



Qualifying Examination



Round 1/2564 (1/2021)

27 August 2021

Name:.....

Student ID:.....

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 8 questions. (Each questions is for 10 points/marks)
2. Write down detailed and clear solution to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	
7	
8	

Name:.....

1 Vector Calculus

Question 1. The fluid has its density function in the form of $\rho(x, y, z, t)$ moving with velocity $\vec{v}$ and it doesn't increase or decrease its mass. Show that

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad (1)$$

where $\vec{J} = \rho \vec{v}$ (Hint: Find the fluid continuity equation). [10 marks]

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2 Linear Algebra

Question 2. Solve the system of equations

$$x - z = 5 \quad (1)$$

$$-2x + 3y = 1 \quad (2)$$

$$x - 3y + 2z = -10 \quad (3)$$

Hint: Use matrix multiplication $Mr = k$, so $r = M^{-1}k$ and, $M^{-1} = \frac{1}{\det M} C^T$ where $C_{ij} = \text{cofactor of } m_{ij}$. [10 marks]

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3 Group Theory/Lie Algebra

Question 3. Given $U \in SU(N)$, where $SU(N)$ is a group of the unitary $N \times N$ matrices, i.e., $U^\dagger U = I$ and $\det U = 1$.

3.1) Consider a transformation of N -vector valued field Ψ under $SU(N)$ such that $\Psi' = U\Psi$. Show that $\Psi^\dagger \Psi$ is invariant under an $SU(N)$ transformation. [2 marks]

3.2) The element U can be expressed as an exponential

$$U(\lambda_1, \lambda_2, \dots, \lambda_{N^2-1}) = e^{i \sum_{j=1}^{N^2-1} \lambda_j G_j},$$

where λ_j are real-valued and continuous, and G_j are the generators of the group. Show that, in order to make U unitary, all generators must be Hermitian [4 marks].

3.3) Show that, in order to make U possessing unit determinant, all generators must be traceless. [2 marks]

3.4) Given the relation $U = U_2^{-1} U_1^{-1} U_2 U_1 \in SU(N)$. Show that all generators form the Lie-algebra $[T_a, T_b] = i f_{abc} T_c$, where $f_{abc} \in \mathbb{R}$ are anti-symmetric structure constant. [2 marks]

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4 Group Theory/Lie Algebra

Question 4. Give $u \in SU(2)$, where

$$\begin{pmatrix} a & b \\ -b & a^* \end{pmatrix} \quad (1)$$

with $|a|^2 + |b|^2 = 1$.

4.1) Show that the manifold associated with $SU(2)$ is 3-sphere S^3 . [2 marks]

4.2) Show that $SU(2)$ possesses a set of generators is $\{\sigma_1, \sigma_2, \sigma_3\}$, where σ_j are the Pauli matrices given by [2 marks]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

4.3) Show that, with a given set of generators in 4.2), one can construct any element $u \in SU(2)$. [6 marks]

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5 Function of Complex Variables

Question 5. We consider the Feynman propagator Δ

$$\Delta(x - y) = \langle 0 | \mathbf{T} \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle , \quad (1)$$

where $\mathbf{T}$ is time-ordering operator such that

$$\mathbf{T} \hat{\phi}(x) \hat{\phi}(y) = \theta(x^0 - y^0) \hat{\phi}(x) \hat{\phi}(y) + \theta(y^0 - x^0) \hat{\phi}(y) \hat{\phi}(x) , \quad (2)$$

and $\theta(t)$ is the Heaviside function such that

$$\theta(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} . \quad (3)$$

Here $\hat{\phi}(x)$ is the field operator given by

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (\hat{a}_{\mathbf{p}} e^{-ip \cdot x} + \hat{a}_{\mathbf{p}}^\dagger e^{ip \cdot x}) , \quad \text{where } \omega_p = |\sqrt{\mathbf{p}^2 + m^2}| , \quad (4)$$

and $\hat{a}_{\mathbf{p}}^\dagger$ and $\hat{a}_{\mathbf{p}}$ are creation and annihilation operators, respectively. Using the Wightman function

$$D(x - y) = \langle 0 | \hat{\phi}(x) \hat{\phi}(y) \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip(x-y)} , \quad (5)$$

the Feynman propagator (1) can be expressed as

$$\Delta(x - y) = \theta(x^0 - y^0) D(x - y) + \theta(y^0 - x^0) D(y - x) . \quad (6)$$

Show that in the momentum-energy space, the Feynman propagator $\Delta(p)$ given by

$$\Delta(p) = \frac{1}{p^2 - m^2 + i\epsilon} , \quad (7)$$

where ϵ is an infinitesimal positive real number, is completely consistent with the space-time coordinate form (6) [10marks].

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Hint: The relation between $\Delta(x - y)$ and $\Delta(p)$ is given by

$$\Delta(x - y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \Delta(p) \ .$$

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6 Integral Transforms

Question 6. (6.1). Find the Fourier transform, $F(t)$, of the function $f(x) = e^{-K|x|}$, where $K > 0$ is a constant. [2 marks]

(6.2). Find the Fourier transform, $G(t)$, of the delta function $\delta(x)$. [2 marks]

(6.3). A function $f(x)$ and $f(\pm\infty) = 0$, the corresponding Fourier Transform is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx.$$

Suppose the higher order derivatives of the function and the corresponding Fourier transforms are $\frac{d^n f(x)}{dx^n}$ and $g_n(\omega)$, respectively, where n is positive integer.

Prove that

$$g_n(\omega) = (-i\omega)^n g(\omega).$$

Note that you need to use integration by part and present the first, second, third and n-th order derivatives as well as the corresponding conditions in the process. [6 marks]

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7 Integral Transforms

Question 7. Using the derivative form of Fourier transform to solve the following questions.

(7.1). A heat flow partial differential equation is given by

$$\frac{\partial \psi(x, t)}{\partial t} = a^2 \frac{\partial^2 \psi(x, t)}{\partial x^2},$$

where $\psi(x, t)$ is the temperature at position x and time t , solve $\psi(x, t)$. [5 marks]

(7.2). A 1-dimensional neutron diffusion equation with a source is

$$-D \frac{d^2 \psi(x)}{dx^2} + K^2 D \psi(x) = Q \delta(x),$$

where $\psi(x)$ is the neutron flux, $\delta(x)$ is the delta function, and D and K^2 are constants, solve $\psi(x)$. [5 marks]

hint (7.1) and (7.2): One may use the results in the Question 6. to solve the equation in the transform space, and transform your solution back to find the solutions.

hint (7.1): One need to use the Gaussian integral $\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2} dx$.

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8 Ordinary Differential Equations

Question 8. A linear, second-order, homogeneous ODE of the general form

$$y'' + P(x)y' + Q(x)y = 0,$$

where $y = y(x)$ and the prime means the derivative to x . Let y_1 and y_2 be two independent solutions, then the Wronskian is given by definition as

$$W = y_1 y_2' - y_1' y_2.$$

(8.1). Show that $W' = -P(x)W$. [3 marks]

(8.2). Show that the relation between y_1 and y_2 shall be

$$y_2(x) = y_1(x) \frac{y_2(b)}{y_1(b)} + y_1(x) W(a) \int_b^x \frac{e^{-\int_a^{x_2} P(x_1) dx_1}}{[y_1(x_2)]^2} dx_2,$$

where a and b are arbitrary constants. [7 marks]

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