathbf{r}/(2\pi)^3\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r})\delta(\mathbf{r})$, and then modify Eq. (3.5) to read $\langle\delta(\mathbf{k})\delta(\mathbf{k}')\rangle \equiv \delta_D(\mathbf{k}+\mathbf{k}')\,P(k)$, which leads to $4\pi k^3 P(k)$ being the contribution per logarithmic wavenumber to the variance, rather than $k^3P(k)/(2\pi^2)$ as in our case.

The Wick Theorem for Gaussian Fields

The power spectrum is a well defined quantity for almost all homogeneous random fields. This concept becomes however extremely fruitful when one considers a *Gaussian* field. It means that any joint distribution of local densities is Gaussian distributed. Any ensemble average of product of variables can then be obtained by product of ensemble averages of pairs. We write explicitly this property for the Fourier modes as it will be used extensively in this work,

$$\langle \delta(\mathbf{k}_1)\ldots\delta(\mathbf{k}_{2p+1})\rangle = 0 \tag{3.9}$$

$$\langle \delta(\mathbf{k}_1)\ldots\delta(\mathbf{k}_{2p})\rangle = \sum_{\text{all pair associations}} \prod_{p \text{ pairs (i,j)}} \langle \delta(\mathbf{k}_i)\delta(\mathbf{k}_j)\rangle \tag{3.10}$$

This is the *Wick theorem*, a fundamental theorem for classic and quantum field theories.

The statistical properties of the random variables $\delta(\mathbf{k})$ are then entirely determined by the shape and normalization of $P(k)$. A specific cosmological model will eventually be determined e.g. by the power spectrum in the linear regime, by Ω_m and Ω_Λ only as long as one is only interested in the dark matter behavior.

As mentioned in the previous section, in the case of an inflationary scenario the initial energy fluctuations are expected to be distributed as a Gaussian random field. This is a consequence of the commutation rules given by

$$\left[a_{\mathbf{k}}, a^\dagger_{-\mathbf{k}'}\right] = (2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}'), \tag{3.11}$$

for the creation and annihilation operators for a free quantum field. They imply that

$$\left[\left(a_{\mathbf{k}} + a^\dagger_{-\mathbf{k}}\right), \left(a_{\mathbf{k}'} + a^\dagger_{-\mathbf{k}'}\right)\right] = (2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}'). \tag{3.12}$$

As a consequence of this, the relations in Eqs. (3.9)-(3.10) are verified for $\varphi_{\mathbf{k}}$ for all modes that exit the Hubble radius, which long afterwards come back in as classical stochastic perturbations. These properties obviously apply also to any quantities linearly related to $\varphi_{\mathbf{k}}$.

Higher-Order Correlators: Diagrammatics

In general it is possible to define higher-order correlation functions. They are defined as the *connected* part (denoted with subscript c) of the joint ensemble average of the density in an arbitrarily number of locations. They can be formally written,

$$\begin{aligned}\xi_N(\mathbf{x}_1,\ldots,\mathbf{x}_N) &= \langle \delta(\mathbf{x}_1),\ldots,\delta(\mathbf{x}_N)\rangle_c \\ &\equiv \langle \delta(\mathbf{x}_1),\ldots,\delta(\mathbf{x}_N)\rangle - \sum_{\mathcal{S}\in\mathcal{P}(\{\mathbf{x}_1,\ldots,\mathbf{x}_n\})} \prod_{s_i\in\mathcal{S}} \xi_{\#s_i}(\mathbf{x}_{s_i(1)},\ldots,\mathbf{x}_{s_i(\#s_i)})\end{aligned} \tag{3.13}$$

where the sum is made over the proper partitions (any partition except the set itself) of $\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}$ and s_i is thus a subset of $\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}$ contained in partition $\mathcal{S}$. When the average of $\delta(\mathbf{x})$ is defined as zero, only partitions that contain no singlets contribute.

The decomposition in connected and non-connected parts can be easily visualized. It means that any ensemble average can be decomposed in a product of connected parts. They are defined for instance in Figure. 1. The tree-point moment is "written" in Figure. 2 and the four-point moment in Figure. 4.

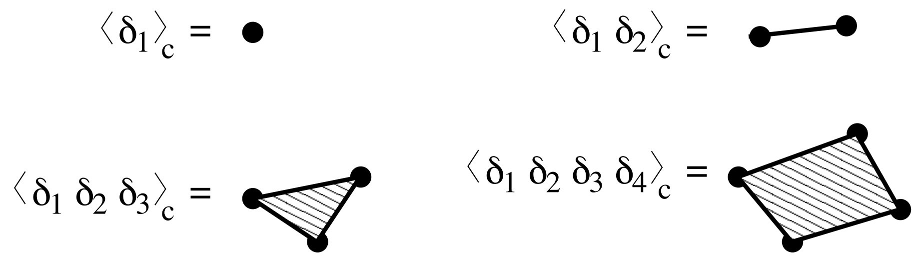

Figure 1: Representation of the connected part of the moments.

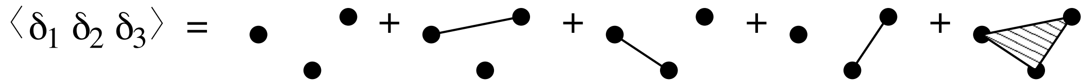

Figure 2: Writing of the three-point moment in terms of connected parts.

In case of a Gaussian field all connected correlation functions are zero except ξ_2. This is a consequence of Wick's theorem. As a result the only non-zero connected part is the two-point correlation function. An important consequence is that the statistical properties of any field, not necessarily linear, built from a Gaussian field δ can be written in terms of combinations of two-point functions of δ. Note that in a diagrammatic representation the connected moments of any of such field is represented by a *connected* graph. This is illustrated in Figure. 3 for the field $\delta = \phi^2$: the connected part of the 2-point function of this field is obtained by all the diagrams that explicitly join the two points. The other ones contribute to the moments, but not to its connected part.

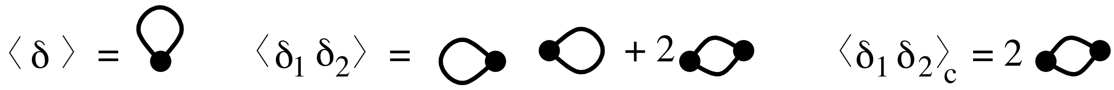

Figure 3: Disconnected and connected part of the two-point function of the field δ assuming it is given by $\delta = \phi^2$ with ϕ Gaussian.

The connected part has the important property that it vanishes when one or more points are separated by infinite separation. In addition, it provides a useful way of characterizing the statistical properties, since unlike unconnected correlation functions, each connected correlation provides independent information.

These definitions can be extended to Fourier space. Because of homogeneity of space $\langle\delta(\mathbf{k}_1)\ldots\delta(\mathbf{k}_N)\rangle_c$ is always proportional to $\delta_D(\mathbf{k}_1 + \cdots + \mathbf{k}_N)$. Then we can define $P_N(\mathbf{k}_1, \ldots, \mathbf{k}_N)$ with

$$\langle\delta(\mathbf{k}_1)\ldots\delta(\mathbf{k}_N)\rangle_c = \delta_D(\mathbf{k}_1 + \cdots + \mathbf{k}_N)\, P_N(\mathbf{k}_1, \ldots, \mathbf{k}_N). \tag{3.14}$$

One particular case that will be discussed in the following is for $n = 3$, the bispectrum, which is usually denoted by $B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$.

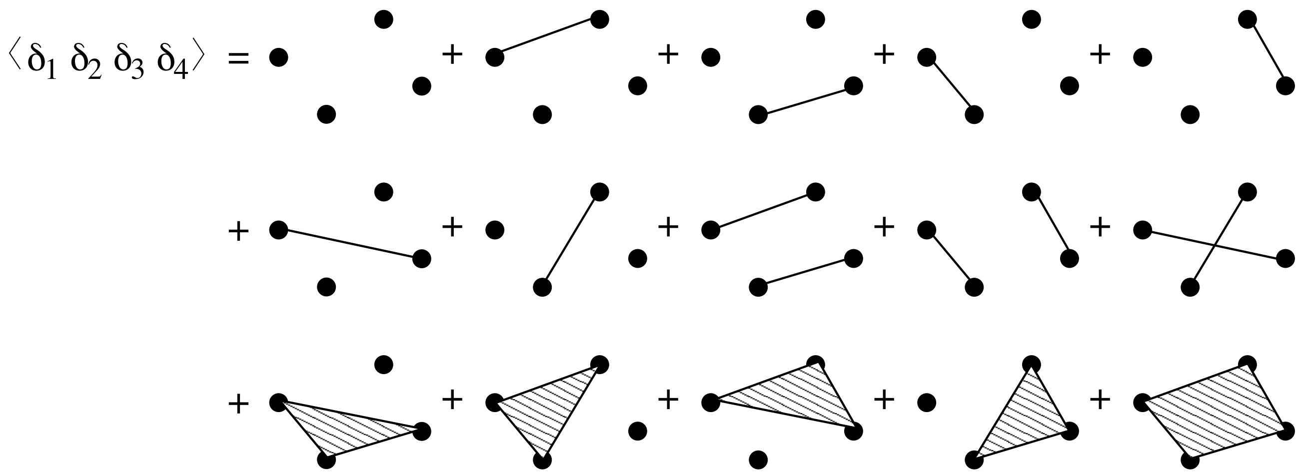

Figure 4: Writing of the three-point moment in terms of connected parts.

Probabilities and Correlation Functions

Correlation functions are directly related to the multi-point probability function, in fact they can be defined from them. Here we illustrate this for the case of the density field, as these results are frequently used in the literature. The physical interpretation of the two-point correlation function is that it measures the excess over random probability that two particles at volume elements $\mathrm{d}V_1$ and $\mathrm{d}V_2$ are separated by distance $x_{12} \equiv |\mathbf{x}_1 - \mathbf{x}_2|$,

$$\mathrm{d}P_{12} = n^2[1 + \xi(x_{12})]\mathrm{d}V_1\mathrm{d}V_2, \tag{3.15}$$

where n is the mean density. If there is no clustering (random distribution), $\xi = 0$ and the probability of having a pair of particles is just given by the mean density squared, independently of distance. Since the probability of having a particle in $\mathrm{d}V_1$ is $n\mathrm{d}V_1$, the conditional probability that there is a particle at $\mathrm{d}V_2$ given that there is one at $\mathrm{d}V_1$ is

$$\mathrm{d}P(2|1) = n[1 + \xi(x_{12})]\mathrm{d}V_2. \tag{3.16}$$

The nature of clustering is clear from this expression; if objects are clustered ($\xi(x_{12}) > 0$), then the conditional probability is enhanced, whereas if objects are anti-correlated ($\xi(x_{12}) < 0$) the conditional probability is suppressed over the random distribution case, as expected. Similarly to Eq. (3.15), for the three-point case the probability of having three objects is given by

$$\mathrm{d}P_{123} = n^3[1 + \xi(x_{12}) + \xi(x_{23}) + \xi(x_{31}) + \xi_3(x_{12}, x_{23}, x_{31})]\mathrm{d}V_1\mathrm{d}V_2\mathrm{d}V_3, \tag{3.17}$$

where ξ_3 denotes the three-point (connected) correlation function. If the density field were Gaussian, $\xi_3 = 0$, and all probabilities are determined by $\xi(r)$ alone. Analogous results hold for higher-order correlations.

Moments, Cumulants and their Generating Functions

One particular case for Eq. (3.13) is when all points are at the same location. Because of statistical homogeneity $\xi_p(\mathbf{x}, \ldots, \mathbf{x})$ is independent on the position $\mathbf{x}$ and it reduces to the cumulants of the one-point density probability distribution functions, $\langle \delta^p \rangle_c$. The relation (3.13) tells us also how the cumulants are related to the moments $\langle \delta^p \rangle$. For convenience we write here the first few terms,

$$\begin{aligned}
\langle \delta \rangle_c &= \langle \delta \rangle \\
\langle \delta^2 \rangle_c &= \sigma^2 = \langle \delta^2 \rangle - \langle \delta \rangle_c^2 \\
\langle \delta^3 \rangle_c &= \langle \delta^3 \rangle - 3\langle \delta^2 \rangle_c \langle \delta \rangle_c - \langle \delta \rangle_c^3 \\
\langle \delta^4 \rangle_c &= \langle \delta^4 \rangle - 4\langle \delta^3 \rangle_c \langle \delta \rangle_c - 3\langle \delta^2 \rangle_c^2 - 6\langle \delta^2 \rangle_c \langle \delta \rangle_c^2 - \langle \delta \rangle_c^4 \\
\langle \delta^5 \rangle_c &= \langle \delta^5 \rangle - 5\langle \delta^4 \rangle_c \langle \delta \rangle_c - 10\langle \delta^3 \rangle_c \langle \delta^2 \rangle_c - 10\langle \delta^3 \rangle_c \langle \delta \rangle_c^2 - 15\langle \delta^2 \rangle_c^2 \langle \delta \rangle_c \\
&\quad -10\langle \delta^2 \rangle_c \langle \delta \rangle_c^3 - \langle \delta \rangle_c^5
\end{aligned} \tag{3.18}$$

In most cases $\langle \delta \rangle = 0$ and the above equations simplify considerably. In the following we usually denote σ^2 the local second order cumulant. The Wick theorem then implies that in case of a Gaussian field σ^2 is the only non-vanishing cumulant.

It is important to note that the local PDF is essentially characterized by its *cumulants* which constitute a set of independent quantities. This is important since in most of applications that follow the higher-order cumulants are small compared to their associated moments. Finally, let's note that a useful mathematical property of cumulants is that $\langle (b\delta)^n \rangle_c = b^n \langle \delta^n \rangle_c$, and $\langle (b + \delta)^n \rangle_c = \langle \delta^n \rangle_c$ where b is an ordinary number.

The density distribution is usually smoothed with a filter W_R of a given size, R, commonly a top-hat or a Gaussian window. Indeed, this is required by the discrete nature of galaxy catalogs and N-body experiments used to simulate them. Moreover, we shall see later that the scale-free nature of gravitational clustering implies some remarkable properties about the scaling behavior of the smoothed density distribution. The quantities of interest are then the moments $\langle \delta_R^p \rangle$ and the cumulants $\langle \delta_R^p \rangle_c$ of the smoothed density field

$$\delta_R(\mathbf{x}) = \int W_R(\mathbf{x}' - \mathbf{x})\delta(\mathbf{x}')\mathrm{d}^3\mathbf{x}'. \tag{3.19}$$

Note that for the top hat window,

$$\langle \delta_R^p \rangle_c = \int_{v_R} \xi_p(\mathbf{x}_1, \ldots, \mathbf{x}_p) \frac{\mathrm{d}^{\mathcal{D}}\mathbf{x}_1 \ldots \mathrm{d}^{\mathcal{D}}\mathbf{x}_p}{v_R^p} \tag{3.20}$$

(where $\mathcal{D} = 2$ or 3 is the dimension of the field) is nothing but the average of the N-point correlation function over the corresponding cell of volume v_R.

Power Spectrum Evolution in Linear Perturbation Theory

The simplest (trivial) application of the gravitational perturbation theory is the leading order contribution to the evolution of the power spectrum. Since we are dealing with the two-point function in Fourier space ($N = 2$), only linear theory is required, that is, the connected part is just given by a single line joining the two points.

In this lecture we are concerned about time evolution of the cosmic fields during the matter domination epoch. In this case, as we discussed previously, diffusion effects are negligible and the evolution can be cast in terms of perfect fluid equations that describe conservation of mass and momentum. In this case, the evolution of the density field is given by a simple time-dependent scaling of the "linear" power spectrum

$$P(k, \tau) = [D_1^{(+)}(\tau)]^2 \, P_L(k) \tag{3.21}$$

where $D_1^{(+)}(\tau)$ is the growing part of the linear growth factor. One must note, however, that the "linear" power spectrum specified by $P_L(k)$[3] derives from the linear evolution of density fluctuations through the radiation domination era and the resulting decoupling of matter from radiation. This evolution must be followed by using general relativistic Boltzmann numerical codes, although analytic techniques can be used to understand quantitatively the results. The end result is that

$$P_L(k) \propto k^{n_s} \, T^2(k) \tag{3.22}$$

where n_s is the primordial spectral index ($n_s = 1$ denotes the canonical scale-invariant spectrum[4]), $T(k)$ is the transfer function that describes the evolution of the density field perturbations through decoupling ($T(0) \equiv 1$). It depends on cosmological parameters in a complicated way, although in simple cases (where the baryonic content is negligible) it can be approximated by a fitting function that depends on the shape parameter $\Gamma \equiv \Omega_m h$. For the adiabatic cold dark matter (CDM) scenario, $T^2(k) \to \ln^2(k)/k^4$ as $k \to \infty$, due to the suppression of fluctuations growth during the radiation dominated era.

[3] We denote the linear power spectrum interchangeably by $P_L(k)$ or by $P^{(0)}(k)$.

[4] This corresponds to fluctuations in the gravitational potential at the Hubble radius scale that have the same amplitude for all modes, i.e. the gravitational potential has a power spectrum $P_\varphi \sim k^{-3}$, as predicted by inflationary models.

Exercise

1. Prove Eq. (1.4). Why does a minus sign appear?

2. Show that for a Gaussian random field, δ, and an integer $n \geq 1$,

$$\begin{aligned}\langle \delta^{2n} \rangle &\equiv \frac{\int_{-\infty}^{\infty} \mathrm{d}\delta\, \delta^{2n} \exp\left(-\frac{1}{2}a\delta^2\right)}{\int_{-\infty}^{\infty} \mathrm{d}\delta\, \exp\left(-\frac{1}{2}a\delta^2\right)} \\ &= \frac{1}{a^n}(2n-1)(2n-3)\ldots 5 \cdot 3 \cdot 1 \\ &= (2n-1)(2n-3)\ldots 5 \cdot 3 \cdot 1\, \langle \delta^2 \rangle^n,\end{aligned}$$

   and

$$\langle \delta^{2n-1} \rangle \equiv \frac{\int_{-\infty}^{\infty} \mathrm{d}\delta\, \delta^{2n-1} \exp\left(-\frac{1}{2}a\delta^2\right)}{\int_{-\infty}^{\infty} \mathrm{d}\delta\, \exp\left(-\frac{1}{2}a\delta^2\right)} = 0.$$

   Hence, justify the validity of the Wick theorem.

3. Write down the diagrammatic connecting parts for 5-point connected correlation functions. Then verify with the last equation in (3.18).

4. The top-hat window function in 3-dimensional space is given by

$$f(\mathbf{r}) = \begin{cases} \dfrac{3}{4\pi R^3}, & \text{if } \mathbf{r} \leq R, \\ 0, & \text{otherwise .} \end{cases}$$

   Show that the Fourier transform of the top-hat window function is given by

$$W(kR) = 3j_1(kR) = 3\frac{\sin(kR) - (kR)\cos(kR)}{(kR)^3}.$$

   $j_\nu(kR)$ is the spherical Bessel function of the first kind.